Nodal solutions to quasilinear elliptic problems involving the 1-laplacian operator via variational and approximation methods

Giovany M. Figueiredo
E-mail adress: giovany@unb.br
Departamento de Matemática, Universidade de Brasília - UNB
CEP: 70910-900, Brasilia-DF, Brazil,
Marcos T. O. Pimenta
E-mail address: marcos.pimenta@unesp.br
Departamento de Matemática e Computação, Universidade Estadual Paulista - Unesp
CEP: 19060-900, Presidente Prudente - SP, Brazil.

Abstract
In this work we use two different methods to get nodal solutions to quasilinear elliptic problems involving the 1−Laplacian operator. In the first one, we develop an approach based on a minimization of the energy functional associated to a problem involving the 1−Laplacian operator in $\mathbb{R}^N$, on a subset of the Nehari set which contains just sign-changing functions. In the second part we obtain a nodal solution to a quasilinear elliptic problem involving the 1−Laplacian operator in a bounded domain, through a thorough analysis of the sequence of solutions of the $p$−Laplacian problem associated to it, as $p \to 1^+$. In both cases, several technical difficulties appear in comparison with the related results involving signed solutions.

Keywords: 1-Laplacian operator, Nehari method, Nodal solutions

2010 Mathematics Subject Classification: 35J62, 35J75

1 Introduction

As far as works involving the 1−Laplacian operator are regarded, the pioneering works involving this operator were written by F. Andreu, C. Ballesteler, V. Caselles and J.M. Mazón in a series of papers (among them [2, 3, 4]), which gave rise to the monograph [5]. Indeed, in [2], the authors characterize the imprecise quotient $\frac{Du}{|Du|}$ (when $Du$ is just a Radon measure, rather than an $L^1$ function), through the Pairing Theory of G. Anzellotti (see [6] and also [5]). This theory allows them to introduce a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ which plays the role of $\frac{Du}{|Du|}$. Among the very first works on this issue we could cite the works of Kawohl [19] and also Demengel [16], where in the later, the author used the symmetry of the domain to get nodal solutions to problems involving the 1−Laplacian operator and a nonlinearity with critical growth.

Since the works mentioned above, the interest of the mathematical community in problems involving the 1−Laplacian operator had grown a lot. Among the motivations to study such kind of problems, we could cite the method developed by Rudin, Osher and Fatemi in [26] to tackle on problems involving image restoration.
In [22, 23], the authors study the linear problem to the $1$–laplacian operator, given by

$$
\begin{aligned}
-\text{div} \left( \frac{D u}{|D u|} \right) &= f(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
$$

(1.1)

with $f$ belonging to some Lebesgue spaces. In these works they reached some unexpected conclusions, as for example the fact that bounded variation solutions to (1.1) just exist when $f$ is small enough. Also, uniqueness does not hold at all, since if $u \in BV(\Omega)$ satisfies (in some sense) (1.1), then $g(u)$ also satisfies for all smooth increasing function $g : \mathbb{R} \to \mathbb{R}$. In [20] and [21], the authors dealt with the problem

$$
\begin{aligned}
-\text{div} \left( \frac{D u}{|D u|} \right) + |Du| &= f(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
$$

where $f \in L^q(\Omega)$, with $q = 1$ in [20] and $q > N$ in [21]. In [10], the authors study the Gelfand problem associated to an operator that can be view as a convex combination of the laplacian and the $\infty$–laplacian operators. In this work, a comparison principle to this operator is proved, which seems to be interesting by its own.

Since the pioneering works of Nehari, there were several articles using the so called Nehari method to find solutions of semilinear and quasilinear elliptic problems. Actually this method has the advantage of providing ground-state solutions (rather than just nontrivial ones) and can be found in [25]. In [9], the authors adapted the Nehari method to get sign-changing solutions to problems involving the laplacian operator by minimizing the energy functional on a subset of the Nehari manifold. In [17] the authors were the first to show the existence of ground-state solutions to problems involving the $1$–Laplacian operator (and also the $1$–biharmonic operator) through the minimization of the energy functional on a natural constraint of the Nehari type. Taking these works into account, one could wonder whether it is possible to apply the approach proposed in [9], to get nodal solutions of problems involving the $1$–Laplacian operator. This is not a trivial question, since in the definition of the candidate to be the natural constraint, it is involved the following directional derivatives,

$$
\Phi'(u)u^\pm,
$$

where $\Phi$ is the energy functional associated to the problem involving the $1$–Laplacian operator and $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. In fact, even though $u^\pm \in BV(\Omega)$ if $u \in BV(\Omega)$, in order to have well defined such directional derivatives, it is necessary to verify some conditions involving $|Du|^*$ and $|Du^+|^*$, the singular part of the total variation of the derivatives of $u$ and $u^+$, respectively.

The purpose of the the first part of this work is to prove the existence of sign-changing solutions to a problem involving the $1$–Laplacian operator, based on the analysis of the Nehari set associated to it. We apply this method to find nodal solutions of the following problem involving the $1$–Laplacian operator in $\mathbb{R}^N$,

$$
-\text{div} \left( \frac{D u}{|D u|} \right) + \frac{u}{|u|} = f(u) \quad \text{in } \mathbb{R}^N,
$$

(1.2)

where $N \geq 2$ and $f$ satisfies the following set of assumptions:

$(f_1)$ $f \in C^0(\mathbb{R})$;

$(f_2)$ $\lim_{|s| \to 0} f(s) = 0$;

$(f_3)$ there exist constants $c_1, c_2 > 0$ and $q \in (1, 1^*)$ such that

$$
|f(s)| \leq c_1 + c_2 |s|^{q-1}, \quad s \in \mathbb{R},
$$

where $1^* = N/(N - 1)$;
Remark 1.1 In the last result, by saying that $\partial$ signed solutions to the following problem involving this operator, we apply such method to deal with (1.2), we believe it could be applied for several other problems.

Then there exists $u_0 \in BV(\mathbb{R}^N)$ such that $u_0$ changes sign, then such a solution has the lowest energy level among all the nodal ones. Although $u_0 \in BV(\mathbb{R}^N)$ such that $\|u\|_\infty \leq 1$, $\|\gamma\|_\infty \leq 1$, and

\[
\begin{cases}
-\text{div} \ z + \gamma = f(u_0) \text{ in } D'(\mathbb{R}^N), \\
(z, Du_0) = |Du_0| \text{ in the sense of measures}, \\
\gamma u_0 = |u_0| \text{ a.e. in } \mathbb{R}^N.
\end{cases}
\]

In the proof of this result we shall prove that the minimum of the energy functional $\Phi(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx - \int_{\mathbb{R}^N} F(u) dx$, on the following set

$\mathcal{N}^\pm = \{ u \in BV_{rad}(\mathbb{R}^N); u^\pm \neq 0 \text{ and } \Phi'(u)u^\pm = 0 \}$,

is in fact a sign-changing bounded variation solution of (1.2). Moreover, since $\mathcal{N}^\pm$ contains all the sign-changing solutions, then such a solution has the lowest energy level among all the nodal ones. Although we apply such method to deal with (1.2), we believe it could be applied for several other problems involving this operator.

In order to motivate the second part of this paper, note that in [24], S. Segura and A. Molino obtain signed solutions to the following problem

\[
\begin{cases}
-\text{div} \left( \frac{Du}{|Du|} \right) = f(x, u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where $f$ has a superlinear and subcritical growth at infinity. Their approach is based on an approximation of (1.3) by similar problems involving the $p$–Laplacian operator and several technical difficulties have been faced. By studying this problem, one could raise the question if it is possible to use the approximation method to show the existence of sign-changing solutions to (1.3). In the second part of this paper we tackle on this question and prove the following result.

