On the Cauchy problem of the full Navier-Stokes equations for three-dimensional compressible viscous heat-conducting flows subject to large external potential forces

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Abstract. In this paper, we consider the Cauchy problem of the full Navier-Stokes equations for three-dimensional compressible viscous heat-conducting flows subject to external potential forces in the whole space \( \mathbb{R}^3 \). For discontinuous data with small energy and vacuum, the global “intermediate weak” solutions with large oscillations and large external potential forces are obtained, provided the unique steady state is strictly away from vacuum. Moreover, if \( \| \nabla \rho_0 \|_{L^2(\mathbb{R}^3)} \), \( \| \nabla u_0 \|_{L^3} \) and \( \| \nabla \theta_0 \|_{L^2} \) are bounded, then the weak solution becomes a strong one belonging to a class of functions in which the uniqueness can be shown to hold, when the density is strictly away from vacuum and the viscosity coefficients satisfy \( 7\mu > \lambda \) additionally.

Keywords. compressible Navier-Stokes equations; external potential forces; large oscillations; “intermediate weak” solutions; uniqueness

AMS Subject Classifications (2000). 35Q35, 35B65, 76N10

1 Introduction

The motion of compressible, viscous, heat-conductive, and Newtonian polytropic fluids occupying a spatial domain \( \Omega \) is governed by the full Navier-Stokes equations:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}S + \rho \nabla f, \\
(\rho E)_t + \text{div}(\rho E u) + \text{div}(Pu) &= \text{div}(\kappa \nabla \theta) + \text{div}(Su) + \rho u \cdot \nabla f,
\end{align*}
\]

(1.1)

where \( S \) and \( E \) are respectively the viscous stress tensor and the total energy given by

\[
S = \mu \left( \nabla u + (\nabla u)^\top \right) + \lambda (\text{div}u)I \quad \text{and} \quad E = e + \frac{1}{2} |u|^2.
\]

(1.2)

Here, the unknown functions \( \rho, u = (u^1, u^2, u^3)^\top, P(\rho, e), e \) and \( \theta \) are the fluid density, velocity, pressure, specific internal energy and absolute temperature, respectively, and \( f = f(x) \) denotes the external force. The viscosity coefficients \( \mu, \lambda \) are constants satisfying the physical restrictions for Newtonian fluids:

\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0.
\]

(1.3)
The heat-conductivity coefficient $\kappa > 0$ is a positive constant, and the pressure $P = P(\rho, e)$ is determined through the equations of state for ideal polytropic fluids:

$$P(\rho, e) = (\gamma - 1)\rho e = R\rho \theta, \quad e = \frac{R\theta}{\gamma - 1} = c_v \theta,$$

(1.4)

where $\gamma > 1$ is the adiabatic exponent, $R > 0$ is the perfect gas constant, and $c_v > 0$ is the specific heat at constant volume.

There is a lot of literature on the mathematical theory of multi-dimensional compressible Navier-Stokes equations, due to its physical importance and mathematical challenge. Local well-posedness of strong/classical solutions was studied in [23, 26, 29] and [2, 3] for the non-vacuum and vacuum case, respectively. The global-in-time existence of smooth solutions was firstly obtained by Matsumura-Nishida [21], when the initial perturbation around the non-vacuum equilibrium is sufficiently small in $H^3$. Later, Hoff [11, 12, 13] extended the Matsumura-Nishida’s result [21] to weak solutions with discontinuous data. For the case of generally large data, the most important breakthrough is due to Lions [20] (see also [8]), who proved the global existence of finite-energy weak solutions to the isentropic Navier-Stokes equations, when the adiabatic exponent $\gamma$ is suitably large (i.e., $\gamma > 3/2$). Based on the techniques developed in [20, 8], Feireisl [6, 7], obtained the global “variational” weak solutions to the non-isentropic equations for viscous compressible heat-conducting flows satisfying some specific pressure laws. Recently, Huang-Li-Xin [16] and Huang-Li [14] established the global well-posedness of classical solutions to the Cauchy problem of isentropic and non-isentropic compressible Navier-Stokes equations (without external forces), respectively, when the initial data are of small energy but possibly large oscillations; in particular, the initial density is allowed to vanish, even has compact support.

In reality, the fluid motion is often driven by the external forces, which will significantly affect the dynamic stability of flows when the external forces are large. When both the initial perturbations and the external forces are sufficiently small, the existence of global solution and its large-time behavior of the compressible Navier-Stokes equations have been studied by many people, see, for example, [4, 5, 11, 27, 30], and among others. However, if the external force could be arbitrarily large, some seriously mathematical difficulties will arise. To the best of our knowledge, the existing mathematical results about the large external forces are only known for the barotropic flows with potential forces. In particular, the authors [9, 25] showed that if the adiabatic exponent $\gamma > 3/2$ and there exists a unique steady state, then the density of weak solution converges to the steady density in $L^\gamma$ as time goes to infinity. If the adiabatic exponent $\gamma$ is close enough to 1 and the external potential forces decay suitably fast at infinity, Matsumura-Yamagata [22] proved the convergence in $L^p$-norm with $2 < p \leq \infty$, provided the initial perturbations are sufficiently small in $L^2 \cap L^\infty$ for density (non-vacuum) and in $H^1$ for velocity. This result was later improved by Li-Matsumura [17] by removing the smallness condition on $|\gamma - 1|$ and the far-field decay condition of potential force, and then was extended to the vacuum case by the authors [18].

Due to the complicate mathematical structure and strong nonlinearities, it is more difficult to deal with the full Navier-Stokes system for compressible heat-conducting fluids, compared with the one for barotropic/isentropic fluids. Indeed, up to now, there isn’t any literature about the global theory of (1.1)–(1.4) with large external potential force, although some results have been achieved for the barotropic/isentropic case (cf. [9, 25, 22, 17, 18]). So, the first purpose of this paper is to investigate the existence of global solutions and its asymptotic stability to the Cauchy problem of (1.1)–(1.4), when the external potential force could be arbitrarily large.

Let $\Omega = \mathbb{R}^3$. We consider the Cauchy problem of (1.1)–(1.4) with the far-field behavior

$$\left(\rho, u, \theta\right)(x, t) \to (\rho_\infty, 0, \theta_\infty), \quad as \ |x| \to \infty,$$

(1.5)
and the initial data
\[(\rho, \rho u, \rho \theta)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0)(x), \quad x \in \mathbb{R}^3, \tag{1.6}\]
where \(\rho_\infty, \theta_\infty > 0\) are fixed positive constants, and \(\rho_0, \theta_0 \geq 0\) are non-negative.

Before stating the main result, we first introduce some notations and conventions which will be used throughout this paper. Let
\[
\int f dx \triangleq \int_{\mathbb{R}^3} f dx
\]
For \(1 \leq p \leq \infty\) and \(0 \leq k \in \mathbb{Z}\), the following simplified notations denote the standard homogeneous and nonhomogeneous Sobolev spaces:
\[
\begin{align*}
L^p & \triangleq L^p(\mathbb{R}^3), \quad W^{k,p} \triangleq W^{k,p}(\mathbb{R}^3), \quad H^k \triangleq W^{k,2}, \\
D^1 & \triangleq \{u \in L^1_{\text{loc}} \mid \|\nabla u\|_{L^2} < \infty\}, \quad D^{1,p} \triangleq \{u \in L^1_{\text{loc}} \mid \|\nabla u\|_{L^p} < \infty\}.
\end{align*}
\]

Without loss of generality, assume that \(\rho_\infty = \theta_\infty = 1\). The first theorem of this paper is concerned with the global existence of weak solutions, which are fundamental and important both in the physical theory of nonequilibrium thermodynamics and in the mathematical theory of inviscid models for compressible flows. A weak solution of (1.1)–(1.6) is defined in the same way as in [13, 14].

**Definition 1.1** A pair of functions \((\rho, u, \theta)\) with \(E = |u|^2/2 + c_\sqrt{\theta}\) is said to be a weak solution to the Cauchy problem (1.1)–(1.6), if it holds that
\[
\rho - 1 \in L^\infty_{\text{loc}}([0, \infty); L^2 \cap L^\infty), \quad (u, \theta - 1) \in L^2(0, \infty; H^1), \tag{1.7}
\]
and that for all test functions \(\phi \in \mathcal{D}(\mathbb{R}^3 \times (-\infty, \infty))\), the equations in (1.1) are satisfied in the sense of distributions:
\[
\begin{align*}
\int \rho_0 \phi(\cdot, 0) dx + \int_0^\infty \int (\rho \phi_t + \rho u \cdot \nabla \phi) dx dt &= 0, \tag{1.8} \\
\int \rho_0 u_0^j \phi(\cdot, 0) dx + \int_0^\infty \int (\rho u^j \phi_t + \rho u^j u \cdot \nabla \phi + P(\rho, \theta) \phi_{x_j}) dx dt &= \int_0^\infty \int \left[ (\mu \nabla u^j \cdot \nabla \phi + (\mu + \lambda)(\text{div} u) \partial_j \phi) - \rho f_{x_j} \phi \right] dx dt \tag{1.9}
\end{align*}
\]
for \(j = 1, 2, 3\), and
\[
\begin{align*}
\int \left( \frac{1}{2} \rho_0 |u_0|^2 + c_\sqrt{\rho_0 \theta_0} \right) \phi(\cdot, 0) dx + \int_0^\infty \int [\rho E \phi_t + (\rho E + P) u \cdot \nabla \phi] dx dt &= \int_0^\infty \int \left[ (\kappa \nabla \theta + \frac{1}{\mu} \nabla(|u|^2) + \mu u \cdot \nabla u + \lambda \text{div} u \nabla \phi) \cdot \nabla \phi - \rho u \cdot \nabla f \phi \right] dx dt. \tag{1.10}
\end{align*}
\]

In order to formulate the existence theorem precisely, we need to consider the stationary problem of (1.1). It is easy to check that the pair of functions \((\rho_s(x), 0, 1)\) is the steady solution, where the steady state density \(\rho_s(x)\) is uniquely determined by
\[
\nabla P_s = \rho_s \nabla f \quad \text{and} \quad \rho_s(x) \to 1 \quad \text{as} \quad |x| \to \infty \quad \tag{1.11}
\]
with \(P_s = P(\rho_s, 1) = \mathcal{R}\rho_s\) being the steady pressure. Solving (1.11), we obtain
Proposition 1.1 For $f \in H^2 \cap W^{1,\infty}$, there exists a unique solution $\rho_s(x) = \exp\{f(x)/R\}$ of the problem (1.11), satisfying
\[
\begin{cases}
0 < \rho \leq \inf_{x \in \mathbb{R}^3} \rho_s(x) \leq \sup_{x \in \mathbb{R}^3} \rho_s(x) \leq \bar{\rho} < \infty, \\
\rho_s - 1 \in H^2 \cap W^{1,\infty}, \quad \|\nabla \rho_s\|_{H^1 \cap L^\infty} \leq C_f,
\end{cases}
\] (1.12)
where $\rho, \bar{\rho}$ are positive constants depending only on $R$ and $\|f\|_{L^\infty}$, and $C_f$ is a positive constant depending only on $\|f\|_{H^2 \cap L^\infty}$.

In order to measure the size of the data, we define the initial energy $C_0$ as follows:
\[
C_0 \triangleq \int \left[ \frac{1}{2} \rho_0|u_0|^2 + R \rho_s \left( \frac{\rho_0}{\rho_s} \ln \frac{\rho_0}{\rho_s} - \frac{\rho_0}{\rho_s} + 1 \right) \right] \, dx.
\] (1.13)
Then, our first result of this paper can be stated as follows.

Theorem 1.1 Let the conditions of Proposition 1.1 hold. For given positive numbers $\tilde{\rho}, \tilde{\theta} > 1$ and $M > 0$ (not necessarily small), assume that
\[
\begin{cases}
0 \leq \inf_{x \in \mathbb{R}^3} \rho_0(x) \leq \sup_{x \in \mathbb{R}^3} \rho_0(x) < \tilde{\rho}, \\
0 \leq \inf_{x \in \mathbb{R}^3} \theta_0(x) \leq \sup_{x \in \mathbb{R}^3} \theta_0(x) \leq \tilde{\theta}, \\
\|\nabla u_0\|_{L^2} \leq M.
\end{cases}
\] (1.14)
There exists a positive constant $\varepsilon$, depending on $\mu, \lambda, \kappa, R, cv, \tilde{\rho}, \tilde{\theta}, M$ and $f$, such that if
\[
C_0 \leq \varepsilon,
\] (1.15)
then the Cauchy problem (1.1)–(1.6) has a global weak solution $(\rho, u, E)$ on $\mathbb{R}^3 \times (0, \infty)$ in the sense of Definition 1.1, satisfying
\[
\begin{cases}
0 \leq \rho(x, t) \leq 2\tilde{\rho}, \quad \theta(x, t) \geq 0 \quad \text{a.e.} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty), \\
\rho - 1 \in C([0, \infty); L^2 \cap L^p), \quad \left( pu, \rho |u|^2, \rho (\theta - 1) \right) \in C([0, \infty); H^{-1}), \\
u \in C([0, \infty); L^2 \cap L^2(0, \infty; D^1)), \quad \theta - 1 \in C([0, \infty); W^{1,r} \cap L^2(0, \infty; D^1)), \\
(u, \omega, G, \nabla \theta)(\cdot, t) \in H^1, \quad \left\langle u \right\rangle_{\mathbb{R}^3 \times [t, \infty)}, \left\langle \theta \right\rangle_{\mathbb{R}^3 \times [t, \infty)} < \infty \quad \text{for} \quad t > 0,
\end{cases}
\] (1.16)
and for any $p \in (2, \infty)$ and $r \in [2, 6)$,
\[
\lim_{t \to \infty} \left( \|\rho(\cdot, t) - \rho_s\|_{L^p} + \|u(\cdot, t)\|_{L^p \cap L^\infty} + \|\nabla \theta(\cdot, t)\|_{L^r} \right) = 0.
\] (1.17)
Here, $G \triangleq (2\mu + \lambda)\text{div} u - (P(\rho, \theta) - P(\rho_s, 1))$ and $\omega \triangleq \nabla \times u$ are the “effective viscous flux” and the vorticity, respectively, and $\langle g \rangle_{1, 2}$ denotes the standard Hölder norm.

Remark 1.1 The weak solutions obtained in Theorem 1.1 are the so-called “intermediate weak” ones, which were firstly suggested by Hoff [11, 13]. Analogously to the arguments in [14], the following weak form of (2.1) holds for any test function $\phi \in \mathcal{D}(\mathbb{R}^3 \times (\infty, \infty))$,
\[
cv \int \rho_0 \phi_0(\cdot, 0) \, dx + cv \int_0^\infty \int_0^\infty \rho \theta (\phi_t + u \cdot \nabla \phi) \, dx \, dt - R \int_0^\infty \int \rho \theta (\text{div} u) \phi \, dx \, dt
\]
\[
= \kappa \int_0^\infty \int \nabla \theta \cdot \nabla \phi \, dx \, dt - \int_0^\infty \int (2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div} u)^2) \phi \, dx \, dt,
\] (1.18)
where \( \mathcal{D}(u) \triangleq (\nabla u + (\nabla u)^\top)/2 \) is the deformation tensor.

Although the “intermediate weak” solutions obtained in Theorem 1.1 have more regularity than the usual finite-energy/variational weak solutions (see [20, 8, 6]), the question that whether the uniqueness of such weak solutions holds or not is still unclear. Here, as usual, a solution is called a strong one if and only if it satisfies the equations almost everywhere and belongs to certain class of functions in which the uniqueness can be shown to hold.

