GLOBAL STABILITY OF RAREFACTION WAVE FOR THE OUTFLOW PROBLEM GOVERNED BY THE RADIATIVE EULER EQUATIONS

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Abstract. This paper is devoted to the study of the initial-boundary value problem for the radiative full Euler equations, which are a fundamental system in the radiative hydrodynamics with many practical applications in astrophysical and nuclear phenomena. It turns out that the pattern of the asymptotic states is not unique and depends on the data both on the boundary and at the far field. In this paper, we focus our attention on the outflow problem when the flow velocity on the boundary is negative, and give a rigorous proof of the asymptotic stability of the rarefaction wave without restrictions on the smallness of the wave strength. Different from our previous work on the inflow problem for the radiative Euler equations in [6], lack of boundary conditions on the density and velocity prevents us from applying the integration by part to derive the energy estimates directly. So the outflow problem is more challenging in mathematical analysis than the inflow problem studied in [6]. New weighted energy estimates are introduced and the trace of the density and velocity on the boundary are handled by some subtle analysis. The weight is chosen based on the new observation on the key decay properties of the smooth rarefaction wave. Our investigations on the inflow and outflow problem provide a good understanding on the radiative effect and boundary effect in the setting of rarefaction wave.

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1. Introduction

The radiative Euler equations are a fundamental system to describe the motion of the compressible gas with the radiative heat transfer phenomena, which has many applications in astrophysics and nuclear explosions. Mathematically, the radiative Euler equations in the Eulerian coordinates can be modelled as a hyperbolic-elliptic coupled system of the following form:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= 0, \\
\left\{ \rho \left( e + \frac{|u|^2}{2} \right) \right\}_t + \text{div} \left\{ \rho u \left( e + \frac{|u|^2}{2} \right) + pu \right\} + \text{div} q &= 0, \\
-\nabla\text{div} q + aq + b\nabla\theta^4 &= 0,
\end{align*}
\]

where \( \rho, u, p, e \) and \( \theta \) are respectively the density, velocity, pressure, internal energy and absolute temperature of the gas, and \( q \) is the radiative heat flux. Positive constants \( a \) and \( b \) depend only on the gas itself. Like the classic compressible Euler equations, the first three equations in (1.1) stand for the conservation of the mass, momentum and energy respectively. The fourth equation in (1.1) is related to the radiative heat transfer phenomenon, and one can refer [1, 9, 23, 29, 40, 46] for more details. System (1.1) can also be derived by the non-relativistic limit (speed of light tending to \(+\infty\)) from a hyperbolic-kinetic system, and rigorous mathematical derivation can be found in [14].

As far as we know, so far most of the existing results concern the analysis of the global-in-time existence and stability of the elementary wave for the one-dimensional case. The global-in-time existence of solutions around a constant state was shown in [15]. For the analysis of the rarefaction wave, if the initial data is a small perturbation of a given rarefaction wave with small strength, it was proved in [19] that the solutions converge to the rarefaction wave as \( t \to +\infty \). Then in [11], the authors showed that when the absorption coefficient \( \alpha \) tends to \(+\infty\), the solutions converge to the rarefaction wave with the convergence rate \( \alpha^{-\frac{1}{3}} |\ln \alpha|^2 \), where the absorption coefficient \( \alpha \) is defined by the relationship \( a = 3\alpha^2 \) and \( b = 4\alpha \sigma \) for positive constants \( a, b \) and the Stefan-Boltzmann constant \( \sigma \). The asymptotic stability of a single viscous contact wave was proved in [42, 43]. The existence and stability for zero mass perturbation of the small amplitude shock profile were respectively studied in [20] and [21]. The authors in [30] showed the nonlinear orbital asymptotic stability of small amplitude shock profiles for general hyperbolic-elliptic coupled systems of the type modeling the radiative gas. Analysis of large amplitude shock profiles was given in [2, 24]. Finally, for the case of composite waves, the stability of the composite wave of rarefaction waves and a viscous contact wave was investigated in [34, 45]. Recently the authors in [5] studied the unique global-in-time existence and the asymptotic stability of the composite wave of two viscous shock waves by employing the anti-derivative method.
Due to the complexity and difficulties resulted from the boundary effect, up to now, only one rigorous mathematical result on the well-posedness of the initial-boundary value problem governed by the one-dimensional radiative Euler equations in the Lagrangian coordinates was established by the authors in [6]. Precisely speaking, in [6] we investigated the inflow problem where the velocity of the inward flow on the boundary is given as a positive constant. We gave a rigorous proof of the asymptotic stability of the rarefaction wave, provided that the data on the boundary is supersonic. It is the first rigorous result on the initial-boundary value problem for the radiative Euler equations.

In this paper, we will continue to study the outflow problem, which is the second one of our series of papers on the initial-boundary value problem for the one-dimensional radiative Euler equations. Precisely speaking, we will consider the outflow problem governed by the one-dimensional radiative Euler equations in the Eulerian coordinates

$$
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= 0, \\
\left\{ \rho \left( e + \frac{u^2}{2} \right) \right\}_t + \left\{ \rho u \left( e + \frac{u^2}{2} \right) + pu \right\}_x + q_x &= 0, \\
-q_{xx} + aq + b(\theta^4)_x &= 0.
\end{align*}
$$

That is, we will investigate the initial-boundary value problem of system (1.2) on $0 \leq x < \infty$ and $0 \leq t < \infty$ with the initial data

$$
(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), \quad \text{and} \quad \inf_{x \in \mathbb{R}^+} (\rho_0, \theta_0)(x) > 0, \quad \text{for} \quad x \geq 0,
$$

the asymptotic boundary condition at the far field $x = +\infty$

$$
(\rho, u, \theta, q)(+\infty, t) = (\rho_+, u_+, \theta_+, 0), \quad t \geq 0
$$

and the condition on the boundary $x = 0$

$$
\theta(0, t) = \theta_-, \quad q(0, t) = 0, \quad t \geq 0,
$$

where $\rho_+, u_+, \theta_+ > 0$ are given constants. Moreover, the initial value (1.3) and the boundary condition (1.5) satisfy the compatibility condition $\theta_0(0) = \theta_-$ at the origin $(0, 0)$.

Throughout this paper, we will concentrate on the ideal polytropic gases:

$$
p = R\rho\theta, \quad e = \frac{R}{\gamma - 1}\theta,
$$

where $\gamma > 1$ is the adiabatic exponent and $R > 0$ is the specific gas constant.
We expect that solutions of (1.2) eventually converge toward the solutions of the following system by dropping the term $-q_{xx}$ in the last equation of (1.2):

$$
\begin{cases}
\bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\
(\bar{\rho}\bar{u})_t + (\bar{\rho}\bar{u}^2 + p(\bar{\rho}, \bar{\theta}))_x = 0, \\
\{\bar{\rho}\left(e(\bar{\theta}) + \frac{\bar{u}^2}{2}\right)\}_t + \{\bar{\rho}\bar{u}\left(e(\bar{\theta}) + \frac{\bar{u}^2}{2}\right) + p(\bar{\rho}, \bar{\theta})\bar{u}\}_x + \bar{q}_x = 0,
\end{cases}
$$

which is equivalent to the following hyperbolic-parabolic coupled system if $\bar{\rho} > 0$:

$$
\begin{cases}
\bar{\rho}_t + \bar{\rho}_x + 2\bar{\rho}\bar{u}_x = 0, \\
\bar{u}_t + \bar{u}\bar{u}_x + \frac{R\bar{\theta}}{\bar{\rho}}\bar{\rho}_x = -R\bar{\theta}_x, \\
C_v\bar{\theta}_t - \frac{4b}{a\bar{\rho}}(\bar{\theta}^3\bar{\theta}_x)_x = -C_v\bar{u}\bar{\theta}_x - R\bar{\theta}\bar{u}_x
\end{cases}
$$

with

$$\bar{q} = -\frac{4b\bar{\theta}^3}{a}\bar{\theta}_x. \quad (1.9)$$

There are two eigenvalues of the hyperbolic part (the first two equations) of (1.8)

$$\bar{\lambda}_1 = \bar{u} - \sqrt{R\bar{\theta}}, \quad \bar{\lambda}_2 = \bar{u} + \sqrt{R\bar{\theta}}. \quad (1.10)$$

They are different from the ones of (1.2), for which there are three eigenvalues of the hyperbolic part (the first three equations)

$$\lambda_1 = u - \sqrt{\gamma R\theta}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{\gamma R\theta}. \quad (1.11)$$

As we know, in order to make the initial boundary value problem well-posed, the boundary conditions given on the boundary depend on the sign of the eigenvalues. Because the eigenvalues of the hyperbolic part of equations (1.2) and equations (1.7) are different, we need to pay more attention on the boundary for the local existence and the asymptotic behaviours of solutions at the same time.

![Figure 1](image-url)

It is well-known that different type of boundary states yields different possible configurations of the asymptotic states (which could consist of the rarefaction wave,
viscous shock wave, viscous contact wave, stationary boundary layer, or some of
them) and different type of boundary conditions. By (1.10) and (1.11), there are six
possible cases for the boundary condition (three for the inflow problem and three
for the outflow problem). In this paper, let us consider the case that the boundary
data is in the shaded area as shown in Figure 1, i.e.,
\[-\sqrt{\gamma R\theta} = -\sqrt{\gamma R\theta(0, t)} < u(0, t) < -\sqrt{R\theta(0, t)} = -\sqrt{R\theta}. \tag{1.12}\]

Under assumption (1.12), we cannot assign boundary conditions for \(\rho\) and \(u\). Lack of these boundary conditions makes the outflow problem more challenging in
mathematical analysis than the inflow problem studied in [6]. In order to overcome
new difficulties for the outflow problem, we try to emphasize the following three
aspects:

(i) Different from the inflow problem studied in [6], the lack of boundary conditions
on density and velocity prevents us from applying the integration by parts directly,
so the trace of density and velocity on the boundary need to be handled by some
subtle analysis. In particular, more regularity on the solutions is required to close the
energy estimates. Thus the outflow problem is more challenging in mathematical
analysis than the inflow problem in [6]. Moreover, due to the lack of boundary
condition of velocity, we cannot investigate the outflow problem in the Lagrangian
coordinates to simplify the first three equations to simplify the calculations as done
in [6]. So we investigate the outflow problem in the Eulerian coordinates directly.
(ii) We find a weighted term \(\int_0^t \int_{\infty}^0 (|\tilde{\theta}_{xx}| + \tilde{\varepsilon}^2)\xi dx d\tau\) (see (4.20)), which plays a
key role in the energy estimates. Here it is pointed out ahead that \(\tilde{\theta}\) arises in the
regularization of the rarefaction wave and \(\xi\) denotes the perturbation of temperature
around the regularized rarefaction wave. Compared to the previous works on the
radiative Euler equations, the weighted term is completely new due to the boundary
effect. In order to estimate the weighted term, additional decay properties on the
smooth rarefaction wave are given in Lemma 2.3. As far as we know, this kind of
decay properties is firstly established in this paper.
(iii) In order to handle the estimates on the boundary, we frequently take good
advantage of the properties of subsonic domain, i.e., assumption (1.12) (see (4.15),
(4.33), (4.48), (4.83), (4.97) and so on).

For the introduction of the inflow and outflow problems, one can refer to the paper
by Matsumura [25] for more details. There are also many results on the study of
the inflow or outflow problem governed by other systems such as the Navier-Stokes
system (see [12, 13, 17, 26, 31–33, 39, 41]). However, the dissipation of the radiative
Euler system is much weaker than the Navier-Stokes system, it is the main reason
that there are few results on the initial-boundary value problem of the radiative
Euler system, although there are many results on the inflow or outflow problem
governed by the Navier-Stokes system. Our investigations on the inflow and outflow
problem provide a good understanding to radiative effect and boundary effect in the
setting of rarefaction wave. We remark that even for the scalar equation such as
Burgers equation, behaviors of solutions with boundary effect corresponding to the rarefaction waves are also complex and need subtle techniques to handle. Detailed discussions can be found in [22].

We need to mention that we are also motivated by the related investigations on the simplified model (Hamer model), which gives a good approximation to the fundamental system in a certain physical situation, c.f. [8, 18]. The investigations on the simplified model provide a good understanding on the radiative effect. The exhaustive literature list is beyond the scope of the paper, and thus only few closely related results on the rarefaction waves are mentioned, c.f. [3, 7, 16, 35–38]. Interested readers can refer to them and references therein.

The rest of the paper is organized as follows. In Section 2, the smooth rarefaction wave is constructed based on the Riemann problem of the full Euler equations. Properties of smooth rarefaction waves which will be frequently used in this paper and the main theorem of this paper are given. In Section 3, we reformulate the system and establish the local existence of the reformulated problem. The \textit{a priori} estimates, and then our main theorem are proved in Section 4.

2. Rarefaction wave and main results

In this section, we will introduce the smooth rarefaction wave which will be used as the background solution in this paper. Then several properties of the smooth rarefaction wave and main theorem of this paper will be given.

2.1. Rarefaction wave. It is well known that the 3-rarefaction wave curve through the right-hand side state \((\rho_+, u_+, \theta_+)\) is

\[
R_3(\rho_+, u_+, \theta_+) = \{(\rho^r, u^r, \theta^r): 0 < \rho^r < \rho_+, (\rho^r)^{1-\gamma} \theta^r = \rho_+^{1-\gamma} \theta_+, \]
\[
u^r = u_+ + \frac{2}{\gamma-1} \sqrt{R\gamma \rho_+^{1-\gamma} \theta_+ \left[ (\rho^r)^{\frac{\gamma-1}{2}} - \rho_+^{\frac{\gamma-1}{2}} \right]} \}.
\]

(2.1)

For any given \(\theta_- < \theta_+\), there exist unique \(\rho_-\) and \(u_-\) such that \((\rho_-, u_-, \theta_-) \in R_3(\rho_+, u_+, \theta_+)\). The 3-rarefaction wave \((\rho^r, u^r, \theta^r)(\xi)\) connecting \((\rho_-, u_-, \theta_-)\) and \((\rho_+, u_+, \theta_+)\) is a global-in-time weak solution to the following Riemann problem of Euler system:

\[
\begin{cases}
\rho^r_t + (\rho^r u^r)_x = 0, \\
(\rho^r u^r)_t + [\rho^r (u^r)^2 + p^r]_x = 0, \\
\rho^r \left[ e^r + \frac{(u^r)^2}{2} \right]_t + \left[ \rho^r u^r (e^r + \frac{(u^r)^2}{2}) + p^r u^r \right]_x = 0, \\
(\rho^r, u^r, \theta^r)(x, 0) = \begin{cases}
(\rho_-, u_-, \theta_-), & x < 0, \\
(\rho_+, u_+, \theta_+), & x > 0.
\end{cases}
\end{cases}
\]

(2.2)

Here it is easy to see that \(\theta_- < \theta_+, \rho_- < \rho_+, \) and \(u_- < u_+\). Next, in order to give the details of the large-time behavior of the solutions to the outflow problem, it is necessary to construct a smooth approximation solution \((\bar{\rho}, \bar{u}, \bar{\theta})(x, t)\) from \((\rho^r, u^r, \theta^r)(\xi)\).
As done in [13], firstly let us define \( \tilde{w}(\frac{x}{t}) \) to be the solution of
\[
\begin{aligned}
&\begin{cases}
\tilde{w}_t + \tilde{w}\tilde{w}_x = 0, \\
\tilde{w}(x,0) = \tilde{w}_0(x) = \frac{1}{2}(w_+ + w_-) + \tilde{w}K_\nu \int_0^{\infty \frac{dx}{1+y^2}},
\end{cases}
\end{aligned}
\tag{2.3}
\]
where \( \tilde{w} = \frac{1}{2}(w_+ - w_-) > 0, \varepsilon > 0 \) and \( K_\nu \) is a constant such that \( K_\nu \int_{-\infty}^{\infty} \frac{dy}{1+y^2} = 1 \) for \( \nu > \frac{3}{2} \). The properties of the solution \( \tilde{w} \) to the regularized problem (2.3) were given in lemma 2.1 of [27, 28, 39, 44] as follows:

Lemma 2.1 (c.f., [27, 28, 39, 44]). The regularized problem (2.3) admits a unique global smooth solution \( \tilde{w}(x,t) \) satisfying the following properties:

(i) \( w_- < \tilde{w}(x,t) < w_+ \), \( \tilde{w}_x(x,t) > 0 \) for each \( (x,t) \in \mathbb{R} \times [0, \infty) \);

(ii) For any \( p \) with \( 1 \leq p \leq \infty \), there exists a constant \( C_{p,\nu} \) depending on \( p \) and \( \nu \) such that
\[
\|\tilde{w}_x(t)\|_{L^p}^p \leq C_{p,\nu} \min(\varepsilon^{p-1} \tilde{w}^p, \tilde{w}^{-p+1}),
\]
\[
\|\tilde{w}_{xx}(t)\|_{L^p}^p \leq C_{p,\nu} \min(\varepsilon^{2p-1} \tilde{w}^p, \varepsilon^{(p-1)(1-\frac{1}{2p})} \tilde{w}^{\frac{p-1}{2p}} \tilde{t}^{-p+\frac{1}{2p}});
\]

(iii) There exists a constant \( C_\nu \) depending on \( \nu \) such that
\[
\int_{\mathbb{R}} \frac{\tilde{w}_{xx}^2}{\tilde{w}_x} \, dx = \left\| \frac{\tilde{w}_{xx}^2}{\tilde{w}_x} \right\|_{L^1} \leq C_\nu \min(\varepsilon^2 \tilde{w}, \varepsilon^{1-\frac{1}{2\nu}} \tilde{w}^{-\frac{1}{2\nu}} \tilde{t}^{1-\frac{1}{2\nu}});
\]

(iv) \( \left| \partial_t^k \partial_x^l \tilde{w} \right|_{\infty} \leq C|w_+ - w_-|^{l+k+1}, l, k \geq 0, l + k \leq 4; \)

(v) \( \sup_{\mathbb{R}}|\tilde{w}(x,t) - w^R(\frac{x}{t})| \to 0, \text{ as } t \to \infty. \)

Here we point out that property (iii) was given in [44].