Theorem 1.2 Assume that $(f_1), (f_3), (f_4) \text{ and } (f_5)$ are satisfied. Moreover, suppose the following condition holds:

$(f_5)'$ there exists $\alpha > 0$ such that $\limsup_{s \to 0} \frac{f(s)}{|s|^\alpha} < +\infty$.

Then there exists $u_0 \in BV(\Omega)$, a sign-changing bounded variation solution of (1.2), i.e., there exists $u_0 \in BV(\Omega)$ such that $u_0$ changes sign, $z \in L^\infty(\Omega, \mathbb{R}^N)$ such that $\|z\|_\infty \leq 1$ and

\[
\begin{cases}
-\text{div} \ z = f(u_0) \text{ a.e. in } \Omega, \\
z, Du_0 = |Du_0| \text{ in the sense of measures}, \\
z, \nu \in \text{sign}(-u_0) \text{ on } \partial \Omega.
\end{cases}
\]

Remark 1.1 In the last result, by saying that $[z, \nu] \in \text{sign}(-u_0)$ we mean that $[z, \nu](-u_0) = |u_0|$ on $\partial \Omega$. 

(f_4) there exists $\kappa > 1$ such that

\[0 < \kappa F(s) \leq f(s)s,\]

for all $s \neq 0$;

(f_5) $f$ is increasing for $s \in \mathbb{R}$.
In the proof of Theorem 1.2, the fact that $\Omega$ has finite Lebesgue measure is used in a crucial way. The application of this approximation technique in problems involving the $1$–Laplacian operator in infinite Lebesgue measure domains is, up to our knowledge, an open question.

This paper is organized as follows. In Section 2 we present some definitions and basic results about functions of bounded variation. In Section 3 we prove Theorem 1.1 by using variational arguments. In Section 4 we prove Theorem 1.2 using arguments based on an approximation of the original problem.

## 2 Basic results in the space of functions of bounded variation

First of all let us introduce the space of functions of bounded variation, $BV(\Omega)$. We say that $u \in BV(\Omega)$, or is a function of bounded variation, if $u \in L^1(\Omega)$, and its distributional derivative $Du$ is a vectorial Radon measure, i.e.,

$$BV(\Omega) = \{ u \in L^1(\Omega); Du \in \mathcal{M}(\Omega, \mathbb{R}^N) \}.$$  

It can be proved that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and

$$\int_{\Omega} |Du| := \sup \left\{ \int_{\Omega} u \text{div} \phi dx; \phi \in C^1_c(\Omega, \mathbb{R}^N), \|\phi\|_{\infty} \leq 1 \right\} < +\infty.$$

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV} := \int_{\Omega} |Du| + \int_{\Omega} |u|dx,$$

which is continuously embedded into $L^r(\Omega)$ for all $r \in [1, 1^*)$, where $1^* = N/(N - 1)$. Since the domain $\Omega$ is bounded, it holds also the compactness of the embeddings of $BV(\Omega)$ into $L^r(\Omega)$ for all $r \in [1, 1^*)$.

As one can see in [7], the space $BV(\Omega)$ has different convergence and density properties than the usual Sobolev spaces. For example, $C^\infty(\overline{\Omega})$ is not dense in $BV(\Omega)$ with respect to the strong convergence, since the closure of $C^\infty(\overline{\Omega})$ in the norm of $BV(\Omega)$ is equal to $W^{1,1}(\Omega)$, which is a proper subspace of $BV(\Omega)$. This is the motivation to define a weaker sense of convergence in $BV(\Omega)$, called the strict convergence. We say that $(u_n) \subset BV(\Omega)$ converges to $u \in BV(\Omega)$ in the sense of the strict convergence if

$$u_n \rightarrow u, \quad \text{in } L^1(\Omega)$$

and

$$\int_{\Omega} |Du_n| \rightarrow \int_{\Omega} |Du|,$$

as $n \rightarrow \infty$. Fortunately, with respect to the strict convergent, $C^\infty(\overline{\Omega})$ is dense in $BV(\Omega)$. In [1] one can see also that it is well defined a trace operator $BV(\Omega) \hookrightarrow L^1(\partial \Omega)$, in such a way that

$$\|u\| := \int_{\Omega} |Du| + \int_{\partial \Omega} |u|d\mathcal{H}^{N-1},$$

is a norm equivalent to $\| \cdot \|_{BV}$.

For a vectorial Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual decomposition stated in the Radon-Nikodym Theorem, where $\mu^a$ and $\mu^s$ are, respectively, the absolutely continuous and the singular parts with respect to the $N$–dimensional Lebesgue measure $\mathcal{L}^N$. We denote by $|\mu|$, the total variation of $\mu$, the scalar Radon measure defined as in [7][pg. 125]. By $\frac{\mu}{|\mu|}(x)$ we denote the usual Lebesgue derivative of $\mu$ with respect to $|\mu|$, given by

$$\frac{\mu}{|\mu|}(x) = \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))},$$
Up to now, given a function \( u \in BV(\Omega) \), we can decompose its distributional derivative as
\[
Du = D^u + D^s u,
\]
where \( D^u \) is absolutely continuous w.r.t. the Lebesgue measure \( \mathcal{L}^N \), while \( D^s u \) is singular w.r.t. the same measure. Nevertheless, it is possible to study deeper the measure \( D^s u \), by dealing with its Jump and Cantor parts, \( D^J u \) and \( D^C u \), respectively. For defining them, we closely follow \([1]\). First of all, for a given \( u \in BV(\Omega) \), we say that \( x \in \Omega \) is a \textit{Lebesgue point} for \( u \) if there exists \( \tilde{u}(x) \) such that
\[
\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x)} |u(y) - \tilde{u}(x)| dy = 0. \tag{2.1}
\]
We denote \( S_u \) the set of points of \( \Omega \) which are not Lebesgue points, i.e., \( x \in \Omega \setminus S_u \) if and only if there exists \( \tilde{u}(x) \) such that (2.1) holds.

For a function \( u \in BV(\Omega) \) we say that \( x \in \Omega \) is an \textit{approximate jump point} if there exist real numbers \( u^-(x) < u^+(x) \) and a vector \( \nu_u \in S^{N-1} \) such that
\[
\lim_{\rho \to 0} \frac{1}{\rho^N} \int_{B_\rho(x)} |u(y) - u^\pm(x)| dy = 0,
\]
where \( B_\rho^+(x) = \{ y \in B_\rho(x); \langle y - x, \nu_u \rangle > 0 \} \) and \( B_\rho^-(x) = \{ y \in B_\rho(x); \langle y - x, \nu_u \rangle < 0 \} \). The set of jump points of \( u \) is denoted by \( J_u \). By Federer-Vol’pert’s Theorem (see \([1]\)[Theorem 3.78]), for every \( u \in BV(\Omega) \), \( S_u \) is countably \( \mathcal{H}^{N-1} \)-rectifiable, i.e., there exist countably many Lipschitz hypersurfaces \( \Gamma_k \) such that
\[
\mathcal{H}^{N-1}(S_u \setminus \bigcup_{k \in \mathbb{N}} \Gamma_k) = 0.
\]
Moreover, \( \mathcal{H}^{N-1}(S_u \setminus J_u) = 0 \). We denote
\[
D^J u = D^s u |_{J_u} \quad \text{and} \quad D^C u = D^s u |_{(\Omega \setminus S_u)},
\]
the Jump and Cantor parts of \( Du \), respectively. The jump part \( D^J u \) can also be characterized as
\[
D^J(u) = (u^+ - u^-)\nu_u \mathcal{H}^{N-1}|_{J_u}.
\]

For a given \( u \in BV(\Omega) \), in several arguments it is very important to work with a representative that arise as the limit of mollifications of \( u \). This is the \textit{precise representative} of \( u \), denoted by \( u^* \) and is defined to be equal to \( \tilde{u}(x) \), for \( x \in \Omega \setminus S_u \) and \( (u^+(x) + u^-(x))/2 \), for \( x \in J_u \). Hence, if \( (\rho_\varepsilon)_\varepsilon \) is an \textit{mollifier} family, then \( u \ast \rho_\varepsilon \to u^* \) pointwisely, as \( \varepsilon \to 0^+ \).