Indeed, assume further that the initial data satisfies \( \rho_0 - 1 \in H^2 \cap W^{2,q}, (u_0, \theta_0 - 1) \in H^2 \), and the following compatibility conditions

\[
\begin{aligned}
-\mu \Delta u_0 - (\mu + \lambda)\nabla \text{div} u_0 + R \nabla (\rho \theta_0) &= \sqrt{\rho_0} g_1 \quad \text{with} \quad g_1 \in L^2, \\
\kappa \Delta \theta_0 + \frac{\kappa}{2} |\nabla u_0 + (\nabla u_0)^\top|^2 + \lambda (\text{div} u_0)^2 &= \sqrt{\rho_0} g_2 \quad \text{with} \quad g_2 \in L^2,
\end{aligned}
\tag{1.19}
\]

Huang-Li [14] (see also [31]) showed that the weak solution obtained becomes a classical one for any positive time. It is worth pointing out that the compatibility conditions as that in (1.19) were also proposed by Cho et al. (cf. [2, 3]) to deal with the effects induced by the presence of vacuum and to prove the local existence of strong solutions. It is easy to see that (1.10), together with (1.1), particularly implies

\[
\lim_{t \to 0^+} (\sqrt{\rho} \hat{u})(x, t) \in L^2 \quad \text{and} \quad \lim_{t \to 0^+} (\sqrt{\rho} \hat{\theta})(x, t) \in L^2,
\tag{1.20}
\]

where \( \hat{f} \triangleq f_t + u \cdot \nabla f \) denotes the material derivative.

Such a kind of compatibility condition as that in (1.19) also plays a very mathematically important role in the regularity issues of solutions, even in the case that the density is bounded below away from vacuum. For example, Valli [30] proved the existence of global strong solutions without vacuum to the compressible barotropic Navier-Stokes equations subject to small \( H^2 \) data under the assumption that

\[
\lim_{t \to 0^+} \hat{u}(x, t) \in L^2 \quad (\sim u_0 \in D^2 \text{ almost, if } u \in C([0, T]; D^2)).
\tag{1.21}
\]

Roughly speaking, if the condition (1.20) holds, then one has (see, for example, [14, Lemma 4.1] and [31, Lemma 3.4])

\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) + \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} \theta\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \, dt \leq C(T),
\tag{1.22}
\]

provided the initial energy is suitably small. This is the essential ingredient for the existence of strong solutions, especially, for the uniqueness of solutions (see, for example, [3]). With the help of (1.22), one can make use of the “effective viscous flux” and the vorticity to get that

\[
\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty} \in L^2(0, T),
\tag{1.23}
\]

which, together with the Beale-Kato-Majda’s inequality (see [1, 15]), shows that \( \|\nabla \rho\|_{L^2 \cap L^6} \) is bounded. With this, (1.22) and (1.23) at hand, one can then derive some (\( t \)-weighted) higher-order estimates of the solutions, which are necessary for the existence of classical solutions. By combining the methods used in [13] and [14, 31], it is easily seen that the same approach as above also works for the non-vacuum case, if (1.21) holds.

In view of the previous results achieved, up to now, the condition (1.20) or (1.21) has been technically used to derive the higher-order estimates of solutions, which play a key role in the existence and uniqueness theory for the compressible Navier-Stokes equations. So, a natural
question arises: Can we remove or relax the condition (1.20) or (1.22) and prove the uniqueness of solutions in a more general class of functions?

Motivated by [10, 18], by careful analysis we observe that if the solution \((\rho, u, \theta)\) belongs to the following class of functions:

\[
\begin{align*}
\rho - 1 &\in L^\infty([0, T]; L^2 \cap L^\infty \cap W^{1,3}), \\
u &\in L^\infty([0, T]; L^2 \cap L^2(0, T; H^1)), \\
\nabla v &\in L^\infty(0, T; L^3) \cap L^1(0, T; L^\infty), \\
\theta - 1 &\in L^\infty(0, T; L^2) \cap L^2(0, T; H^2), \\
(\sqrt{\nu}, \sqrt{\theta}) &\in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \\
\end{align*}
\]  

(1.24)

then \((\rho, u, \theta)\) is unique on \(\mathbb{R}^3 \times [0, T]\). The class of solutions given in (1.24) is more general and wider than the one in [3]. In particular, compared with the one in (1.20)/(1.21) \((\sim u_0 \in D^2 \text{ almost})\), it follows from (1.24) that, to prove uniqueness, it requires much weaker condition on the velocity (i.e., \(\nabla u_0 \in L^2 \cap L^3\), and thus, \(\nabla u \in L^\infty(0, T; L^2 \cap L^3)\)). So, the question now is that whether such a kind of solutions with lower regularity on initial data exists or not? This is the second purpose of the present paper. More precisely, we shall show that

**Theorem 1.2** In addition to the conditions of Theorem 1.1, assume further that

\[
\begin{align*}
\inf_{x \in \mathbb{R}^3} \rho_0(x) > 0, & \quad \|\nabla \rho_0\|_{L^p} \leq M_1 \quad \text{for some} \quad p \in (3, 6), \\
\|\nabla u_0\|_{L^3} \leq M_2, & \quad \|\nabla \theta_0\|_{L^2} \leq M_3, \quad \text{and} \quad 7\mu > \lambda.
\end{align*}
\]  

(1.25)

Then for any \(0 < T < \infty\), the Cauchy problem (1.1)–(1.6) admits a unique global strong solution \((\rho, u, \theta)\) on \(\mathbb{R}^3 \times [0, T]\) such that for any \(2 \leq r < 6\) and \(1 < q < \frac{3p}{3p - r}\),

\[
\begin{align*}
\rho - 1 &\in C([0, T]; H^1 \cap W^{1,p}), \quad \inf \rho(x, t) > 0, \\
u &\in C([0, T]; L^2 \cap L^r), \quad \inf \nabla \theta(x, t) > 0, \\
\theta - 1 &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \cap L^q(0, T; W^{1,\infty}), \\
(\sqrt{\nu}, \sqrt{\theta}) &\in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \\
(\sqrt{\nu}, \sqrt{\theta}) &\in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), \\
\end{align*}
\]  

(1.26)

provided (1.15) holds.

**Remark 1.2** The additional condition \(7\mu > \lambda\) was also assumed in [15, 28], where a blowup criterion, depending only on the upper bound of density, was obtained. Indeed, based on the condition \(7\mu > \lambda\), the authors [15, 28] succeeded in proving the higher-integrability of \(\rho u\) in the form \(\|\rho^{1/q} u\|_{L^\infty(0, T; \mathbb{L}^q)}\) for some \(q > 3\), which was technically used to deal with the nonlinear convection term \(\rho u \cdot \nabla u\) in the momentum equation (1.1)_2. However, here, we shall make use of such a condition to estimate the \(L^3\)-norm of the gradient of velocity, which is completely different from that in [15, 28].

Theorems 1.1 and 1.2 will be shown in Section 5, based on the local existence theorem of smooth solutions with strictly positive density (see, for example, [21, 14]) and the global a priori estimates. First, to prove Theorem 1.1, it suffices to derive some global a priori estimates, which are independent of the time and the lower bound of density. This will be done by modifying the analysis in [14], however, some new difficulties arise and the approach in [14] cannot be applied directly, due to the arbitrariness of the external potential force \(f\). Roughly speaking, to
circumvent the difficulties arising from the large external potential force, we need to make full use of the mathematical structure of the steady state to control the pressure and the external force by the deviations of \((\rho, \theta)\) from the steady state \((\rho_s, 1)\). Moreover, to deal with the term \(\|\nabla u\|_{L^4([0,T;L^4])}\) induced by the convection term \(u \cdot \nabla u\) as well as the quadratic nonlinearities \(|\mathcal{D}(u)|^2\) and \((\text{div} u)^2\) in (2.1), we have to introduce the following modified the modified “effective viscous flux” \(F\) and the modified vorticity \(\tilde{\omega}\):
\[
F \triangleq \rho_s^{-1}((2\mu + \lambda)\text{div} u - (P(\rho, \theta) - P(\rho_s, 1))) \quad \text{and} \quad \tilde{\omega} \triangleq \rho_s^{-1}\nabla \times u, \quad (1.27)
\]
which are completely different from the ones in [14] and are motivated by that in [17, 18]. Indeed, since \(\nabla u\) may be discontinuous across the hypersurface of \(\mathbb{R}^3\), it is difficult to show that \(\nabla^2 u\) is locally integrable, and thus, one cannot expect to control \(\|\nabla u\|_{L^4}\) by \(\|\nabla u\|_{H^1}\) directly. However, it turns out that \(F\) and \(\tilde{\omega}\) defined in (1.27) have better regularity than the velocity and the density/pressure. In fact, since
\[
\rho_s^{-1}(\mu \Delta u + (\mu + \lambda)\nabla \text{div} u) = \nabla F - \nabla \times \omega + \nabla \left((\rho_s^{-1}(P(\rho, \theta) - P(\rho_s, 1))) + \mathcal{O}\right),
\]
where \(\mathcal{O}\) can be well bounded by \(|\nabla \rho_s|\|\nabla u\|\). So, if \(F, \omega \in H^1\) and \(P(\rho, \theta) - P(\rho_s, 1) \in L^4\), one can formally obtain that \(\nabla^2 u \in W^{-1,4}\), which in turn yields the desired bound of \(\|\nabla u\|_{L^4}\). We mention here that the reason for the presence of the multiplier \(\rho_s^{-1}\) in both \(F\) and \(\tilde{\omega}\) is due to the effect of the external potential force \(f\).

The following standard “effective viscous flux” \(G\):
\[
G \triangleq (2\mu + \lambda)\text{div} u - (P(\rho, \theta) - P(\rho_s, 1)),
\]
also plays an important role in the derivations of the global a priori estimates, in particular, in the proof of the \(t\)-independent upper bound of the density. This will be done by slightly modifying the analysis in [14], based on the Gronwall type inequality (cf. Lemma 2.5). The slight difference between [14] and the present paper lies in the fact that the steady pressure is no longer a constant one as that in [14]. Thus, we have to compare the density with the upper bound of the steady density, and it is fortunate that some additional terms have right sign and can be neglected (see Lemma 3.8).

Secondly, to prove Theorem 1.2, we assume that the initial density has a positive lower bound, which is used to exclude the presence of vacuum. This, together with the additional and non-physical condition on the viscosity coefficients \(7\mu > \lambda\), is technically useful for the estimation of the gradient of velocity in \(L^3\)-norm. The bound of \(\|\nabla u\|_{L^3}\) plays an important role in the treatment of the quadratic nonlinear terms \(|\mathcal{D}(u)|^2\) and \((\text{div} u)^2\) in (2.1) and in the derivation of the \(t\)-weighted estimates \(\sqrt{t} \theta \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)\), the latter of which are crucial for the analysis of uniqueness.

The rest of this paper is organized as follows. In Section 2, we collect some known facts and elementary inequalities. In Sections 3 and 4, we derive the global \(t\)-independent a priori estimates and the global \(t\)-dependent estimates of the solutions, respectively. With all the a priori estimates at hand, we prove Theorems 1.1 and 1.2 in Section 5.

2 Preliminaries

To begin, it is easy to check that if the solution \((\rho, u, \theta)\) is a classical one, then the problem (1.1)–(1.6) can be reformulated as follows:
\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
\rho(u_t + u \cdot \nabla u) + \nabla P = \mu \Delta u + (\mu + \lambda)\nabla \text{div} u + \rho \nabla f, \\
c \nu \rho(\theta_t + u \cdot \nabla \theta) + P \text{div} u = \kappa \Delta \theta + 2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div} u)^2,
\end{cases}
(2.1)
\]
with the far-field behavior:
\[(\rho, u, \theta)(x, 0) = (1, 0, 1)(x), \quad \text{as} \quad |x| \to \infty, \quad (2.2)\]
and the initial data:
\[(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), \quad x \in \mathbb{R}^3. \quad (2.3)\]

The well-known Gagliardo-Nirenberg-Sobolev-type inequality (cf. [24]) will be repeatedly used in the derivations of a priori estimates

**Lemma 2.1** For \( p \in [2, 6] \), \( q \in (1, \infty) \) and \( r \in (3, \infty) \), assume that \( h \in H^1 \) and \( g \in L^q \cap D^{1,r} \). Then there exists a generic positive constant \( C \), depending only on \( p \), \( q \) and \( r \), such that

\[
\|h\|_{L^p} \leq C \|h\|_{L^2}^{\frac{6-p}{2}} \|\nabla h\|_{L^2}^{\frac{3q-6}{2}},
\]

\[
\|g\|_{L^{q}} \leq C \|g\|_{L^q}^{\frac{3r+q(r-3)}{6}} \|\nabla g\|_{L^r}^{\frac{3r}{6}}.
\]

Since the steady state density is strictly positive due to Proposition 1.1, the following Poincaré-type inequality can be obtained in a similar manner as that in [14, Lemma 2.3] (see also [18, Lemma 2.4])

**Lemma 2.2** Assume that \( \rho \geq 0 \) and \( \rho - \rho_s \in L^2 \), where \( \rho_s \geq \rho \) is the steady state density given by Proposition 1.1. Then there exists a positive constant \( C \), depending only on \( \rho \), such that for any open set \( \Sigma \subset \mathbb{R}^3 \) and \( v \in \{v \in D^1 \mid \rho|v|^r \in L^1(\Sigma)\} \) with \( r \in [1, 2] \),

\[
\int_\Sigma |v|^r \rho \leq C \int_\Sigma \rho |v|^r + C \|\rho - \rho_s\|_{L^2} \|\nabla v\|_{L^2}.
\]  

*Proof.* Indeed, it follows from (1.12) and Lemma 2.1 that

\[
\rho \int_\Sigma |v|^r \rho \leq \int_\Sigma \rho |v|^r \rho = \int_\Sigma \rho |v|^r \rho - \int_\Sigma (\rho - \rho_s) |v|^r \rho \leq \int_\Sigma \rho |v|^r \rho + C \|\rho - \rho_s\|_{L^2} \|v\|_{L^q}^{r(3-r)} \|v\|_{L^q}^{\frac{3r}{6}} \|v\|_{L^q}^{\frac{3r}{6}} \leq \int_\Sigma \rho |v|^r \rho + C \|\rho - \rho_s\|_{L^2} \|\nabla v\|_{L^2}^{3r} \|\nabla v\|_{L^2}^{3r},
\]

which, together with the Cauchy-Schwarz inequality, proves (2.6).

As aforementioned, the estimate of \( \|\nabla u\|_{L^4} \) relies strongly on the modified “effective viscous flux” \( F \) defined by

\[
F = \rho_s^{-1} ((2 \mu + \lambda) \text{div} u - (P - P_s)) \quad \text{with} \quad P = R \rho \theta \quad \text{and} \quad P_s = R \rho_s. \quad (2.7)
\]

By direct calculations, we infer from (1.11) that

\[
\rho_s^{-1} (\nabla P - \rho \nabla f) = \rho_s^{-1} [\nabla (P - P_s) - \rho_s^{-1} (\rho - \rho_s) \nabla P_s] = \nabla \left[ \rho_s^{-1} (P - P_s) \right] + \rho_s^{-2} [(P - P_s) - R(\rho - \rho_s)] \nabla \rho_s,
\]

and hence, it follows from (2.1) that

\[
\rho_s^{-1} \rho u = (2 \mu + \lambda) \rho_s^{-1} \nabla \text{div} u - \mu \rho_s^{-1} \nabla \times (\nabla \times u) - \rho_s^{-1} (\nabla P - \rho \nabla f) = \nabla F - \mu \nabla \times (\rho_s^{-1} \nabla \times u) - (2 \mu + \lambda) (\text{div} u) \nabla \rho_s^{-1} + \mu \nabla \rho_s^{-1} \times (\nabla \times u) + R \rho (\theta - 1) \nabla \rho_s^{-1},
\]

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where \( \dot{v} \triangleq v_t + u \cdot \nabla v \) and the identity \( \nabla \times (\nabla \times u) = \nabla \text{div} u - \Delta u \) was also used. Thus,
\[
\Delta F = \text{div} \left( \rho_s^{-1} \dot{\rho}u + F_1 + F_2 \right),
\]
and
\[
\mu \Delta (\rho_s^{-1} \nabla \times u) = \nabla \times (\rho_s^{-1} \dot{\rho}u + F_1 + F_2) + \mu \nabla \left( \nabla \rho_s^{-1} \cdot (\nabla \times u) \right),
\]
where
\[
F_1 \triangleq (2\mu + \lambda) (\text{div} u) \nabla \rho_s^{-1} - \mu \nabla \rho_s^{-1} \times (\nabla \times u), \quad F_2 \triangleq -R \rho(\theta - 1) \nabla \rho_s^{-1}.
\]

In view of (2.7)–(2.10) and the standard estimates of elliptic system, we obtain the following important estimates.