Let \( w_{\pm} = \lambda_3(v_{\pm}, u_{\pm}, \theta_{\pm}) \). Then the smooth approximated solution \( \tilde{z}(x,t) = (\tilde{\rho}(x,t), \tilde{u}(x,t), \tilde{\theta}(x,t)) \) is constructed by solving the following equations
\[
\begin{aligned}
&\begin{cases}
S^r(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x,t) = S^r(v_+, u_+, \theta_+), \\
\lambda_3(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x,t) = \tilde{w}(x,1+t), \\
\tilde{u} = u_+ - \int_{v_+}^{\tilde{v}} \lambda_3(\mu, S^r_+) d\mu,
\end{cases}
\end{aligned}
\tag{2.4}
\]
where \( S^r(\tilde{v}, \tilde{u}, \tilde{\theta}) = R\theta \tilde{v}^{-1} \) and \( S^r_+ = S^r(v_+, u_+, \theta_+) = R\theta_+ v_+^{-1} \).
It is easy to check that
\[
\begin{aligned}
\tilde{\rho}_t + (\tilde{\rho}\tilde{u})_x &= 0, \\
(\tilde{\rho}\tilde{u})_t + (\tilde{\rho}\tilde{u}^2 + \tilde{p})_x &= 0, \\
\{\tilde{\rho}(R_{\gamma - 1} - \tilde{\theta} + \tilde{u}^2)\}_t + \{\tilde{\rho}u(R_{\gamma - 1} - \tilde{\theta} + \tilde{u}^2) + \tilde{p}\}_x &= 0,
\end{aligned}
\]
(2.5)

Moreover, for the smooth rarefaction wave \(\tilde{z}(x,t)\), we have the following lemma.

**Lemma 2.2.** (*Property of smooth rarefaction wave*) Smooth rarefaction wave \(\tilde{z}(x,t)\) obtained via (2.3) and (2.4) satisfies

1. \(\tilde{u}_x \geq 0, \quad \text{for } x > 0, t > 0\).
2. For any \(p (1 \leq p \leq +\infty)\), there exists a constant \(C_{p,\gamma}\) such that
   \[
   \left\| \begin{pmatrix} \tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x \end{pmatrix}(t) \right\|_{L^p} \leq C_{p,\gamma} \min \left\{ \delta\epsilon^{1-\frac{1}{p}, \delta\tilde{\theta}^2(1 + t)^{-1+\frac{1}{p}}} \right\},
   \]
3. \(\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^+} \left| \begin{pmatrix} \tilde{\rho}, \tilde{u}, \tilde{\theta} \end{pmatrix}(t, x) - (\rho^r, u^r, \theta^r)(\frac{x}{t}) \right| = 0\).

We omit the proof of Lemma 2.2 since it is standard. For example, one can check lemma 2.2 in [39] for more details.

Define
\[
q^r = -\frac{b}{a} \left\{ (\theta^r)^4 \right\}_x, \quad \tilde{q} = -\frac{b}{a} \left( \tilde{\theta}^4 \right)_x.
\]
(2.7)

In this paper, we will use \((\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{q})(x,t)\) to represent \((\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{q})(x,t)|_{x \geq 0}\) for the notational simplicity.

Finally, for the smooth rarefaction wave \(\tilde{z}(x,t)\), we have the following additional estimate.

**Lemma 2.3.** (*Additional property of smooth rarefaction wave*) The smooth rarefaction wave \(\tilde{z}(x,t) = (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{q})(x,t)\) satisfies

\[
\int_{\mathbb{R}^+} \left( \frac{\tilde{\theta}_x^2}{\tilde{u}_x} + \frac{\tilde{\theta}_x^4}{\tilde{u}_x} \right) (x,t) dx \lesssim \epsilon \tilde{\theta}^2(1 + t)^{-\frac{\gamma}{2}}.
\]
(2.8)
Proof. From (2.4), we have

\[
\begin{align*}
(1 - \gamma)\tilde{\rho}^{\gamma-1}\tilde{\rho}_x + \tilde{\rho}^{1-\gamma}\tilde{\theta}_x &= 0, \\
\tilde{u}_x + \sqrt{\frac{\gamma}{2}}\tilde{\theta}^{-\frac{1}{2}}\tilde{\theta}_x &= \tilde{w}_x, \\
\tilde{u}_x = \sqrt{R\gamma\rho^{1-\gamma}\theta_+^\gamma} \tilde{\rho}_x.
\end{align*}
\]

(2.9)

Combining (2.9)\textsubscript{1} with (2.9)\textsubscript{3}, one has

\[
\frac{1 - \gamma}{\sqrt{R\gamma\rho^{1-\gamma}\theta_+}} \tilde{\rho}\tilde{\theta}_x + \tilde{\rho}^{1-\gamma}\tilde{\theta}_x = 0,
\]

(2.10)

which together (2.9)\textsubscript{1} implies

\[
\left(\frac{\sqrt{R\gamma\rho^{1-\gamma}\theta_+}}{\gamma - 1} \tilde{\rho}^{(1-\gamma)}\tilde{\theta}_x^2 + \frac{\sqrt{R\gamma}}{2}\tilde{\theta}_x^2\right) \tilde{\theta}_x = \tilde{w}_x.
\]

(2.11)

From now on, we use the symbol \(a \sim b\) to describe that there exists a constant \(C > 0\) such that \(C^{-1}a \leq b \leq Ca\). Thus, it follows from (2.9)\textsubscript{3}, (2.10) and (2.11)

\[
\tilde{\rho}_x \sim \tilde{u}_x \sim \tilde{\theta}_x \sim \tilde{w}_x
\]

(2.12)

and

\[
\|\tilde{\rho}_x(t)\|_{L^p} \sim \|\tilde{u}_x(t)\|_{L^p} \sim \|\tilde{\theta}_x(t)\|_{L^p} \sim \|\tilde{w}_x(t)\|_{L^p}
\]

(2.13)

since \(\tilde{\rho}\) and \(\tilde{\theta}\) are bounded.

Differentiating (2.11) with respect to \(x\), we have

\[
\sqrt{R\gamma\rho^{1-\gamma}\theta_+} \tilde{\rho}^{\gamma-1}\tilde{\theta}_x^2 + \sqrt{R\gamma} \tilde{\theta}_x^2
\]

(2.14)

Thus, one has

\[
\tilde{\theta}_{xx} \sim \tilde{w}_{xx} + \tilde{\rho}_x\tilde{\theta}_x + \tilde{\theta}_x^2 \sim \tilde{w}_{xx} + \tilde{\theta}_x^2 \sim \tilde{w}_{xx} + \tilde{w}_x^2
\]

(2.15)

and

\[
\|\tilde{\theta}_{xx}(t)\|_{L^p} \sim \|\tilde{w}_{xx}(t)\|_{L^p} + \|\tilde{w}_x(t)\|_{L^p}^2
\]

(2.16)

since \(\tilde{\rho}\) and \(\tilde{\theta}\) are bounded.

Similarly, one has

\[
\|\tilde{\rho}_{xx}(t)\|_{L^p} \sim \|\tilde{u}_{xx}(t)\|_{L^p} \sim \|\tilde{\theta}_{xx}(t)\|_{L^p}.
\]

(2.17)
Finally, we obtain the following two key estimates

$$\int_{\mathbb{R}^+} \frac{\tilde{\theta}^4_x}{\tilde{u}_x} dx \sim \int_{\mathbb{R}^+} \frac{\tilde{\theta}^4_x}{\tilde{u}_x} dx \sim \left\| \tilde{\theta}_x(t) \right\|_{L^3}^3 \sim \left\| \tilde{w}_x(t) \right\|_{L^3}^3 \sim (1 + t)^{-2}. \quad (2.18)$$

and

$$\int_{\mathbb{R}^+} \frac{\tilde{\theta}^2_{xx}}{\tilde{u}_x} dx \sim \int_{\mathbb{R}^+} \frac{\tilde{w}^2_{xx} + \tilde{w}^4_x}{\tilde{w}_x} dx \sim \int_{\mathbb{R}^+} \frac{\tilde{w}^2_{xx}}{\tilde{w}_x} dx + \left\| \tilde{w}_x(t) \right\|_{L^3}^3 \sim (1 + t)^{-2}. \quad (2.19)$$

This completes the proof of Lemma 2.3. \qed

We remark that Lemma 2.3 plays an important role in the energy estimates below (for example, see (4.20)).

2.2. Main results. We are ready to introduce the main result of this paper in this section. First, we define the solution space as:

$$\mathbb{X}_{\alpha, \beta, M}(0, t) := \{ (\phi, \psi, \xi) \in C([0, t]; H^2(\mathbb{R}^+)), w \in C([0, t]; H^3(\mathbb{R}^+)),
(\phi, \psi, \xi)_t \in C([0, t]; H^1(\mathbb{R}^+)), w_t \in C(0, t; H^2(\mathbb{R}^+)),
(\phi, \psi, \xi)_{tt} \in C([0, t]; L^2(\mathbb{R}^+)), w_{tt} \in C(0, t; H^1(\mathbb{R}^+)),
(\phi, \psi, \xi)_x \in L^2(0, t; H^1(\mathbb{R}^+)), w_x \in L^2([0, t]; H^3(\mathbb{R}^+))
\}
(2.20)$$

$$w_t \in L^2(0, t; H^2(\mathbb{R}^+)), w_{tt} \in L^2(0, t; H^1(\mathbb{R}^+)),
\inf_{[0, t] \times \mathbb{R}^+} \rho(x, t) \geq \alpha, \quad \inf_{[0, t] \times \mathbb{R}^+} \theta(x, t) \geq \beta,$$
$$\sup_{\tau \in [0, t]} \left\{ \left\| (\phi, \psi, \xi)(\tau) \right\|_2 + \left\| (\phi_t, \psi_t, \xi_t, w_t)(\tau) \right\|_1
+ \left\| (\phi_{tt}, \psi_{tt}, \xi_{tt}, w_{tt})(\tau) \right\|
+ \left\| w(\tau) \right\|_3 + \left\| w_t(\tau) \right\|_2 + \left\| w_{tt}(\tau) \right\|_1 \right\} \leq M \}. \}$$

Now we turn to state our main result, that the smooth rarefaction wave constructed in (2.4) and (2.7) is globally stable.

**Theorem 2.1.** Assume \((\rho_-, u_-, \theta_-) \in R_3(\rho_+, u_+, \theta_+)\) and the assumption (1.12) holds. Assume the initial data (1.3) and the boundary data (1.5) satisfy the compatibility condition \(\theta_0(0) = \theta_-\). Moreover, assume that the initial data satisfy

$$\left( \rho_0 - \tilde{\rho}_0, u_0 - \tilde{u}_0, \theta_0 - \tilde{\theta}_0 \right)(x) \in (H^2 \cap L^1)[0, +\infty). \quad (2.21)$$

If there exist constants \(\epsilon_0 > 0\) and \(\eta_0 > 0\) suitably small such that \(\epsilon \lesssim \epsilon_0\) and

$$\left\| (\rho_0 - \tilde{\rho}_0, u_0 - \tilde{u}_0, \theta_0 - \tilde{\theta}_0)(x) \right\|_2 \lesssim \eta_0, \quad (2.22)$$
then the outflow problem (1.2)-(1.5) admits a unique solution $(\rho, u, \theta, q)(x, t)$ satisfying
\[
(\rho - \tilde{\rho}, u - \tilde{u}, \theta - \tilde{\theta}, q - \tilde{q})(x, t) \in X_{\frac{1}{4}\rho, \frac{1}{4}\theta, \rho}[0, +\infty),
\]
where $M$ is a positive constant depending on initial data.

Furthermore, it holds
\[
\sup_{x \geq 0} \left| (\rho, u, \theta, q)(x, t) - (\rho^r, u^r, \theta^r, q^r) \left( \frac{x}{t} \right) \right| \to 0, \quad \text{as } t \to +\infty. \quad (2.24)
\]

3. Mathematical reformulation and the local existence

In this section, we will reformulate the outflow problem mathematically by introducing the difference of the solutions and the smooth rarefaction wave defined by (2.4) and (2.7)
\[
(\varphi, \psi, \xi, w) = (\rho, u, \theta, q) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{q}). \quad (3.1)
\]

Then $(\varphi, \psi, \xi, w)$ satisfies the following equations
\[
\begin{cases}
\varphi_t + u\varphi_x + \rho\psi_x = h_1, \\
\rho(\psi_t + u\psi_x) + (p - \bar{p})_x = h_2, \\
C_v\rho(\xi_t + u\xi_x) + p\psi_x + w_x = h_3, \\
-w_{xx} + aw + 4b\theta^3\xi_x + 4\tilde{\theta}\xi_x \left( \theta^2 + \tilde{\theta}^2 + \tilde{\theta}\xi \right) = \tilde{q}_{xx},
\end{cases}
\]
where
\[
\begin{align*}
 h_1 &:= -\tilde{\rho}\psi - \tilde{u}\phi, \\
 h_2 &:= -\rho\tilde{u}\psi + \frac{\tilde{p}_x}{\rho} \phi = -\rho\tilde{u}\psi + R\tilde{\theta}_x \phi + \frac{R\tilde{\theta}\tilde{\rho}_x}{\rho} \phi, \\
 h_3 &:= -R\rho\xi\tilde{u}_x - C_v\tilde{\theta}_x \rho\psi - \tilde{q}_x,
\end{align*}
\]
with the initial-boundary conditions
\[
\begin{cases}
(\varphi, \psi, \xi)(x, 0) = (\phi_0, \psi_0, \xi_0)(x) \to (0, 0, 0) \quad \text{as } x \to +\infty, \\
\xi(0, t) = 0, \quad w(0, t) = -\tilde{q}(0, t) = \frac{4b}{a} \theta^3 \tilde{\theta}_x(0, t).
\end{cases}
\]

The local-in-time existence of the initial-boundary value problem (3.2)-(3.4) is stated as follows.

**Proposition 3.1. (Local existence)** There exist positive constants $\epsilon_1, \eta_1$ and $\bar{C}, (\bar{C}\eta_1 \leq \eta_0)$ such that if $\eta \lesssim \eta_1$ and $\epsilon \lesssim \epsilon_1$, then for any constant $M \in (0, \eta_1)$, there exists a positive constant $t_0 = t_0(M)$ such that if $\| (\varphi, \psi, \xi)(0) \|_2 \leq M$ and $\inf_{[0, t] \times \mathbb{R}^+} \rho(x, t) \geq \frac{1}{4}\rho_-, \inf_{[0, t] \times \mathbb{R}^+} \theta(x, t) \geq \frac{1}{4}\theta_-$, then problem (3.2)-(3.4) admits a unique solution $(\varphi, \psi, \xi, w)(x, t) \in X_{\frac{1}{4}\rho, \frac{1}{4}\theta, \rho}[0, t_0)$. 
Proof. The local existence can be proven by the fixed point theorem based on the energy estimates and existence of the linearised boundary value problem. For the shortness, we only sketch the main steps, since the argument is standard.

Let $W := (\phi, \psi, \xi)$. Then the first three equations in (3.2) can be rewritten in the following symmetric form:

$$A^0(W)W_t + A(W)W_x = F_0(W, w_x) + F_1(W),$$

(3.5)

where the coefficients matrix are

$$A^0(W) = \begin{pmatrix} \frac{R\theta}{\rho} & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \frac{C_v\rho}{\theta} \end{pmatrix}, \quad A(W) = \begin{pmatrix} \frac{R\theta}{\rho} & R\theta & 0 \\ \frac{R\theta}{\rho} & \rho u & R\rho \\ 0 & R\rho & \frac{C_v\rho}{\theta} \end{pmatrix}$$

(3.6)

and

$$F_0(W, w_x) = \begin{pmatrix} 0 \\ 0 \\ \frac{w_s}{\theta} \end{pmatrix}, \quad F_1(W) = \begin{pmatrix} \frac{R\theta h_1}{\rho} \\ \frac{R\theta h_1}{\rho} \\ \frac{1}{\theta} h_3 \end{pmatrix}.$$  

(3.7)

If $(W_n, w_n)$ is given, then solve the equation

$$A^0(W_n)(W_{n+1})_t + A(W_n)(W_{n+1})_x = F_0(W_n, (w_n)_x) + F_1(W_n),$$

(3.8)

with initial boundary conditions that $\xi_{n+1}(0, t) = 0$ and $W_{n+1}(x, 0) = (\phi_0, \psi_0, \xi_0)$. Multiply (3.8) by $W_{n+1}$ and integrate it on $[0, t] \times \mathbb{R}_+$,

$$\begin{aligned}
\frac{1}{2} \left. \int_{\mathbb{R}_+} W_{n+1} A^0(W_n) W_{n+1} dx \right|_{\tau = t} &- \frac{1}{2} \left. \int_{\mathbb{R}_+} W_{n+1} A^0(W_n) W_{n+1} dx \right|_{\tau = 0} \\
\frac{1}{2} \int_0^t \left. W_{n+1} A(W_n) W_{n+1} d\tau \right|_{x = 0} - \int_0^t \int_{\mathbb{R}_+} W_{n+1} ((A^0(W_n))_x + (A(W_n))_x) W_{n+1} dx d\tau & = \int_0^t \int_{\mathbb{R}_+} (F_0(W_n, (w_n)_x) + F_1(W_n)) dx d\tau.
\end{aligned}$$

Notice that $\xi_{n+1} = 0$ on $x = 0$. So

$$-W_{n+1} A(W_n) W_{n+1} dt \big|_{x = 0} = -\rho u \left( \psi_{n+1} + \frac{R\theta \phi_{n+1}}{\rho u} \right)^2 - \frac{R\theta (u^2 - R\theta)}{\rho u} \phi_{n+1}^2 \geq C \left( \phi_{n+1}^2 + \psi_{n+1}^2 \right).$$

