

On Fixed Point Free Maps of the Complex Flag Manifold

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§1. Introduction. Let $T^n \subset U_n$ be a maximal torus. The homogeneous space

$$F(n) = \frac{U_n}{T^n}$$

is a complex manifold of complex dimension $n(n - 1)/2$. Now $F(n)$ can be regarded as the space of orthogonal decompositions of \mathbf{C}^n into lines, so permutation of lines gives a free action of the symmetric group Σ_n on $F(n)$. Hence $F(n)$ admits a set of $n! - 1$ fixed-point-free maps, no two of which have a coincidence. In this paper we show that any set of $n!$ or more fixed-point-free maps on $F(n)$ contains a pair with a coincidence: in the language of §4, we show that $F(n)$ has noncoincidence index $n!$. The proof uses the Lefschetz fixed point and coincidence theorems, together with a classification of the endomorphisms of $H^*(F(n); \mathbf{Z})$.

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§2. The cohomology ring. As above, we regard $F(n)$ as the set of orthogonal decompositions of \mathbf{C}^n into lines. We evidently have n projection maps

$$\pi_i: F(n) \rightarrow \mathbf{C}P^{n-1}$$

given by taking the i^{th} line. Let $t \in H^2(\mathbf{C}P^{n-1}; \mathbf{Z})$ be the usual generator, and let $t_i = \pi_i^* t \in H^2(F(n); \mathbf{Z})$. Then the following result is well known (see [1]).

Theorem 2.1. *The cohomology ring of $F(n)$ is given by*

$$H^*(F(n); \mathbf{Z}) = \mathbf{Z}[t_1, \dots, t_n] / e_i(t_1, \dots, t_n) = 0, \quad 1 \leq i \leq n$$

where $e_i(t_1, \dots, t_n)$ is the i^{th} elementary symmetric function in t_1, \dots, t_n .

Remark. The free action of Σ_n on $F(n)$ mentioned in §1 evidently induces the action on $H^*(F(n); \mathbf{Z})$ which permutes the t_i .

Since t_i is pulled back from $H^2(\mathbf{C}P^{n-1}; \mathbf{Z})$, $t_i^n = 0$ for $1 \leq i \leq n$. In fact the only two-dimensional cohomology classes with this property are multiples of the

t_i , as we see in the next result. This theorem was first proved in [3]: for an independent and quite different proof, see [2].

Theorem 2.2. *Let $u \in H^2(F(n); \mathbf{Z})$ with $u^n = 0$. Then $u = at_i$ for some $a \in \mathbf{Z}$ and $1 \leq i \leq n$.*

Using Theorem 2.2, we can classify all endomorphisms of $H^*(F(n); \mathbf{Z})$.

Proposition 2.3. *Any endomorphism h of $H^*(F(n); \mathbf{Z})$ has the form*

$$h(t_i) = mt_{\sigma(i)}, \quad 1 \leq i \leq n$$

for some $m \in \mathbf{Z}$ and $\sigma \in \Sigma_n$.

Proof. Define a matrix $A = (A_{ij})_{1 \leq i, j \leq n}$ by

$$h(t_i) = \sum_j A_{ij} t_j.$$

Then since $h(t_i)^n = 0$, we know that at most one of the elements $A_{i1}, A_{i2}, \dots, A_{in}$ is nonzero: i.e., at most one entry in each row of A is nonzero. Now for $e_1 = e_1(t_1, \dots, t_n)$, we have

$$h(e_1) = me_1, \quad \text{some } m \in \mathbf{Z}.$$

But

$$h(e_1) = h\left(\sum_i t_i\right) = \sum_i \sum_j A_{ij} t_j = \sum_j \left(\sum_i A_{ij}\right) t_j$$

and thus

$$(1) \quad \sum_i A_{ij} = m \quad \text{for all } j,$$

i.e., all column sums of A are equal.