**Lemma 2.3** Let \( \rho_s(x) \) be the steady density in Proposition 1.1. Assume that \((\rho, u, \theta)\), satisfying \(0 \leq \rho \leq 2\bar{\rho}\), is a smooth solution of (2.1)–(2.3). Then there exists a positive constant \(C > 0\), depending only on \(\mu, \lambda, R, \underline{\rho}, \bar{\rho}\) and \(\|f\|_{H^2 \cap W^{1, \infty}}\), such that
\[
\|F\|_{L^2} + \|\rho_s^{-1} \nabla \times u\|_{L^2} \leq C(\bar{\rho}) \left( \|\nabla u\|_{L^2} + \|\sqrt{\bar{\rho}}(\theta - 1)\|_{L^2} + \|\rho - \rho_s\|_{L^2} \right),
\]
and
\[
\|\nabla F\|_{L^2} + \|\nabla(\rho_s^{-1} \nabla \times u)\|_{L^2} \leq C(\bar{\rho}) \left( \|\sqrt{\bar{\rho}} u\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} \right).
\]

**Proof.** Indeed, noting that
\[
P(\rho, \theta) - P_s = R \rho(\theta - 1) + R(\rho - \rho_s),
\]
we immediately obtain (2.11) from (2.7) and (1.12). Using (2.10) and (1.12), we see that
\[
\|(F_1, F_2)\|_{L^2} \leq C(\bar{\rho}) \|\nabla \rho_s\|_{H^1 \cap L^\infty} \left( \|\nabla u\|_{L^2} + \|\theta - 1\|_{L^6} \right) \leq C(\bar{\rho}) \left( \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} \right),
\]
which, together with (2.8), (2.9) and the \(L^p\)-estimates of elliptic system, leads to (2.12). \(\square\)

Next, we consider the standard “effective viscous flux” \(G\) and vorticity \(\omega\) defined by
\[
G \triangleq (2\mu + \lambda) \text{div} u - (P - P_s), \quad \omega \triangleq \nabla \times u,
\]
from which, (2.1) and (1.11), we see that
\[
\Delta G = \text{div}(\dot{\rho}u - (\rho - \rho_s) \nabla f), \quad \mu \Delta \omega = \nabla \times (\dot{\rho}u - (\rho - \rho_s) \nabla f).
\]
The main difference between \(F\) and \(G\) lies in the fact that the estimates of \(F\) can be well controlled by the deviations of \((\rho, \theta)\) from the steady state \((\rho_s, 1)\).

Similarly to the proof of Lemma 2.3, we have by (2.13) and (2.14) that

**Lemma 2.4** Let \(\rho_s(x)\) be the steady density in Proposition 1.1. Assume that \((\rho, u, \theta)\), satisfying \(0 \leq \rho \leq 2\bar{\rho}\), is a smooth solution of (2.1)–(2.3). Then there exists a positive constant \(C > 0\), depending only on \(\mu, \lambda, R, \underline{\rho}, \bar{\rho}\) and \(\|f\|_{H^2 \cap W^{1, \infty}}\), such that for any \(p \in [2, 6]\),
\[
\|\nabla G\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C(\bar{\rho}) \left( \|\sqrt{\bar{\rho}} u\|_{L^p} + \|\rho - \rho_s\| \nabla f\|_{L^p} \right),
\]
\[
\|G\|_{L^p} + \|\omega\|_{L^p} \leq C(\bar{\rho}) \left( \|\nabla u\|_{L^2} + \|\theta - 1\|_{L^2} + \|\rho - \rho_s\|_{L^2} \right)^{\frac{6-p}{2p}}
\times \left( \|\sqrt{\bar{\rho}} u\|_{L^2} + \|\rho - \rho_s\|_{L^6} \right)^{\frac{3p-6}{2p}},
\]
and
\[
\|\nabla u\|_{L^p} \leq C(\bar{\rho}) \|\nabla u\|_{L^2} \left( \|\sqrt{\bar{\rho}} u\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\rho - \rho_s\|_{L^6} \right)^{\frac{3p-6}{2p}}.
\]
Proof. In terms of the standard $L^p$-estimate of elliptic system, one obtains (2.15) from (2.14) immediately. By virtue of (2.4), (2.13) and (2.15) with $p = 2$, we have

$$
\|G\|_{L^p} + \|\omega\|_{L^p} \leq C \|G\|_{L^2}^{\frac{6-p}{2p}} \|\nabla G\|_{L^2}^{\frac{6-p}{2p}} + C \|\omega\|_{L^2}^{\frac{6-p}{2p}} \|\nabla \omega\|_{L^2}^{\frac{6-p}{2p}}
$$

$$
\leq C(\tilde{\rho}) \left( \|\nabla u\|_{L^2} + \|P - P_s\|_{L^2} \right) \frac{6-p}{2p} \left( \|\sqrt{\rho} \nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2} \right) \frac{3p-6}{2p}
$$

$$
\leq C(\tilde{\rho}) \left( \|\nabla u\|_{L^2} + \|\rho(\theta - 1)\|_{L^2} + \|\rho - \rho_s\|_{L^2} \right) \frac{6-p}{2p} \left( \|\sqrt{\rho} \nabla u\|_{L^2} + \|\rho - \rho_s\|_{L^2} \right) \frac{3p-6}{2p},
$$

which finishes the proof of (2.16). Similarly,

$$
\|\nabla u\|_{L^p} \leq C \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \|\nabla \omega\|_{L^p}^{\frac{6-p}{2p}}
$$

$$
\leq C \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \left( \|\nabla \omega\|_{L^2} + \|\nabla \omega\|_{L^6} \right) \frac{3p-6}{2p}
$$

$$
\leq C \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \left( \|\nabla G\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\rho(\theta - 1)\|_{L^6} + \|\rho - \rho_s\|_{L^6} \right) \frac{3p-6}{2p}
$$

$$
\leq C(\tilde{\rho}) \|\nabla u\|_{L^2}^{\frac{6-p}{2p}} \left( \|\sqrt{\rho} \nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\rho - \rho_s\|_{L^6} \right) \frac{3p-6}{2p}.
$$

The proof of Lemma 2.5 is therefore complete. □

The global-in time upper bound of the density will be proved by applying the following Gronwall-type inequality, whose proof is elementary and can be found in [14].

**Lemma 2.5** For $T > 0$ and $T_1 \in [0, T]$, assume that $g \in L^p(0, T_1) \cap L^q(T_1, T)$ with $p, q \geq 1$. Let $y \in W^{1,1}(0, T)$ be the solution of the ODE system:

$$
y'(t) + \alpha y(t) \leq g(t) \quad \text{on} \quad [0, T], \quad y(0) = y_0,
$$

(2.18)

where $\alpha > 0$ is a positive constant. Then,

$$
\sup_{0 \leq t \leq T} y(t) \leq |y_0| + (1 + \alpha^{-1}) (\|g\|_{L^p(0, T_1)} + \|g\|_{L^q(T_1, T)}).
$$

(2.19)

Finally, the following Beale-Kato-Majda’s type inequality (cf. [1, 15]) will be used to deal with $\|\nabla u\|_{L^\infty}$, which is essential for the estimate of $\|\nabla \rho\|_{L^p}$ for $3 < p < 6$.

**Lemma 2.6** For any $q \in (3, +\infty)$, assume that $u \in D^1 \cap D^{2, q}$. Then there exists a positive constant $C$, depending only on $q$, such that

$$
\|\nabla u\|_{L^\infty} \leq C (\|\nabla u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln (e + \|\nabla^2 u\|_{L^q}) + C (\|\nabla u\|_{L^2} + 1).
$$

(2.20)

### 3 A Priori Estimates (I): Time-independent Estimates

This section is devoted to the derivations of the global a priori estimates. To do so, let $(\rho, u, \theta)$ be a smooth solution of (2.1)–(2.3) on $\mathbb{R}^3 \times [0, T]$ for some $T > 0$. Similarly to that in [14, 13], to estimate the solution, we define $A_i(T)$ with $i = 1, \ldots, 4$ as follow:

$$
A_1(T) \triangleq \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} \nabla u\|_{L^2}^2 dt,
$$

(3.1)
\[ A_2(T) \triangleq \sup_{0 \leq t \leq T} \| \sqrt{\rho}(\theta - 1) \|_{L^2}^2 + \int_0^T (\| \nabla \theta \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) \, dt, \]  
\[ A_3(T) \triangleq \sup_{0 < t \leq T} \left( \sigma \| \nabla u \|_{L^2}^2 + \sigma^2 \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \sigma^2 \| \nabla \theta \|_{L^2}^2 \right) 
+ \int_0^T \left( \sigma \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \sigma^2 \| \nabla \theta \|_{L^2}^2 \right) \, dt, \]  
\[ A_4(T) \triangleq \sup_{0 < t \leq T} \left( \sigma^4 \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 \right) + \int_0^T \sigma^4 \| \nabla \dot{\theta} \|_{L^2}^2 \, dt, \]  
where \( \dot{v} = v_t + u \cdot \nabla u \) is the material derivative and the weighted function \( \sigma(t) = \min\{1, t\} \) will be used to eliminate the effects of initial layer.

The proof of Theorem 1.1 is based on the following key a priori estimates of \( (\rho, u, \theta) \).

**Proposition 3.1** Let \( f \in H^2 \cap W^{1,\infty} \) and \( \rho_s(x) \) be the ones given in Proposition 1.1. For given positive numbers \( \bar{\rho} > 2\rho, \bar{\theta} > 1 \) and \( M > 0 \) (not necessary small), assume that

\[ 0 < \inf \rho_0 \leq \rho_0 \leq \bar{\rho}, \quad 0 < \inf \theta_0 \leq \theta_0 \leq \bar{\theta}, \quad \| \nabla u_0 \|_{L^2} \leq M. \]  

Then there exist positive constants \( K \) and \( \varepsilon_0 \), depending only on \( \mu, \lambda, R, c, \rho, \bar{\rho}, f \| H^2 \cap W^{1,\infty}, \bar{\rho}, \bar{\theta} \) and \( M \), such that if \( (\rho, u, \theta) \) is a smooth solution of (2.1)-(2.3) on \( \mathbb{R}^3 \times (0, T] \), satisfying

\[ 0 < \rho \leq 2\bar{\rho}, \quad A_1(\sigma(T)) \leq 3K, \quad \text{and} \quad A_i(T) \leq 2C_0^{1/(2i)} \quad \text{with} \quad i = 2, 3, 4, \]  

then the following estimates hold:

\[ 0 < \rho \leq \frac{7\bar{\rho}}{4}, \quad A_1(\sigma(T)) \leq 2K, \quad \text{and} \quad A_i(T) \leq C_0^{1/(2i)} \quad \text{with} \quad i = 2, 3, 4, \]  

provided

\[ C_0 \leq \varepsilon_0, \]  

where \( C_0 \) is the initial energy defined in (1.13).

**Proof.** Proposition 3.1 readily follows from Lemmas 3.2, 3.3, 3.6, 3.8 and 3.9 with \( K \) and \( \varepsilon_0 \) being the same ones determined in (3.21) and (3.95), respectively. \( \square \)

Throughout this section, we always assume that \( C_0 \leq 1 \). For simplicity, we denote by \( C, C_i \ (i = 1, 2, \ldots) \) the generic positive constants, which may depend on \( \mu, \lambda, R, c, \rho, \bar{\rho}, f \| H^2 \cap W^{1,\infty}, \bar{\rho}, \bar{\theta} \) and \( M \), but not on \( T \). We also sometimes write \( C(\alpha) \) to emphasize the dependence on \( \alpha \). We will also use the Einstein convention that repeated indices denote the summation over the indices.

We begin with the following elementary \( L^2 \)-estimate.

**Lemma 3.1** Assume that \( (\rho, u, \theta) \) with \( 0 < \rho \leq 2\bar{\rho} \) is a smooth solution of (2.1)-(2.3) on \( \mathbb{R}^3 \times [0, T] \). Then,

\[ \sup_{0 \leq t \leq T} \left( \| \sqrt{\rho} u(t) \|_{L^2}^2 + \| (\rho - \rho_s)(t) \|_{L^2}^2 \right) \leq C(\bar{\rho})C_0, \]  

and

\[ \| (\theta - 1)(t) \|_{L^2} \leq C(\bar{\rho}) \left( C_0^{1/2} + C_0^{1/3} \| \nabla \theta(t) \|_{L^2} \right), \quad \forall \ t \in (0, T]. \]  

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Proof. First of all, an application of maximum principle to (2.1)$_3$ shows that
\[
\theta(x,t) > 0 \quad \text{for all} \quad x \in \mathbb{R}^3, \ t \in [0,T]. \tag{3.11}
\]

Next, multiplying (2.1)$_1$ and (2.1)$_3$ by $u$ and $1 - \theta^{-1}$ in $L^2$, and adding them together, we obtain after integrating by parts that
\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + c \rho (\theta - \ln \theta - 1) \right) dx - \int (\rho u \cdot \nabla f + \theta^{-1} P \div u) dx \\
+ \int \left( \frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\lambda (\div u)^2}{\theta} + 2 \mu |\mathcal{D} u|^2 \right) dx = 0, \tag{3.12}
\]
where we have used the following simple fact that
\[
2 \int |\mathcal{D}(u)|^2 dx = \int (|\nabla u|^2 + (\div u)^2) dx.
\]

Thanks to (2.1)$_1$, (1.11) and (1.12), we have
\[
- \int (\rho u \cdot \nabla f + \theta^{-1} P \div u) dx = -R \int (\rho_s^{-1} \rho u \cdot \nabla \rho_s + \rho \div u) dx \\
= R \int \div (\rho u) (\ln \rho_s - \ln \rho) dx = R \int \rho_t \ln \rho dx \\
= R \frac{d}{dt} \int \rho_s \left( \frac{\rho}{\rho_s} \ln \frac{\rho}{\rho_s} - \frac{\rho}{\rho_s} + 1 \right) dx,
\]
which, inserted into (3.12), gives
\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R \rho_s \left( \frac{\rho}{\rho_s} \ln \frac{\rho}{\rho_s} - \frac{\rho}{\rho_s} + 1 \right) + c \rho (\theta - \ln \theta - 1) \right) (x,t) dx \\
+ \int \left( \frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\lambda (\div u)^2}{\theta} + 2 \mu |\mathcal{D} u|^2 \right) dx ds = 0, \tag{3.13}
\]
so that, one obtains the following entropy-type estimate:
\[
\int \left( \frac{1}{2} \rho |u|^2 + R \rho_s \left( \frac{\rho}{\rho_s} \ln \frac{\rho}{\rho_s} - \frac{\rho}{\rho_s} + 1 \right) + c \rho (\theta - \ln \theta - 1) \right) (x,t) dx \\
+ \int_0^t \int \left( \frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\lambda (\div u)^2}{\theta} + 2 \mu |\mathcal{D} u|^2 \right) dx ds = C_0. \tag{3.14}
\]