In several arguments we use in this work, it is mandatory to have a sort of \textit{Green’s formula} to expressions like \( w \text{ div}(z) \), where \( z \in L^\infty(\Omega, \mathbb{R}^N) \), \( \text{div}(z) \in L^N(\Omega) \) and \( w \in BV(\Omega) \). For this we have to somehow deal with the \textit{product} between \( z \) and \( Dw \), which we denote by \( (z, Dw) \). This can be done through the “pairings theory”, developed by Anzellotti in \([6]\) and independently by Frid and Chen in \([13]\). Despite all the necessary results can be found in \([24]\), we state them here for the sake of completeness.

Let us denote
\[
X_N(\Omega) = \{ z \in L^\infty(\Omega, \mathbb{R}^N); \text{div}(z) \in L^N(\Omega) \}.
\]
For \( z \in X_N(\Omega) \) and \( w \in BV(\Omega) \), we define the distribution \( (z, Dw) \in D'(\Omega) \) as
\[
\langle (z, Dw), \varphi \rangle := - \int_{\Omega} w \varphi \text{ div}(z) dx - \int_{\Omega} wz \cdot \nabla \varphi dx,
\]
for every \( \varphi \in D(\Omega) \). With this definition, it can be proved that \( (z, Dw) \) is in fact a Radon measure such that
\[
\left| \int_B (z, Dw) \right| \leq \| z \|_\infty \int_B |Dw|, \tag{2.2}
\]
for every Borel set $B \subset \Omega$.

In order to define an analogue of the Green’s Formula, it is also necessary to describe a weak trace theory for $z$. In fact, there exists a trace operator $[\cdot, \nu] : X_N(\Omega) \to L^\infty(\partial \Omega)$ such that

$$
\| [z, \nu] \|_{L^\infty(\partial \Omega)} \leq \| z \|_{\infty}
$$

(2.3)

and, if $z \in C^1(\overline{\Omega}_\delta, \mathbb{R}^N)$,

$$
[z, \nu](x) = z(x) \cdot \nu(x) \quad \text{on } \Omega_\delta,
$$

where by $\Omega_\delta$ we denote a $\delta$-neighborhood of $\partial \Omega$. With these definitions, it can be proved that the following Green’s formula hold for every $z \in X_N(\Omega)$ and $w \in BV(\Omega),$

$$
\int_{\Omega} w \div(z) dx + \int_{\Omega} (z, Dw) = \int_{\partial \Omega} [z, \nu] w d\mathcal{H}^{N-1}.
$$

(2.4)

### 3 Existence by minimization on the Nehari nodal set

In this section, we denote $\| \cdot \|$ as the usual norm $\| \cdot \|_{BV(\mathbb{R}^N)}$, i.e.,

$$
\| u \| = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx,
$$

for $u \in BV(\mathbb{R}^N)$.

For a given $u \in BV(\mathbb{R}^N)$, we consider $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$, in such a way that $u = u^+ + u^-$. Moreover, note that since $BV(\mathbb{R}^N)$ is a lattice, it follows that $u^\pm \in BV(\mathbb{R}^N)$. In the next result let us state the relation satisfied by $Du$, $Du^+$ and $Du^-.$

**Lemma 3.1** Let $\Omega \subset \mathbb{R}^N$ be a domain (bounded or unbounded). If $u \in BV(\Omega)$, then for every Borel set $F \subset \Omega$,

$$
\int_F |Du| = \int_F |Du^+| + \int_F |Du^-|.
$$

**Proof.** First of all let us recall the coarea formula for $BV(\Omega)$ functions (see [1][Theorem 3.40] or [7][Theorem 10.3.3]), which states that for $u \in BV(\Omega)$ and $F \subset \Omega$ a Borel set, we have that

$$
\int_F |Du| = \int_{-\infty}^{+\infty} \int_F |D\chi_{E_t}| dt, \quad \text{(3.1)}
$$

where $E_t := \{x \in \Omega; u(x) > t\}$. Moreover, let us denote $E_t^+ := \{x \in \Omega; u^+(x) > t\}$ and $E_t^- := \{x \in \Omega; u^-(x) > t\}$. Then

$$
\int_F |Du| = \int_{-\infty}^{+\infty} \int_F |D\chi_{E_t^+}| dt
$$

$$
= \int_{-\infty}^{0} \int_F |D\chi_{E_t^+}| dt + \int_{0}^{+\infty} \int_F |D\chi_{E_t^+}| dt
$$

$$
= \int_{-\infty}^{+\infty} \int_F |D\chi_{E_t^+}| dt + \int_{-\infty}^{+\infty} \int_F |D\chi_{E_t^-}| dt
$$

$$
= \int_F |Du^+| + \int_F |Du^-|,
$$

6
where the third equality follows since
\[ \text{if } t < 0, E^-_t = E_t \text{ and } E^+_t = \emptyset \]
and
\[ \text{if } t > 0, E^-_t = \emptyset \text{ and } E^+_t = E_t, \]
while the last one follows from (3.1).

\[ \square \]

**Remark 3.1** From the last result we can easily see that if \( u \in BV(\mathbb{R}^N) \), then
\[ \|u\| = \|u^+\| + \|u^-\|. \]

Let us define the following functionals on \( BV(\mathbb{R}^N) \), \( J, F : BV(\mathbb{R}^N) \to \mathbb{R} \), given by
\[ J(u) = \|u\| \]
and
\[ F(u) = \int_{\mathbb{R}^N} F(u) dx. \]

Also, let us consider \( \Phi : BV(\mathbb{R}^N) \to \mathbb{R} \) given by
\[ \Phi(u) = J(u) - F(u). \]

Note that, from (f3), \( F \) is well defined and in fact, \( F \in C^1(BV(\mathbb{R}^N)) \). Then \( \Phi \) is written as the difference between a convex and Lipschitz functional \( J \), and the \( C^1 \) one \( F \). Then, it is well defined in the sense of the theory developed by Chang in [12] and Clarke in [14], the subdifferential of \( \Phi \), \( \partial \Phi \). We say that \( u \in BV(\mathbb{R}^N) \) is a critical point of \( \Phi \) if \( 0 \in \partial \Phi(u) \). Hence, since \( F \) is smooth, \( 0 \in \partial \Phi(u) \) is equivalent to \( F'(u) \in \partial J(u) \). Finally, since \( J \) is convex, then this is equivalent to
\[ J(v) - J(u) \geq F'(u)(v - u), \quad \forall v \in BV(\mathbb{R}^N). \tag{3.2} \]

Although \( \Phi \) is not a smooth functional, it is possible to calculate some directional derivatives of \( \Phi \), as one can see in [6]. More precisely, \( J'(u)v \) is well defined for every \( v \in BV(\mathbb{R}^N) \) such that \( D^a v \) vanishes a.e. in the set where \( D^a u \) vanishes, \( |D^a v| \) is absolutely continuous with respect to \( |D^a u| \) and \( v(x) = 0 \), a.e. in the set \( \{ x \in \mathbb{R}^N; u(x) = 0 \} \). Moreover, we have that
\[ J'(u)v = \int_{\Omega} \frac{D^a u D^a v}{|D^a u|} dx + \int_{\Omega} \frac{D u}{|D u|} (x) \frac{D v}{|D v|} (x) |D^a v| + \int_{\mathbb{R}^N} \text{sign}(u) v dx, \tag{3.3} \]
where we define \( \text{sign}(u) \) as a measurable function such that \( \text{sign}(u) u = |u| \).