Therefore, by the Gronwall inequality, there exists a time $t_0$ such that when $t \leq t_0$,

$$\begin{aligned}
&\int_{\mathbb{R}_+} (\phi_{n+1}^2 + \psi_{n+1}^2 + \xi_{n+1}^2) \big|_{\tau = t} dx + \int_0^t (\phi_{n+1}^2 + \psi_{n+1}^2) \big|_{x = 0} d\tau \\
\lesssim &\int_{\mathbb{R}_+} (\phi_{0}^2 + \psi_{0}^2 + \xi_{0}^2) dx + \int_0^t ((W_n)_x(\tau))_\infty + \|((W_n)_n(\tau))_\infty \|^2_\infty) d\tau + \int_0^t \int_{\mathbb{R}_+} ((w_n)_x)^2 dx d\tau \\
\lesssim &\int_{\mathbb{R}_+} (\phi_{0}^2 + \psi_{0}^2 + \xi_{0}^2) dx + \int_0^t \int_{\mathbb{R}_+} ((w_n)_x)^2 dx d\tau.
\end{aligned}$$

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\[ + \int_0^t \int_{\mathbb{R}^+} \left( |(W_n)_\tau|^2 + |(W_n)_x|^2 + |W_n|^2 + |(W_n)_{x\tau}|^2 + |(W_n)_{xx}|^2 \right) dx d\tau. \]

Taking \( \partial_\tau \) or \( \partial_x^2 \) on (3.8), similarly, we have the estimates

\[ \int_{\mathbb{R}^+} \left( (\phi_{n+1})_\tau^2 + (\psi_{n+1})_\tau^2 + (\xi_{n+1})_\tau^2 \right) dx + \int_{\mathbb{R}^+} \left( (\phi_{n+1})_\tau^2 + (\psi_{n+1})_\tau^2 \right) dx + \int_{\mathbb{R}^+} \left( (\phi_{n+1})_\tau^2 + (\psi_{n+1})_\tau^2 \right) dx \]

Because matrix \( A(W_n) \) is invertible, by (3.8) or by taking \( \partial_\tau \) or \( \partial_x \) on (3.8), we have

\[ \int_{\mathbb{R}^+} \left( (\phi_{n+1})_x^2 + (\psi_{n+1})_x^2 + (\xi_{n+1})_x^2 \right) dx + \int_{\mathbb{R}^+} \left( (\phi_{n+1})_x^2 + (\psi_{n+1})_x^2 \right) dx + \int_{\mathbb{R}^+} \left( (\phi_{n+1})_x^2 + (\psi_{n+1})_x^2 \right) dx \]

\[ \leq \int_{\mathbb{R}^+} \left( (\psi_0)_x^2 + (\psi_0)_x^2 + (\psi_0)_x^2 + (\phi_0)_x^2 + (\phi_0)_x^2 \right) dx \quad (3.9) \]

So \( W_{n+1} \) exists by applying the standard Galerkin approximations easily (see [4, page 401] for a Galerkin approximations argument). Then we solve the fourth equation of (3.2) to obtain \( w_{n+1} \), where \( \theta \) and \( \rho \) are replaced by \( \theta_{n+1} = \theta + \zeta_{n+1} \) and
\( \rho_{n+1} = \bar{\rho} + \phi_{n+1}, \) i.e., \( w_{n+1} \) solves equation
\[
- (w_{n+1})_{xx} + aw_{n+1} = -4b \theta^3 \xi_x - 4b \tilde{\theta} x \xi (\theta^2 + \bar{\theta}^2 + \tilde{\theta}^2) + \tilde{q}_{xx},
\] (3.10)
where \( w_{n+1}(0, t) = \frac{4b}{a} \theta^3 \tilde{\theta}_x(0, t) \) and \( w_{n+1}(x, t) \to 0 \) as \( x \to +\infty \), for all \( t \geq 0 \).

Multiplying (3.8) by \( w_{n+1}, (w_{n+1})_{xx}, w_{tt}, \) or \( w_{txx} \), and integrating by part, and then applying (3.9) and the Gronwall inequality, by a cumbersome computation, we have
\[
(\phi_{n+1}, \psi_{n+1}, \xi_{n+1}, w_{n+1})(t, x) \in X_{1, \theta, \frac{1}{2} \theta, C} M(0, t),
\] (3.11)
provided that \( t \leq t_0 \) for a given small \( t_0 \) which does not depend on the data.

So \( W_{n+1} \) and \( w_{n+1} \) uniquely exist.

Next, in order to apply the Banach fixed point theorem, for any integers \( n \) and \( m \), let \( (\check{W}, \check{w}) := (W_{n+1} - W_{m+1}, w_{n+1} - w_{m+1}) \). Then \( \check{W} \) satisfies equation
\[
A^0(W_n)(W)_t + A(W_n)(W)_x
= F_0(w_n) - F_0(w_m) + F_1(W_n) - F_1(W_m)
+ (W_{m+1}t)(A^0(W_m) + A^0(W_n)) - (W_{m+1}t)(A(W_m) + A(W_n)),
\]
with initial boundary conditions that \( \xi(0, t) = 0 \) and \( \check{W}(x, 0) = (0, 0, 0) \). Also \( \check{w} \) satisfies that
\[
- \check{q}_{xx} + a \check{q} = b((\theta^3_{n+1} + \theta^3_n + \theta_{n+1}^2 + \theta_n^2) \check{\theta}_x),
\] (3.12)
where \( \check{w}(0, t) = 0 \) and \( \check{w}(x, t) \to 0 \) as \( x \to +\infty \). Note that \( \| (\phi_{n+1}, \psi_{n+1}, \xi_{n+1})(\tau) \|_2 + \| w_{n+1}(\tau) \|_3 \leq M \). Hence following the similar argument for (3.11) above, (but only for the estimate of lower order), we easily have
\[
\| (\check{W}, \check{w})(\tau) \|_1 \leq \frac{1}{2} \| (W_n - W_m, w_n - w_m)(\tau) \|_1
\] (3.13)
if \( M \) is sufficiently small. Because the argument is standard, we omit the details for the shortness. Then the local existence follows by the Banach fixed point theorem.

\[ \square \]

4. Energy estimates and global-in-time existence

Based on Proposition 3.1, the global-in-time existence can be established with \textit{a priori} estimates obtained in this section. Suppose that solutions \( (\phi, \psi, \xi, w)(x, t) \) of problem (3.2)-(3.4) has been extended to the time \( T > t \), we will derive the following \textit{a priori} estimates.

**Proposition 4.1.** (\textit{A priori estimate}) Under the assumptions stated in Theorem 2.1, there exist positive constants \( \eta_2 \leq \eta_1, \epsilon_2 \leq \min \{ \epsilon_1, 1 \} \) and \( C \), such that for any \( t < T \), if \( (\phi, \psi, \xi, w) \in X([0, t]) \) with satisfying \( \epsilon \leq \epsilon_2 \) and
\[
N(t) := \sup_{0 \leq \tau \leq t} \{ \| (\phi, \psi, \xi)(\tau) \|_2 + \| (\phi_t, \psi_t, \xi_t)(\tau) \|_1
+ \| (\phi_{tt}, \psi_{tt}, \xi_{tt})(\tau) \| + \| w(\tau) \|_3 + \| w_t(\tau) \|_2 + \| w_{tt}(\tau) \|_1 \} \lesssim \eta_2,
\] (4.1)
then it holds the estimate that
\[
\sup_{0 \leq \tau \leq t} \left\{ \|(\phi, \psi, \xi)(\tau)\|^2 + \|(\phi_t, \psi_t, \xi_t)(\tau)\|^2 + \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(\tau)\|^2 \right\} \\
+ \sup_{0 \leq \tau \leq t} \left\{ \|w(\tau)\|^2_3 + \|w_t(\tau)\|^2_1 + \|w_{tt}(\tau)\|^2_1 \right\} + \int_0^t \|(w_x, w_{xx}, w_{xxx})\|^2(0, \tau) d\tau \\
+ \int_0^t \|(\phi, \psi, \phi_x, \psi_x, \xi, \psi_{xx}, \xi_{xx}, \phi_{tx}, \psi_{tx}, \xi_{tx}, \phi_t, \psi_t, \phi_{tt}, \psi_{tt})\|^2(0, \tau) d\tau \\
+ \int_0^t \left( \|(\phi_x, \psi_x, \xi_x)(\tau)\|^2_1 + \|w(\tau)\|^2_3 + \|w_t(\tau)\|^2_1 + \|w_{tt}(\tau)\|^2_1 \right) d\tau
\]
\[\lesssim \|(\phi_0, \psi_0, \xi_0)\|^2_2 + \frac{2}{N(t)} \int_0^\infty \sqrt{\|\xi_x(\phi, \psi, \xi)(\tau)\|^2} d\tau.
\]

Here both \(\eta_1\) and \(\epsilon_1\) are the same positive constants as in Proposition 3.1.

Once Proposition 4.1 is proved, we can extend the local solution \((\phi, \psi, \xi, w)(x, t)\) which we have obtained in Proposition 3.1 to the time \(t = \infty\) by the standard continuation argument. Moreover, the estimate (4.2) with passing the limit \(t \to \infty\) implies that

\[\int_0^\infty \left( \|(\phi_x, \psi_x, \xi_x, w_x)(t)\|^2 + \frac{d}{dt} \|(\phi_x, \psi_x, \xi_x, w_x)(t)\|^2 \right) d\tau < +\infty.
\]

Combining the Sobolev inequality, we can easily get the asymptotic behavior (2.24), that concludes the proof of Theorem 2.1. Therefore, the remaining task is to show the \(a \ priori\) estimate in Proposition 4.1.

4.1. Basic energy estimates. At first, we will show the basic energy estimate for the perturbation \((\phi, \psi, \xi, w)(x, t)\). Let

\[E = R\tilde{\theta}\omega \left( \frac{\tilde{\rho}}{\rho} \right) + \frac{\psi^2}{2} + \frac{R}{\gamma - 1} \tilde{\theta} \omega \left( \frac{\theta}{\tilde{\theta}} \right), \quad \omega(s) = s - 1 - \ln s. \quad (4.3)
\]

By the definition of \(\omega\), we see that

\[\omega(s) \sim (s - 1)^2.
\]

Lemma 4.1. Under the same assumptions listed in Proposition 4.1, if \(\epsilon\) and \(N(t)\) are suitably small, it holds

\[\|(\phi, \psi, \xi)(t)\|^2 + \int_0^t \left( \|(\phi, \psi)\|^2(0, \tau) + \|w(\tau)\|^2_1 \right) d\tau \\
+ \int_0^t \left\| \sqrt{u_x}(\phi, \psi, \xi)(\tau) \right\|^2 d\tau
\]
\[\lesssim \|(\phi_0, \psi_0, \xi_0)\|^2_2 + \epsilon^{\frac{3}{8}} + N(t) \int_0^t \|\xi_x(\tau)\|^2 d\tau.
\]
Proof. Multiplying (3.2)_2 by $\psi$, we get
\[
\left(\frac{\rho \psi^2}{2}\right)_t + \left(\frac{\rho u \psi^2}{2}\right)_x + (p - \bar{p})_x \psi = -\rho \tilde{u}_x \psi^2 + R\tilde{\theta}_x \phi \psi + \frac{R\tilde{\theta}_{xx}}{\rho} \phi \psi. \tag{4.5}
\]
Multiplying (3.2)_1 by $\frac{R\tilde{\theta}}{\rho} \phi$, we have
\[
\left\{R\rho \tilde{\theta} \omega \left(\frac{\bar{\rho}}{\rho}\right)\right\}_t + \left\{R\rho u \tilde{\theta} \omega \left(\frac{\bar{\rho}}{\rho}\right)\right\}_x + R\tilde{\theta} \phi \psi = -R\rho \tilde{\theta} \bar{u}_x (\gamma - 1) \omega \left(\frac{\bar{\rho}}{\rho}\right) - R\rho \tilde{\theta}_x \psi \ln \left(\frac{\bar{\rho}}{\rho}\right) - R\tilde{\theta}_x \phi \psi - \frac{R\tilde{\theta}_{xx}}{\rho} \phi \psi. \tag{4.6}
\]
Since $C_v \left(\tilde{\theta}_t + \bar{u} \tilde{\theta}_x\right) = -R\tilde{\theta} \bar{u}_x$, multiplying (3.2)_3 by $\frac{\xi}{\tilde{\theta}}$, we obtain
\[
\left\{C_v \rho \tilde{\theta} \omega \left(\frac{\theta}{\tilde{\theta}}\right)\right\}_t + \left\{C_v \rho u \tilde{\theta} \omega \left(\frac{\theta}{\tilde{\theta}}\right)\right\}_x + \frac{\xi}{\tilde{\theta}} w_x + (p - \bar{p}) \psi_x - R\tilde{\theta} \phi \psi_x = -R\rho \tilde{\theta} \bar{u}_x \omega \left(\frac{\theta}{\tilde{\theta}}\right) - C_v \rho \tilde{\theta}_x \psi \ln \left(\frac{\theta}{\tilde{\theta}}\right) - \tilde{q}_x \frac{\xi}{\tilde{\theta}}. \tag{4.7}
\]
Combining the foregoing analysis, it holds
\[
\left(\rho E\right)_t + I_{1x} + \frac{\xi}{\tilde{\theta}} w_x = G_1 - \tilde{q}_x \frac{\xi}{\tilde{\theta}}, \tag{4.8}
\]
where
\[
I_1 := \rho u E + (p - \bar{p}) \psi,
\]
\[
G_1 := -\rho \tilde{\theta} \bar{u}_x \left\{\frac{\psi^2}{\tilde{\theta}} + R\omega \left(\frac{\bar{\rho}}{\rho}\right) + R(\gamma - 1) \omega \left(\frac{\theta}{\tilde{\theta}}\right) + \frac{R(\gamma - 1)}{\sqrt{R\gamma \tilde{\theta}}} \psi \ln \left(\frac{\bar{\rho}}{\rho}\right) + \frac{R\psi}{\sqrt{R\gamma \tilde{\theta}}} \ln \left(\frac{\theta}{\tilde{\theta}}\right)\right\}.
\]
Here we used $\tilde{\theta}_x = \frac{R(\gamma - 1)}{\sqrt{R\gamma \tilde{\theta}}} \bar{u}_x$.

By direct calculations, one has
\[
\lambda - 1 - \ln \lambda \geq \frac{1}{3} \ln^2 \lambda, \quad as \quad |\lambda - 1| \leq \frac{1}{4}. \tag{4.9}
\]
Based on the above analysis, we have
\[
\frac{R(\gamma - 1)}{\sqrt{R\gamma \theta}} \psi \ln \left( \frac{\tilde{\rho}}{\rho} \right) \leq \frac{7R(\gamma - 1)}{24} \ln^2 \left( \frac{\tilde{\rho}}{\rho} \right) + \frac{6R(\gamma - 1)}{7R\gamma} \psi^2, \tag{4.10}
\]
\[
\frac{R\psi}{\sqrt{R\gamma \theta}} \ln \left( \frac{\tilde{\theta}}{\theta} \right) \leq \frac{7R}{24} \ln^2 \left( \frac{\tilde{\theta}}{\theta} \right) + \frac{6R}{7R\gamma} \psi^2.
\]
Thus, by Lemma 2.2, we see that
\[
G_1 \leq -\frac{1}{9} \rho \tilde{\theta} \tilde{u}_x \left\{ \psi^2 + R\omega \left( \frac{\tilde{\rho}}{\rho} \right) + R(\gamma - 1)\omega \left( \frac{\tilde{\theta}}{\theta} \right) \right\} < 0. \tag{4.11}
\]
Multiplying (3.2) by \( \frac{w^4}{4b\theta^4} \), we get
\[
\frac{1}{4b\theta^4} (aw^2 + w_x^2) - \left( \frac{wq_x}{4b\theta^4} \right)_x + \left( \frac{1}{4b\theta^4} \right)_x wq_x
\]
\[
+ \frac{\xi_x}{\theta} w + \frac{\theta^2 + \tilde{\theta} + \tilde{\theta}^2}{\theta^4} \theta_x \xi w = -\frac{1}{4b\theta^4} \tilde{q}_x w_x. \tag{4.12}
\]
Combining (4.11) with (4.12), we obtain
\[
(\rho E)_t + \left( I_1 + \frac{\xi}{\theta} w - \frac{wq_x}{4b\theta^4} \right)_x + \frac{1}{4b\theta^4} (aw^2 + w_x^2)
\]
\[
+ \left( \frac{1}{4b\theta^4} \right)_x w\theta_x + \left( \frac{\theta^2 + \tilde{\theta} + \tilde{\theta}^2}{\theta^4} \right) \tilde{\theta}_x \xi w
\]
\[
= G_1 - \frac{1}{4b\theta^4} \tilde{q}_x w_x - \left( \frac{1}{4b\theta^4} \right)_x w\tilde{q}_x - \frac{\theta^2}{\theta^2} \xi w - \frac{\tilde{q}_x}{\theta} \xi, \tag{4.13}
\]
where we have used
\[
\left( \frac{\xi}{\theta} w \right)_x = \frac{\xi}{\theta} w_x + \frac{\xi_x}{\theta} w - \frac{\theta_x}{\theta^2} \xi w.
\]
Due to \( \xi(0, t) = 0 \), one has
\[
-I_1(0, t) := \{ -\rho u E - (p - \tilde{p})\psi \} (0, t)
\]
\[
= -\rho u \left\{ \frac{R\tilde{\theta}}{2} \left( \frac{\tilde{\rho}}{\rho} - 1 \right)^2 + \frac{\psi^2}{2} + \frac{R}{\gamma - 1} \left( \frac{\theta}{\theta} - 1 \right)^2 \right\} (0, t)
\]
\[
- R \left( \rho \theta - \tilde{\rho} \tilde{\theta} \right) \psi(0, t) + O(1)(\epsilon + N(t)) |(\phi, \psi, \xi)|^2 (0, t)
\]
\[ \begin{aligned}
&= \left\{ \frac{R\tilde{\theta}(-\tilde{u})}{2\tilde{\rho}} \phi^2 - R\tilde{\theta}\phi\psi + \frac{(-\tilde{u})\tilde{\rho}}{2}\psi^2 \right\} (0, t) \\
&+ O(1)(\epsilon + N(t))|\langle \phi, \psi \rangle|^2 (0, t) \\
&= \frac{R\tilde{\theta}(-\tilde{u})}{2\tilde{\rho}} \left( \phi + \frac{\tilde{\rho}}{\tilde{u}} \psi \right)^2 (0, t) + \frac{\tilde{\rho}}{2(-\tilde{u})} \psi^2 (0, t) \\
&+ O(1)(\epsilon + N(t))|\langle \phi, \psi \rangle|^2 (0, t). \\
\end{aligned} \]