Based on the Taylor theorem, similarly to that in [14], we infer from (3.14) and (1.12) that
\[
\int |\rho - \rho_s|^2 dx \leq C(\bar{\rho}) \int \rho_s \left( \frac{\rho}{\rho_s} \ln \frac{\rho}{\rho_s} - \frac{\rho}{\rho_s} + 1 \right) dx \leq C(\bar{\rho}) C_0 \tag{3.15}
\]
and
\[
\int (\rho(\theta - 1) \chi_{\{\theta > 1\}} + \rho(\theta - 1)^2 \chi_{\{\theta < 3\}}) dx \leq C \int \rho(\theta - \ln \theta - 1) dx \leq CC_0, \tag{3.16}
\]
where $\chi_A$ is the characteristic function of the set $A$, and
\[
\{\theta(x,t) > 2\} \triangleq \{x \in \mathbb{R}^3 \mid \theta(x,t) > 2\}, \quad \{\theta(x,t) < 3\} \triangleq \{x \in \mathbb{R}^3 \mid \theta(x,t) < 3\}.
\]
With the help of (3.16), it is easily deduced from Lemma 2.2 that
\[
\|\theta - 1\|_{L^1(\{\theta(t) > 2\})} \leq C\|\rho(\theta - 1)\|_{L^1(\{\theta(t) > 2\})} + C\|\rho - \rho_s\|_{L^2}^{5/3}\|\nabla \theta\|_{L^2}
\leq C(\bar{\rho}) \left( C_0 + C_0^{5/6}\|\nabla \theta\|_{L^2} \right),
\]
and consequently,
\[
\|\theta - 1\|_{L^2(\{\theta(t) > 2\})} \leq \|\theta - 1\|_{L^1(\{\theta(t) > 2\})}^{2/5}\|\theta - 1\|_{L^2(\{\theta(t) > 2\})}^{3/5}
\leq C(\bar{\rho}) \left( C_0 + C_0^{5/6}\|\nabla \theta\|_{L^2} \right)^{2/5}\|\nabla \theta\|_{L^2}^{3/5}
\tag{3.17}
\]
Analogously, we also have
\[
\|\theta - 1\|_{L^2(\{\theta(t) < 3\})} \leq C\|\rho(\theta - 1)\|_{L^2(\{\theta(t) < 3\})} + C\|\rho - \rho_s\|_{L^2}^{2/3}\|\nabla \theta\|_{L^2}
\leq C(\bar{\rho}) \left( C_0^{1/2} + C_0^{1/3}\|\nabla \theta\|_{L^2} \right).
\tag{3.18}
\]
Therefore, collecting (3.14), (3.15), (3.17) and (3.18) together finishes the proofs of (3.9) and (3.10).

The next lemma is concerned with the short-time estimate of $A_1(\sigma(T))$.

**Lemma 3.2** Assume that $(\rho, u, \theta)$ is a smooth solution of (2.1)–(2.3) on $\mathbb{R}^3 \times [0, T]$, satisfying
\[
0 < \rho \leq 2\bar{\rho} \quad \text{and} \quad A_2(\sigma(T)) \leq 2C_0^{1/4}.
\tag{3.19}
\]
Then there exist positive constants $K \geq M + 1$ and $\varepsilon_1$, depending only on $\mu$, $\lambda$, $\kappa$, $R$, $c_Y$, $\rho$, $\bar{\rho}$, $\|f\|_{H^2 \cap W^{1, \infty}}$, $\bar{\rho}$, $\bar{\theta}$ and $M$, such that
\[
A_1(\sigma(T)) \leq 2K,
\tag{3.20}
\]
provided
\[
A_1(\sigma(T)) \leq 3K \quad \text{and} \quad C_0 \leq \varepsilon_1.
\tag{3.21}
\]
**Proof.** Thanks to (1.11), we can write (2.1)$_2$ in the form:
\[
\rho \dot{u} - \mu \Delta u - (\mu + \lambda) \nabla \text{div}u + \nabla (P - P_s) = (\rho - \rho_s) \nabla f
\tag{3.22}
\]
which, multiplied by $u_t$ in $L^2$ and integrated by parts over $\mathbb{R}^3$, gives
\[
\frac{1}{2} \frac{d}{dt} \left( \mu \|\nabla u\|_{L^2}^2 + (\mu + \lambda) \|\text{div} u\|_{L^2}^2 \right) + \|\sqrt{\rho} \dot{u}\|_{L^2}^2
\]
\[= - \int u_t \cdot \nabla (P - P_s) dx + \int (\rho - \rho_s) u_t \cdot \nabla f dx + \int \rho u \cdot \nabla u \cdot \dot{u} dx
\]
\[
= \frac{d}{dt} \int (\text{div}u)(P - P_s) dx + \frac{d}{dt} \int (\rho - \rho_s) u \cdot \nabla f dx - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int (P - P_s)^2 dx
\]
\[= - \frac{1}{2\mu + \lambda} \int G(P - P_s) dx + \int \text{div} ((\rho - \rho_s) u + \rho_s u) u \cdot \nabla f dx + \int \rho u \cdot \nabla u \cdot \dot{u} dx
\]
\[= \frac{d}{dt} \int \left( (\text{div}u)(P - P_s) + (\rho - \rho_s) u \cdot \nabla f - \frac{1}{2(2\mu + \lambda)} (P - P_s)^2 \right) dx + \sum_{i=1}^{3} I_i,
\]
where we have also used (2.1) and (2.13) to get that
\[
\text{div} u = (2\mu + \lambda)^{-1} (\nabla (G + (P - P_s)) \quad \text{and} \quad \rho_t + \text{div} ((\rho - \rho_s)u) + \text{div}(\rho_s u) = 0.
\]

To deal with the most difficult term \(I_1\), we first infer from (2.1) that
\[
\frac{\text{cv}}{R} (P - P_s)_t = -\frac{\text{cv}}{R} \text{div}(P u) - P \text{div} u + \kappa \Delta \theta + 2\mu |\nabla u|^2 + \lambda (\text{div} u)^2,
\]
which, inserted into \(I_1\) and integrated by parts, yields (noting that \(P = R\rho(\theta - 1) + R\rho\))
\[
I_1 \leq C(\hat{\rho}) \sqrt{\rho}(\theta - 1) \|\nabla \theta\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\nabla G\|_{L^2} + C(\hat{\rho}) \sqrt{\rho} \|\nabla u\|_{L^2} \|\nabla G\|_{L^2}
+ C(\hat{\rho}) \|\nabla u\|_{L^2} \|\nabla G\|_{L^2} + C(\hat{\rho}) \|\nabla u\|_{L^2} \|\nabla G\|_{L^2}
+ C(\hat{\rho}) \|\nabla u\|_{L^2} \|\nabla G\|_{L^2} + C(\hat{\rho}) \|\nabla u\|_{L^2} \|\nabla G\|_{L^2}
\]
\[
\leq C(\hat{\rho}) C_0^{1/16} \|\nabla \theta\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\nabla G\|_{L^2} + C(\hat{\rho}) C_0^{1/2} \|\nabla G\|_{L^2}
\]
\[
+ C(\hat{\rho}) \|\nabla u\|_{L^2} \|\nabla G\|_{L^2} + C(\hat{\rho}) \|\nabla u\|_{L^2} \|\nabla G\|_{L^2}
\]
where we have used (3.19), Lemmas 2.1 and 3.1. In terms of Lemmas 2.4 and 3.1, one has
\[
\|\nabla G\|_{L^2} + \|\nabla \omega\|_{L^2} \leq C(\hat{\rho}) \left( \|\sqrt{\rho} \hat{u}\|_{L^2} + C_0^{1/3} \right),
\]
\[
\|G\|_{L^2} + \|\omega\|_{L^2} \leq C(\hat{\rho}) \left( \|\nabla u\|_{L^2} + C_0^{1/3} \|\nabla \theta\|_{L^2} + C_0^{1/2} \right),
\]
and
\[
\|\nabla u\|_{L^6} \leq C(\hat{\rho}) \left( \|\sqrt{\rho} \hat{u}\|_{L^2} + \|\nabla \theta\|_{L^2} + C_0^{1/6} \right),
\]
so that, putting (3.27)–(3.29) into (3.26), we infer from the Cauchy-Schwarz inequality that
\[
I_1 \leq \delta \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + C(\delta, \hat{\rho}) \left( C_0^{1/3} + \|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 \right).
\]

Integrating by parts and using Lemma 2.1, (1.12), (3.9) and (3.19), we have
\[
I_2 \leq C \left( 1 + \|\rho - \rho_s\|_{L^2}^6 + \|\nabla \rho_s\|_{L^2} \right) \left( \|u\|_{L^6} \|\nabla u\|_{L^2} + \|u\|_{L^6}^2 \right) \|f\|_{L^2}
\]
\[
\leq C(\hat{\rho}) \|\nabla u\|_{L^2}^2.
\]

Using (3.29) and the Cauchy-Schwarz inequality, we easily get that
\[
I_3 \leq C(\hat{\rho}) \|\sqrt{\rho} \hat{u}\|_{L^2} \|u\|_{L^6} \|\nabla u\|_{L^1} \leq C(\hat{\rho}) \|\sqrt{\rho} \hat{u}\|_{L^2} \|\nabla u\|_{L^2}^{3/2} \|\nabla u\|_{L^6}^{1/2}
\]
\[
\leq C(\hat{\rho}) \|\sqrt{\rho} \hat{u}\|_{L^2} \|\nabla u\|_{L^2}^{3/2} \left( \|\sqrt{\rho} \hat{u}\|_{L^2} + \|\nabla \theta\|_{L^2} + C_0^{1/6} \right)^{1/2}
\]
\[
\leq \delta \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + C(\delta, \hat{\rho}) \left( C_0^{1/3} + \|\nabla u\|_{L^2}^6 + \|\nabla \theta\|_{L^2}^2 \right).
\]

Finally, due to (3.9) and (3.19), it is easy to see that
\[
\int \left( (\text{div} u)(P - P_s) \right) dx + (\rho - \rho_s) u \cdot \nabla f - \frac{1}{2(2\mu + \lambda)} (P - P_s)^2 \right) dx
\]
\[
\leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C(\rho(\theta - 1)) \|\nabla f\|_{L^2}^2 + C(\rho - \rho_s) \|\nabla f\|_{L^2}^2 \left( 1 + \|\nabla f\|_{L^2}^2 \right)
\]
\[
\leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C(\hat{\rho}) C_0^{1/4}.
\]
Now, substituting (3.30)–(3.32) into (3.24) and choosing $\delta > 0$ sufficiently small, by (3.19) and (3.33) we deduce after integrating it over $(0, \sigma(T))$ that

$$A_1(\sigma(T)) \leq CM + C(\bar{\rho})C_0^{1/4} + C(\bar{\rho}) \int_0^{\sigma(T)} \|\nabla u\|^6_{L^2} dt$$

$$\leq K + C(\bar{\rho})C_0^{1/4} [A_1(\sigma(T))]^2,$$

where $K \triangleq 1 + CM + C(\bar{\rho})$. Thus if $A_1(\sigma(T)) \leq 3K$ and $C_0 \leq \varepsilon_1 \triangleq \min\{1, (9C(\bar{\rho})K)^{-4}\}$, then (3.20) readily follows from (3.34).

The next step is to estimate $A_2(T)$, which is an energy-type estimate of the solutions.

**Lemma 3.3** Let $(\rho, u, \theta)$ be a smooth solution of (2.1)–(2.3) on $\mathbb{R}^3 \times [0, T]$, satisfying (3.6) with $K > 0$ being the same one as in (3.21). Then, there exists a positive constant $\varepsilon_2$, depending only on $\mu, \lambda, \kappa, R, c_V, \rho, \bar{\rho}, \|f\|_{H^2 \cap W^{1, \infty}}, \bar{\rho}, \theta$ and $M$, such that

$$A_2(T) \leq C_0^{1/4},$$

provided $C_0 \leq \varepsilon_2$.

**Proof.** First, multiplying (3.22) by $u$ in $L^2$ and integrating by parts, by Lemma 3.1 we obtain

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u\|^2_{L^2} + (\mu \|\nabla u\|^2_{L^2} + (\mu + \lambda) \|\text{div} u\|^2_{L^2})$$

$$\leq C(\bar{\rho}) (\|\theta - 1\|_{L^2} + \|\rho - \rho_s\|_{L^2} + \|\rho - \rho_s\|_{L^2} \|\nabla f\|_{L^2}) \|\nabla u\|_{L^2}$$

$$\leq C(\bar{\rho}) C_0^{2/3} + C(\bar{\rho}) C_0^{1/3} (\|\nabla \theta\|^2_{L^2} + \|\nabla u\|^2_{L^2}).$$

Similarly, multiplying (2.1)3 by $\theta - 1$ in $L^2$ and integrating by parts, we have

$$\frac{c_V}{2} \frac{d}{dt} \|\sqrt{\rho} (\theta - 1)\|^2_{L^2} + \kappa \|\nabla \theta\|^2_{L^2}$$

$$\leq C \int \rho \theta |\text{div} u| |\theta - 1| dx + C \int |\nabla u|^2 |\theta - 1| dx$$

$$\leq C(\bar{\rho}) \int |\text{div} u| |\theta - 1|^2 dx + C(\bar{\rho}) \int |\text{div} u| |\theta - 1| dx + C \int |\nabla u|^2 |\theta - 1| dx$$

$$\triangleq I_1 + I_2 + I_3.$$

In terms of Lemma 2.1, (3.6) and (3.10), the term $I_1$ can be bounded as follows:

$$I_1 \leq C(\bar{\rho}) \|\nabla u\|_{L^2} \|\theta - 1\|_{L^2}^{1/2} \|\theta - 1\|_{L^6}^{3/2}$$

$$\leq C(\bar{\rho}) \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}^{3/2} \left( C_0^{1/4} + C_0^{1/6} \|\nabla \theta\|_{L^2}^{1/2} \right)$$

$$\leq C(\bar{\rho}, M) C_0^{1/2} + C(\bar{\rho}, M) C_0^{1/6} (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}),$$

where we have also used (3.6) to get that

$$\sup_{0 \leq t \leq T} \|\nabla u(t)\|^2_{L^2} \leq A_1(\sigma(T)) + A_3(T) \leq C(\bar{\rho}, M).$$

Similarly, it follows from (3.10), (3.29) and (3.39) that

$$I_2 \leq C(\bar{\rho}) \|\nabla u\|_{L^2} \left( C_0^{1/2} + C_0^{1/3} \|\nabla \theta\|_{L^2} \right)$$

$$\leq C(\bar{\rho}, M) C_0^{1/2} + C(\bar{\rho}, M) C_0^{1/6} (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}),$$

$$I_3 \leq C(\bar{\rho}, M) C_0^{1/2} + C(\bar{\rho}, M) C_0^{1/6} (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}).$$
\[ I_3 \leq C \| \nabla u \|_{L^2} \| \nabla u \|_{L^6} \| \theta - 1 \|_{L^2}^{1/2} \| \nabla \theta \|_{L^2}^{1/2} \]
\[ \leq C(\bar{\rho}, M) \left( \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \| \nabla \theta - 1 \|_{L^2}^2 + 1 \right) \left( C_0^{1/4} + C_0^{1/6} \| \nabla \theta \|_{L^2}^{1/2} \right) \| \nabla \theta \|_{L^2}^{1/2} \]
\[ \leq C(\bar{\rho}, M) \left( \delta + C_0^{1/6} \right) \| \nabla \theta \|_{L^2}^2 + C(\bar{\rho}, M, \delta) C_0^{1/3} \left( 1 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \right). \]

Now, inserting (3.38), (3.40) and (3.41) into (3.37), and then adding it to (3.37), we find
\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho} u \|_{L^2}^2 + c \| \sqrt{\rho} (\theta - 1) \|_{L^2}^2 \right) + \left( \mu \| \nabla u \|_{L^2}^2 + (\mu + \lambda) \| \text{div} u \|_{L^2}^2 + \kappa \| \nabla \theta \|_{L^2}^2 \right)
\[
\leq C(\bar{\rho}, M) \left( \delta + C_0^{1/6} \right) \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) + C(\bar{\rho}, M, \delta) C_0^{1/3} \left( 1 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \right). \]

Hence, if \( C_0 \) and \( \delta \) are chosen to be such that
\[
C_0 \leq \varepsilon_{e_1} \triangleq \min \left\{ \varepsilon_1, \left( (4C(\bar{\rho}, M)^{-1} \min \{\mu, \kappa \} \right) \right\} \quad \text{and} \quad \delta \leq (4C(\bar{\rho}, M))^{-1} \min \{\mu, \kappa \},
\]
then by (3.6) we deduce from (3.42) that
\[
\sup_{0 \leq t \leq \sigma(T)} \left( \| \sqrt{\rho} u \|_{L^2}^2 + \| \sqrt{\rho} (\theta - 1) \|_{L^2}^2 \right) + \int_0^{\sigma(T)} \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) dt
\leq C(\bar{\rho}, M) C_0^{1/3} \int_0^{\sigma(T)} \left( 1 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \right) dt \leq C(\bar{\rho}, M) C_0^{1/3},
\]
since it holds that
\[
\| \sqrt{\rho_0} (\theta_0 - 1) \|_{L^2}^2 \leq C(\bar{\theta}) \int \rho_0 (\theta_0 - \ln \theta_0 - 1) dx,
\]
due to the fact that \( 0 < \theta_0 \leq \bar{\theta} \).