In order to define what we call the Nehari nodal set, it is going to be crucial to calculate the directional derivatives \( \Phi'(u)u^\pm \), for a given function \( u \in BV(\mathbb{R}^N) \). In order to do so, we need to prove that
\[ |D^a u^\pm| \text{ is absolutely continuous with respect to } |D^a u|, \tag{3.4} \]
since the other two conditions are trivially satisfied. In order to prove (3.4), note that, if \( F \subset \mathbb{R}^N \) is a Borel set such that \( \int_F |D^a u| = 0 \), then by Lemma 3.1,
\[
\int_F |D^a u| dx + \int_F |D^a u| = \int_F |Du| = \int_F |Du^+| + \int_F |Du^-| = \int_F |D^a u^+| dx + \int_F |D^a u^-| dx + \int_F |D^a u^-| dx + \int_F |D^a u^-|. \tag{3.5}
\]
Moreover, note that, if \( u \) and \( u^+ \) and \( u^- \) have disjoint supports, it follows that
\[
\int_F |D^a u| dx = \int_F |D^a u^+| dx + \int_F |D^a u^-| dx.
\] (3.6)

Then, from (3.5) and (3.6) and since \( \int_F |D^a u| = 0 \), it follows that
\[
0 = \int_F |D^a u^+| + \int_F |D^a u^-|,
\]
what implies that \( \int_F |D^a u^\pm| = 0 \) and shows that (3.4) holds.

In order to deal with the lack of compactness of the Sobolev embeddings of \( BV(\mathbb{R}^N) \), we use a result of [18], in which it is proved the compactness of the embeddings of the space of functions of bounded variation in \( \mathbb{R}^N \) which are radially symmetric, \( BV_{rad}(\mathbb{R}^N) \), into the Lebesgue spaces \( L^r(\mathbb{R}^N) \), for \( 1 < r < 1^* \). In the light of this fact, we consider from now on all the functionals \( J, F \) and \( \Phi \), restricted to \( BV_{rad}(\mathbb{R}^N) \).

Let us define the **Nehari set**, which consists in a set containing all the nontrivial solutions of (1.2),
\[
\mathcal{N} = \{ u \in BV_{rad}(\mathbb{R}^N); u \neq 0 \text{ and } \Phi'(u)u = 0 \}.
\]

It is well known (see [17]) that for each \( u \in BV_{rad}(\mathbb{R}^N) \), \( u \neq 0 \), there exists a unique \( t_u > 0 \) such that
\[
\Phi(t_u u) = \max_{t>0} \Phi(tu). \tag{3.7}
\]

Moreover, it also holds that \( t_u u \in \mathcal{N} \). Hence, if \( u \in \mathcal{N} \), then \( t_u = 1 \).

Taking into account (3.4), it is well defined the following set, which we call the **nodal Nehari set**,
\[
\mathcal{N}^\pm = \{ u \in BV_{rad}(\mathbb{R}^N); u^\pm \neq 0 \text{ and } \Phi'(u)u^\pm = 0 \}.
\]

By (3.3) we can also write,
\[
\mathcal{N}^\pm = \left\{ u \in BV_{rad}(\mathbb{R}^N); u^\pm \neq 0 \text{ and } \|u^\pm\| = \int_{\mathbb{R}^N} f(u^\pm) u^\pm dx \right\}.
\]

The next result proves that all sign-changing critical points of \( \Phi \) belong to \( \mathcal{N}^\pm \).

**Lemma 3.2** If \( u \in BV_{rad}(\mathbb{R}^N) \) is such that \( u^\pm \neq 0 \) and \( 0 \in \partial \Phi(u) \), then \( u \in \mathcal{N}^\pm \).

**Proof.** First of all, note that by Remark 3.1,
\[
J(u) = J(u^+) + J(u^-), \quad \forall u \in BV(\mathbb{R}^N). \tag{3.8}
\]
Moreover, note that, if \( u \in BV(\mathbb{R}^N) \), then
\[
\begin{align*}
(u + tu^+)^+ &= u^+ + tu^+ & \text{if } t > -1, \\
(u + tu^+)^- &= u^- & \text{if } t \in \mathbb{R}, \\
(u + tu^-)^+ &= u^+ & \text{if } t \in \mathbb{R}, \\
(u + tu^-)^- &= u^- + tu^- & \text{if } t > -1.
\end{align*} \tag{3.9}
\]

Now, suppose that \( u \in BV_{rad}(\mathbb{R}^N) \) is such that \( u^\pm \neq 0 \) and \( 0 \in \partial \Phi(u) \). Then, by taking in (3.2) \( v = u + tu^+ \) as a test function, and taking into account (3.8) and (3.9) we have that:
\[
\text{if } t \to 0^+, \\
\lim_{t \to 0^+} \frac{J(u^+ + tu^+) - J(u^+)}{t} \geq \int_{\mathbb{R}^N} f(u^+) u^+ dx.
\]

8
if $t \to 0^-$,
\[
\lim_{t \to 0^-} \frac{J(u^+ + tu^+) - J(u^+)}{t} \leq \int_{\mathbb{R}^N} f(u^+) u^+ dx.
\]
Then,
\[
\|u^+\| = \int_{\mathbb{R}^N} f(u^+) u^+ dx.
\]

Analogously, one can check that, by taking $v = u + tu^-$ as a test function in (3.2) and doing $t \to 0^+$,
\[
\|u^-\| = \int_{\mathbb{R}^N} f(u^-) u^- dx,
\]
which implies that in fact $u \in \mathcal{N}^\pm$.

**Lemma 3.3** Let $u \in BV_{rad}(\mathbb{R}^N)$ such that $u^\pm \neq 0$. Then there exist a unique pair $(t_u, s_u) \in (0, +\infty) \times (0, +\infty)$ such that $t_u u^+ + s_u u^- \in \mathcal{N}^\pm$. Moreover,
\[
\Phi(t_u u^+ + s_u u^-) = \max_{t, s > 0} \Phi(tu^+ + su^-).
\]
Finally, if $u \in \mathcal{N}^\pm$, then $t_u = s_u = 1$.

**Proof.** First of all, note that by Lemma 3.1 and since $u^+$ and $u^-$ have disjoint supports, it follows that
\[
\Phi(tu^+ + su^-) = \Phi(tu^+) + \Phi(su^-),
\]
for every $u \in BV_{rad}(\mathbb{R}^N)$ and $t, s > 0$. Then, the result follows just observing that, for $u \in BV(\mathbb{R}^N)$ such that $u^\pm \neq 0$,
\[
\max_{t, s > 0} \Phi(tu^+ + su^-) = \max_{t, s > 0} \left( \Phi(tu^+) + \Phi(su^-) \right)
\]
\[
= \Phi(tu^+ + su^-),
\]
where $t_u u^+$ and $s_u u^-$ are such that (3.7) holds for $u^+$ and $u^-$, respectively. Moreover, since $u \in \mathcal{N}^\pm$ if and only if $u^\pm \in \mathcal{N}$, then if $u \in \mathcal{N}^\pm$, then $t_u = s_u = 1$.