Now let

$$p_2 = \sum_i t_i^2.$$

Then since h must preserve the ideal generated by the symmetric functions,

$$h(p_2) = Rp_2 + Se_2 + \sum N_i t_i e_1$$

for some integers R , S , and N_i for $1 \leq i \leq n$. But

$$h(p_2) = \sum_i h(t_i^2) = \sum_i \sum_j A_{ij}^2 t_j^2 = \sum_j \left(\sum_i A_{ij}^2\right) t_j^2$$

so we have

$$\sum_i A_{ij}^2 = R + N_j \quad \text{for all } j$$

and

$$0 = S + N_i + N_j \quad \text{for all } i \neq j.$$

Since the case $n = 2$ is trivial, we can assume $n \geq 3$. Then $S = -N_1 - N_2$, and from the relations

$$0 = S + N_1 + N_j, \quad j > 2$$

we get $N_2 = N_3 = \dots = N_n$. But then

$$0 = S + N_2 + N_3$$

implies $N_3 = N_1$, so all the N_i are equal. Thus

$$\sum_i A_{ij}^2 = R + N_1$$

for all j . If $R + N_1 = 0$, then A is the zero matrix (and the conclusion holds with $m = 0$). If $R + N_1 > 0$, then A has at least one nonzero entry in each column. But A has at most one nonzero entry in each row, so A must have precisely one nonzero entry in each row and column. All these entries are m by (1), so $A = mP$ for a permutation matrix P . The conclusion then follows.

Remark. We shall denote the endomorphism $t_i \mapsto mt_{\sigma(i)}$ by h_m^σ .

§3. A formula for Lefschetz number. In this section we develop a formula for $L(h_m^\sigma)$, where

$$L(f^*) = \sum_{i \geq 0} (-1)^i \text{Tr}(f^* : H^i(F(n); \mathbf{Q}) \odot)$$

is the Lefschetz number. Since $H^*(F(n); \mathbf{Q}) = H^*(F(n); \mathbf{Z}) \otimes \mathbf{Q}$ is concentrated in even dimensions and is generated by the t_i ,

$$L(h_m^\sigma) = \sum_{i \geq 0} \text{Tr}(h_m^\sigma : H^{2i}(F(n); \mathbf{Q}) \odot) = \sum_{i \geq 0} m^i \text{Tr}(h_1^\sigma : H^{2i}(F(n); \mathbf{Q}) \odot)$$

so $L(h_m^\sigma)$ is a polynomial in m . Now

$$h_1^{\tau\sigma\tau^{-1}} = (h_1^\tau)(h_1^\sigma)(h_1^\tau)^{-1},$$

so $L(h_m^\sigma)$ evidently depends only on the conjugacy class of σ in Σ_n . The conjugacy class of σ is determined by its cycle-type

$$c(\sigma) = \lambda_1 \geq \lambda_2 \geq \dots,$$

where λ_1 is the length of the longest cycle in σ , λ_2 is the length of the next longest cycle, etc. ($c(\sigma)$ is a partition of n). Our formula for $L(h_m^\sigma)$ is

$$(2) \quad L(h_m^\sigma) = \frac{(1 - m)(1 - m^2) \dots (1 - m^n)}{(1 - m^{\lambda_1})(1 - m^{\lambda_2}) \dots}.$$

To prove (2), we begin with the polynomial ring

$$P = \mathbf{Q}[t_1, \dots, t_n].$$

For each $\sigma \in \Sigma_n$, there is an endomorphism h^σ of P given by

$$h^\sigma(t_i) = t_{\sigma(i)}.$$

Now give P a grading by assigning each t_i grade 2, so

$$P = \bigoplus_{i \geq 0} P_{2i}$$

and define a generating function

$$G(h^\sigma, x) = \sum_{i \geq 0} x^i \text{Tr}(h^\sigma : P_{2i} \rightarrow P_{2i}).$$