For the case of \( t \in [\sigma(T), T] \), it suffices to show that \( \theta(x, t) \) is uniformly bounded on \( \mathbb{R}^3 \times [\sigma(T), T] \), since it is easily deduced from (3.3), (3.43) and the Taylor theorem that
\[
\sup_{\sigma(T) \leq t \leq T} \left( \| \sqrt{\rho} u \|_{L^2}^2 + \| \sqrt{\rho} (\theta - 1) \|_{L^2}^2 \right) + \int_\sigma(T)^T \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right) dt \leq C(\bar{\rho}, M) C_0^{1/3}. \]

Next, we claim that if \( C_0 \) is chosen to be small enough, then
\[
\frac{1}{2} \leq \theta(x, t) \leq \frac{3}{2}, \quad \forall (x, t) \in \mathbb{R}^3 \times [\sigma(T), T]. \]

Indeed, due to (3.29) and (3.6), one has
\[
\sigma \| \nabla u \|_{L^6} \leq C(\bar{\rho}) \sigma \left( \| \sqrt{\rho} \dot{u} \|_{L^2} + \| \nabla \theta \|_{L^2} + C_0^{1/6} \right) \leq C(\bar{\rho}) C_0^{1/12}, \]
and hence, by (3.6) we infer from (2.1)_3 and the \( L^2 \)-theory of elliptic system that
\[
\sigma^2 \| \nabla^2 \theta \|_{L^2} \leq C(\bar{\rho}) \sigma^2 \left( \| \rho \theta \|_{L^2} + \| \theta \nabla u \|_{L^2} + \| \nabla u \|_{L^4}^2 \right)
\leq C(\bar{\rho}) \sigma^2 \left( \| \rho \theta \|_{L^2} + \| \nabla u \|_{L^2} + \| (\theta - 1) \nabla u \|_{L^2} + \| \nabla u \|_{L^2}^2 \right)
\leq C(\bar{\rho}) C_0^{1/16} + C(\bar{\rho}) \sigma^2 \left( \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2}^{1/2} \| \nabla u \|_{L^6}^{1/2} + \| \nabla u \|_{L^2}^{1/2} \| \nabla u \|_{L^6}^{3/2} \right) \]
\leq C(\bar{\rho}) C_0^{1/16}, \]
which, together with (3.6) and Lemma 2.1, yields
\[
\sup_{0 \leq t \leq T} \sigma^2 \| \theta - 1 \|_{L^\infty} \leq C \sup_{0 \leq t \leq T} \sigma^2 \left( \| \nabla \theta \|_{L^2} + \| \nabla^2 \theta \|_{L^2} \right) \leq C(\bar{\rho})C_0^{1/16} \leq \frac{1}{2}, \tag{3.48}
\]
provided that \( C_0 \) is small enough such that
\[
C_0 \leq \epsilon_{2,2} \overset{\Delta}{=} \min \{ \epsilon_{2,1}, (2C(\bar{\rho}))^{-16} \}.
\]
It is easily seen from (3.48) that (3.45) holds, and thus, (3.44) follows. In view of (3.43) and (3.44), we immediately obtain (3.35) by choosing \( C_0 \leq \epsilon_{2,2} \overset{\Delta}{=} \min \{ \epsilon_{2,2}, C(\bar{\rho}, M)^{-12} \} \).

In order to estimate \( A_3(T) \), we first prove the following preliminary lemma.

**Lemma 3.4** Let \((\rho, u, \theta)\) be a smooth solution of (2.1)–(2.3) on \(\mathbb{R}^3 \times [0, T]\), satisfying (3.6) with \( K > 0 \) being the same one as in (3.21). Then for any \( \delta \in (0, 1) \), one has
\[
\frac{d}{dt} \left( \sigma \left( \mu \| \nabla u \|_{L^2}^2 + (\mu + \lambda)(\| \text{div} u \|_{L^2}^2) \right) \right) + \sigma \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \leq 2 \frac{d}{dt} \int \sigma ((P - P_s) \text{div} u + (\rho - \rho_s) u \cdot \nabla f) \, dx + 2\delta \sigma^2 \| \sqrt{\rho} \theta \|_{L^2}^2
\]
\[+ C(\bar{\rho})C_0^{1/4} \sigma' + C(\delta, \bar{\rho}, M) \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 + \sigma^2 \| \nabla u \|_{L^4}^4 \right), \tag{3.49}
\]
and
\[
\frac{d}{dt} \left( \sigma^2 \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \right) + \sigma^2 \left( \mu \| \nabla \dot{u} \|_{L^2}^2 + (\mu + \lambda)(\| \text{div} \dot{u} \|_{L^2}^2) \right)
\]
\[\leq C(\bar{\rho})\sigma \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + C(\bar{\rho}, M)\sigma^2 \left( \| \sqrt{\rho} \theta \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right)
\]
\[+ C(\bar{\rho}, M) \left( \| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \right). \tag{3.50}
\]

**Proof.** First, multiplying (3.22) by \( \sigma \dot{u} \) in \( L^2 \) gives
\[
\int \sigma \rho \dot{u} dx = \int \sigma (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u) \cdot \dot{u} dx
\]
\[- \int \sigma \dot{u} \cdot \nabla (P - P_s) dx + \int \sigma (\rho - \rho_s) \dot{u} \cdot \nabla f dx \overset{\Delta}{=} \sum_{i=1}^{3} I_i. \tag{3.51}
\]

Similarly to that in [11, 14, 18], we have from integration by parts that
\[
I_1 \leq -\frac{1}{2} \frac{d}{dt} \int \sigma (\mu \| \nabla u \|_{L^2}^2 + (\mu + \lambda)(\| \text{div} u \|_{L^2}^2) \, dx + C\sigma' \| \nabla u \|_{L^2}^2 + C\sigma \| \nabla u \|_{L^3}^3
\]
\[\leq -\frac{1}{2} \frac{d}{dt} \int \sigma (\mu \| \nabla u \|_{L^2}^2 + (\mu + \lambda)(\| \text{div} u \|_{L^2}^2) \, dx + C (\| \nabla u \|_{L^2}^2 + \sigma^2 \| \nabla u \|_{L^4}^4) \tag{3.52}
\]
Noting that
\[
(P - P_s)_t + \text{div}(u(P - P_s)) = R \rho \dot{\theta} - \text{div}(uP_s), \tag{3.53}
\]
so that, using (1.12), (2.4), (3.6), (3.9), (3.35), (3.39) and (3.46) we deduce after integrating by
parts that for any $\delta \in (0,1)$ (to be chosen later),

\[
I_2 = \frac{d}{dt} \int \sigma(P - P_s)(\text{div} u)dx - \sigma' \int (P - P_s)(\text{div} u)dx \\
- \int \sigma \left( R\rho \theta (\text{div} u) - \text{div} (uP_s) \text{div} u - (P - P_s) \partial_t u^i \partial_j u^j \right) dx \\
\leq \frac{d}{dt} \int \sigma(P - P_s)(\text{div} u)dx + C(\bar{\rho})\sigma' \left( \|\sqrt{\rho} (\theta - 1)\|_{L^2}^2 + \|\rho - \rho_s\|_{L^2}^2 \right) \\
+ \delta \sigma^2 \|\sqrt{\rho} \theta\|_{L^2}^2 + C(\delta, \rho) \|\nabla u\|_{L^2}^2 + C(\bar{\rho})\|\theta - 1\|_{L^6} \|\nabla u\|_{L^2}^3/2 \|\nabla u\|_{L^2}^{1/2} \\
\leq \frac{d}{dt} \int \sigma(P - P_s)(\text{div} u)dx + \delta \sigma^2 \|\sqrt{\rho} \theta\|_{L^2}^2 + C(\bar{\rho})C_0^{1/4} \sigma' \\
+ C(\delta, \rho, M) \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right),
\]

(3.54)

By virtue of (1.12), (3.9) and (3.24), we have

\[
I_3 \leq \frac{d}{dt} \int \sigma(\rho - \rho_s) u \cdot \nabla f dx + C(\bar{\rho})C_0^{1/2} \sigma' + C(\bar{\rho})\|\nabla u\|_{L^2}^2.
\]

(3.55)

Thus, substituting (3.52), (3.54) and (3.55) into (3.51), we arrive at (3.49).

In order to prove (3.50), operating $\sigma^m \tilde{u}^j (\partial_t + \text{div}(u \cdot))$ with $m \geq 1$ to the $j$-th equation of (3.22) and integrating it over $\mathbb{R}^3$, we obtain after adding them together that

\[
\frac{d}{dt} \left( \frac{\sigma^m}{2} \|\sqrt{\rho} \tilde{u}\|_{L^2}^2 \right) - \mu \int \sigma^m \tilde{u}^j \left( \Delta u^i_t + \partial_k(u^k \Delta u^j) \right) dx \\
= \int \sigma^m \tilde{u}^j \left( \partial_j \text{div} u + \partial_k(u^k \partial_j \text{div} u) \right) dx \\
= \frac{m}{2} \sigma^{m-1} \sigma' \|\sqrt{\rho} \tilde{u}\|_{L^2}^2 - \int \sigma^m \tilde{u}^j \left( (\partial_j P)_t + \partial_k \left( u^k \partial_j (P - P_s) \right) \right) dx \\
+ \int \sigma^m \tilde{u}^j \left( \rho_t \partial_j f + \partial_k \left( u^k (\rho - \rho_s) \partial_j f \right) \right) dx 
\]

(3.56)

Based upon integration by parts, it is easy to get that

\[
- \mu \int \sigma^m \tilde{u}^j \left( \Delta u^i_t + \partial_k(u^k \Delta u^j) \right) dx \geq \frac{7\mu}{8} \sigma^m \|\nabla \tilde{u}\|_{L^2}^2 - C\sigma^m \|\nabla u\|_{L^4}^4,
\]

(3.57)

and similarly,

\[
- (\mu + \lambda) \int \sigma^m \tilde{u}^j \left( \partial_j \text{div} u + \partial_k(u^k \partial_j \text{div} u) \right) dx \\
\geq \sigma^m \left( (\mu + \lambda) \|\text{div} \tilde{u}\|_{L^2}^2 - \frac{\mu}{8} \|\nabla \tilde{u}\|_{L^2}^2 \right) - C\sigma^m \|\nabla u\|_{L^4}^4.
\]

(3.58)

It follows from (3.53) that

\[
(\partial_j P)_t + \partial_k \left( u^k \partial_j (P - P_s) \right) = \partial_j \left( R\rho \theta - \text{div} (uP_s) \right) - \text{div} (\partial_j u(P - P_s)),
\]

and hence, after integrating by parts we have by (1.12) and (3.6) that

\[
J_1 \leq C(\bar{\rho})\sigma^m \|\nabla \tilde{u}\|_{L^2} \left( \|\sqrt{\rho} \tilde{\theta}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^1} \|\theta - 1\|_{L^6} \right) \\
\leq \frac{\mu}{8} \sigma^m \|\nabla \tilde{u}\|_{L^2}^2 + C(\bar{\rho})\sigma^m \left( \|\sqrt{\rho} \tilde{\theta}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \right).
\]

(3.59)
Analogously, by (2.1) we find
\[
J_2 = \int \sigma^m \dot{\theta} (\mu \theta \partial_{j}^2 f - \text{div} (\rho, u \partial_{j} f)) \, dx
\]
\[
\leq C(\bar{\rho}) \sigma^m \left( ||\sqrt{\rho} \dot{\theta}||_{L^2}^{1/2} ||\dot{\theta}||_{L^2} ||u||_{L^6} ||\nabla^2 f||_{L^2} + ||\nabla \dot{\theta}||_{L^2} ||u||_{L^6} ||\nabla f||_{L^3} \right)
\]
\[
\leq \frac{\mu}{8} \sigma^m ||\nabla \dot{\theta}||_{L^2}^2 + C(\bar{\rho}) \sigma^m (||\sqrt{\rho} \dot{\theta}||_{L^2}^2 + ||\nabla u||_{L^2}^2).
\]

Now, substituting (3.57)–(3.60) into (3.56) gives
\[
\frac{d}{dt} \left( \sigma^m ||\sqrt{\rho} \dot{\theta}||_{L^2}^2 \right) + \sigma^m \left( \mu ||\nabla \dot{\theta}||_{L^2}^2 + (\mu + \lambda) ||\text{div} \dot{u}||_{L^2}^2 \right)
\]
\[
\leq m \sigma^{m-1} \sigma' ||\sqrt{\rho} \dot{\theta}||_{L^2}^2 + C(\bar{\rho}) \sigma^m \left( ||\sqrt{\rho} \dot{\theta}||_{L^2}^2 + ||\sqrt{\rho} \dot{\theta}||_{L^2}^2 + ||\nabla u||_{L^2}^2 \right) + C(\bar{\rho}) \sigma^m (||\nabla u||_{L^2}^2 + ||\nabla u||_{L^6} ||\nabla \theta||_{L^2}^2).
\]

Thus, choosing \( m = 2 \) in (3.61), using (3.39) and (3.46), we immediately arrive at (3.50).

Next, we proceed to estimate the first-order derivatives of temperature.