In the following result we show that $\mathcal{N}^\pm$ is a natural constraint to (1.2), i.e., that if $u_0 \in \mathcal{N}^\pm$ minimizes the energy $\Phi$ restricted to $\mathcal{N}^\pm$, then $u_0$ is a critical point of $\Phi$ (restricted to $BV_{rad}(\mathbb{R}^N)$).

**Lemma 3.4** If $u_0 \in \mathcal{N}^\pm$ is such that
\[
\Phi(u_0) = \min_{v \in \mathcal{N}^\pm} \Phi(v),
\]
then $0 \in \partial \Phi(u_0)$.

**Proof.** In this proof we proceed using the ideas of Bartsch, Weth and Willem in [9]. Suppose by contradiction that $0 \not\in \partial \Phi(u_0)$. Then $\beta(u_0) > 0$, where $\beta(u_0) = \inf \{ \|z\|_* ; z \in \partial \Phi(u_0) \}$. Since $u \mapsto \beta(u)$ is lower semicontinuous (see [12]), it follows that there exists $\theta > 0$ such that
\[
\beta(u) > \frac{\beta(u_0)}{2} > 0, \quad \forall u \in B_\theta(u_0).
\]

Let us denote $J = \left[ 1 - \frac{\theta}{4\|u_0\|}, 1 + \frac{\theta}{4\|u_0\|} \right] \times \left[ 1 - \frac{\theta}{4\|u_0\|}, 1 + \frac{\theta}{4\|u_0\|} \right] \subset \mathbb{R}_+^2$ and define $g : J \to BV_{rad}(\mathbb{R}^N)$ by
\[
g(t, s) = tu_0^+ + su_0^-
\]
\[
(3.10)
\]
Denoting $c := \Phi(u_0) = \inf_{v \in N^\pm} \Phi(v)$, note that, by Lemma 3.3

$$\Phi(g(t, s)) < c, \quad \forall (t, s) \neq (1, 1).$$

Moreover, note that

$$\max_{(t, s) \in \partial J} \Phi(g(t, s)) = c_0 < c. \quad (3.11)$$

By using the version of the Deformation Lemma to locally Lipschitz functionals without the Palais-Smale condition (see [15]), there exists $\epsilon > 0$ such that

$$\epsilon < \epsilon_0 := \min \left\{ \frac{c - c_0}{2}, -\frac{\beta(u_0)}{16} \right\},$$

and an homeomorphism $\eta : BV_{rad}(\mathbb{R}^N) \to BV_{rad}(\mathbb{R}^N)$ such that

1. $\eta(u) = u$ for all $u \notin \Phi^{-1}([c - \epsilon_0, c + \epsilon_0]) \cap B_\theta(u_0)$;
2. $\eta(\Phi_c \pm \theta B_\theta(u_0)) \subset \Phi_c - \epsilon$;
3. $\Phi(\eta(u)) \leq \Phi(u)$, for all $u \in BV_{rad}(\mathbb{R}^N)$.

Let us define $h : J \to BV_{rad}(\mathbb{R}^N)$ by $h(t, s) = \eta(g(t, s))$ and two functions, $\Psi_0, \Psi_1 : J \to \mathbb{R}^2$ by

$$\Psi_0(t, s) = (\Phi'(tu_0^+)u_0^+, \Phi'(su_0^-)u_0^-)$$

and

$$\Psi_1(t) = \left( \frac{1}{t} \Phi'(h(t, s)^+)h(t, s)^+, \frac{1}{s} \Phi'(h(t, s)^-)h(t, s)^- \right).$$

Since by (3.11), for $(t, s) \in \partial J$, $\Phi(g(t, s)) \leq c_0 < c - \epsilon_0$, then $h(t, s) = \eta(g(t, s)) = g(t, s) = tu_0^+ + su_0^-$, for $(t, s) \in \partial J$. Hence

$$\Psi_0(t, s) = \Psi_1(t, s), \quad \forall (t, s) \in \partial J. \quad (3.12)$$

By Degree Theory and Lemma 3.3, $d(\Psi_0, J, (0, 0)) = 1$. Then, taking into account (3.12), we have that $d(\Psi_1, J, (0, 0)) = 1$. Hence there exists $(t, s) \in J$ such that $h(t, s) \in N^\pm$. This implies that

$$c \leq \Phi(h(t, s)) = \Phi(\eta(g(t, s))).$$

But note that $\Phi(g(t, s)) < c + \epsilon$ and also $g(J) \subset B_\theta(0)$. Then, by ii)

$$\Phi(\eta(g(t, s))) < c - \epsilon,$$

which contradicts the last inequality. Then the result follows. $\square$

In the next lemma, we prove some technical results that are going to be used in the proof of Theorem 1.1.

**Lemma 3.5** Under the assumptions $(f_1) - (f_5)$, it holds:

(i) For all $(u_n)_{n \in \mathbb{N}} \subset N$ such that $\|u_n\| \to +\infty$ as $n \to +\infty$, it holds that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.

(ii) There exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in N$ and also $\|u^\pm\| \geq \rho$ for all $u \in N^\pm$.

(iii) For all sequence $(u_n)_{n \in \mathbb{N}} \subset N^\pm$, it holds that

$$\lim_{n \to +\infty} \int_{\Omega} |u_n^\pm|^q dx > 0,$$

where $q$ is as in $(f_3)$.  

10
Proof. By the definition of $N$, if $u \in N$, by $(f_4)$ we have that

$$\Phi(u) = \Phi(u) - \frac{1}{\kappa} \langle \Phi'(u), u \rangle \geq \left(1 - \frac{1}{\kappa}\right) \|u\|.$$  

From the last expression, we can see that if $(u_n)_{n \in \mathbb{N}} \subset N$ is such that $\|u_n\| \to +\infty$ as $n \to +\infty$, then $\lim_{n \to +\infty} \Phi(u_n) = +\infty$, what proves $(i)$. 

In order to prove $(ii)$, let us observe that, by $(f_2)$ and $(f_3)$, for any $\epsilon > 0$, there exists a positive constant $C_\epsilon > 0$ such that

$$|f(s)s| \leq \epsilon |s| + C_\epsilon |s|^q, \text{ for all } s \in \mathbb{R}. \quad (3.13)$$

Since for every $u \in N$, we have that

$$\|u\| = \int_{\mathbb{R}^N} f(u)udx, \quad (3.14)$$

from (3.13) and the Sobolev embeddings, it holds that

$$(1 - \epsilon)\|u\| \leq C_\epsilon \|u\|^q. \quad (3.15)$$

Hence, since $q > 1$, there exists $\rho > 0$ such that $\|u\| \geq \rho$. This proves the first part of $(ii)$. For the second one, we just have to note that if $u \in N^\pm$, then from the definition of $N^\pm$, we can see that $u^\pm \in N$. 

Finally, $(iii)$ follows straightforwardly once we take into account $(ii)$ and (3.15).  