(4.14)

Since \((\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in \Omega^\text{-sub} \cap \Omega^\text{-super} \), \(\tilde{u} < 0, R\tilde{\theta} < \tilde{u}^2 < \gamma R\tilde{\theta}\), it holds

\[-I_1(0, t) \geq (c - O(1)(\epsilon + N(t)))|\langle \phi, \psi \rangle(0, t)|^2.\]

(4.15)

Integrating (4.13) over \([0, t] \times [0, +\infty)\), choosing \(\epsilon\) and \(N(t)\) suitable small, we have

\[
\begin{aligned}
\int_0^\infty \rho E dx + \int_0^t \left( |\langle \phi, \psi \rangle|^2 (0, \tau) + \| w(\tau) \|^2_1 \right) d\tau \\
\lesssim \| (\phi_0, \psi_0, \xi_0) \|^2 + \int_0^t \int_0^\infty G_1 dx d\tau + \int_0^t |w(w_x + \bar{q}_x)(0, \tau)| d\tau \\
+ \int_0^t \int_{\mathbb{R}^+} \left( |(\bar{\theta}_x, \xi_x)| |w\bar{q}_x| + |\theta_x w\xi| + |\bar{q}_x w_x| + |\bar{q}_x \xi| \right) dx d\tau,
\end{aligned}
\]

where

\[
\begin{aligned}
\int_0^t \int_{\mathbb{R}^+} |\theta_x w\xi| dx d\tau = \int_0^t \int_{\mathbb{R}^+} |\xi_x w\xi| dx d\tau + \int_0^t \int_{\mathbb{R}^+} |\bar{\theta}_x w\xi| dx d\tau \\
\lesssim \frac{1}{8} \int_0^t \| w(\tau) \|^2 d\tau + N(t) \int_0^t \| \xi_x(\tau) \|^2 d\tau + \int_0^t \int_{\mathbb{R}^+} |\bar{\theta}_x w\xi| dx d\tau
\end{aligned}
\]

(4.17)

and

\[
\begin{aligned}
\int_0^t \int_{\mathbb{R}^+} \left( |(\bar{\theta}_x, \xi_x)| |w w_x| + |\bar{\theta}_x w\xi| + |\bar{q}_x w_x| + |\bar{q}_x \xi| \right) dx d\tau \\
\lesssim \left( \frac{1}{8} + \epsilon + N(t) \right) \int_0^t \| w(\tau) \|^2 d\tau + \int_0^t \int_{\mathbb{R}^+} \left( \bar{\theta}_x^2 \xi^2 + \bar{q}_x^2 + |\bar{q}_x \xi| \right) dx d\tau.
\end{aligned}
\]

(4.18)

Noticing that \(a\bar{q} = -4b\bar{\theta}_x^3\bar{\theta}_x\), it holds

\[
\bar{q}_x = -\frac{4b}{a} \left( \bar{\theta}_x^3 \bar{\theta}_{xx} + 3\bar{\theta}_x^2 \bar{\theta}_{x}^2 \right).
\]

(4.19)
By the properties of rarefaction wave in Lemma 2.3, we get
\[
\int_0^t \int_{R^+} \left( |\tilde{\theta}_{xx}| + \tilde{\theta}_x^2 \right) \xi \, dx \, d\tau \\
\lesssim \frac{1}{8} \int_0^t \left( \| \sqrt{u} \xi(\tau) \|_2^2 \, d\tau + \int_0^t \int_{R^+} \left( \frac{\tilde{\theta}_x^2}{u_x} + \frac{\tilde{\theta}_x^4}{u_x} \right) \, dx \, d\tau \right) \\
\lesssim \frac{1}{8} \int_0^t \left( \| \sqrt{u} \xi(\tau) \|_2^2 \, d\tau + \int_0^t \epsilon^{\frac{1}{8}} (1 + \tau)^{-\frac{9}{8}} \, d\tau \right) \\
\lesssim \frac{1}{8} \int_0^t \| \sqrt{u} \xi(\tau) \|_2^2 \, d\tau + \epsilon^{\frac{1}{8}}.
\]
(4.20)

By the properties of rarefaction wave in Lemma 2.2, we see that
\[
\| (\tilde{v}_x, \tilde{u}_x, \tilde{\theta}_x)(t) \|_2^2 \lesssim \epsilon^{\frac{1}{8}} (1 + t)^{-\frac{7}{8}},
\]
\[
\| (\tilde{v}_{xx}, \tilde{u}_{xx}, \tilde{\theta}_{xx})(t) \|_{L^2} \lesssim \epsilon^{\frac{1}{8}} (1 + t)^{-\frac{7}{8}},
\]
(4.21)

and
\[
\int_0^t \int_{R^+} \tilde{\theta}_{xx}^2 \, dx \, d\tau \lesssim \int_0^t \int_{R^+} \left( |\tilde{\theta}_{xx}|^2 + \tilde{\theta}_x^4 \right) \, dx \, d\tau
\]
\[
\lesssim \frac{1}{8} \int_0^t \left( \| \tilde{\theta}_{xx}(\tau) \|_2^2 + \| \tilde{\theta}_x(\tau) \|_2^2 \right) \, d\tau \lesssim \epsilon^{\frac{1}{8}} \int_0^t (1 + \tau)^{-\frac{7}{8}} \, d\tau \lesssim \epsilon^{\frac{1}{8}},
\]
(4.22)

Putting (4.17), (4.18), (4.20), (4.22) and (4.23) into (4.16) and choosing \( \epsilon \) is small enough, we get (4.4). This completes the proof of Lemma 4.1. \( \square \)
4.2. **First order energy estimates.** Differentiate (3.2) with respect to \( t \), the reformed equations can be written as

\[
\begin{aligned}
&\phi_{tt} + u\phi_{tx} + \rho\psi_{tx} = \tilde{h}_1, \\
&\rho(\psi_{tt} + u\psi_{tx}) + (p - \bar{p})_{tx} = \tilde{h}_2, \\
&C_v\rho(\xi_{tt} + u\xi_{tx}) + p\psi_{xt} + w_{tx} = \tilde{h}_3, \\
&-w_{txx} + aw_t + 4b\beta^3\xi_{tx} + 4b\tilde{\theta}_x\xi_t \left( \theta^2 + \theta\tilde{\theta} + \tilde{\theta}^2 \right) = \tilde{h}_4,
\end{aligned}
\]

(4.24)

where

\[
\begin{aligned}
\tilde{h}_1 &:= h_{1t} - u_t \phi_x - \rho_t \psi_x, \\
\tilde{h}_2 &:= h_{2t} - \rho_t(\psi_t + w_x) - \rho u_t \psi_x, \\
\tilde{h}_3 &:= h_{3t} - C_v\rho_t(\xi_t + u\xi_x) - C_v\rho u_t \xi_x - p_t \psi_x, \\
\tilde{h}_4 &:= \tilde{q}_{txx} - 12b\beta^2\theta_t \xi_x - 4b\xi \left[ \tilde{\theta}_x \left( \theta^2 + \theta\tilde{\theta} + \tilde{\theta}^2 \right) \right]_t.
\end{aligned}
\]

(4.25)

We first obtain normal estimate.

**Lemma 4.2.** Under the same assumptions listed in Proposition 4.1, if \( \epsilon \) and \( N(t) \) are suitably small, it holds

\[
\begin{aligned}
&\| (\phi_t, \psi_t, \xi_t)(t) \|^2 + \int_0^t \left( \| (\phi_t, \psi_t) \|^2(0, \tau) + \| w_t(\tau) \|^2 \right) \, d\tau \\
&\lesssim \| (\phi_0, \psi_0, \xi_0) \|^2 + \| (\phi_t, \psi_t, \xi_t)(0) \|^2 + \epsilon^{\frac{1}{8}} \\
&\quad + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x, w_{tx})(\tau) \|_1^2 \, d\tau.
\end{aligned}
\]

(4.26)

**Proof.** Multiplying (4.24)\_2 by \( \psi_t \), we get

\[
\begin{aligned}
\left( \frac{\rho}{2} \psi_t^2 \right)_t + \left[ \frac{\rho u}{2} \psi_t^2 + (p - \bar{p})_t \psi_t \right]_x - R\theta \phi_t \psi_{tx} - R\rho \xi_t \psi_{tx} \\
= \tilde{h}_2 \psi_t + R \left( \rho \xi + \tilde{\theta}_t \phi \right) \psi_{tx}.
\end{aligned}
\]

(4.27)

Multiplying (4.24)\_1 by \( \frac{R\theta}{\rho} \phi_t \), we have

\[
\begin{aligned}
\left( \frac{R\theta}{2\rho} \phi_t^2 \right)_t + \left( \frac{R\theta u}{2\rho} \phi_t^2 \right)_x + R\theta \phi_t \psi_{tx} = \left\{ \left( \frac{R\theta}{2\rho} \right)_t + \left( \frac{R\theta u}{2\rho} \right)_x \right\} \phi_t^2 + \frac{R\theta}{\rho} \phi_t \tilde{h}_1.
\end{aligned}
\]

(4.28)
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Multiplying \((4.24)_3\) by \(\frac{\xi_t}{\theta}\), we obtain
\[
\left(\frac{C_v\rho}{2\theta} \xi_t^2\right)_t + \left(\frac{C_v\rho u}{2\theta} \xi_t^2\right)_x + R\rho \xi_t \psi_{tx} + \frac{\xi_t}{\theta} w_{tx} = \frac{\xi_t}{\theta} h_3 + \left\{ \left(\frac{C_v\rho}{2\theta}\right)_t + \left(\frac{C_v\rho u}{2\theta}\right)_x \right\} \xi_t^2.
\]
(4.29)

Multiplying \((4.24)_4\) by \(\frac{w_t}{4\theta^4}\), we get
\[
\left(-\frac{w_t}{4\theta^4} w_{tx}\right)_x + \left(\frac{1}{4\theta^4}\right) w_t w_{tx} + \frac{w_t^2}{4\theta^4} + \frac{aw_t^2}{4\theta^4} + w_t \xi_{tx} = \frac{w_t}{4\theta^4} \left\{ h_4 - 4b\theta_x \xi_t \left( \theta^2 + \bar{\theta} + \bar{\theta}^2 \right) \right\}.
\]
(4.30)

Combining \((4.27)-(4.30)\), one has
\[
\left(\frac{R\theta}{2\rho} \phi_t^2 + \frac{\rho}{2} \psi_t^2 + \frac{C_v\rho}{2\theta} \xi_t^2\right)_t + \frac{w_t^2}{4\theta^4} + \frac{aw_t^2}{4\theta^4} + \left(\frac{R\theta u}{2\rho} \phi_t^2 + \frac{\rho u}{2} \psi_t^2 + (p - \bar{p})_t \psi_t + \frac{C_v\rho u}{2\theta} \xi_t^2 - \frac{w_t}{4\theta^4} w_{tx} + \frac{\xi_t}{\theta} w_t \right)_x
\]
\[
\begin{aligned}
&= \frac{R\theta}{\rho} \phi_t h_1 + \left\{ \left(\frac{R\theta}{2\rho}\right)_t + \left(\frac{R\theta u}{2\rho}\right)_x \right\} \phi_t^2 - \left(\frac{1}{\theta}\right)_x \xi_t w_t - \left(\frac{1}{4\theta^4}\right)_x w_t w_{tx} \quad (4.31) \\
&+ \tilde{h}_2 \psi_t + R\left( \rho_t \xi + \bar{\rho}_t \phi \right) \psi_{tx} + \frac{\xi_t}{\theta} h_3 + \left\{ \left(\frac{C_v\rho}{2\theta}\right)_t + \left(\frac{C_v\rho u}{2\theta}\right)_x \right\} \xi_t^2 \\
&+ \frac{w_t}{4\theta^4} \left\{ h_4 - 4b\theta_x \left[ \bar{\phi}_x \left( \theta^2 + \bar{\theta} + \bar{\theta}^2 \right) \right]_x \right\}.
\end{aligned}
\]

Integrating \((4.31)\) over \([0, +\infty) \times [0, t]\), choosing \(\epsilon, N(t)\) suitable small, we have
\[
\| (\phi_t, \psi_t, \xi_t)(t) \|^2 + \int_0^t \| w_t(\tau) \|^2 d\tau + \int_0^t I_2(0, \tau) d\tau
\]
\[
\lesssim \| (\phi_0, \psi_0, \xi_0) \|_1^2 + \| (\phi_t, \psi_t, \xi_t)(0) \|^2 + \epsilon^\frac{1}{8}
+ (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x)(\tau) \| d\tau
+ \int_0^t \int_0^\infty \left| (\bar{\phi}_x, \bar{\phi}_{xx}) \right|^2 |(\phi, \psi, \xi)|^2 dx d\tau.
\]
(4.32)
Noted that \((p - \bar{p})_t = R\theta\phi_t + R\rho\xi_t + R\left(\bar{\theta}_t\phi + \bar{\rho}_t\xi\right)\), it yields

\[
I_2(0,t) = -\left\{ \frac{R\theta u}{2\rho} \phi_t^2 + \frac{\rho u}{2} \psi_t^2 + (p - \bar{p})_t \psi_t - \frac{w_t}{4b\theta^4} w_{tx} \right\} (0,t)
\]

\[
= -\left( \frac{R\theta u}{2\rho} \phi_t^2 + \frac{\rho u}{2} \psi_t^2 + R\theta\phi_t \psi_t \right) (0,t)
\]

\[
-\left( \bar{R}\bar{\theta}_t\phi \psi_t + \frac{w_t}{4b\theta^4} w_{tx} \right) (0,t)
\]

\[
\geq c \left( \phi_t^2 + \psi_t^2 \right) (0,t) - \left( R\bar{\theta}_t\phi \psi_t + \frac{w_t}{4b\theta^4} w_{tx} \right) (0,t)
\]

where we have used \(u < 0, R\theta < u^2 < \gamma R\theta\) and \(\partial_i\xi (0,t) = 0 (i = 0, 1, 2)\). Therefore, we get

\[
\|(\phi_t, \psi_t, \xi_t) (t)\|^2 + \int_0^t \left( \|(\phi_t, \psi_t)^2 (0, \tau) + \|(w_t, w_{tx}) (\tau)\|^2 \right) d\tau
\]

\[
\lesssim \|(\phi_0, \psi_0, \xi_0)\|^2 + \|(\phi_t, \psi_t, \xi_t) (0)\|^2 + \varepsilon^\frac{1}{8}
\]

\[
+ (\varepsilon + N(t)) \int_0^t \|(\phi_x, \psi_x, \xi_x) (\tau)\|_1 d\tau + \int_0^t \|w_t w_{tx}\| (0, \tau) d\tau
\]

\[
+ \int_0^t \int_{R^+} \left| (\bar{\theta}_x, \bar{\phi}_{xx}) \right|^2 \|\phi, \psi, \xi\|^2 dx d\tau.
\]

The last two terms on the right hand side of (4.34) can be estimated as follows.

\[
\int_0^t \|w_t w_{tx}\| (0, \tau) d\tau \lesssim \int_0^t \|\bar{q}_t (\tau)\|_\infty \|w_{tx} (\tau)\|_\infty d\tau
\]

\[
\lesssim \int_0^t \varepsilon^\frac{1}{8} (1 + \tau)^{-\frac{3}{8}} \|w_{tx} (\tau)\|_\frac{3}{2} \|w_{txx} (\tau)\|_\frac{3}{2} d\tau
\]

\[
\lesssim \varepsilon^\frac{1}{8} \int_0^t \|w_{tx} (\tau)\|^2 d\tau + \frac{1}{8} \int_0^t (1 + \tau)^{-\frac{5}{8}} \|w_{txxx} (\tau)\|_\frac{5}{4} d\tau
\]

\[
\lesssim \varepsilon^\frac{1}{8} \int_0^t \|(w_{tx}, w_{txxx}) (\tau)\|^2 d\tau + \varepsilon^\frac{1}{8} \int_0^t (1 + \tau)^{-\frac{5}{8}} d\tau
\]

\[
\lesssim \varepsilon^\frac{1}{8} \int_0^t \|(w_{tx}, w_{txxx}) (\tau)\|^2 d\tau + \varepsilon^\frac{1}{8}
\]

and

\[
\int_0^t \int_{R^+} \left| (\bar{\theta}_x, \bar{\phi}_{xx}) \right|^2 \|\phi, \psi, \xi\|^2 dx d\tau
\]
\[ \lesssim \int_0^t \left\| (\phi, \psi, \xi) \right\|_\infty^2 \| (\tilde{\theta}_x, \tilde{\theta}_{xx})(\tau) \|^2 d\tau \tag{4.36} \]

\[ \lesssim \epsilon^{1/8} \int_0^t (1 + \tau)^{-7/4} \left\| (\phi, \psi, \xi)(\tau) \right\|_1^2 d\tau \]

\[ \lesssim \epsilon^{1/8} N(t) \int_0^t (1 + \tau)^{-7/4} d\tau \lesssim \epsilon^{1/8}. \]

Substituting (4.36) and (4.35) into (4.34), we get (4.26). This completes the proof of Lemma 4.2. \qed

Then for tangential derivatives, we have the following lemma.