**Lemma 3.5** Let \((\rho, u, \theta)\) be a smooth solution of (2.1)–(2.3) on \(\mathbb{R}^3 \times [0, T]\), satisfying (3.6) with \(K > 0\) being the same one as in (3.21). Then for \(m \geq 1\),
\[
\frac{d}{dt} \left( \kappa \sigma^m ||\nabla \theta||_{L^2}^2 - 2 \sigma^m \int \theta (\text{div} u)^2 + 2 \mu ||\mathcal{D}(u)||^2 \, dx \right) + \sigma^m ||\sqrt{\rho} \dot{\theta}||_{L^2}^2
\]
\[
\leq C \sigma^{m-1} \sigma' \left( ||\nabla \theta||_{L^2}^2 + \int \theta ||\nabla u||_{L^2}^2 \, dx \right) + C(\eta, \bar{\rho}, M) \sigma^m \left( ||\theta \nabla u||_{L^2}^2 + ||\nabla u||_{L^4}^4 \right) + C(\eta, \bar{\rho}, M) ||\nabla \theta||_{L^2}^2.
\]

**Proof.** Similarly to the proof of (3.49), multiplying (2.1) by \(\sigma^m \dot{\theta}\) with \( m \geq 1 \) in \(L^2\) and integrating by parts, we have
\[
\frac{\kappa}{2} \frac{d}{dt} \left( \sigma^m ||\nabla \theta||_{L^2}^2 \right) - \frac{\kappa}{2} m \sigma^{m-1} \sigma' ||\nabla \theta||_{L^2}^2 + C \sigma^m \sigma' ||\sqrt{\rho} \dot{\theta}||_{L^2}^2
\]
\[
= -\kappa \int \sigma^m \nabla \theta \cdot \nabla (u \cdot \nabla \theta) \, dx + \lambda \int \sigma^m (\text{div} u)^2 \dot{\theta} \, dx
\]
\[
+ 2 \mu \int \sigma^m |\mathcal{D}(u)|^2 \dot{\theta} \, dx - R \int \sigma^m \rho (\text{div} u) \dot{\theta} \, dx \triangleq \sum_{i=1}^{4} I_i.
\]

After integrating by parts, we have by (2.4), (3.39) and (3.47) that
\[
I_1 \leq C \sigma^m ||\nabla u||_{L^2} ||\nabla \theta||_{L^4}^2 \leq C \sigma^m ||\nabla u||_{L^2} ||\nabla \theta||_{L^2}^{1/2} ||\nabla^2 \theta||_{L^2}^{3/2}
\]
\[
\leq C(\bar{\rho}) \sigma^m ||\nabla u||_{L^2} ||\nabla \theta||_{L^2}^{1/2} \left( ||\sqrt{\rho} \dot{\theta}||_{L^2} + ||\theta \nabla u||_{L^2} + ||\nabla u||_{L^4}^2 \right)^{3/2}
\]
\[
\leq \frac{C \nu}{4} \sigma^m ||\sqrt{\rho} \dot{\theta}||_{L^2}^2 + C(\bar{\rho}, M) \sigma^m \left( ||\nabla \theta||_{L^2}^2 + ||\theta \nabla u||_{L^2}^2 + ||\nabla u||_{L^4}^4 \right).
\]
By direct calculations, we see that for any \( \eta \in (0, 1) \),

\[
I_2 = \lambda \frac{d}{dt} \int \sigma^m (\text{div} u)^2 \theta dx - \lambda m \sigma^{m-1} \sigma' \int (\text{div} u)^2 \theta dx \\
- \lambda \int \sigma^m \left( 2\theta (\text{div} u)(\text{div} \hat{u}) - 2\theta (\text{div} u) \partial_i u^k \partial_k u^i + \theta (\text{div} u)^3 \right) dx \\
\leq \lambda \frac{d}{dt} \int \sigma^m (\text{div} u)^2 \theta dx - \lambda m \sigma^{m-1} \sigma' \int (\text{div} u)^2 \theta dx + \eta \sigma^m \| \nabla u \|_{L^2}^2 \\
+ C(\eta) \sigma^m \left( \| \theta \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right),
\]

and similarly,

\[
I_3 \leq 2\mu \frac{d}{dt} \int \sigma^m |\mathcal{D}(u)|^2 \theta dx - 2\mu m \sigma^{m-1} \sigma' \int |\mathcal{D}(u)|^2 \theta dx + \eta \sigma^m \| \nabla u \|_{L^2}^2 \\
+ C(\eta) \sigma^m \left( \| \theta \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^4}^4 \right).
\]

It is easy to get that

\[
I_4 \leq \frac{cv}{4} \sigma^m \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 + C(\rho) \| \theta \nabla u \|_{L^2}^2.
\]

Thus, putting (3.64)–(3.67) into (3.63) immediately leads to (3.62).

With the help of Lemmas 3.4 and 3.5, we can now close the estimate of \( A_3(T) \).

**Lemma 3.6** Let \( (\rho, u, \theta) \) be a smooth solution of (2.1)–(2.3) on \( \mathbb{R}^3 \times [0, T] \), satisfying (3.6) with \( K > 0 \) being the same one as (3.21). Then, there exists a positive constant \( \varepsilon_3 \), depending only on \( \mu, \lambda, \kappa, R, c_v, \varrho, \bar{\rho}, \| f \|_{H^2} \), \( \| \nabla \|_{W^{1, \infty}}, \bar{\rho}, \bar{\theta} \) and \( M \), such that

\[
A_3(T) \leq C_0^{1/6},
\]

provided \( C_0 \leq \varepsilon_3 \).

**Proof.** It follows from (3.39) and (3.46) that

\[
\sigma^{1/2} \| \nabla u \|_{L^2} \leq \sigma (\| \theta - 1 \|_{L^2} + \| \nabla u \|_{L^2}) \\
\leq \sigma^{1/2} \left( \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2}^{3/2} + \| \nabla u \|_{L^2} \right) \\
\leq C(\rho, M) \left( \| \nabla \theta \|_{L^2} + \| \nabla u \|_{L^2} \right),
\]

and hence,

\[
\sigma^2 \int (\lambda \text{div} u)^2 + 2\mu |\mathcal{D}(u)|^2 \, dx \leq C\sigma^2 \| \theta \nabla u \|_{L^2} \| \nabla u \|_{L^2} \\
\leq \frac{\kappa}{8} \sigma^2 \| \nabla \theta \|_{L^2}^2 + C(\rho, M) \sigma \| \nabla u \|_{L^2}^2.
\]

So, choosing \( m = 2 \) in (3.62), multiplying it by a suitably large number, adding the resulting inequality to (3.50), and then taking \( \eta > 0 \) suitably small, by virtue of (3.69) and (3.70) we obtain after integrating over \((0, T)\) that

\[
\sup_{0 \leq t \leq T} \sigma^2 \left( \| \nabla \theta \|_{L^2}^2 + \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 \right) + \int_0^T \sigma^2 \left( \| \sqrt{\rho} \dot{\theta} \|_{L^2}^2 + \| \nabla \dot{u} \|_{L^2}^2 \right) dt \\
\leq C(\rho, M) \sup_{0 \leq t \leq T} (\sigma \| \nabla u \|_{L^2}^2) + C(\rho, M) \int_0^T \sigma \| \sqrt{\rho} \dot{u} \|_{L^2}^2 dt \\
+ C(\rho, M) \int_0^T \sigma^2 \| \nabla u \|_{L^4}^4 dt + C(\rho, M) C_0^{1/4},
\]

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where we have used (3.6) and (3.69) to get that
\[
\int_0^T \sigma \int_0^T \theta |\nabla u|^2 \, dx \, dt \leq C \int_0^T (\sigma^2 \|\theta \nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \, dt
\]
\[
\leq C(\bar{\rho}, M) \int_0^T (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) \, dt \leq C(\bar{\rho}, M) C_0^{1/4}.
\]
Due to (3.6) and (3.9), we have
\[
\sigma \int 0^T (\|\nabla u\|^2_{L^2} + \|\rho - \rho_s\|_{L^2} \|\nabla u\|_{L^2}) \, dt
\]
\[
\leq C(\bar{\rho}) \sigma (\|\sqrt{\rho}(\theta-1)\|_{L^2} + \|\rho - \rho_s\|_{L^2}) \|\nabla u\|_{L^2}
\]
\[
\leq C(\bar{\rho}) C_0^{1/8} \sigma \|\nabla u\|_{L^2} \leq \frac{\mu}{4} \sigma \|\nabla u\|^2_{L^2} + C(\bar{\rho}) C_0^{1/4},
\]
and thus, multiplying (3.49) by a suitably large number, adding it to (3.71), and then choosing \(\delta > 0\) in (3.49) suitably small, we find
\[
A_3(T) \leq C(\bar{\rho}, M) C_0^{1/4} + C(\bar{\rho}, M) \int_0^T \sigma^2 \|\nabla u\|^4_{L^4} \, dt. \tag{3.72}
\]
It remains to deal with \(\|\nabla u\|^4_{L^4}\). Indeed, due to (3.6) and (3.39), one has
\[
\|F\|_{L^2} + \|\rho_s^{-1} \nabla \times u\|_{L^2} \leq C(\bar{\rho}) (\|\nabla u\|_{L^2} + \|\sqrt{\rho}(\theta-1)\|_{L^2} + \|\rho - \rho_s\|_{L^2}) \leq C(\bar{\rho}, M)
\]
so that, the standard \(L^p\) estimates, together with (1.12), (2.4) and Lemma 2.3, show
\[
\|\nabla u\|^4_{L^4} \leq C (\|\nabla u\|^4_{L^4} + \|\rho_s^{-1} \nabla \times u\|^4_{L^4} + \|P - P_s\|^4_{L^4})
\]
\[
\leq C(\bar{\rho}) (\|F\|^2_{L^2} \|\nabla F\|^2_{L^2} + \|\rho_s^{-1} \nabla \times u\|_{L^2} \|\nabla (\rho_s^{-1} \nabla \times u)\|_{L^2})
\]
\[
+ C(\bar{\rho}) (\|\sqrt{\rho}(\theta-1)\|_{L^2} \|\nabla \theta\|^2_{L^2} + \|\rho - \rho_s\|^2_{L^2}) \tag{3.73}
\]
\[
\leq C(\bar{\rho}, M) (\|\sqrt{\rho_s} \nabla u\|^3_{L^2} + \|\nabla u\|^3_{L^2} + \|\nabla \theta\|^3_{L^2} + \|\rho - \rho_s\|^4_{L^2}),
\]
and analogously,
\[
\|F\|^4_{L^4} + \|\rho(\theta-1)\|^4_{L^4} \leq C(\bar{\rho}, M) (\|\sqrt{\rho_s} \nabla u\|^3_{L^2} + \|\nabla u\|^3_{L^2} + \|\nabla \theta\|^3_{L^2}). \tag{3.74}
\]
We are now in a position of estimating the right-hand side of (3.73). On one hand, it is easily derived from (3.6) and (3.39) that
\[
\int_0^T \sigma^2 \left(\|\sqrt{\rho_s} \nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}\right) \, dt
\]
\[
\leq C(\bar{\rho}, M) C_0^{1/4} + [A_3(T)]^{1/2} \int_0^T (\sigma \|\sqrt{\rho_s} \nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) \, dt \tag{3.75}
\]
\[
\leq C(\bar{\rho}, M) C_0^{1/4} + C C_0^{1/12} \left(C_0^{1/6} + C_0^{1/4}\right) \leq C(\bar{\rho}, M) C_0^{1/4}.
\]
On the other hand, to estimate \(\|\rho - \rho_s\|^4_{L^4}\), we use (2.7) to rewrite (2.1) in the form:
\[
(\rho - \rho_s) + \frac{R \rho_s}{2\mu + \lambda} (\rho - \rho_s) = -\text{div}(u(\rho - \rho_s)) - u \cdot \nabla \rho_s - \frac{R \rho_s}{2\mu + \lambda} \rho(\theta - 1) - \frac{\rho_s^2 F}{2\mu + \lambda}.
\]
which, multiplied by $4(\rho - \rho_s)^2$ and integrated by parts over $\mathbb{R}^3$, yields

$$
\frac{d}{dt}\|\rho - \rho_s\|_{L^2}^4 + \|\rho - \rho_s\|_{L^2}^4 \leq C(\tilde{\rho}) \left( \|\nabla u\|_{L^2}^2 + \|\rho(\theta - 1)\|_{L^2}^4 + \|F\|_{L^4}^4 \right),
$$

(3.76)

where we have also used (1.12) and (3.6). Thus, multiplying (3.76) by $\sigma^2$ and integrating it over $[0, T]$, by (3.7), (3.76) and (3.75) we obtain

$$
\int_0^T \sigma^2\|\rho - \rho_s\|_{L^4}^4 dt \leq C(\tilde{\rho}, M)C_0^{1/4}.
$$

(3.77)

Thus, by virtue of (3.73), (3.75) and (3.77) we easily infer from (3.72) that

$$
A_3(T) \leq C(\tilde{\rho}, M)C_0^{1/4} \leq C_0^{1/6}.
$$

provided $C_0 \leq \varepsilon_3 \triangleq \min\{\varepsilon_2, C(\tilde{\rho}, M)^{-12}\}$. This finishes the proof of (3.68).

The following lemma plays an important role in both the $t$-independent upper bound of density and the $t$-dependent estimate of the gradient of density as well (see Lemma 4.1).

**Lemma 3.7** Let $(\rho, u, \theta)$ be a smooth solution of (2.1)–(2.3) on $\mathbb{R}^3 \times [0, T]$, satisfying (3.6) with $K > 0$ being the same one as in (3.21). Then,

$$
\sup_{0 \leq t \leq T} \sigma \left( \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \int_0^t \sigma \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 \right) dt \right) \leq C(\tilde{\rho}, M),
$$

(3.78)

provided $C_0 \leq \varepsilon_3$.

**Proof.** Similarly to the derivation of (3.70), by (3.39) and (3.69) we have

$$
\sigma \int \theta \left( \lambda(\text{div} u)^2 + 2\mu |\mathcal{D}(u)|^2 \right) dx \leq \frac{K}{8} \sigma \|\nabla \theta\|_{L^2}^2 + C(\tilde{\rho}, M),
$$

so that, choosing $m = 1$ in (3.61) and (3.62), using (3.6), (3.39) and (3.46) we deduce in a manner similar to the derivation of (3.71) that

$$
\sup_{0 \leq t \leq T} \sigma \left( \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \int_0^T \sigma \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 \right) dt \right)
$$

$$
\leq C(\tilde{\rho}, M) + C(\tilde{\rho}, M) \int_0^{\sigma(T)} \int |\nabla u|^2 dx dt + C(\tilde{\rho}, M) \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^4}^4 dt
$$

$$
\leq C(\tilde{\rho}, M) + C(\tilde{\rho}, M) \int_0^{\sigma(T)} \sigma \|\nabla u\|_{L^4}^4 dt,
$$

(3.79)

where we have used (3.6), (3.29) and (3.39) to get that

$$
\int_0^{\sigma(T)} \int |\nabla u|^2 dx dt \leq C \int_0^{\sigma(T)} \left( \|\theta - \|_{L^6} \|\nabla u\|_{L^{12/5}}^2 + \|\nabla u\|_{L^2}^2 \right) dt
$$

$$
\leq C(\tilde{\rho}, M) + C(\tilde{\rho}, M) \int_0^{\sigma(T)} \|\nabla \theta\|_{L^2} \|\nabla u\|_{L^2}^{3/2} \|\nabla u\|_{L^6}^{1/2} dt
$$

$$
\leq C(\tilde{\rho}, M) + C(\tilde{\rho}, M) \int_0^{\sigma(T)} \left( \|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 \right) dt
$$
\[ \leq C(\tilde{\rho}, M). \]

In view of (3.6), one gets
\[ \int_0^T \sigma \left( ||\sqrt{\rho}u||_{L^2}^2 + ||\nabla \theta||_{L^2}^2 + ||\nabla u||_{L^2}^2 \right) dt \leq C(\tilde{\rho}, M)[A_3(T)]^{1/2} \int_0^T \left( ||\sqrt{\rho}u||_{L^2}^2 + ||\nabla \theta||_{L^2}^2 + ||\nabla u||_{L^2}^2 \right) dt \leq C(\tilde{\rho}, M)C_0^{1/12}, \]
and hence, it is easily derived from (3.74) that
\[ \int_0^T \sigma \left( ||F||_{L^2}^4 + ||\rho(\theta - 1)||_{L^2}^4 \right) dt \leq C(\tilde{\rho}, M)C_0^{1/12}, \]
which, together with (3.73) and (3.76), yields
\[ \int_0^T \sigma \left( ||\rho - \rho_s||_{L^2}^4 + ||\nabla u||_{L^4}^4 \right) dt \leq C(\tilde{\rho}, M)C_0^{1/12}. \quad (3.80) \]

Similarly to the derivation of (3.47), one deduces from (3.6), (3.46) and (3.80) that
\[ \int_0^T \sigma ||\nabla^2 \theta||_{L^2}^2 dt \leq C(\tilde{\rho}, M) \int_0^T \sigma \left( ||\sqrt{\rho}u||_{L^2}^2 + ||\nabla \theta||_{L^2}^2 + ||\nabla u||_{L^2}^2 ||\nabla \theta||_{L^2}^2 + ||\nabla u||_{L^4}^4 \right) dt \leq C(\tilde{\rho}, M) + C(\tilde{\rho}, M) \int_0^T \sigma \left( ||\sqrt{\rho}u||_{L^2}^2 + ||\nabla \theta||_{L^2}^2 ||\nabla u||_{L^2}^2 ||\nabla \theta||_{L^2}^2 \right) dt \leq C(\tilde{\rho}, M) + C(\tilde{\rho}, M) \int_0^T \sigma ||\sqrt{\rho}u||_{L^2}^2 dt, \quad (3.81) \]
so that, substituting (3.80) into (3.79) and integrating it over \([0, T]\), by (3.81) we immediately arrive at the desired estimate stated in (3.78).