Proof of Theorem 1.1. Let $(u_n)_{n \in \mathbb{N}} \subset N^\pm$ be a minimizing sequence to $\Phi$ on $N^\pm$, i.e., such that

$$\Phi(u_n) = c + o_n(1),$$

where $c = \inf_{N^\pm} \Phi$. Since from Lemma 3.5, $\Phi$ is coercive on $N$ and $N^\pm$, it follows that $(u_n)_{n \in \mathbb{N}}$ is bounded in $BV_{rad}(\mathbb{R}^N)$. Since $(u_n^\pm)_{n \in \mathbb{N}}$ is also bounded, by using the compactness of the Sobolev embeddings $BV_{rad}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, for $1 < r < 1^*$ (see [18]), there exist $v_0, w_1, w_2 \in BV_{rad}(\mathbb{R}^N)$ such that

$$u_n \to v_0, \quad u_n^+ \to w_1, \quad u_n^- \to w_2 \quad \text{in } L^q(\mathbb{R}^N), \text{ as } n \to +\infty.$$ 

Moreover, since $w \mapsto w^+$ and $w \mapsto w^-$ are continuous functions in $L^q(\mathbb{R}^N)$ (see [11][Lemma 2.3]), it follows that

$$u_n \to v_0, \quad u_n^+ \to v_0^+, \quad u_n^- \to v_0^- \quad \text{in } L^q(\mathbb{R}^N), \text{ as } n \to +\infty. \quad (3.16)$$

Note also that from Lemma 3.5, it follows that $v_0^+ \neq 0$ and consequently $v_0 = v_0^+ + v_0^-$ is a sign-changing function.

Since $v_0^+ \neq 0$, by Lemma 3.3, there exist $t, s > 0$ such that

$$tv_0^+ + sv_0^- \in N^\pm. \quad (3.17)$$

In the next step we show that $\Phi(tv_0^+ + sv_0^-) = c$. Note that, from the lower semicontinuity of the norm in $BV(\mathbb{R}^N)$ with respect to the $L^q(\mathbb{R}^N)$ convergence, (3.16), (3.17) and Lemma 3.3, we have that

$$c \leq \Phi(tv_0^+ + sv_0^-) \leq \liminf_{n \to +\infty} \Phi(tv_n^+ + sv_n^-) \leq \lim_{n \to +\infty} \Phi(u_n) = c.$$
Hence, we have proved that in fact $\Phi(tv_0^+ + sv_0^+) = c$ and then $u_0 = tv_0^+ + sv_0^-$ is a minimizer of $\Phi$ in $\mathcal{N}$. Then, from Lemma 3.4, we have that $u_0$ is such that $0 \in \partial \Phi(u_0)$, which implies that (3.2) is satisfied for all $v \in BV_{rad}(\mathbb{R}^N)$. Now, by using the version of the Symmetric Criticality Principle of Palais in [27], it follows that (3.2) holds for every $v \in BV(\mathbb{R}^N)$ and then $u_0$ is a bounded variation solution of (1.2).

By the characterization of the subdifferential of $J$ developed in [5], it is possible to show that there exist $z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $\gamma \in L^\infty(\mathbb{R}^N)$ such that $\|z\|_\infty \leq 1$, $\|\gamma\|_\infty \leq 1$ and
\[
\begin{aligned}
- \text{div } z + \gamma &= f(u_0) \text{ in } D'(\mathbb{R}^N), \\
(z, Du_0) &= |Du_0| \text{ in the sense of measures,} \\
\gamma u &= |u| \text{ a.e. in } \mathbb{R}^N.
\end{aligned}
\]

Then, such a function $u_0$ is in fact a nodal bounded variation solution of (1.2) which has the lowest energy level among all the sign-changing solutions.

\[\square\]

4 Existence by approximation scheme

In this section, in comparison with Section 3, we apply a different approach to prove Theorem 1.2. The approach we follow here is based on an approximation of the original problem through similar ones involving the $p$-Laplacian operator.

First of all, let us define what we understand as a bounded variation solution for (1.2), since this equation contains some expressions which are not well defined (for instance the ratio $Du/|Du|$, when $u \in BV(\Omega)$). We say that $u \in BV(\Omega)$ is a bounded variation solution of (1.2) if there exists $z \in X_N(\Omega)$ such that $\|z\|_\infty \leq 1$ and
\[
\begin{aligned}
- \text{div } z &= f(u) \text{ in } D'(\Omega), \\
(z, Du) &= |Du| \text{ in the sense of measures,} \\
[z, \nu] &= \text{sign}(-u) \mathcal{H}^{N-1}-\text{a.e. in } \partial \Omega.
\end{aligned}
\]

For $p > 1$, let us consider the problem
\[
\begin{aligned}
- \text{div } (|\nabla u|^{p-2}\nabla u) &= f(u) \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]
whose weak form is given by the following integral identity
\[
\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega f(u)v dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.2}
\]

It is well known that weak solutions of (4.1) are critical points of the energy functional $I_p : W_0^{1,p}(\Omega) \to \mathbb{R}$, given by
\[
I_p(u) := \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega F(u) dx,
\]
which is well defined for $1 < p < N$, since by ($f_3$),
\[
\int_\Omega F(u) dx \leq c_1\|u\|_1 + c_3\|u\|_q^q,
\]
where $q < N/(N-1) < pN/(N-p)$.
Let us consider the functional \( \Phi_p : W^{1,p}_0(\Omega) \to \mathbb{R} \) given by
\[
\Phi_p(u) := I_p(u) + \frac{p-1}{p} |\Omega|
\]
and note that, for each \( u \) such that \( \Phi_p(u) \) is well defined, \( (\Phi_p(u))_{p>1} \) is non-decreasing. Indeed, if \( 1 < p_1 \leq p_2 < N \), then by Young inequality with exponents \( p_2/p_1 \) and \( p_2/(p_2 - p_1) \), it follows that
\[
\int_{\Omega} |\nabla u|^{p_1} dx \leq \frac{p_1}{p_2} \int_{\Omega} |\nabla u|^{p_2} dx + \frac{p_2 - p_1}{p_2} |\Omega|.
\]
Hence, it follows that
\[
\Phi_{p_1}(u) = \frac{1}{p_1} \int_{\Omega} |\nabla u|^{p_1} dx + \frac{p_1 - 1}{p_1} |\Omega| - \int_{\Omega} F(u) dx
\]
\[
\leq \frac{1}{p_1} \left( \frac{p_1}{p_2} \int_{\Omega} |\nabla u|^{p_2} dx + \frac{p_2 - p_1}{p_2} |\Omega| \right) + \frac{p_1 - 1}{p_1} |\Omega| - \int_{\Omega} F(u) dx
\]
\[
= \Phi_{p_2}(u).
\]
(4.4)

Since \( \Phi_p \) and \( I_p \) differs just by a constant, we can study (4.1) from a variational point of view by dealing with either \( I_p \) or \( \Phi_p \).