**Lemma 4.3.** Under the same assumptions in Proposition 4.1, if \( \epsilon \) and \( N(t) \) are suitably small, then it holds for \( t \in [0, T] \),

\[ \| (\phi_x, \psi_x, \xi_x)(t) \|^2 + \int_0^t \left( \| (\phi_x, \psi_x, \xi_x, w_x)(0, \tau) + \| w_x(\tau) \|_1^2 \right) dx \]

\[ \lesssim \| (\phi_0, \psi_0, \xi_0) \|^2 + \| (\phi_t, \psi_t, \xi_t)(0) \|^2 + \epsilon^{1/8} \]

\[ + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x, w_t)(\tau) \|_1^2 d\tau. \tag{4.37} \]

**Proof.** Multiplying (3.2)\(_{1x} \) by \( R\theta \phi_x \), we get

\[ \left( \frac{R\theta}{2} \phi_x^2 \right)_t + \left( \frac{R\theta u \phi_x^2}{2} + p\phi_x \psi_x \right)_x - p\psi_x \phi_{xx} \]

\[ = \frac{R}{2} (\theta_t + u\theta_x - \theta u_x) \phi_x^2 + R\theta_x \phi_x \psi_x + R\theta \phi_x h_{1x}. \tag{4.38} \]

Multiplying (3.2)\(_{2x} \) by \( \rho \psi_x \), we have

\[ \left( \frac{\rho^2}{2} \psi_x^2 \right)_t + \left( \frac{\rho^2 u \psi_x^2}{2} \right)_x + p\psi_x \phi_{xx} + R\rho^2 \psi_x \xi_{xx} \]

\[ = - \frac{3}{2} \rho^2 u_x \psi_x^2 - R \psi \left( \tilde{\theta}_{xx} \phi + \tilde{\rho}_{xx} \xi + 2\phi_x \theta_x + 2\tilde{\rho}_x \xi_x \right) \psi_x \]

\[ + \rho_x \psi_x (p - \tilde{p})_x - \rho_x \psi_x h_2 + \rho \psi_x h_{2x}. \tag{4.39} \]

Multiplying (3.2)\(_{3} \) by \( -\frac{\xi}{\theta} \xi_{xx} \), we obtain

\[ \left( C_v \frac{\rho^2}{2 \theta^2} \xi_{xx}^2 \right)_t - C_v \left( \frac{\rho^2}{2 \theta} \xi_t \xi_x + \frac{\rho^2 u}{2 \theta} \xi_{xx}^2 \right)_x - R \rho^2 \psi_x \xi_{xx} - \frac{\rho}{\theta} w_x \xi_{xx} \]

\[ = - \left( \frac{\rho}{\theta} \xi \xi_{3x} \right)_x + \xi_x \left( \frac{\rho}{\theta} h_3 \right)_x + C_v \left\{ \left( \frac{\rho^2}{\theta} \right)_t - \left( \frac{\rho^2 u}{\theta} \right)_x \right\} \frac{\xi_{3x}^2}{2} - C_v \left( \frac{\rho^2}{\theta} \right)_x \xi_x \xi_t. \tag{4.40} \]
Multiplying (3.2)_4x by $\frac{\rho}{4b\theta^4}w_x$, we get
\[
\begin{align*}
-\left(\frac{\rho}{4b\theta^4}w_xw_{xx}\right)_x + \left(\frac{\rho}{4b\theta^4}w_x\right)_x w_{xx} + \frac{\alpha\rho}{4b\theta^4}w_x^2 + \frac{\rho}{\theta}w_x\xi_{xx} \\
+ \frac{3\rho}{\theta^2}\theta_x\xi_x w_x + \frac{\rho}{4b\theta^4}w_x \left\{4b\tilde{\theta}_x\xi \left(\theta^2 + \theta\tilde{\theta} + \tilde{\theta}^2\right) - \tilde{q}_{xx}\right\}_x = 0.
\end{align*}
\] (4.41)

Combining (4.38)-(4.41), one has
\[
\begin{align*}
\left(\frac{R\theta}{2}\phi_x^2 + \frac{\rho^2}{2}\psi_x^2 + C_v\frac{\rho^2}{2}\xi_x^2\right)_t + \left(\frac{\rho}{4b\theta^4}w_x\right)_x w_{xx} + \frac{\alpha\rho}{4b\theta^4}w_x^2 + I_3 \\
= \frac{R}{2} \left(\theta_t + u\theta_x - \theta u_x\right) \phi_x^2 + R\rho\theta_x\phi_x\psi_x + R\theta\phi_xh_{1x} \\
- \frac{3}{2}\frac{\rho^2}{\theta^2}u\psi_x^2 - R\rho \left(\tilde{\theta}_{xx}\phi + \tilde{\psi}_{xx}\xi + 2\phi_x\theta_x + 2\tilde{\psi}_x\xi_x\right) \psi_x  \\
+ \rho_x\psi_x(p - \tilde{p})_x - \rho_x\psi_xh_2 + \rho\psi_xh_{2x} + \xi_x \left(\frac{\rho}{\theta}h_3\right)_x \\
+ C_v \left\{ \left(\frac{\rho^2}{\theta}\right)_t - \left(\frac{\rho^2}{\theta}u\right)_x \right\} \frac{\xi_x^2}{2} - C_v \left(\frac{\rho^2}{\theta}\right)_x \xi_x \xi_t \\
- \frac{3\rho}{\theta^2}\theta_x\xi_x w_x - \frac{\rho}{4b\theta^4}w_x \left\{4b\tilde{\theta}_x\xi \left(\theta^2 + \theta\tilde{\theta} + \tilde{\theta}^2\right) - \tilde{q}_{xx}\right\}_x,
\end{align*}
\] (4.42)

where
\[
I_3 := \frac{R\theta u}{2}\phi_x^2 + p\phi_x\psi_x + \frac{\rho^2}{2}\psi_x^2  \\
- C_v\frac{\rho^2}{\theta}\xi_x - C_v\frac{\rho^2}{2\theta^2}\xi_x^2 - \frac{\rho}{4b\theta^4}w_xw_{xx} + \frac{\rho}{\theta}\xi_x h_3.
\] (4.43)

Integrating (4.42) over $[0, +\infty) \times [0, t]$, choosing $\epsilon$ and $N(t)$ suitable small, we obtain
\[
\begin{align*}
\|&(\phi_x, \psi_x, \xi_x)(t)\|^2 + \int_0^t \left(\|w_x(\tau)\|^2_1 - I_3(0, \tau)\right) d\tau \\
\lesssim &\|&(\phi_0, \psi_0, \xi_0)\|^2_1 + (\epsilon + N(t)) \int_0^t \|&(\phi_x, \psi_x, \xi_x)(\tau)\|^2 d\tau \\
&+ \int_0^t \int_{\mathbb{R}^+} \left|\left(\theta_x, \tilde{\theta}_{xx}\right)\right|^2 \|(\phi, \psi, \xi)\|^2 dxd\tau.
\end{align*}
\]
Here
\[-I_3(0, t) = \left( -\frac{R\theta u}{2} \phi_x^2 - p\phi_x\psi_x - \frac{\rho^2 u}{2} \psi_x^2 \right) (0, t) \]
\[+ \left( C_v \frac{\rho^2 u}{2\theta} \xi_x^2 + \frac{\rho}{4b\theta^4} w_x w_{xx} - \frac{\rho}{\theta} \xi_x h_3 \right) (0, t). \tag{4.44} \]

It holds from the third and fourth equations in (3.2)
\[
\begin{cases}
  w_x(0, t) = -C_v (\rho u \xi_x)(0, t) - (p \psi_x)(0, t) + h_3(0, t), \\
  w_{xx}(0, t) = a w(0, t) + 4b\theta^3 \xi_x(0, t) - \tilde{q}_{xx}(0, t),
\end{cases} \tag{4.45}
\]
which implies
\[
\left( \frac{\rho}{4b\theta^4} w_x w_{xx} \right) (0, t) = \left( \frac{\rho}{\theta} w_x \xi_x \right) (0, t) + \left( \frac{3\rho}{4b\theta^4} w_x w \right) (0, t) - \left( \frac{\rho}{4b\theta^4} w_x \tilde{q}_{xx} \right) (0, t) \]
\[= - \left( C_v \frac{\rho^2 u}{\theta} \xi_x^2 \right) (0, t) - (R \rho^2 \psi_x \xi_x)(0, t) + \left( \frac{\rho}{\theta} \xi_x h_3 \right) (0, t) \]
\[+ \left( \frac{3\rho}{4b\theta^4} w_x w \right) (0, t) - \left( \frac{\rho}{4b\theta^4} w_x \tilde{q}_{xx} \right) (0, t). \tag{4.46} \]

Thus, we have
\[-I_3(0, t) = \left( -\frac{R\theta u}{2} \phi_x^2 - p\phi_x\psi_x - \frac{\rho^2 u}{2} \psi_x^2 \right) (0, t) \]
\[= \left( -\frac{R\theta u}{2} \phi_x^2 - p\phi_x\psi_x - \frac{\rho^2 u}{2} \psi_x^2 \right) (0, t) \]
\[+ \left( C_v \frac{\rho^2 u}{2\theta} \xi_x^2 + R \rho^2 \psi_x \xi_x \right) (0, t) + \left\{ \frac{\rho}{4b\theta^4} \left( a w_x w - w_x \tilde{q}_{xx} \right) \right\} (0, t) \tag{4.47} \]

Since \(-\sqrt{\gamma R\theta} < u < -\sqrt{R\theta}\), one has
\[\left( \frac{\rho}{4b\theta^4} w_x w_{xx} \right) (0, t) \leq C \left( \phi_x^2 + \psi_x^2 \right) \tag{4.48} \]

and it holds by (4.36)
\[
\| (\phi_x, \psi_x, \xi_x)(t) \|^2 + \int_0^t \left\{ |(\phi_x, \psi_x)(0, \tau)|^2 + w_x(\tau) \right\} d\tau \\
\lesssim \| (\phi_0, \psi_0, \xi_0) \|^2_{L^1} + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x)(\tau) \|^2 d\tau \\
+ \int_0^t \int_{\mathbb{R}^+} \left( |\tilde{q}_{xx}| + |\tilde{q}_x| \right) |(\phi, \psi, \xi)|^2 dx d\tau + \int_0^t \xi_x^2(0, \tau) d\tau 
\]
\[ + \int_0^t (|w w_x| + |w_x \tilde{q}_{xx}|) (0, \tau) d\tau \]  

\[ \lesssim \|(\phi_0, \psi_0, \xi_0)\|_1^2 + \epsilon^{\frac{1}{8}} + \int_0^t \xi_x^2 (0, \tau) d\tau \]  

\[ + (\epsilon + N(t)) \int_0^t \|(\phi_x, \psi_x, \xi_x)(\tau)\|_1^2 d\tau. \]

Since \((p - \tilde{p})_x = R \rho \xi_x + R \theta \phi_x + R \tilde{\rho}_x \xi + R \tilde{\theta}_x \phi\), we get from (3.2)

\[ \begin{aligned}
  u \phi_x + \rho \psi_x &= h_1 - \phi_t, \\
  \rho u \psi_x + R \rho \xi_x + R \theta \phi_x &= h_2 - R \tilde{\rho}_x \xi - R \tilde{\theta}_x \phi - \rho \psi_t := H_2 - \rho \psi_t, \\
  C_v \rho u \xi_x + p \psi_x + w_x &= h_3 - C_v \rho \xi_t
\end{aligned} \]  

and

\[ \begin{aligned}
  u \phi_x &= h_1 - \phi_t - \rho \psi_x, \\
  \rho (u^2 - R \theta) \psi_x + R \rho u \xi_x &= h_2 - R \tilde{\theta} h_1 + R \theta \phi_t - \rho \psi_t, \\
  C_v \rho u \xi_x + p \psi_x + w_x &= h_3 - C_v \rho \xi_t
\end{aligned} \]  

Furthermore, we obtain

\[ \left( C_v \rho u - \frac{R \rho u}{u^2 - R \theta} \right) \xi_x + w_x \]

\[ = C_v \rho u \frac{u^2 - \gamma R \theta}{u^2 - R \theta} \xi_x + w_x \]

\[ = h_3 - C_v \rho \xi_t - \frac{R \theta}{u^2 - R \theta} (H_2 - R \theta h_1 + R \theta \phi_t - \rho \psi_t) \]

\[ = O(1) |(h_1, H_2, h_3, \phi_t, \psi_t, \xi_t)|. \]

Therefore, it holds

\[ \xi_x^2 (0, t) \lesssim O(1) |(h_1, H_2, h_3, \phi_t, \psi_t)|^2 (0, t) + w_x^2 (0, t) \]

\[ \lesssim |(\phi, \psi, \phi_t, \psi_t)|^2 (0, t) + w_x^2 (0, t). \]

Based on Lemmas 4.1-4.2, we can control \( \int_0^t |(\phi, \psi, \phi_t, \psi_t)|^2 (0, \tau) d\tau \) on the boundary and obtain

\[ \|(\phi_x, \psi_x, \xi_x)(t)\|^2 + \int_0^t (|w_x(\tau)|^2 + |(\phi_x, \psi_x)|^2 (0, \tau)) d\tau \]

\[ \lesssim \|(\phi_0, \psi_0, \xi_0)\|^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|^2 + \epsilon^{\frac{1}{8}} \]  

(4.54)
\[ + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x, w_{tx})(\tau) \|^2_1 d\tau + \int_0^t w_x^2(0, \tau) d\tau, \]

and

\[ \int_0^t w_x^2(0, \tau) d\tau \lesssim \int_0^t \| w_x(\tau) \|^2_\infty d\tau \lesssim \int_0^t \| w_x(\tau) \| \| w_{xx}(\tau) \| d\tau \]
\[ \lesssim \frac{1}{8} \int_0^t \| w_{xx}(\tau) \|^2 d\tau + \int_0^t \| w_x(\tau) \|^2 d\tau. \] (4.55)

Thus, we get (4.37) and complete the proof of Lemma 4.3. \qed

\textbf{Remark 4.1.} By (3.1)_4, it holds

\[ \int_0^t \| \xi_x(\tau) \|^2 d\tau \lesssim \int_0^t \| (w, w_{xx})(\tau) \|^2 d\tau + \int_0^t \int_{\mathbb{R}^+} \left( \tilde{\theta}_x^2 \xi^2 + \tilde{q}_{xx}^2 \right) dx d\tau \] (4.56)

and

\[ \int_0^t w_{xx}^2(0, \tau) d\tau \lesssim \int_0^t \left( w^2 + \tilde{q}_{xx}^2 \right) (0, \tau) d\tau + \int_0^t \xi_x^2(0, \tau) d\tau. \] (4.57)

Thus, we get

\[ \int_0^t \left( w_{xx}^2(0, \tau) + \| \xi_x(\tau) \|^2 \right) d\tau \]
\[ \lesssim \|(\phi_0, \psi_0, \xi_0)\|^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|^2 + \epsilon^{\frac{1}{8}} \]
\[ + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x, w_{tx})(\tau) \|^2_1 d\tau. \] (4.58)

Combining the foregoing results, we have

\[ \| (\phi, \psi, \xi)(t) \|^2 + \int_0^t \left( \|(\phi, \psi, \phi_x, \psi_x, \xi_x, w_x, w_{xx})\|^2(0, \tau) + \| w(\tau) \|^2_2 \right) d\tau \]
\[ + \| (\phi_t, \psi_t, \xi_t)(t) \|^2 + \int_0^t \left( \| (\phi_t, \psi_t)\|^2(0, \tau) + \| (\xi_x, w_t, w_{tx})(\tau) \|^2 \right) d\tau \]
\[ \lesssim \|(\phi_0, \psi_0, \xi_0)\|^2 + \| (\phi_t, \psi_t, \xi_t)(0)\|^2 + \epsilon^{\frac{1}{8}} \]
\[ + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x, w_{tx})(\tau) \|^2_1 d\tau. \] (4.59)
4.3. Second order energy estimates. Differentiate (4.24) with respect to \( t \), the reformed equations can be written as

\[
\begin{cases}
\phi_{ttt} + u \phi_{ttx} + \rho \psi_{tx} = \tilde{g}_1, \\
\rho (\psi_{ttt} + u \psi_{tx}) + (p - \tilde{p})_{ttx} = \tilde{g}_2, \\
C_v \rho (\xi_{ttt} + u \xi_{tx}) + p \psi_{txx} + w_{tx} = \tilde{g}_3, \\
-w_{txxx} + aw_{tx} + 4b\theta^3 \xi_{txx} + \left\{ 4b\theta_x \xi_t \left( \theta^2 + \bar{\theta} + \bar{\theta}^2 \right) \right\}_t = \tilde{g}_4,
\end{cases}
\]

where

\[
\tilde{g}_1 := \tilde{h}_{1t} - u_t \phi_{tx} - \rho_t \psi_{tx},
\]
\[
\tilde{g}_2 := \tilde{h}_{2t} - \rho_t \psi_{tt} - (\rho u)_t \psi_{tx},
\]
\[
\tilde{g}_3 := \tilde{h}_{3t} - C_v \rho_t \xi_{tt} - (\rho u)_t \xi_{tx} - p_t \psi_{xx},
\]
\[
\tilde{g}_4 := \tilde{h}_{3t} - 4b (\theta^3)_t \xi_{tx}.
\]

Lemma 4.4. Under the same assumptions listed in Proposition 4.1, if \( \epsilon \) and \( N(t) \) are suitably small, it holds

\[
\| (\phi_{tt}, \psi_{tt}, \xi_{tt}) (t) \|_2^2 + \int_0^t \left( \| (\phi_{tt}, \psi_{tt}) (0, \tau) \|_1^2 + \| w_{tx} (\tau) \|_1^2 \right) d\tau \\
\lesssim \left( \| (\phi_0, \psi_0, \xi_0) \|_1^2 + \| (\phi_t, \psi_t, \xi_t) (0) \|_2^2 + \| (\phi_{tt}, \psi_{tt}, \xi_{tt}) (0) \|_2^2 + \epsilon \frac{1}{\tilde{N}} \right) \\
+ (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x) (\tau) \|_1^2 d\tau + \epsilon \frac{1}{\tilde{N}} \sup_{\tau \in [0, t]} \| w_{tx} (\tau) \|_1^2.
\]