Now, based on Lemma 2.5, we can derive the upper bound of density.

**Lemma 3.8** Let \((\rho, u, \theta)\) be a smooth solution of (2.1)–(2.3) on \(\mathbb{R}^3 \times [0, T]\), satisfying (3.6) with \(K > 0\) being the same one as in (3.21). Then, there exists a positive constant \(\varepsilon_4\), depending only on \(\mu, \lambda, \kappa, R, c_V, \bar{\rho}, \tilde{\rho}, \bar{\theta}, \theta\) and \(M\), such that
\[ \sup_{0 \leq t \leq T} ||\rho(t)||_{L^\infty} \leq \frac{3}{2} \tilde{\rho}, \quad (3.82) \]
provided \(C_0 \leq \varepsilon_4\).

**Proof.** It follows from (2.1) and (2.13) that
\[ (2\mu + \lambda)D_t(\rho - \tilde{\rho}) = -(2\mu + \lambda)\rho \text{div} u = -\rho (G + P - P_s) = -\rho G - R\rho^2(\theta - 1) - R\tilde{\rho}(\rho - \tilde{\rho}) - R(\rho - \tilde{\rho})^2 - R\rho(\rho - \rho_s) \leq -\rho G - R\rho^2(\theta - 1) - R\tilde{\rho}(\rho - \tilde{\rho}), \]
since the last two terms in the second identity are nonnegative due to (1.12). Thus,
\[ (2\mu + \lambda)D_t(\rho - \tilde{\rho}) + R\tilde{\rho}(\rho - \tilde{\rho}) \leq C(\tilde{\rho}) \left( ||G||_{L^\infty} + ||\theta - 1||_{L^\infty} \right). \quad (3.83) \]
In order to apply Lemma 2.5, we need to handle the right-hand side of (3.83). First, with the help of (2.5), (3.6) and (3.78), we have

$$\int_0^{\sigma(T)} \| \theta - 1 \|_{L^\infty} dt \leq C \int_0^{\sigma(T)} \| \nabla \theta \|_{L^2}^{1/2} \left( \sigma \| \nabla^2 \theta \|_{L^2}^2 \right)^{1/4} \sigma^{-1/4} dt$$

$$\leq C \left( \int_0^{\sigma(T)} \| \nabla \theta \|_{L^2}^2 dt \right)^{1/4} \left( \int_0^{\sigma(T)} \sigma \| \nabla^2 \theta \|_{L^2}^2 dt \right)^{1/4} \left( \int_0^{\sigma(T)} \sigma^{-1/2} dt \right)^{1/2}$$

(3.84)

and

$$\int_{\sigma(T)}^T \| \theta - 1 \|_{L^\infty} dt \leq C \int_{\sigma(T)}^T \| \nabla \theta \|_{L^2}^2 \| \nabla^2 \theta \|_{L^2}^2 dt$$

$$\leq CA_3(T) \int_{\sigma(T)}^T \| \nabla^2 \theta \|_{L^2}^2 dt \leq C(\tilde{\rho}, M) C_0^{1/6}.$$  

(3.85)

Lemma 2.4, together with the fact that \( f \in H^2 \cap W^{1,\infty} \), shows that

$$\| \nabla G \|_{L^2} \leq C(\tilde{\rho}) \left( \| \sqrt{\rho \tilde{u}} \|_{L^2} + \| \rho - \rho_s \|_{L^4} \right)$$

(3.86)

and

$$\| \nabla G \|_{L^4} \leq C(\tilde{\rho}) \left( \| \sqrt{\rho \tilde{u}} \|_{L^2}^{1/4} \| \nabla \tilde{u} \|_{L^2}^{3/4} + \| \rho - \rho_s \|_{L^4} \right),$$

(3.87)

which, combined with Lemma 2.1, (3.6), (3.9) and (3.78), gives

$$\| G \|_{L^\infty} \leq C \| \nabla G \|_{L^2}^{1/3} \| \nabla G \|_{L^4}^{2/3}$$

$$\leq C(\tilde{\rho}) \left( \| \sqrt{\rho \tilde{u}} \|_{L^2}^{1/3} + C_0^{1/12} \right) \left( \| \nabla \tilde{u} \|_{L^2}^{1/6} \| \nabla \tilde{u} \|_{L^2} + C_0^{1/6} \right)$$

$$\leq C(\tilde{\rho}) C_0^{1/3} + C(\tilde{\rho}) C_0^{1/6} \left( \sigma \| \sqrt{\rho \tilde{u}} \|_{L^2} \right)^{1/3} \sigma^{-1/3}$$

$$+ C(\tilde{\rho}) C_0^{1/12} \left( \sigma \| \nabla \tilde{u} \|_{L^2}^2 \right)^{1/4} \sigma^{-1/3}$$

$$+ C(\tilde{\rho}) \left( \sigma \| \sqrt{\rho \tilde{u}} \|_{L^2} \right)^{1/4} \left( \sigma \| \nabla \tilde{u} \|_{L^2}^2 \right)^{1/4} \sigma^{-5/8}$$

$$\leq C(\tilde{\rho}, M) C_0^{1/3} + C(\tilde{\rho}, M) C_0^{1/6} \sigma^{-1/3} + C(\tilde{\rho}, M) C_0^{1/12} \left( \sigma \| \nabla \tilde{u} \|_{L^2}^2 \right)^{1/4} \sigma^{-1/3}$$

$$+ C(\tilde{\rho}, M) C_0^{1/48} \left( \sigma \| \nabla \tilde{u} \|_{L^2} \right)^{1/4} \sigma^{-5/8},$$

and consequently, it readily follows from (3.78) that

$$\int_0^{\sigma(T)} \| G \|_{L^\infty} dt \leq C(\tilde{\rho}, M) C_0^{1/48}.$$  

(3.88)

Similarly, using (3.6), (3.80), (3.86), (3.87) and the Cauchy-Schwarz inequality, we obtain

$$\int_{\sigma(T)}^T \| G \|_{L^\infty} dt \leq C \int_{\sigma(T)}^T \| \nabla G \|_{L^2}^{4/3} \| \nabla G \|_{L^4}^{8/3} dt$$

$$\leq C(\tilde{\rho}) \int_{\sigma(T)}^T \left( \| \sqrt{\rho \tilde{u}} \|_{L^2}^{4/3} + \| \rho - \rho_s \|_{L^4}^{4/3} \right) \left( \| \sqrt{\rho \tilde{u}} \|_{L^2}^{2/3} \| \nabla \tilde{u} \|_{L^2}^2 + \| \rho - \rho_s \|_{L^4}^{8/3} \right) dt$$

(3.89)

$$\leq C(\tilde{\rho}) \int_{\sigma(T)}^T \left( \| \rho - \rho_s \|_{L^4}^4 + \| \sqrt{\rho \tilde{u}} \|_{L^2}^4 + \| \sqrt{\rho \tilde{u}} \|_{L^2}^2 \| \nabla \tilde{u} \|_{L^2}^2 + \| \rho - \rho_s \|_{L^4}^2 \| \nabla \tilde{u} \|_{L^2}^2 \right) dt$$

$$\leq C(\tilde{\rho}, M) C_0^{1/12} + C(\tilde{\rho}, M) [A_3(T)]^2 + C(\tilde{\rho}, M) C_0^{1/2} \sigma_{A_3(T)} \leq C(\tilde{\rho}, M) C_0^{1/12}.$$
Now, if we set
\[ y = \rho - \bar{\rho}, \quad \alpha = \frac{R\bar{\rho}}{2\mu + \lambda}, \quad g(t) = C(\bar{\rho}) (||G||_{L^\infty} + ||\theta - 1||_{L^\infty}), \quad T_1 = \sigma(T) \]
in Lemma 2.5, then by (3.84), (3.85), (3.88) and (3.89) we deduce from (2.19) that
\[
||\rho(t)||_{L^\infty} \leq \bar{\rho} + \bar{\rho} + C (||g||_{L^1(0, \sigma(T))} + ||g||_{L^1(\sigma(T), T)}) \leq \bar{\rho} + \bar{\rho} + C(\bar{\rho}, M) C_0^{1/48} \leq \frac{3\bar{\rho}}{2},
\]
provided \( C_0 \) is chosen to be such that
\[
C_0 \leq \varepsilon_4 \triangleq \left\{ \varepsilon_3, \left( \frac{\bar{\rho} - 2\bar{\rho}}{2C(\bar{\rho}, M)} \right)^{48} \right\}.
\]
The proof of Lemma 3.8 is therefore complete. \( \square \)

The final step is to close the estimate of \( A_4(T) \), which can be done in a manner similar to that in [14].

**Lemma 3.9** Assume that \((\rho, u, \theta)\) is a smooth solution of (2.1)–(2.3) on \( \mathbb{R}^3 \times [0, T] \), satisfying (3.6) with \( K > 0 \) being the same one as in (3.21). Then, there exists a positive constant \( \varepsilon_0 \), depending only on \( \mu, \lambda, \kappa, C, V, \bar{\rho}, \rho, \theta \) and \( M \), such that
\[
A_4(T) \leq C_0^{1/8},
\]
provided \( C_0 \leq \varepsilon_0 \).

**Proof.** For completeness, we sketch the proofs. By tedious but direct calculations, we obtain after applying the operator \( \partial_t + \text{div}(u \cdot \cdot) \) to both sides of (2.3) that
\[
c_v \rho \left( \partial_t \bar{\theta} + u \cdot \nabla \bar{\theta} \right) - \kappa \Delta \bar{\theta} = \kappa (\text{div}u \Delta \bar{\theta} - \partial_i (\partial_i u \cdot \nabla \bar{\theta}) - \partial_i \bar{\theta} \cdot \nabla \bar{\theta}) - R\rho \left( \bar{\theta} \text{div}u + \theta \text{div}\bar{u} - \theta \partial_k u^k \partial_l u^l \right) + \text{div} (\lambda (\text{div}u)^2 + 2\mu |\nabla u|^2) + 2\lambda \text{div} (\text{div}\bar{u} - \partial_k u^k \partial_l u^l) + \mu (\partial_i u^i + \partial_j u^j) \left( \partial_i \bar{u}^i + \partial_j \bar{u}^j - \partial_i u^i \partial_k u^k - \partial_j u^j \partial_k u^k \right) \triangleq \sum_{i=1}^{5} I_i,
\]
which, multiplied by \( \bar{\theta} \) in \( L^2 \) and integrated by parts, gives
\[
\frac{c_v}{2} \frac{d}{dt} ||\sqrt{\bar{\theta}}||_{L^2}^2 + \kappa ||\nabla \bar{\theta}||_{L^2}^2 = \sum_{i=1}^{5} \langle I_i, \bar{\theta} \rangle \triangleq \sum_{i=1}^{5} J_i,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the standard \( L^2 \)-inner product.

The right-hand side of (3.91) can be estimated term by term as follows, using Lemma 2.1.

\[
\begin{align*}
J_1 & \leq C \left( ||\nabla u||_{L^2} ||\nabla^2 \theta||_{L^2} ||\bar{\theta}||_{L^6} + ||\nabla u||_{L^6} ||\nabla \theta||_{L^6} ||\nabla \bar{\theta}||_{L^2} \right) \\
& \leq \eta ||\nabla \bar{\theta}||_{L^2}^2 + C(\eta) ||\nabla u||_{L^6} ||\nabla^2 \theta||_{L^6}^2,
J_2 & \leq C(\bar{\rho}) ||\sqrt{\bar{\theta}}||_{L^2}^2 ||\nabla \theta||_{L^6} + C(\bar{\rho}) ||\theta||_{L^6} ||\sqrt{\bar{\theta}}||_{L^2}^2 \left( ||\sqrt{\bar{u}}||_{L^2}^2 + ||\nabla u||_{L^6}^2 \right) \\
& \leq \eta ||\nabla \theta||_{L^2}^2 + C(\eta, \bar{\rho}) \left( ||\nabla u||_{L^6}^2 + ||\theta||_{L^6} \right) \left( ||\sqrt{\bar{\theta}}||_{L^2}^2 + ||\sqrt{\bar{u}}||_{L^2}^2 + ||\nabla u||_{L^6}^2 \right),
J_3 & \leq C ||\nabla u||_{L^2} ||\nabla u||_{L^6}^2 ||\theta||_{L^6} \leq \eta ||\nabla \theta||_{L^2}^2 + C(\eta) ||\nabla u||_{L^2}^2 ||\nabla u||_{L^6}^2.
\end{align*}
\]
and similarly,

\[ J_4 + J_5 \leq C\|\nabla u\|_{L^2}^2 \|\nabla \dot{u}\|_{L^2} \|\dot{\theta}\|_{L^6} + C\|\nabla u\|_{L^2} \|\nabla u\|_{L^6}^2 \|\dot{\theta}\|_{L^6} \]

\[ \leq \eta\|\nabla \dot{\theta}\|_{L^2}^2 + C(\eta) \left( \|\nabla u\|_{L^3}^2 \|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^6}^4 \right). \]

In view of (3.6) and (3.46), it is easy to see that

\[ \sigma^2 \|\nabla u\|_{L^3}^2 \leq C \left( \sigma \|\nabla u\|_{L^2} \right) \|\nabla u\|_{L^6} \leq C(\tilde{\rho}, M) C_0^{1/6}, \quad \forall \ t \in [0, T], \quad (3.92) \]

and moreover, by (3.6) we infer from (3.47) that

\[ \int_0^T \sigma^2 \|\nabla \theta\|_{L^2}^2 \, dt \leq C \int_0^T \sigma^2 \left( \|\rho \dot{\theta}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\theta - 1\| \nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 \right) \, dt \]

\[ \leq C(\tilde{\rho}, M) C_0^{1/6} + C(\tilde{\rho}, M) \int_0^T \sigma^2 \|\nabla u\|_{L^2} \|\nabla u\|_{L^2}^4 \, dt \quad (3.93) \]

\[ \leq C(\tilde{\rho}, M) C_0^{1/6}. \]

where we have also used (3.75) and (3.77) to get from (3.73) that

\[ \int_0^T \sigma^2 \|\nabla u\|_{L^4}^4 \, dt \leq C(\tilde{\rho}, M) C_0^{1/4}. \quad (3.94) \]

Thus, multiplying (3.91) by \( \sigma^4 \), integrating it over \([0, T]\), using (3.6), (3.39), (3.46), (3.48) and (3.92–3.94), we obtain by choosing \( \eta > 0 \) small enough that

\[ A_4(T) \leq C(\tilde{\rho}) C_0^{1/6} + C(\tilde{\rho}, M) \sup_{0 \leq t \leq T} \left( \sigma^2 \|\nabla u\|_{L^3}^2 \right) \int_0^T \sigma^2 \|\nabla \theta\|_{L^2}^2 \, dt \]

\[ + C(\tilde{\rho}, M) \int_0^T \sigma^2 \left( \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 \right) \, dt \]

\[ \leq C(\tilde{\rho}, M) C_0^{1/6} \leq C_0^{1/8}, \]

provided \( C_0 \) is chosen to be such that

\[ C_0 \leq \varepsilon_0 \triangleq \min \{\varepsilon_4, C(\tilde{\rho}, M)^{-24}\}. \quad (3.95) \]

The proof of (3.90) is therefore complete. \hfill \Box

As an easy consequence, we have the following estimates, which imply that both velocity and temperature are Hölder continuous strictly away from \( t = 0 \).