By \( (f_2)' \), for \( p > 1 \) sufficiently close to 1, the function \( f \) satisfies
\[
\lim_{s \to 0} \sup \frac{f(s)}{|s|^p} = 0.
\]
Then, for each such \( p \), the Nehari manifold associated to (4.1) is well defined and is given by
\[
\mathcal{N}_p^\pm = \{ u \in W^{1,p}_0(\Omega) ; u^\pm \neq 0 \text{ and } \Phi'_p(u^\pm)u^\pm = 0 \}.
\]
It is well know that there exists \( u_p \in \mathcal{N}_p^\pm \) (see for instance [8]) such that
\[
\Phi_p(u_p) = \min_{v \in \mathcal{N}_p^\pm} \Phi_p(v)
\]
(4.5)
and \( u_p \) is a sign-changing solution to (4.1), which has the least energy among all of nodal solutions. It is also well known that
\[
\Phi_p(u_p) = \max_{t,s>0} \Phi_p(tu_p^+ + su_p^-).
\]
(4.6)
Moreover, for each \( u \in W^{1,p}_0(\Omega) \) such that \( u^\pm \neq 0 \), there exist \( t,s > 0 \) such that
\[
tu_p^+ + su_p^- \in \mathcal{N}_p^\pm.
\]

**Lemma 4.1** The family \( (\Phi_p(u_p))_p \) is non-decreasing for \( p \in (1, N) \).

**Proof.** Let \( 1 < p_1 \leq p_2 < N \) and \( u_{p_1} \in W^{1,p_1}_0(\Omega) \) and \( u_{p_2} \in W^{1,p_2}_0(\Omega) \) satisfying (4.5). Since \( p_2 \geq p_1 \), then \( W^{1,p_2}_0(\Omega) \subset W^{1,p_1}_0(\Omega) \), in such a way that, since \( u_{p_2}^\pm \neq 0 \), there exist \( t,s > 0 \) such that
\[
tu_{p_2}^+ + su_{p_2}^- \in \mathcal{N}_{p_1}^\pm.
\]
(4.7)
Then, from (4.4), (4.5), (4.6) and (4.7), it follows that
\[
\Phi_{p_2}(u_{p_2}) \geq \Phi_{p_2}(tu_{p_2}^+ + su_{p_2}^-)
\]
\[
\geq \Phi_{p_1}(tu_{p_2}^+ + su_{p_2}^-)
\]
\[
\geq \Phi_{p_1}(u_{p_1}).
\]
Lemma 4.2 For $\overline{p} \in (1, N)$, the family $(u_p)_{1 < p < \overline{p}}$ is bounded in $BV(\Omega)$.

Proof. From Lemma 4.1, it follows that for $\overline{p} \in (1, N)$ fixed,

$$\Phi_p(u_p) \leq \Phi_{\overline{p}}(u_{\overline{p}}) =: C,$$

for all $p \in (1, \overline{p})$. Hence, there exists $C > 0$ such that

$$\int_{\Omega} |\nabla u_p|^p dx \leq C, \quad \forall p \in (1, \overline{p}).$$

In fact, by ($f_4$) and (4.8), note that

$$C \geq \Phi_p(u_p) = \Phi_p(u_p) - \frac{1}{\kappa} \Phi'(u_p) u_p$$

$$= \left( \frac{1}{p} - \frac{1}{\kappa} \right) \int_{\Omega} |\nabla u_p|^p dx + \int_{\Omega} \left( \frac{1}{\kappa} f(u_p) u_p - F(u_p) \right) dx$$

$$\geq \left( \frac{1}{p} - \frac{1}{\kappa} \right) \int_{\Omega} |\nabla u_p|^p dx$$

$$\geq \left( \frac{1}{ \overline{p} - \kappa} \right) \int_{\Omega} |\nabla u_p|^p dx. \quad (4.10)$$

Then, by using Young inequality and (4.10),

$$\|u_p\| = \int_{\Omega} |\nabla u_p| dx$$

$$\leq \frac{1}{p} \int_{\Omega} |\nabla u_p|^p dx + \frac{p - 1}{p} |\Omega|$$

$$\leq C_1.$$

From the last result and the compact Sobolev embeddings, it follows that there exists $u_0 \in BV(\Omega)$ such that

$$u_p \to u_0, \quad \text{in } L^r(\Omega), \text{ for all } 1 \leq r < 1^*, \quad (4.11)$$

and

$$u_p \to u_0, \quad \text{a.e. in } \Omega, \quad (4.12)$$

as $p \to 1^+$.

Moreover, Lemma 4.2 also implies that (see [3][Proposition 3]) there exists $z \in L^\infty(\Omega, \mathbb{R}^N)$, such that $\|z\|_{\infty} \leq 1$ and

$$|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z, \quad (4.13)$$

in $L^r(\Omega, \mathbb{R}^N)$, as $p \to 1^+$, for all $1 \leq r < \infty$. In particular

$$|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup \text{div } z \quad \text{in } D'(\Omega) \quad (4.14)$$

as $p \to 1^+$, what follows from taking $\nabla \varphi$ as a test function in (4.13), where $\varphi \in D(\Omega)$.

Note that, from (4.1), (4.11) and (4.14) and applying the Lebesgue Dominated Convergence Theorem, it follows that

$$-\text{div } z = f(u_0), \quad \text{in } D'(\Omega). \quad (4.15)$$
Lemma 4.3 The function \( u_0 \) and the vector field \( z \) satisfy the following equality in the sense of measures in \( \Omega \),

\[
(z, Du_0) = |Du_0|.
\]

Proof. First of all, note that, since \( \|z\|_\infty \leq 1 \), it follows that, \( (z, Du_0) \leq |Du_0| \). Indeed, for all Borel set \( B \subset \Omega \), by (2.2),

\[
\int_B (z, Du_0) \leq \left| \int_B (z, Du_0) \right| \leq \|z\|_\infty \int_B |Du_0| \leq \int_B |Du_0|.
\]

Hence, it is enough to show the opposite inequality, i.e., that for all \( \varphi \in C^1_0(\Omega), \varphi \geq 0, \)

\[
\langle (z, Du_0), \varphi \rangle \geq \int_\Omega \varphi |Du_0|.
\] (4.16)

In order to do so, let us consider \( u_0 \varphi \in W^{1,p}_0(\Omega) \) as a test function in (4.2). Then we obtain,

\[
\int_\Omega \varphi |\nabla u_0|^p dx + \int_\Omega u_0 |\nabla u_0|^{p-2} |\nabla u_0| \varphi dx = \int_\Omega f(u_0) u_0 \varphi dx.
\] (4.17)

Now we shall calculate the \( \lim \inf \) as \( p \to 1^+ \) in both sides of (4.17). Before it, note that,

\[
\int_\Omega \varphi |Du_0| \leq \liminf_{p \to 1^+} \int_\Omega |\nabla u_0|^p dx.
\] (4.18)

Indeed, by the Young inequality,

\[
\int_\Omega \varphi |Du_0| \leq \liminf_{p \to 1^+} \int_\Omega |\nabla u_0|^p dx \leq \liminf_{p \to 1^+} \left( \frac{1}{p} \int_\Omega \varphi |\nabla u_0|^p dx + \frac{p-1}{p} \int_\Omega \varphi dx \right) = \liminf_{p \to 1^+} \int_\Omega \varphi |\nabla u_0|^p dx.
\]

Moreover, by (4.14), it follows that

\[
\lim_{p \to 1^+} \int_\Omega u_0 |\nabla u_0|^{p-2} |\nabla u_0| \varphi dx = \int_\Omega u_0 z \nabla \varphi dx.
\] (4.19)

Finally, Lebesgue Convergence Dominated Theorem and (4.11) imply that

\[
\lim_{p \to 1^+} \int_\Omega f(u_0) u_0 \varphi dx = \int_\Omega f(u_0) u_0 \varphi dx.
\] (4.20)
Then, from (4.15), (4.18), (4.19) and (4.20), it follows that
\[
\langle (z, Du_0), \varphi \rangle = -\int_\Omega \varphi u_0 \text{div} z - \int_\Omega u_0 z \nabla \varphi dx
\]
\[
= \int_\Omega f(u_0)u_0 \varphi dx - \int_\Omega u_0 z \nabla \varphi dx
\]
\[
= \lim_{p \to 1^+} \left( \int_\Omega f(u_p)u_p \varphi dx - \int_\Omega u_p |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi dx \right)
\]
\[
= \lim \inf_{p \to 1^+} \int_\Omega \varphi |\nabla u_p|^p dx
\]
\[
\geq \int_\Omega \varphi |Du_0|,
\]
which implies in (4.16) and finishes the proof. \(\square\)