Proof. Multiplying (4.60)_2 by \( \psi_{tt} \), we get

\[
\left( \frac{\rho}{2} \psi_{tt}^2 \right)_t + \left\{ \frac{\rho u}{2} \psi_{tt}^2 + (p - \tilde{p})_{ttx} \psi_{tx} \right\}_x - R\theta \phi_{tt} \psi_{ttx} - R\rho \xi_{tt} \psi_{txx} \\
= \tilde{g}_2 \psi_{tt} - R \left\{ \theta_t \phi_t + \rho_t \xi_t + \left( \tilde{\rho}_t + \tilde{\theta}_t \phi \right)_t \right\} \psi_{tt}.
\]

Multiplying (4.60)_1 by \( \frac{R\theta}{\rho} \phi_{tt} \), we have

\[
\left( \frac{R\theta}{2\rho} \phi_{tt}^2 \right)_t + \left( \frac{R\theta u}{2\rho} \phi_{tt}^2 \right)_x + R\theta \phi_{tt} \psi_{txx} \\
= \left\{ \left( \frac{R\theta}{2\rho} \right)_t + \left( \frac{R\theta u}{2\rho} \right)_x \right\} \phi_{tt}^2 + \frac{R\theta}{\rho} \phi_{tt} \tilde{g}_1.
\]
Multiplying (4.60) by $\xi u_\theta$, we obtain
\[
\left(\frac{C_v \rho}{2\theta} \xi_{tt}^2\right)_t + \left(\frac{C_v \rho u}{2\theta} \xi_{tt}\right)_x + R p \xi_{tt} \psi_{tx} + \frac{\xi_{tt}}{\theta} w_{tx}
= \frac{\xi_{tt}}{\theta} g_3 + \left\{ \left(\frac{C_v \rho}{2\theta}\right)_t + \left(\frac{C_v \rho u}{2\theta}\right)_x \right\} \xi_{tt}^2.
\] (4.65)

Multiplying (3.2) by $\frac{w_{tt}}{4b\theta^4}$, we get
\[
- \left(\frac{w_{tt}}{4b\theta^4} w_{tx}\right)_x + \left(\frac{w_{tt}}{4b\theta^4}\right)_x w_{tx} + a \frac{w_{tt}^2}{4b\theta^4} + \frac{w_{tt}}{\theta} \xi_{tx} = \frac{w_{tt}}{4b\theta^4} \left\{ \tilde{g}_4 - \left[ 4b \tilde{\theta}_x \xi \left( \theta^2 + \tilde{\theta} \tilde{\delta}^2 \right) \right]_t \right\}.\] (4.66)

Combining (4.63)-(4.66), one has
\[
\left(\frac{R \theta}{2\rho} \phi_{tt}^2 + \rho \frac{u^2}{2} \psi_{tt}^2 + \frac{C_v \rho}{2\theta} \xi_{tt}^2\right)_t + \left(\frac{w_{tt}}{4b\theta^4}\right)_x w_{tx} + a \frac{w_{tt}^2}{4b\theta^4} + I_4
= \frac{R \theta}{\rho} \phi_{tt} g_1 + \left\{ \left(\frac{R \theta}{2\rho}\right)_t + \left(\frac{R \theta u}{2\rho}\right)_x \right\} \phi_{tt}^2 - \left(\frac{1}{\theta}\right)_x \xi_{tt} w_{tt}
+ \tilde{g}_2 \psi_{tt} - R \left\{ \tilde{\theta}_t \phi_t + \rho t \xi_t + \left(\tilde{\rho}_t \xi + \tilde{\theta}_t \phi\right) \right\} \psi_{tt} + \frac{\xi_{tt}}{\theta} g_3
+ \left\{ \left(\frac{C_v \rho}{2\theta}\right)_t + \left(\frac{C_v \rho u}{2\theta}\right)_x \right\} \xi_{tt}^2 + \frac{w_{tt}}{4b\theta^4} \left\{ \tilde{g}_4 - \left(4b \tilde{\theta}_x \xi \left( \theta^2 + \tilde{\theta} \tilde{\delta}^2 \right) \right)_t \right\}.
\] (4.67)

where
\[
I_4 := \frac{R \theta u}{2\rho} \phi_{tt}^2 + \frac{\rho u^2}{2} \psi_{tt}^2 + (p - \tilde{p})_t \psi_{tt} + \frac{C_v \rho u}{2\theta} \xi_{tt}^2 + \frac{\xi_{tt}}{\theta} w_{tt} - \frac{w_{tt}}{4b\theta^4} w_{tx}
\] (4.68)

and
\[
(p - \tilde{p})_t = R \theta \phi_{tt} + R \theta_t \phi_t + (R \rho \xi_t)_t + R \left( \tilde{\theta}_t \phi + \tilde{\rho}_t \xi \right)_t.
\]

Furthermore, we get
\[
-I_4(0, t) = \left( -\frac{R \theta u}{2\rho} \phi_{tt}^2 - R \theta \phi_{tt} \psi_{tt} - \frac{\rho u^2}{2} \psi_{tt}^2 \right)(0, t)
- \left\{ R \theta_t \phi_t \psi_{tt} + R \left( \tilde{\theta}_t \phi \right)_t \psi_{tt} + \frac{w_{tt}}{4b\theta^4} w_{tx} \right\}(0, t)
\geq c \left( \phi_{tt}^2 + \psi_{tt}^2 \right)(0, t) - C \left( \phi_t^2 + \phi^2 + \frac{w_{tt}}{4b\theta^4} w_{tx} \right)(0, t).
\] (4.69)
Integrating (4.67) over $[0, +\infty) \times [0, t]$, we used $\partial_i^2 \xi(0, t) = 0 (i = 0, 1, 2, 3)$ and (4.69) to obtain
\[
\| (\phi_{tt}, \psi_{tt}, \xi_{tt})(t) \|^2 + \int_0^t \left( \| (\phi_{tt}, \psi_{tt}) \|^2(0, \tau) + \| w_{tt}(\tau) \|_1^2 \right) d\tau
\lesssim \| (\phi_0, \psi_0, \xi_0) \|^2 + \| (\phi_t, \psi_t, \xi_t)(0) \|^2 + \| (\phi_{tt}, \psi_{tt}, \xi_{tt})(0) \|^2 + \epsilon^\frac{1}{8}
\]
\[
+ (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x, \xi_x, w_{tx})(\tau) \|^2 d\tau - \int_0^t \| (w_{tt} w_{txx})(0, \tau) \| d\tau.
\]
(4.70)

The last term in (4.70) can be estimated as follows.
\[
\int_0^t (w_{tt} w_{txx})(0, \tau) d\tau
\lesssim \int_0^t \{ (w_{tt} w_{tx})_t - w_{tttt} w_{tx} \}(0, \tau) d\tau
\]
\[
\lesssim (w_{tt} w_{tx})(0, t) - (w_{tt} w_{tx})(0, 0) - \int_0^t (w_{tttt} w_{tx})(0, \tau) d\tau
\]
\[
\lesssim |w_{tt} w_{tx}|(0, t) + |w_{tt} w_{tx}(0, 0)| + \epsilon^\frac{1}{8} \int_0^t \| w_{tx}(\tau) \|_1^2 d\tau + \epsilon^\frac{1}{8}.
\]
(4.71)

On the other hand, we see that
\[
w_{tt}(0, t) = -\tilde{q}_{tt}(0, t) = b \left( \frac{a}{\theta^4} \right)_{txx}(0, t)
\]
\[
= \frac{4b}{a} \left( \tilde{\theta}_t \tilde{\theta}_{txx} + 6 \tilde{\theta}^2 \tilde{\theta}_t \tilde{\theta}_{tx} + 6 \tilde{\theta} \tilde{\theta}_t^2 \tilde{\theta}_t + 3 \tilde{\theta}^2 \tilde{\theta}_t \tilde{\theta}_tt \right)(0, t).
\]
(4.72)

Therefore, it holds
\[
|w_{tt} w_{tx}|(0, t) \lesssim \left| \tilde{\theta}_t \tilde{\theta}_{txx} + \tilde{\theta}^2 \tilde{\theta}_t \tilde{\theta}_{tx} + \tilde{\theta} \tilde{\theta}_t^2 \tilde{\theta}_t + \tilde{\theta}^2 \tilde{\theta}_t \tilde{\theta}_tt \right|(0, t) \| w_{tx}(t) \|_\infty
\lesssim \epsilon^\frac{1}{8} (1 + t)^{-\frac{7}{2}} \| w_{tx}(t) \|_2 \| w_{txx}(t) \|_2
\lesssim \epsilon^\frac{1}{8} \| w_{tx}(t) \|_2 \| w_{txx}(t) \| + \epsilon^\frac{1}{8} (1 + t)^{-\frac{7}{2}}
\]
\[
\lesssim \epsilon^\frac{1}{8} \| (w_{tx}, w_{txx})(t) \|^2 + \epsilon^\frac{1}{8}
\]
(4.73)

and
\[
|w_{tt} w_{tx}|(0, 0) \lesssim \sup_{\tau \in [0, t]} \{ |w_{tt} w_{tx}|(0, \tau) \}
\lesssim \epsilon^\frac{1}{8} \sup_{\tau \in [0, t]} \{ \| (w_{tx}, w_{txx})(\tau) \|^2 \} + \epsilon^\frac{1}{8}.
\]
(4.74)
Substituting (4.71) into (4.70), we get (4.62). This completes the proof of Lemma 4.4.

□

Lemma 4.5. Under the same assumptions listed in Proposition 4.1, if $\epsilon$ and $N(t)$ are suitably small, it holds

$$\left\| (\phi_{tx}, \psi_{tx}, \xi_{tx})(t) \right\|^2 + \int_0^t \left( \left\| (\phi_{tx}, \psi_{tx}, \xi_{tx}, w_{tx}) \right\|^2(d\tau) + \left\| w_{tx}(\tau) \right\|^2 \right) d\tau$$

$$\lesssim \left\| (\phi_0, \psi_0, \xi_0) \right\|^2 + \left\| (\phi_t, \psi_t, \xi_t)(0) \right\|^2 + \left\| (\phi_{tt}, \psi_{tt}, \xi_{tt})(0) \right\|^2 + \epsilon^\frac{1}{8}$$

$$+ (\epsilon + N(t)) \int_0^t \left\| (\phi_x, \psi_x, \xi_x)(\tau) \right\|^2 d\tau + \epsilon^\frac{1}{8} \sup_{\tau \in [0, t]} \left\| w_{tx}(\tau) \right\|^2. \tag{4.75}$$

Proof. Since

$$(p - \tilde{p})_{txx} = R\theta \phi_{txx} + R\rho \xi_{txx} + \left\{ R \left( \tilde{\theta}_x \phi + \tilde{\rho}_x \xi \right) \right\}_{tx}$$

$$+ R\theta_t \phi_{xx} + R\rho_t \xi_{xx} + (R\theta_x \phi_x + R\rho_x \xi_x)_t,$$

multiplying (4.24) by $\psi_{tx}$, we get

$$\left( \frac{\rho}{2} \psi_{tx}^2 \right)_t + \left( \frac{\rho u}{2} \psi_{tx}^2 \right)_x + R\rho \xi_{txx} \psi_{tx} + R\theta \phi_{txx} \psi_{tx}$$

$$= - \left\{ R\theta_t \phi_{xx} + R\rho_t \xi_{xx} + (R\theta_x \phi_x + R\rho_x \xi_x)_t + R \left( \tilde{\rho}_x \psi + \tilde{\theta}_x \phi \right)_{tx} \right\} \psi_{tx}$$

$$+ \left\{ \tilde{h}_{tx} - \rho_x \psi_{tt} - (\rho u)_x \psi_{tx} \right\} \psi_{tx}. \tag{4.76}$$

Multiplying (4.24) by $\frac{\rho}{\rho} \phi_{xt}$, we have

$$\left( \frac{R\theta}{2\rho} \phi_{xt}^2 \right)_t + \left( \frac{R\theta u}{2\rho} \phi_{xt}^2 \right)_x + R\theta \phi_{xt} \psi_{txx}$$

$$= \left\{ \left( \frac{R\theta}{2\rho} \right)_t + \left( \frac{R\theta u}{2\rho} \right)_x \right\} \phi_{xt}^2 + \frac{R\theta}{\rho} \phi_{xt} \left( \tilde{h}_{tx} - u_x \phi_{xt} - \rho_x \psi_{tx} \right). \tag{4.77}$$

Multiplying (4.24) by $\frac{\xi_{tx}}{\theta}$, we obtain

$$\left( \frac{C_v\rho}{2\theta} \xi_{tx}^2 \right)_t + \left( \frac{C_v\rho u}{2\theta} \xi_{tx}^2 \right)_x + R\rho \xi_{txx} \psi_{txx} + \frac{\xi_{tx}}{\theta} w_{txx}$$

$$= \left\{ \left( \frac{C_v\rho}{2\theta} \right)_t + \left( \frac{C_v\rho u}{2\theta} \right)_x \right\} \xi_{tx}^2 + \frac{\xi_{tx}}{\theta} \left\{ \tilde{h}_{3x} - C_v \rho_x \xi_{tx} - C_v (\rho u)_x \xi_{tx} - R (\rho \theta)_x \psi_{tx} \right\}. \tag{4.78}$$
Combining (4.76)-(4.79), one has

\[
\left( -\frac{w_{tx}}{4b\theta^4}w_{txx} \right)_x + \left( \frac{1}{4b\theta^4} \right)_x w_{tx}w_{txx} + \frac{w_{tx}^2}{4b\theta^4} + \frac{aw_{tx}^2}{4b\theta^4} + \frac{w_{tx}}{\theta} \xi_{txx} = \frac{w_{tx}}{4b\theta^4} \left\{ \tilde{h}_4 - 4b\tilde{\theta}_x \xi_{t} \left( \theta^2 + \tilde{\theta}^2 + \overline{\theta}^2 \right) \right\}_x - \frac{3\theta_x}{\theta^2} \xi_{tx}w_{tx},
\]

(4.79)

Combining (4.76)-(4.79), one has

\[
\left( \frac{R\theta}{2\rho} \phi_{xx}^2 + \frac{\rho}{2} \psi_{xx}^2 + \frac{C_v\rho}{2\theta} \xi_{tx}^2 \right)_t + \frac{w_{tx}^2}{4b\theta^4} + \left( \frac{1}{4b\theta^4} \right)_x w_{tx}w_{txx} + \frac{aw_{tx}^2}{4b\theta^4}
\]

\[
+ \left( \frac{R\theta u}{2\rho} \phi_{xx}^2 + \frac{\rho u}{2} \psi_{xx}^2 + \frac{C_v\rho u}{2\theta} \xi_{tx}^2 + R\rho \xi_{tx} \psi_{tx} + R\theta \phi_{tx} \psi_{tx} - \frac{w_{tx}}{4b\theta^4}w_{txxx} + \frac{w_{tx}}{\theta} \xi_{txx} \right)_x
\]

\[
= R\rho_x \xi_{tx} \psi_{tx} + R\theta_x \phi_{tx} \psi_{tx} + \left( \frac{1}{\theta} \right)_x w_{tx} \xi_{tx} + \frac{w_{tx}}{4b\theta^4} \left\{ \tilde{h}_4 - 4b\tilde{\theta}_x \xi_{t} \left( \theta^2 + \tilde{\theta}^2 + \overline{\theta}^2 \right) \right\}_x
\]

(4.80)

\[
+ \left\{ \left( \frac{R\theta}{2\rho} \right)_t + \left( \frac{R\theta u}{2\rho} \right)_x \right\} \phi_{xx}^2 + \frac{R\theta}{\rho} \phi_{xx} \left( \tilde{h}_{1x} - u_x \phi_{xx} - \rho_x \psi_{tx} \right)
\]

\[
- \left\{ R\theta_x \phi_{xx} + R\rho_x \xi_{xx} + (R\theta_x \phi_x + R\rho_x \xi_x)_t + R \left( \tilde{\rho}_x \xi + \tilde{\theta}_x \phi \right) \right\}_x \psi_{tx}
\]

\[
+ \left\{ \tilde{h}_{2x} - \rho_x \psi_{tt} - (\rho u)_x \psi_{xx} \right\} \psi_{tx} - \frac{3\theta_x}{\theta^2} \xi_{tx} w_{tx}
\]

\[
+ \left\{ \left( \frac{C_v\rho}{2\theta} \right)_t + \left( \frac{C_v\rho u}{2\theta} \right)_x \right\} \xi_{tx}^2 + \frac{\xi_{tx}}{\theta} \left\{ \tilde{h}_{3x} - C_v \rho_x \xi_{tt} - C_v (\rho u)_x \xi_{tx} - R(\rho\theta)_x \psi_{tx} \right\}
\]

Integrating (4.80) over \([0, +\infty) \times [0, t] , \) we have

\[
\| (\phi_{xx}, \psi_{xx}, \xi_{tt}) (t) \|^2 + \int_0^t \| (w_{txxx}, w_{txx}) (\tau) \|^2 d\tau - \int_0^t I_5 (0, \tau) d\tau
\]

\[
\lesssim \| (\phi_0, \psi_0, \xi_0) \|^2_2 + \| (\phi_t, \psi_t, \xi_t) (0) \|^2_2 + \| (\phi_{tt}, \psi_{tt}, \xi_{tt}) (0) \|^2_2 + \epsilon^\frac{1}{2}
\]

(4.81)

\[
+ (\epsilon + N(t)) \int_0^t \| (\phi_{xx}, \psi_{xx}, \xi_{tt}) (\tau) \|^2 d\tau + \int_0^t \int_{\mathbb{R}^+} | \tilde{\theta}_{xx, \xi_{xx}} |^2 | (\phi, \psi, \xi) |^2 dx d\tau,
\]

where

\[
-I_5 (0, t) = - \left( \frac{R\theta u}{2\rho} \phi_{xx}^2 + \frac{\rho u}{2} \psi_{xx}^2 + R\theta \phi_{tx} \psi_{tx} \right) (0, t)
\]

\[
- \left( \frac{C_v\rho u}{2\theta} \xi_{xx}^2 + R\rho \xi_{tx} \psi_{tx} - \frac{w_{tx}}{4b\theta^4}w_{txxx} + \frac{w_{tx}}{\theta} \xi_{txx} \right) (0, t).
\]