Proposition 3.1. For $\sigma \in \Sigma_n$ with cycle-type $c(\sigma) = \lambda_1 \geq \lambda_2 \geq \dots$,

$$G(h^\sigma, x) = \frac{1}{(1 - x^{\lambda_1})(1 - x^{\lambda_2}) \dots}.$$

Proof. Suppose first that $\sigma = (12 \dots n)$. Then the only monomials in P left fixed by h^σ are $1, t_1 t_2 \dots t_n, t_1^2 t_2^2 \dots t_n^2$, etc. Thus

$$G(h^\sigma, x) = 1 + x^n + x^{2n} + \dots = \frac{1}{1 - x^n}.$$

For the general case, write σ as a product of disjoint cycles, apply the special case to each cycle, and note that the generating functions are multiplicative on tensor products.

Now the spectral sequence of the fibration

$$F(n) \rightarrow BT^n \rightarrow BU_n$$

collapses for degree reasons, so

$$H^*(BT^n; \mathbf{Q}) = H^*(BU_n; \mathbf{Q}) \otimes H^*(F(n); \mathbf{Q})$$

additively. Since $H^*(BT^n; \mathbf{Q}) = P$ as a graded ring and $h^\sigma : P \rightarrow P$ restricts to the identity on

$$H^*(BU_n; \mathbf{Q}) = \mathbf{Q}[t_1, \dots, t_n]^{\Sigma_n} \subset P,$$

and gives rise to h_1^σ on the quotient

$$H^*(F(n); \mathbf{Q}) = P / e_i = 0, \quad 1 \leq i \leq n,$$

the additive tensor product decomposition above implies

$$G(h^\sigma, x) = G(\text{id}, x) G(h_1^\sigma, x)$$

for

$$G(\text{id}, x) = \sum_{i \geq 0} x^i \dim H^{2i}(BU_n; \mathbf{Q})$$

$$G(h_1^\sigma, x) = \sum_{i \geq 0} x^i \text{Tr}(h_1^\sigma : H^{2i}(F(n); \mathbf{Q}) \circ).$$

As is well known,

$$G(\text{id}, x) = \frac{1}{(1-x) \dots (1-x^n)}$$

so we have

$$\frac{1}{(1-x^{\lambda_1})(1-x^{\lambda_2}) \dots} = \frac{1}{(1-x) \dots (1-x^n)} G(h_1^\sigma, x)$$

or

$$(3) \quad G(h_1^\sigma, x) = \frac{(1-x) \dots (1-x^n)}{(1-x^{\lambda_1})(1-x^{\lambda_2}) \dots}$$

for $c(\sigma) = \lambda_1 \geq \lambda_2 \geq \dots$. We then have the desired result.

Proposition 3.2. *Formula (2) holds for all $\sigma \in \Sigma_n$ and $m \in \mathbf{Z}$.*

Proof. Note that $L(h_m^\sigma) = G(h_1^\sigma, m)$ and use formula (3).

§4. Noncoincidence index of $F(n)$. Let M be a connected manifold. We define the noncoincidence index of M , $\#M$, as follows. If M admits k fixed-point-free maps, no two of which have a coincidence, we put $\#M \geq k + 1$. If $\#M \geq \ell$ for all ℓ , we put $\#M = \infty$; otherwise, $\#M = \ell$ if $\#M \geq \ell$ and $\#M \neq \ell + 1$.

Remark. Evidently $\#M = 1$ if and only if M has the fixed point property. On the other hand, $\#M = \infty$ if M is a (nontrivial) Lie group.

Computation of $\#M$ is made feasible in many cases by the Lefschetz fixed point and coincidence theorems. The following statement is sufficient for our purposes: for proof see [4].