**Lemma 4.4** The function \(u_0\) satisfies \([z, \nu] \in \text{sign}(-u_0)\) on \(\partial \Omega\).

**Proof.** We shall prove that
\[
|u_0| + [z, \nu] u_0 = 0 \text{ on } \partial \Omega.
\]
(4.21)

But since, by (2.3)
\[
| [z, \nu] u_0 | \leq ||z||_\infty |u_0| \leq |u_0|,
\]
it follows that
\[
|u_0| + [z, \nu] u_0 \geq 0 \text{ on } \partial \Omega.
\]
(4.22)

Now, for \(\varphi \in C^1_0(\Omega)\), let us consider \((u_p - \varphi) \in W^{1,p}_0(\Omega)\) as test function in (4.2). Then we get
\[
\int_\Omega |\nabla u_p|^p dx = \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi dx + \int_\Omega f(u_p)(u_p - \varphi)dx.
\]
(4.23)

From Young's inequality, Green's Formula, (4.14), (4.15), Lemma 4.3 and (4.23), we have that, as \(p \to 1^+\),
\[
p \int_\Omega |\nabla u_p|^p dx = \int_\Omega |\nabla u_p|^p dx + (p - 1)|\Omega|
\]
\[
= \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi dx + \int_\Omega f(u_p)(u_p - \varphi)dx + (p - 1)|\Omega|
\]
\[
= \int_\Omega z \cdot \nabla \varphi dx + \int_\Omega f(u_0)(u_0 - \varphi)dx + o(1)
\]
\[
= -\int_\Omega (u_0) \text{div} z - \int_\Omega f(u_0) \varphi dx + \int_\Omega f(u_0) u_0 dx + o(1)
\]
\[
= \int_\Omega f(u_0) u_0 dx + o(1)
\]
\[
= -\int_\Omega u_0 \text{div} z + o(1)
\]
\[
= \int_\Omega (z, Du_0) - \int_{\partial \Omega} [z, \nu] u_0 d\mathcal{H}^{N-1}
\]
\[
= \int_\Omega |Du_0| - \int_{\partial \Omega} [z, \nu] u_0 d\mathcal{H}^{N-1}.
\]
(4.24)

Hence, from (4.24) and the lower semicontinuity of the norm in \(BV(\Omega)\) with respect to the \(L^1(\Omega)\) convergence, it follows that
\[
\int_{\partial \Omega} (|u_0| + [z, \nu] u_0) d\mathcal{H}^{N-1} \leq 0.
\]
(4.25)
Then, taking into account (4.22) and (4.25), it follows that
\[ |u_0| + [z, \nu] u_0 = 0 \quad \text{on } \partial \Omega, \]
from which follows the result. \( \square \)

Note that up to now, from (4.15), and Lemmas 4.3 and 4.4, we have found \( u_0 \in BV(\Omega) \), for which there exists \( z \in X_N(\Omega) \) such that \( ||z||_\infty \leq 1 \) and
\[
\begin{cases}
-\text{div}(z) = f(u_0) \text{ in } D'(\Omega), \\
(z, Du_0) = |Du_0| \text{ in the sense of measures}, \\
[z, \nu] \in \text{sign}(-u_0) \text{ on } \partial \Omega.
\end{cases}
\]
(4.26)

Now, what is left to do is to show that \( u_0 \neq 0 \). In order to do so, we should introduce, as in Section 3, the energy functional \( \Phi : BV(\Omega) \to \mathbb{R} \) given by
\[
\Phi(u) = \|u\| - \int_{\Omega} F(u)dx.
\]
First of all, note that by Young’s inequality, \( \Phi(u) \leq \Phi_p(u) \) for every \( u \in W^{1,p}_0(\Omega) \). Moreover,
\[
\lim_{p \to 1^+} \Phi_p(u_p) = \Phi(u_0). \quad \text{(4.27)}
\]
Indeed, since \( u_0 \) satisfies (4.26), note that, as \( p \to 1^+ \),
\[
\|u_0\| = \int_{\Omega} |Du_0| + \int_{\partial \Omega} |u_0|dH^{N-1} = \int_{\Omega} (z, Du_0) - \int_{\partial \Omega} u_0 [z, \nu] dH^{N-1} = -\int_{\Omega} u_0 \text{div}z \, dx = \int_{\Omega} f(u_0)u_0 \, dx = \frac{1}{p} \int_{\Omega} f(u_p)u_p \, dx + o(1) = \frac{1}{p} \int_{\Omega} |\nabla u_p|^p \, dx + o(1). \quad \text{(4.28)}
\]
Moreover, by \( (f_3) \), (4.11) and the Lebesgue Dominated Convergence Theorem, as \( p \to 1^+ \),
\[
\int_{\Omega} F(u_0) \, dx = \int_{\Omega} F(u_p) \, dx + o(1). \quad \text{(4.29)}
\]
Then, (4.28) and (4.29) imply in (4.27).

Note also that, by \( (f_2)' \) and \( (f_3) \), for all \( \epsilon > 0 \), there exists a positive constant \( C_\epsilon > 0 \) such that
\[
|f(s)s| \leq \epsilon |s| + C_\epsilon |s|^q, \quad \forall s \in \mathbb{R}.
\]
Then,
\[
\Phi(u) \geq (1 - \epsilon)\|u\| - C\|u\|^q.
\]
Let us consider \( \epsilon > 0 \) small enough such that \( 1 - \epsilon > 1/2 \). Then, if \( \|u\| \leq \rho \), where \( 0 < \rho < \left( \frac{(1 - \epsilon) - 1/2}{C_\epsilon} \right)^{1/q} \), it follows that
\[
\Phi(u) \geq \frac{\|u\|}{2}. \quad \text{(4.30)}
\]
Then, since the fact that \( u_p \) belongs to the Nehari nodal set \( \mathcal{N}_p^\pm \) implies that \( u_p^\pm \in \mathcal{N}_p \), it follows that \( s = 1 \) is the maximum of the function \( s \mapsto \Phi_p(su_p^\pm) \). Then, for all \( p \in (1, \overline{p}) \),

\[
\Phi_p(u_p^\pm) \geq \Phi \left( \frac{\rho u_p^\pm}{\| u_p^\pm \|} \right) \quad (4.31)
\]

\[
\geq \Phi \left( \frac{\rho u_p^\pm}{\| u_p^\pm \|} \right) \quad (4.32)
\]

\[
\geq \frac{\rho}{2}, \quad (4.33)
\]

where we have used (4.30) and also the fact that \( u_p \in W^{1,p}(\Omega) \rightarrow BV(\Omega) \). Hence

\[
\frac{\rho}{2} \leq \Phi_p(u_p^\pm) = \int_\Omega \left( \frac{1}{p} f(u_p^\pm) u_p^\pm - F(u_p^\pm) \right) dx \quad (4.34)
\]

for all \( p \in (1, \overline{p}) \). Then, if \( u_p^\pm \rightarrow 0 \) in \( BV(\Omega) \), then \( (f_3) \) and Lebesgue Dominated Convergence Theorem would imply that (4.34) would not hold. Hence, \( u_0^\pm \neq 0 \). This finishes the proof of Theorem 1.2.

Acknowledgments: Marcos T.O. Pimenta is partially supported by FAPESP 2019/14330-9, CNPq 303788/2018-6, Brazil and FAPDF. Giovany M. Figueiredo is partially supported by FAPDF, CNPq and CAPES, Brazil.

References