(4.82)
Similarly, since $-\sqrt{\gamma R\theta} < u < -\sqrt{R\theta}$, we get

$$
- \left( \frac{R\theta u}{2\rho} \phi_{xt}^2 + \frac{\rho u}{2} \psi_{xt}^2 + R\theta \phi_{tx} \psi_{tx} \right) (0, t) \geq c \left( \phi_{xt}^2 + \psi_{xt}^2 \right) (0, t). \tag{4.83}
$$

Since

$$
-w_{txx} + 4b\theta^3 \xi_{tx} = \tilde{h}_4 - aw_t - 4b\tilde{\theta}_x \xi_t \left( \theta^2 + \tilde{\theta} + \tilde{\theta}^2 \right),
$$

we have

$$
\left( - \frac{w_{tx} w_{txx}}{4b\theta^4} + \frac{\xi_{tx}}{\theta} w_{tx} \right) (0, t) = \left( \frac{\tilde{h}_4 w_{tx}}{4b\theta^4} - \frac{aw_t w_{tx}}{4b\theta^4} \right) (0, t). \tag{4.84}
$$

Therefore, it holds

$$
\Vert (\phi_{xt}, \psi_{xt}, \xi_{tx}) (t) \Vert^2 + \int_0^t \left( \Vert (\phi_{xt}, \psi_{xt})^2 (0, \tau) + \Vert w_{tx} (\tau) \Vert^2 \right) d\tau
\leq \Vert (\phi_0, \psi_0, \xi_0) \Vert^2 + \Vert (\phi_t, \psi_t, \xi_t) (0) \Vert^2 + \Vert (\phi_{tt}, \psi_{tt}, \xi_{tt}) (0) \Vert^2 + e^\frac{1}{2}
+ \int_0^t \xi_{tx}^2 (0, \tau) d\tau + (\epsilon + N(t)) \int_0^t \Vert (\phi_x, \psi_x, \xi_x) (\tau) \Vert^2 d\tau
+ \int_0^t \left( |\tilde{h}_4 w_{tx}| + |w_t w_{txx}| \right) (0, \tau) d\tau. \tag{4.85}
$$

By (4.24), we see that

$$
\begin{align*}
\begin{cases}
  u\phi_{tx} + \rho \psi_{tx} = \tilde{h}_1 - \phi_{tt}, \\
  \rho \psi_{tx} + R\rho \xi_{tx} + R\theta \phi_{tx} = \tilde{H}_2 - \rho \psi_{tt}, \\
  C_v \rho u \xi_{tx} + p \psi_{xt} + w_{tx} = \tilde{h}_3 - C_v \rho \xi_{tt},
\end{cases}
\end{align*}
\tag{4.86}
$$

where

$$
\tilde{H}_2 := \tilde{h}_2 - R (\rho_t \xi_x + \theta_t \phi_x) - R \left( \tilde{h}_1 + \tilde{h}_2 - \phi_{tt} \right) t.
$$

By (4.86)1 and (4.86)2, it holds

$$
\begin{align*}
u \phi_{tx} &= -\rho \psi_{tx} + \tilde{h}_1 - \phi_{tt}, \\
R \rho u \xi_{tx} &= \rho \left( R\theta - u^2 \right) \psi_{xt} + u \left( \tilde{h}_2 - \rho \psi_{tt} \right) - R\theta \left( \tilde{h}_1 - \phi_{tt} \right). \tag{4.87}
\end{align*}
$$
By (4.86)3, we get
\[ C_v \rho u \frac{\gamma R \theta - u^2}{u^2 - R \theta} \xi_{xt} + w_{xt} \]
\[ = \tilde{h}_3 - C_v \rho \xi_{tt} - \frac{Pu}{u^2 - R \theta} \left( \tilde{H}_2 - \rho \psi_{tt} \right) + \frac{R^2 \theta^2}{u^2 - R \theta} \left( \tilde{h}_1 - \phi_{tt} \right) \]
\[ = O(1) \left| \left( \tilde{h}_1, \tilde{H}_2, \tilde{h}_3, \phi_{tt}, \psi_{tt}, \xi_{tt} \right) \right| . \]

Thus, it holds
\[ \int_0^t \xi_{xt}^2(0, \tau) d\tau \]
\[ \lesssim \int_0^t w_{xt}^2(0, \tau) d\tau + \int_0^t \left| \left( \tilde{h}_1, \tilde{H}_2, \tilde{h}_3, \phi_{tt}, \psi_{tt} \right) \right|^2 (0, \tau) d\tau \]
\[ \lesssim \frac{1}{8} \int_0^t \| w_{txx}(\tau) \|^2 d\tau + \int_0^t \| w_{tx}(\tau) \|^2 d\tau + \epsilon^{\frac{1}{8}} \]  (4.89)
\[ + \int_0^t \left( \phi, \psi, \phi_x, \psi_x, \xi_x, w_x, w_{xx}, \phi_t, \psi_t, \phi_{tt}, \psi_{tt} \right) (0, \tau) d\tau \]
and
\[ \int_0^t w_{tx}^2(0, \tau) d\tau \lesssim \int_0^t \| w_{tx}(\tau) \|^2_{\infty} d\tau \lesssim \frac{1}{8} \int_0^t \| w_{txx}(\tau) \|^2 d\tau + \int_0^t \| w_{tx}(\tau) \|^2 d\tau . \]

Consequently, we obtain (4.75). This completes the proof of Lemma 4.5. \[ \square \]

Next, we get the higher order estimate on \((\phi, \psi, \xi)\).

**Lemma 4.6.** Under the same assumptions listed in Proposition 4.1, if \(\epsilon\) and \(N(t)\) are suitably small, it holds
\[ \| (\phi_{xx}, \psi_{xx}, \xi_{xx})(t) \|^2 + \int_0^t \| (\phi_{xx}, \psi_{xx}, \xi_{xx}, w_{xxx}) \|^2(0, \tau) d\tau \]
\[ + \int_0^t \| (\xi_{xx}, w_{xx}, w_{xxx})(\tau) \|^2 d\tau \]
\[ \lesssim \| (\phi_0, \psi_0, \xi_0) \|^2_2 + \| (\phi_t, \psi_t, \xi_t)(0) \|^2_2 + \| (\phi_{tt}, \psi_{tt}, \xi_{tt})(0) \|^2 + \epsilon^{\frac{1}{8}} \]
\[ + (\epsilon + N(t)) \int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 d\tau + \epsilon^{\frac{1}{8}} \sup_{\tau \in [0, t]} \| w_{tx}(\tau) \|^2_1. \]  (4.90)
Proof. Multiplying (3.2)_{1xx} by $R\theta \phi_{xx}$, we get

$$
\left( \frac{R\theta}{2} \phi_{xx}^2 \right)_t + \left( \frac{R \theta \phi_{xx}^2}{2} + p \psi_{xx} \phi_{xx} \right)_x - p \psi_{xx} \phi_{xxx} = \frac{R}{2} \left\{ \theta_t - (u \theta)_x \right\} \phi_{xx}^2 + R \rho \theta \phi_{xx} \psi_{xx} - R \theta \phi_{xx} (\phi_x u_{xx} + 2 u_x \phi_{xx} + \psi_x \rho_{xx} + \rho_x \psi_{xx}) + R \theta \phi_{xx} h_{1xx}.
$$

Multiplying (3.2)_{2xx} by $\rho \psi_{xx}$, we have

$$
\left( \frac{\rho^2}{2} \psi_{xx}^2 \right)_t + \left( \frac{\rho^2 u \psi_{xx}^2}{2} \right)_x + p \psi_{xx} \phi_{xxx} + R \rho^2 \psi_{xx} \xi_{xxx} = - R \rho \psi_{xx} \left( 3 \phi_x \theta_{xx} + 3 \phi_{xx} \theta_x \xi_x + \tilde{\rho}_{xxx} \xi + \tilde{\theta}_{xxx} \phi + 3 \tilde{\rho}_{xx} \xi_x + 3 \tilde{\theta}_{xx} \xi_{xx} \right) - \left( \frac{5}{2} \rho^2 u_x + 2 \rho_x u \right) \psi_{xx}^2 + \rho \psi_{xx} \psi_{xxx} - \rho \psi_{xx} \left\{ \rho_{xx} \psi_t + 2 \rho_x \psi_t + (\rho u)_{xx} \psi_x \right\}. \tag{4.92}
$$

Multiplying (3.2)_{3x} by $-\frac{\rho}{\theta} \xi_{xxx}$, we obtain

$$
\left( C_v \frac{\rho^2}{2 \theta} \xi_{xx}^2 \right)_t - C_v \left( \frac{\rho^2}{\theta} \xi_{xx} \xi_{tx} + \frac{\rho^2 u}{2 \theta} \xi_{xx}^2 \right)_x - \frac{\rho}{\theta} \xi_{xxx} w_{xx} = - R \rho \psi_{xx} \left( C_v \rho_x \xi_t + C_v (\rho u)_x \xi_x + p_x \psi_x \right) \left\{ \rho_{xx} \psi_t + 2 \rho_x \psi_t + (\rho u)_{xx} \psi_x \right\} \tag{4.93}
$$

Multiplying (3.2)_{4xx} by $\frac{\rho w_{xx}}{4 \theta^4}$, we get

$$
- \left( \frac{\rho w_{xx}}{4 \theta^4} w_{xxx} \right)_x + \left( \frac{\rho w_{xx}}{4 \theta^4} \right)_x w_{xxx} + \frac{\alpha \rho}{4 \theta^4} w_{xx}^2 + \frac{\rho}{\theta} \xi_{xxx} w_{xx} + \frac{\rho w_{xx}}{\theta^4} \left\{ \left( \theta^3 \right)_{xx} \xi_x + \left( \theta^3 \right)_x \xi_{xx} \right\} \tag{4.94}
$$

$$
= \frac{\rho w_{xx}}{4 \theta^4} \left\{ \tilde{q}_{xx} - 4b \tilde{\theta}_x \xi \left( \theta^2 + \tilde{\theta} + \tilde{\theta}^2 \right) \right\}_{xx}.
$$
Combining (4.91)-(4.94), one has
\[
\frac{R\theta}{2} \frac{\phi_{xx}^2}{\psi_{xx}^2} + \frac{\rho^2}{2} \psi_{xx}^2 + C_u \rho^2 \frac{\xi_{xx}^2}{2} \right) \right)_t + \left( \frac{\rho w_{xx}}{4b\theta^4} \right)_x w_{xxx} + \frac{a\rho}{4b\theta^4} w_{xx}^2 - I_{6x} \\
= O(1)(\epsilon + N(t)) \left| (\phi_x, \psi_x, \xi_x, w_x, \phi_{xx}, \psi_{xx}, \xi_{xx}, w_{xx}) \right|^2 \\
+ O(1) \left| (\tilde{\theta}_x, \tilde{\theta}_{xx}, \tilde{\theta}_{xxx}) \right|^2 \left| (\phi, \psi, \xi) \right|^2,
\]
where
\[
I_6 := - \left( - R\theta \frac{u\phi_{xx}^2}{2} - p\phi_{xx}\psi_{xx} - \frac{\rho^2 u\psi_{xx}^2}{2} \right) \right)(0, t) \\
- \left( C_v \rho^2 \frac{u}{2\theta} \xi_{xx}^2 + R\rho^2 \psi_{xx}^2 \xi_{xx} \right) \right)(0, t) \\
- \frac{\rho w_{xx}}{4b\theta^4} \left\{ a w_x + (4b\theta^3 \xi_x)_x + 4b\theta \xi_x \left( \theta^2 + \tilde{\theta} \theta + \tilde{\theta} \right) - \tilde{q}_{xx} \right\} \right)(0, t).
\]
By (3.2)_3, we see that
\[
C_v \rho_x \xi_t + C_v (\rho u)_x \xi_x + p_x \psi_x + w_{xx} \\
= - C_v \rho \xi_{tx} - C_v \rho u \xi_{xx} - p \psi_{xx}.
\]
Therefore, it holds
\[
I_6(0, t) = \left( - R\theta \frac{u\phi_{xx}^2}{2} - p\phi_{xx}\psi_{xx} - \frac{\rho^2 u\psi_{xx}^2}{2} \right) (0, t) \\
- \left( C_v \rho^2 \frac{u}{2\theta} \xi_{xx}^2 + R\rho^2 \psi_{xx}^2 \xi_{xx} \right) (0, t) \\
- \frac{\rho w_{xx}}{4b\theta^4} \left\{ a w_x + (4b\theta^3 \xi_x)_x + 4b\theta \xi_x \left( \theta^2 + \tilde{\theta} \theta + \tilde{\theta} \right) - \tilde{q}_{xx} \right\} (0, t).
\]
Similarly, since \( R\theta < u^2 < \gamma R\theta \), we obtain
\[
I_6(0, t) \geq c_1 \left| (\phi_{xx}, \psi_{xx}) \right|^2(0, t) - c_2 \xi_{xx}^2(0, t) \\
- c_3 \left( w_{xx}^2 + w_x^2 + \xi_x^2 + \tilde{q}_{xx}^2 \right) (0, t).
\]
Integrating (4.95) over \([0, +\infty) \times [0, t]\), choosing \(\epsilon\) and \(N(t)\) suitable small, we have
\[
\left\| (\phi_{xx}, \psi_{xx}, \xi_{xx})(t) \right\|^2 + \int_0^t \left( \left\| (\phi_{xx}, \psi_{xx}) \right\|^2(0, \tau) + \left\| w_{xx}(\tau) \right\|_1^2 \right) d\tau \\
\lesssim \left\| (\phi_0, \psi_0, \xi_0) \right\|_2^2 + \left( \left\| (\phi, \psi, \xi)(0) \right\|_1^2 + \left\| (\phi_{tt}, \psi_{tt}, \xi_{tt})(0) \right\|_1^2 \right) + \epsilon \frac{t}{4} \\
+ (\epsilon + N(t)) \int_0^t \left\| (\phi_x, \psi_x, \xi_x)(\tau) \right\|_1^2 d\tau + \int_0^t \xi_{xx}^2(0, \tau) d\tau.
\]
+ \int_0^t \int_{\mathbb{R}^+} \left| \left( \tilde{\theta}_x, \tilde{\theta}_{xx}, \tilde{\theta}_{xxx} \right) \right|^2 |(\phi, \psi, \xi)|^2 \, dx \, d\tau. \tag{4.98}

At last, we will deal with \( \int_0^t \varepsilon_{xx}^2(0, t) \, dt \). By (4.60), we see that

\[
\begin{cases}
  w\phi_{xx} + \rho \psi_{xx} = \tilde{g}_1 - \phi_{tx}, \\
  \rho u \psi_{xx} + R\rho \xi_{xx} + R\theta \phi_{xx} = \tilde{g}_2 - \rho \psi_{tx}, \\
  C_v \rho u \xi_{xx} + p \psi_{xx} + w_{xx} = \tilde{g}_3 - C_v \rho \xi_{tx}.
\end{cases}
\tag{4.99}
\]

We also get

\[
C_v \rho u \frac{u^2 - \gamma R \theta}{u^2 - R \theta} \xi_{xx} + w_{xx} = O(1) |(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \phi_{tx}, \psi_{tx}, \xi_{tx})|. \tag{4.100}
\]

Thus, we obtain

\[
\int_0^t \varepsilon_{xx}^2(0, \tau) \, d\tau
\leq \int_0^t w_{xx}^2(0, \tau) \, d\tau + \int_0^t |(\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \phi_{tx}, \psi_{tx}, \xi_{tx})|^2(0, \tau) \, d\tau
\leq \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|_1^2 + \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|_2^2 + \epsilon^{\frac{1}{8}} \tag{4.101}
\]

\[
+ (\epsilon + N(t)) \int_0^t \|(\phi_x, \psi_x, \xi_x)(\tau)\|^2_1 \, d\tau.
\]

In addition, we have from (3.2)$_{4x}$

\[
w_{xxx} = a w_x + 4b \theta^3 \xi_{xx} + 12b \theta^2 \theta_x \xi_x + 4b \left\{ \tilde{\theta}_x \xi \left( \theta^2 + \theta \tilde{\theta} + \tilde{\theta}^2 \right) \right\}_x - \tilde{q}_{xxx}.
\]

Consequently, one has

\[
\int_0^t w_{xx}^2(0, \tau) \, d\tau
\leq \int_0^t \left( w_x^2 + \varepsilon_{xx}^2 + \xi_x^2 + \tilde{\theta}_x^2 \xi_x^2 + \tilde{q}_{xxx}^2 \right)(0, \tau) \, d\tau
\leq \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|_1^2 + \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|_2^2 + \epsilon^{\frac{1}{8}} \tag{4.102}
\]

\[
+ (\epsilon + N(t)) \int_0^t \|(\phi_x, \psi_x, \xi_x)(\tau)\|^2_1 \, d\tau
\]

and

\[
\int_0^t \|\xi_{xx}(\tau)\|^2 \, d\tau
\]
Multiplying (3) by \(\theta\) and integrating, we obtain
\[
\int_t^0 \left( \frac{R\theta}{\rho} \phi_x \right)_t \, dt = \frac{R}{2} \int_t^0 \left( \frac{\phi_{xx}}{\rho} \right)_x \, dx.
\]

Proof. (4.105) by multiplication of \(R\theta\phi_x\), we get
\[
\left( \frac{R\theta}{2} \phi_x \right)_t + \left( \frac{R}{2} \theta u \phi_x + \frac{p}{2} \psi \psi_x \right)_x = \frac{R^2\phi_x^2}{2\rho} - \frac{p}{2} \psi_x^2.
\]
\[
= - R\xi_x R\theta \phi_x^2 + \frac{R}{2} \left( \theta_t - u \theta_x + u_x \theta \right) \phi_x \psi - \frac{R}{2} \theta \psi \left( u \phi_x + \rho \psi \right)
\]
\[
+ \frac{1}{2} p_x \psi_x \psi_x + \frac{R}{2} \psi h_{1x} + \frac{R}{2} \phi_x h_2 - \frac{R^2\phi_x^2}{2\rho} \left( \theta_x \phi + \rho \xi \right).
\]