Theorem 4.1. *Let M be a compact, closed, oriented n -manifold. For maps $f, g : M \rightarrow M$, define $\Theta_i : H^i(M; \mathbf{Q}) \rightarrow H^i(M; \mathbf{Q})$ to be the composition*

$$H^i(M; \mathbf{Q}) \xrightarrow{f^*} H^i(M; \mathbf{Q}) \xrightarrow{\approx} H_{n-i}(M; \mathbf{Q}) \xrightarrow{g_*} H_{n-i}(M; \mathbf{Q}) \xrightarrow{\approx} H^i(M; \mathbf{Q})$$

(where \approx denotes Poincaré duality) and put

$$L(f, g) = \sum_{i=0}^n (-1)^i \text{Tr } \Theta_i.$$

Then if $L(f, g) \neq 0$, f and g have a coincidence.

Remark. It is immediate that $L(f, \text{id}) = L(f)$, the Lefschetz number of f , so the above theorem implies the Lefschetz fixed point theorem for M . Since $L(f, g)$ depends only on f^* and g^* , we shall sometimes write $L(f^*, g^*)$.

We collect some elementary facts about the Lefschetz coincidence number in the next proposition.

Proposition 4.2. *Let M, f, g be as above.*

1. *If $f^* = g^*$ and $f_*[M] = k[M]$, then $L(f, g) = k\chi(M)$.*
2. *If $g^*: H^*(M; \mathbf{Z}) \circlearrowleft$ has inverse \bar{g}^* , then $L(f^*, g^*) = \pm L(\bar{g}^* f^*)$.*

Proof. Routine.

For the proof of the main result, we require two lemmas.

Lemma 4.3. *Let $f, g: F(n) \rightarrow F(n)$ be maps with $f^* = g^*$. Then if $f_*[F(n)] \neq 0$, f and g have a coincidence.*

Proof. Since $\chi(F(n)) = n!$, (1) of 4.2 gives $L(f, g) \neq 0$. Now use 4.1.

Lemma 4.4. *Suppose $H, K \subset \Sigma_n$ so that for every $\sigma \in H, \tau \in K, \tau^{-1}\sigma$ is not a product of $\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix} \right]$ disjoint transpositions. Then $|H| + |K| \leq n!$.*

Proof. Let $\omega \in \Sigma_n$ be a product of $\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix} \right]$ disjoint transpositions. Then for any $\tau \in K, \tau\omega \notin H$. But this means $|H| \leq n! - |K|$.

Now we prove the main result.

Theorem 4.5. $\#F(n) = n!$.

Proof. Since $F(n)$ has a free Σ_n -action, it suffices to prove $\#F(n) \leq n!$. Let S be a set of fixed-point-free maps on $F(n)$, no two of which have a coincidence. By examination of formula (2), we see $L(f) = 0$ if and only if

$$f^* = \begin{cases} h_1^\sigma, & \sigma \neq \text{id} \\ h_{-1}^\sigma, & \sigma \neq \text{product of } \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix} \right] \text{ disjoint transpositions.} \end{cases}$$

Thus, every element of $\bar{S} = \{f^* : f \in S\}$ has this form. It follows that $f_*[F(n)] \neq 0$ for $f \in S$, so $|S| = |\bar{S}|$ by 4.3. Let

$$H = \{\sigma \in \Sigma_n : h_1^\sigma \in \bar{S}\} \cup \{\text{id}\}$$

$$K = \{\sigma \in \Sigma_n : h_{-1}^\sigma \in \bar{S}\}$$

so $|H| + |K| = |\bar{S}| + 1$. Since

$$L(h_1^\sigma, h_{-1}^{\tau^{-1}\sigma}) = \pm L(h_{-1}^{\tau^{-1}\sigma})$$

by (2) of 4.2, we see that $\tau^{-1}\sigma$ cannot be a product of $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint transpositions

if $\sigma \in H$ and $\tau \in K$. Then $|H| + |K| \leq n!$ by 4.4, so $|S| = |\bar{S}| \leq n! - 1$ and the conclusion follows.

Corollary 4.6. *If a group G acts freely on $F(n)$, then $|G| \leq n!$*

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