Multiplying (3.2) by \(\psi\), we have
\[
(C_v \rho \xi \psi_t) - (C_v \rho \xi \psi_t) + p \psi_x^2 + C_v \rho u \xi \psi_x + C_v \rho u \xi \psi_x
\]
\[
+ C_v \rho \xi \psi_t + w_x \psi_x = h_3 \psi_x.
\]

Combining (4.106) with (4.107), we obtain
\[
\left( \frac{R\theta}{2} \phi_x \psi + C_v \rho \xi \psi_t \right)_t + \left( \frac{R}{2} \theta u \phi_x + \frac{p}{2} \psi \psi_x - C_v \rho \xi \psi_t \right)_x + \frac{R^2\phi_x^2}{2\rho} + \frac{p}{2} \psi_x^2
\]
\[
= - R\theta^2 \xi_x \frac{\phi_x^2}{2} + \frac{R}{2} \left( \theta_t - u \theta_x + u_x \theta \right) \phi_x \psi - \frac{R}{2} \theta \psi \left( u \phi_x + \rho \psi \right).
\]
\[
+ \frac{1}{2} p_x \psi_x - \frac{R^2 \theta}{2 \rho} \phi_x \left( \theta_x \phi + \tilde{\theta}_x \xi \right) - w_x \psi_x + \frac{R}{2} \psi h_{1x} + \frac{R \theta}{2 \rho} \phi_x h_2 + h_3 \psi_x
\]
\[
= O(1) \left( |\tilde{\theta}_x| + N(t) \right) |(\phi_x, \psi_x, \xi_x)|^2 + O(1) \left( |\tilde{\theta}_x, \tilde{\theta}_{xx}| \right) \|(\phi, \psi, \xi)\|(\phi, \psi, \xi_x)|
\]
\[
+ O(1) |\xi_x \phi_x + w_x \psi_x|.
\]
\[\text{(4.108)}\]

Integrating (4.108) over \([0, +\infty) \times [0, t]\), it holds
\[
\int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 d\tau \leq \|(\phi_0, \psi_0, \xi_0)\|^2_1 + \|(\phi_t, \psi_t, \xi_t)(0)\|^2 + \left( \frac{1}{4} + \epsilon + N(t) \right) \int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 d\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^+} \left( |(\tilde{\theta}_x, \tilde{\theta}_{xx})|^2 |(\phi, \psi, \xi)|^2 + \xi_x^2 + w_x^2 \right) dx \ d\tau
\]
\[
+ \int_0^t (\phi^2 + \psi^2 + \phi_x^2 + \psi_x^2)(0, \tau)d\tau.
\]
\[\text{(4.109)}\]

Using Lemmas 4.2-4.6, we obtain
\[
\int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 d\tau \leq \|(\phi_0, \psi_0, \xi_0)\|^2_1 + \|(\phi_t, \psi_t, \xi_t)(0)\|^2_1
\]
\[
+ \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|^2 + \epsilon^\frac{1}{4} + \epsilon^\frac{1}{4} \sup_{\tau \in [0, t]} \| w_{tx}(\tau) \|^2_1.
\]
\[\text{(4.110)}\]

Similarly, Multiplying (3.2)_{1xx} by \( R \theta \frac{\phi_{xx}}{2} \) and (3.2)_{3x} by \( \psi_{xx} \), we can also get
\[
\int_0^t \| (\phi_{xx}, \psi_{xx})(\tau) \|^2 d\tau \leq \|(\phi_0, \psi_0, \xi_0)\|^2_1 + \|(\phi_t, \psi_t, \xi_t)(0)\|^2_1
\]
\[
+ \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|^2 + \epsilon^\frac{1}{8} + \epsilon^\frac{1}{8} \sup_{\tau \in [0, t]} \| w_{tx}(\tau) \|^2_1.
\]
\[\text{(4.111)}\]

Using Lemmas 4.2-4.6, it yields (4.104). This completes the proof of Lemma 4.7. \[\Box\]

4.4. Estimates on perturbation \( w \) of the radiative term. Finally, we turn to estimate perturbation \( w \) of the radiative term. First, combining Lemmas 4.2–4.7 together, one has
\[
\sup_{0 \leq \tau \leq t} \left\{ \| (\phi, \psi, \zeta)(\tau) \|^2_2 + \| (\phi_t, \psi_t, \zeta_t)(\tau) \|^2_1 + \| (\phi_{tt}, \psi_{tt}, \zeta_{tt})(\tau) \|^2 \right\}
\]
\[
+ \int_0^t |(\phi, \psi, \phi_x, \psi_x, \xi_x, \phi_t, \psi_t, \phi_{tt}, \psi_{tt}, \phi_{tx}, \psi_{tx}, \phi_{xx}, \psi_{xx}, \phi_{xx}, \psi_{xx}, \xi_{xx})|^2 (0, \tau) d\tau
\]
+ \int_0^t \left( \| (\phi_x, \psi_x, \zeta_x)(\tau) \|_1^2 + \| w(\tau) \|_2^2 + \| w_t(\tau) \|_2^2 + \| w_{tt}(\tau) \|_2^2 \right) d\tau \\
+ \int_0^t |(w_x, w_{xx}, w_{xxx}, w_{tx})|^2(0, \tau)d\tau \\
\lesssim \| (\phi_0, \psi_0, \zeta_0) \|_2^2 + \| (\phi_t, \psi_t, \zeta_t)(0) \|_1^2 + \| (\phi_{tt}, \psi_{tt}, \zeta_{tt})(0) \|^2 \\
+ \epsilon^{\frac{1}{8}} + \epsilon^{\frac{3}{8}} \sup_{\tau \in [0, t]} \| w_{tx}(\tau) \|_1^2. \quad (4.112)

Based on (4.112), we have to estimate \| w(t) \|_3, \| w_t(t) \|_2 and \| w_{tt}(t) \|_1 as follows.

**Lemma 4.8.** Under the same assumptions listed in Proposition 4.1, if \( \epsilon, N(t) \) are suitably small, it holds

\[
\sup_{\tau \in [0, t]} \{ \| w(\tau) \|_3^2 + \| w_t(\tau) \|_2^2 + \| w_{tt}(\tau) \|_1^2 \} \\
\lesssim \| (\phi_0, \psi_0, \zeta_0) \|_2^2 + \| (\phi_t, \psi_t, \zeta_t)(0) \|_1^2 + \| (\phi_{tt}, \psi_{tt}, \zeta_{tt})(0) \|^2 + \epsilon^{\frac{1}{8}}. \quad (4.113)
\]

**Proof.** Multiplying (3.2) by \( w \), we get

\[-(w_x w)_x + aw^2 + w_x^2 + 4b\theta^3 \xi_x w + 4b\tilde{\theta}_x w = \tilde{q}_{xx} w. \quad (4.114)\]

Integrating (4.114) over \([0, +\infty)\), choosing \( \epsilon, N(t) \) suitable small, we have

\[
\int_{\mathbb{R}^+} (aw^2 + w_x^2)(x, t) dx \\
\lesssim \int_{\mathbb{R}^+} (\xi^2_x + \tilde{\theta}_x^2 \xi^2 + \tilde{q}_{xx}^2)(x, t) dx + \| w_x w \|(0, t) \quad (4.115)
\]

and

\[
\| w_x w \|(0, t) \lesssim |\tilde{\phi}_x w_x|(0, t) \lesssim \| \tilde{\phi}_x(t) \|_{\infty} \| w_x(t) \|_{\infty} \\
\lesssim \epsilon^{\frac{1}{8}}(1 + t)^{-\frac{7}{8}} \| w_x(t) \|_{1}^{\frac{1}{2}} \| w_{xx}(t) \|_{1}^{\frac{1}{2}} \\
\lesssim \epsilon^{\frac{1}{8}}(\| w_{xx}(t) \|_{1}^{2} + \| w_x(t) \|_{1}^{2}) + \epsilon^{\frac{1}{8}}. \]

Thus, we obtain

\[
\| w(t) \|_{1}^2 \lesssim \epsilon^{\frac{1}{8}} \| w_{xx}(t) \|_{1}^2 + \| \xi(t) \|_{1}^2 + \epsilon^{\frac{1}{8}}. \quad (4.116)
\]

On the other hand, we see

\[
-w_{xxxx} + aw_{xx} + 4b\theta^3 \xi_{xx} + 24b\theta^2 \theta_{xx} \xi_{xx} + 4b \left( \theta^3 \right)_{xx} \xi_x \\
= \left\{ \tilde{q}_{xx} - 4b\tilde{\theta}_x \xi \left( \theta^2 + \tilde{\theta} \tilde{\theta} + \tilde{\theta}^2 \right) \right\}_{xx}. \quad (4.117)
\]
Multiplying (4.117) by $w_{xx}$, we get
\[-(w_{xx} w_{xxx} - 4b \theta^3 \xi_{xx} w_{xx})_x + w_{xxx}^2 + aw_{xx}^2 - 4b \theta^3 \xi_{xx} w_{xxx} = -12b \theta^2 \theta_x \xi_{xx} w_{xx} - 4b (\theta^2)_{xx} \xi_x w_{xx} + \left\{ q_{xx} - 4b \ddot{\theta}_x \xi (\theta^2 + \ddot{\theta} + \ddot{\theta}^2) \right\}_{xx} w_{xx}, \tag{4.118} \]

Since
\[w_{xx} = aw + 4b \theta^3 \xi_x + 4b \ddot{\theta}_x \xi (\theta^2 + \ddot{\theta} + \ddot{\theta}^2) - q_{xx},\]
and
\[w_{xxx} - 4b \theta^3 \xi_{xx} = aw_x + 12b \theta^2 \theta_x \xi_x + \left\{ 4b \ddot{\theta}_x \xi (\theta^2 + \ddot{\theta} + \ddot{\theta}^2) \right\}_x - q_{xxx},\]
one has
\[w_{xx} (w_{xxx} - 4b \theta^3 \xi_{xx}) (0, t) \]
\[= \left\{ aw_x + 12b \theta^2 \theta_x \xi_x + \left[ 4b \ddot{\theta}_x \xi (\theta^2 + \ddot{\theta} + \ddot{\theta}^2) \right]_x - q_{xxx} \right\} (aw + 4b \theta^3 \xi_x - q_{xx}) (0, t) \]
\[= O(1) \left( w^2 + w_{x}^2 + \xi_{x}^2 + \ddot{\theta}_x \xi_{x}^2 + \ddot{\theta}_x \xi_{xx}^2 + \ddot{\theta}_x q_{xx}^2 + \ddot{\theta}_x q_{xxx} \right) (0, t).\]

Integrating (4.118) over $[0, +\infty)$, choosing $\epsilon, N(t)$ suitable small, we have
\[\int_{\mathbb{R}^+} (aw_{xx}^2 + w_{xxx}^2) (x, t) dx \]
\[\lesssim \int_{\mathbb{R}^+} \left( \xi_{xx}^2 + \ddot{\theta}_x \xi_{xx}^2 + q_{xx}^2 \right) (x, t) dx \tag{4.119} \]
\[+ \left( w^2 + w_x^2 + \xi_x^2 + \ddot{\theta}_x \xi_x^2 + \ddot{\theta}_x \xi_{xx}^2 + \ddot{\theta}_x q_{xx}^2 + \ddot{\theta}_x q_{xxx} \right) (0, t),\]
where
\[\xi_x^2 (0, t) \lesssim \|\xi_x (t)\|_\infty^2 \lesssim \|\xi_x (t)\|_1^2, \tag{4.120} \]
\[w_x^2 (0, t) \lesssim \|w_x (t)\|_\infty^2 \lesssim \|w_x (t)\| \|w_{xx} (t)\| \]
\[\lesssim \frac{1}{8} \|w_{xx} (t)\|^2 + \|w_x (t)\|^2. \tag{4.121} \]
Combining (4.116), (4.119)-(4.121) with (4.112), we obtain
\[\|w(t)\|_3^2 \lesssim \|\xi (t)\|_2^2 + \epsilon^\frac{1}{2} \]
\[\lesssim \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|_1^2 \tag{4.122} \]
\[+ \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|^2 + \epsilon^\frac{1}{2} + \epsilon^\frac{1}{2} \sup_{\tau \in [0, t]} \|w_{tx}(\tau)\|_1^2. \]
On the other hand, multiplying \((3.2)_t\) by \(w_t\), we get
\[- (w_t w_{tx})_x + w_{tx}^2 + aw_t^2 + 4b\theta^3 w_t \xi_{tx}\]
\[= \left\{ \tilde{h}_4 - 4b\tilde{\theta}_x \xi_t \left( \theta^2 + \tilde{\theta} + \bar{\theta}^2 \right) \right\} w_t. \tag{4.123} \]

Multiplying \((4.24)_x\) by \(w_{tx}\), we have
\[- (w_t w_{tx} + 4b\theta^3 w_{tx} \xi_{tx})_x + w_{tx}^2 + aw_{tx}^2 - 4b\theta^3 w_{txx} \xi_{tx}\]
\[= w_{tx} \left\{ \tilde{h}_4 - 4b\tilde{\theta}_x \xi_t \left( \theta^2 + \tilde{\theta} + \bar{\theta}^2 \right) \right\}. \tag{4.124} \]

Similarly, we see that
\[w_{xx} = aw + 4b\theta^3 \xi_x + 4b\tilde{\theta}_x \xi \left( \theta^2 + \tilde{\theta} + \bar{\theta}^2 \right) - \tilde{q}_{xx}\]
and
\[w_{txx} - 4b\theta^3 \xi_{tx} = aw_t + 12b\theta^2 \theta_t \xi_x + \left\{ 4b\tilde{\theta}_x \xi \left( \theta^2 + \tilde{\theta} + \bar{\theta}^2 \right) \right\}_t - \tilde{q}_{txx}. \]

Thus, we obtain
\[w_{tx} \left( w_{txx} - 4b\theta^3 \xi_{tx} \right) (0, t)\]
\[= w_{tx} \left( aw_t + 12b\theta^2 \theta_t \xi_x - \tilde{q}_{txx} \right) (0, t)\]
\[= O(1) \left( w_t w_{tx} + \tilde{\theta}_t \xi_x w_{tx} + \tilde{q}_{txx} w_{tx} \right) (0, t). \]

Integrating \((4.123), (4.124)\) over \([0, +\infty)\), choosing \(\epsilon\) and \(N(t)\) suitable small, it holds by \((4.112)\)
\[
\|w_t(t)\|_2^2 \lesssim \int_{\mathbb{R}^+} \left( \xi_x^2 + \tilde{\theta}_x^2 \xi_t^2 + \tilde{\theta}_x^2 \xi_{tx} + \tilde{h}_4^2 + \tilde{h}_{4x}^2 \right) \, dx \tag{4.125} \]
\[+ \left| w_t w_{tx} + \xi_x^2 + \tilde{\theta}_t^2 w_{tx}^2 + \tilde{q}_{xx} w_{tx} \right| (0, t), \]

where \(\xi_x^2(0, t) \lesssim \|\xi_x(t)\|_1^2\) and
\[
\left| \left( w_t + \tilde{\theta}_t^2 + \tilde{q}_{txx} \right) w_{tx} \right| (0, t)
\lesssim \epsilon^{\frac{1}{3}} (1 + t)^{-\frac{7}{2}} \|w_{tx}(t)\|_{\infty} \lesssim \epsilon^{\frac{1}{3}} (1 + t)^{-\frac{7}{8}} \|w_{tx}(t)\|^{\frac{1}{2}} \|w_{txx}(t)\|^{\frac{1}{2}}
\lesssim \epsilon^{\frac{1}{3}} \|w_{tx}(t)\| \|w_{txx}(t)\| + \epsilon^{\frac{1}{3}} (1 + t)^{-\frac{5}{8}}
\lesssim \epsilon^{\frac{1}{3}} (\|w_{tx}(t)\|^2 + \|w_{txx}(t)\|^2) + \epsilon^{\frac{1}{3}}. \tag{4.126} \]

Thus, we get
\[
\|w_t(t)\|_2^2 \lesssim \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|_1^2 + \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|^2 + \epsilon^{\frac{1}{3}}. \tag{4.127} \]
At last, multiplying (4.24)\(_t\) by \(w_{tt}\), we have
\[
(-w_{tt}w_{tx} + 4b\theta^3\xi_{tt}w_{tt})_x + w_{tt}^2 + aw_{tt}^2 + 12b\theta^2\theta_x\xi_{tt}w_{tt}
\]
\[
= (4b\theta^3w_{tt})_x\xi_{tt} + w_{tt}\left\{\tilde{h}_4 - \left[4b\theta_x\xi\left(\theta^2 + \theta\bar{\theta} + \bar{\theta}^2\right)\right]_t\right\}.
\]
(4.128)
Integrating (4.128) over \([0, +\infty)\), choosing \(\epsilon\) and \(N(t)\) suitable small, we obtain from (4.112)
\[
\|w_{tt}(t)\|_1^2 \lesssim \int_{\mathbb{R}^+} \left(\xi_{tx}^2 + \xi_{tt}^2 + \xi_t^2 + \xi^2 + \tilde{h}_{4t}\right)(x, t)dx + |w_{tt}w_{tx}|(x, 0)
\]
\[
\lesssim \|(\phi_0, \psi_0, \xi_0)\|_2^2 + \|(\phi_t, \psi_t, \xi_t)(0)\|_1^2 + \|(\phi_{tt}, \psi_{tt}, \xi_{tt})(0)\|_2^2 + \epsilon^{\frac{1}{8}}.
\]
(4.129)
Combining (4.119), (4.127) with (4.129), one has (4.113). This completes the proof of Lemma 4.8.

Finally, combining (4.112) and (4.113), we get (4.2) and complete the proof of Proposition 4.1.

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REFERENCES