**Lemma 3.10** Let the conditions of Proposition 3.1 be in force. There exists a positive constant \( C \), depending only on \( \mu, \lambda, \kappa, R, c_V, \rho_0, \tilde{\rho}, \tilde{\theta}, \|f\|_{H^2 \cap W^{1,\infty}}, \tilde{\rho}, \tilde{\theta}, \) and \( M \), such that

\[ \sup_{0 \leq t \leq T} \sigma^2 \left( \|\nabla u\|_{L^6}^2 + \|G\|_{H^1}^2 + \|\omega\|_{H^1}^2 + \sigma^2 \|\theta - 1\|_{H^2}^2 \right) \]

\[ + \int_0^T \sigma^2 \left( \|\rho - \rho_0\|_{L^4}^4 + \|\nabla u\|_{L^4}^4 + \|\nabla \theta\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \sigma^2 \|\theta_t\|_{H^1}^2 \right) \, dt \leq C C_0^{1/8}. \quad (3.96) \]
Proof. In terms of Lemma 2.4, (3.6), (3.46), (3.47), (3.80) and (3.93), one gets that
\[
\sup_{0 \leq t \leq T} (\sigma^2 \|\nabla u\|_{L^6}^2 + \sigma^2 \|G\|_{H^1}^2 + \sigma^2 \|\omega\|_{H^1}^2 + \sigma^4 \|\theta - 1\|_{H^2}^2)
\]
\[
+ \int_0^T \sigma^2 (\|\nabla u\|_{L^4}^4 + \|\nabla \theta\|_{H^1}^2 + \|\rho - \rho_s\|_{L^4}^4) \, dt \leq CC_0^{1/8}. \tag{3.97}
\]

Using Lemmas 2.1–2.2, (3.6), (3.39) and (3.46), we deduce
\[
\int_0^T \sigma^2 \|u_t\|_{L^2}^2 \, dt \leq \int_0^T \sigma^2 (\|\dot{u}\|_{L^2}^2 + \|u\|_{L^4}^4) \, dt
\]
\[
\leq C \int_0^T \sigma^2 (\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^6}^2) \, dt \leq CC_0^{1/6}, \tag{3.98}
\]
and similarly,
\[
\int_0^T \sigma^4 \|\theta_t\|_{H^1}^2 \, dt \leq \int_0^T \sigma^4 \left(\|\dot{\theta}\|_{L^2}^2 + \|u\|_{L^4}^4 + \|\nabla \theta\|_{L^2}^2 + \|\nabla (u \cdot \nabla \theta)\|_{L^2}^2\right) \, dt
\]
\[
\leq C \int_0^T \sigma^4 \left(\|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + \|\nabla \dot{\theta}\|_{L^2}^2 + \|\nabla u\|_{L^6}^2 \|\nabla \theta\|_{L^3}^2\right) \, dt
\]
\[
+ C \int_0^T \sigma^4 \left(\|\nabla \dot{\theta}\|_{L^2}^2 + \|\nabla u\|_{L^6}^2 \|\nabla^2 \theta\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|\nabla^2 \theta\|_{L^2}^2\right) \, dt
\]
\[
\leq CC_0^{1/8},
\]
which, together with (3.97) and (3.98), finishes the proof of (3.96). \qed

4 A Priori Estimates (II): Time-dependent Estimates

This section aims to prove the time-dependent a priori estimates of the first-order derivatives of the solutions, which are necessary for the existence of strong solutions on \(\mathbb{R}^3 \times (0, T]\) for any \(0 < T < \infty\). To do this, from now on, we assume that Proposition 3.1 and the conditions of Theorem 1.2 hold. For simplicity, we denote by \(C\) or \(C_i\) \((i = 1, 2, \ldots)\) the various positive constants which may depend on the initial data, \(\mu, \lambda, \kappa, R, c_V, \bar{\rho}, \rho, \tilde{\rho}, \bar{\theta}, M, f\) and \(T\) as well.

We begin with the following key observation, which is indeed an immediate consequence of Lemma 3.7 and plays an important role in the subsequent analysis.

Lemma 4.1 For any \(T > 0\), there exists a positive constant \(C(T)\), depending on \(T\), such that
\[
\int_0^T \left(\|\nabla \theta\|_{L^p}^q + \|\sqrt{\rho} \dot{u}\|_{L^p}^q + \|\theta - 1\|_{L^\infty}^q + \|\text{div} u\|_{L^\infty}^q + \|\nabla \times u\|_{L^\infty}^q\right) \, dt \leq C(T), \tag{4.1}
\]
where \(p\) and \(q\) satisfy
\[
3 < p < 6, \quad 1 < q < \frac{4p}{5p - 6}. \tag{4.2}
\]
Proof. Thanks to Lemma 3.7, one infers from Lemma 2.1 that

\[
\int_0^T \left( \|\nabla \theta\|_{L^p}^q + \|\sqrt{p} \nabla \mathbf{u}\|_{L^p}^q \right) dt \\
\leq C \int_0^T \left( \|\nabla \theta\|_{L^2}^{q(6-p)/2p} \|\nabla^2 \theta\|_{L^2}^{q(3p-6)/2p} \right) dt + C \int_0^T \|\sqrt{p} \nabla \mathbf{u}\|_{L^p}^{q(6-p)/2p} \|\nabla \mathbf{u}\|_{L^p}^{q(3p-6)/2p} dt \\
\leq C \sup_{0 \leq t \leq T} \left( t\|\nabla \theta(t)\|_{L^2}^2 \right)^{q(6-p)/4p} \int_0^T t^{-\frac{q}{2}} \left( t\|\nabla^2 \theta\|_{L^2}^2 \right)^{q(3p-6)/4p} dt \\
+ C \sup_{0 \leq t \leq T} \left( t\|\sqrt{p} \nabla \mathbf{u}(t)\|_{L^2}^2 \right)^{q(6-p)/4p} \int_0^T t^{-\frac{q}{2}} \left( t\|\nabla \mathbf{u}\|_{L^2}^2 \right)^{q(3p-6)/4p} dt \\
\leq C \left( \int_0^T t^{-\frac{2pq}{4p-3pq+6q}} dt \right)^{4p-3pq+6q} \left( \int_0^T \|\nabla^2 \theta\|_{L^2}^2 dt \right)^{q(3p-6)/4p} \left( \int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 dt \right)^{q(3p-6)/4p} \\
\leq C(T),
\]

since $3 < p < 6$ and $1 < q < \frac{4p}{op-6} < 2$ imply that

\[
0 < \frac{2pq}{4p - 3pq + 6q} < 1 \quad \text{and} \quad 0 < \frac{q(3p-6)}{4p} < 1.
\]

Using Lemma 2.4 and Proposition 3.1, one deduces from (2.2) that

\[
\|\theta - 1\|_{L^\infty} + \|\text{div}\mathbf{u}\|_{L^\infty} + \|\nabla \times \mathbf{u}\|_{L^\infty} \\
\leq C (1 + \|\theta - 1\|_{L^\infty} + \|G\|_{L^\infty} + \|\nabla \times \mathbf{u}\|_{L^\infty}) \\
\leq C (1 + \|\nabla \theta\|_{L^2} + \|\nabla \theta\|_{L^p} + \|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^p} + \|\nabla (\nabla \times \mathbf{u})\|_{L^p}) \\
\leq C (1 + \|\nabla \theta\|_{L^2} + \|\nabla \theta\|_{L^p} + \|\sqrt{p} \nabla \mathbf{u}\|_{L^p}),
\]

which, combined with (4.3) and (3.35), leads to the desired estimates stated in (4.1). \(\square\)

With the help of Lemma 4.1, we can now estimate the gradient of density, based on the Beale-Kato-Majda type inequality (cf. Lemma 2.6).

**Lemma 4.2** For any $T > 0$, there exists a positive constant $C(T)$, depending on $T$, such that

\[
\sup_{0 \leq t \leq T} \left( \|\nabla \rho\|_{L^2 \cap L^p} + \|\rho_t\|_{L^2} \right) + \int_0^T \left( \|\nabla^2 \mathbf{u}\|_{L^p}^q + \|\nabla \mathbf{u}\|_{L^\infty}^q \right) dt \leq C(T),
\]

where $3 < p < 6$ and $q > 1$ are the same ones as in (4.2).

**Proof.** Applying $\nabla$ to both sides of (2.1), and multiplying the resulting equation by $|\nabla \rho|^{p-2} \nabla \rho$ with $2 \leq p \leq 6$, we obtain after integrating by parts that

\[
\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C (\|\nabla \rho\|_{L^\infty} \|\nabla \rho\|_{L^p} + \|\nabla^2 \mathbf{u}\|_{L^p}) .
\]

Recalling that $\mathcal{L} = -\mu \Delta - (\mu + \lambda) \nabla \text{div}$ is a strong elliptic operator (cf. [2]), we deduce from (2.1) that

\[
\|\nabla^2 \mathbf{u}\|_{L^p} \leq C (\|\sqrt{p} \nabla \mathbf{u}\|_{L^p} + \|\nabla \mathbf{P}\|_{L^p} + 1) \\
\leq C (1 + \|\sqrt{p} \nabla \mathbf{u}\|_{L^p} + \|\nabla \theta\|_{L^p} + \|\theta\|_{L^\infty} \|\nabla \rho\|_{L^p}) ,
\]

(4.6)
which, inserted into (4.5), gives
\[
\frac{d}{dt}\|\nabla \rho\|_{L^p} \leq C (\|\nabla u\|_{L^\infty} + \|\theta\|_{L^\infty}) \|\nabla \rho\|_{L^p} + C \left(1 + \|\sqrt{\rho}u\|_{L^p} + \|\nabla \theta\|_{L^p}\right). \tag{4.7}
\]

Lemma 2.6, together with (3.39) and (4.6), implies that for 3 < p < 6,
\[
\|\nabla u\|_{L^\infty} \leq C (\|\text{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla^2 u\|_{L^p}) + C \|\nabla u\|_{L^2} + C
\]
\[
\leq C + C (\|\text{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\nabla \rho\|_{L^p})
\]
\[
+ C (\|\text{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\sqrt{\rho}u\|_{L^p} + \|\nabla \theta\|_{L^p} + \|\theta\|_{L^\infty}). \tag{4.8}
\]

Thus, it follows from (4.7) and (4.8) that for 3 < p < 6,
\[
\frac{d}{dt} (e + \|\nabla \rho\|_{L^p}) \leq C \Lambda(t) (e + \|\nabla \rho\|_{L^p}) \ln(e + \|\nabla \rho\|_{L^p}),
\]
that is,
\[
\frac{d}{dt} \ln(e + \|\nabla \rho\|_{L^p}) \leq C \Lambda(t) \ln(e + \|\nabla \rho\|_{L^p}),
\]
where
\[
\Lambda(t) \triangleq (\|\text{div} u\|_{L^\infty} + \|\nabla \times u\|_{L^\infty}) \ln(e + \|\sqrt{\rho}u\|_{L^p} + \|\nabla \theta\|_{L^p} + \|\theta\|_{L^\infty})
\]
\[
+ C \left(1 + \|\sqrt{\rho}u\|_{L^p} + \|\nabla \theta\|_{L^p}\right) \in L^1(0, T),
\]
due to (4.1) and the simple inequality \(\ln(e + y) \leq (e + y)^\delta\) for any \(y \geq 0\) and \(\delta > 0\). So,
\[
\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^p} \leq C(T), \quad \forall \ p \in (3, 6). \tag{4.9}
\]

By virtue of (4.6) and (4.9), one gets from (4.1) that for any \(p, q\) as in (4.2),
\[
\int_0^T (\|\nabla^2 u\|_{L^p} + \|\nabla u\|_{L^\infty}^q) \, dt \leq C(T), \tag{4.10}
\]
which, combined with (4.5), (4.6) for \(p = 2\) and the Gronwall’s inequality, also shows that
\[
\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^2} \leq C(T), \tag{4.11}
\]
and moreover, it follows from (2.1), and (3.39) that
\[
\|\rho(t)\|_{L^2} \leq C (\|u\|_{L^6} \|\nabla \rho\|_{L^3} + \|\nabla u\|_{L^2}) \leq C(T), \quad \forall \ t \in [0, T]. \tag{4.12}
\]

Therefore, collecting (4.9)–(4.12) together finishes the proof of (4.4). \(\square\)

As an immediate result of Lemmas 4.1 and 4.2, we have

\textbf{Lemma 4.3} For any \(T > 0\), there exists a positive constant \(C(T)\), depending on \(T\), such that
\[
\sup_{0 \leq t \leq T} \left(\frac{1}{2} \|\nabla^2 u(t)\|_{L^2}^2 + \|\sqrt{\rho}u(t)\|_{L^2}^2\right)
\]
\[
+ \int_0^T \left(\|\nabla^2 u\|_{L^2}^2 + \|\sqrt{\rho}u\|_{L^2}^2 + t\|\nabla u\|_{L^2}^2\right) \, dt \leq C(T). \tag{4.13}
\]
Proof. In view of Lemma 3.7 and (4.4), one infers from (4.6) that

$$\sup_{0 \leq t \leq T} (t\|\nabla^2 u(t)\|_{L^2}) + \int_0^T \|\nabla^2 u\|_{L^2}^2 dt \leq C(T). \quad (4.14)$$

Moreover, it is easy to get that

$$\|\sqrt{\rho}u_t\|_{L^2}^2 \leq \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho} \cdot \nabla u\|_{L^2}^2 \leq C \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^1}^2 \right),$$

so that, using (3.20), (3.39), (3.68), (3.78) and (4.14), we find

$$\sup_{0 \leq t \leq T} (t\|\sqrt{\rho}u_t(t)\|_{L^2}) + \int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 dt \leq C, \quad (4.15)$$

and similarly,

$$\int_0^T t\|\nabla u\|_{L^2}^2 dt \leq \int_0^T t\|\nabla u\|_{L^2}^2 dt + \int_0^T t\|\nabla \cdot (u \cdot \nabla u)\|_{L^2}^2 dt$$

$$\leq C + C \int_0^T t\|\nabla u\|_{H^1}^2 dt \leq C.$$ 

This, together with (4.14) and (4.15), finishes the proof of (4.13).

\[\square\]

Remark 4.1 It is easy to check that the estimates obtained in Lemmas 4.1–4.3 are independent of the the lower bound of density.

The next lemma is concerned with the estimates obtained in Lemmas 4.1–4.3 are independent of the the lower bound of density.

Lemma 4.4 Assume that \(\inf_{x \in \mathbb{R}^3} \rho_0(x) \geq \rho_1 > 0\). Then for any \(T > 0\), there exists a positive constant \(c(\rho_1, T)\), depending on \(\rho_1\) and \(T\), such that

$$\rho(x, t) \geq c(\rho_1, T), \quad \forall x \in \mathbb{R}^3, \ t \in [0, T]. \quad (4.16)$$

Assume further that

$$7\mu > \lambda, \quad \|\nabla u_0\|_{L^3} \leq M_1, \quad \|\nabla \theta_0\|_{L^2} \leq M_2. \quad (4.17)$$

Then for any \(T > 0\), there exists a positive constant \(C(\rho_1, T)\), depending on \(\rho_1\) and \(T\), such that

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^3}^3 + \|\nabla \theta\|_{L^2}^2) + \int_0^T \left( \|\nabla u\| L^3 \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 \right) dt \leq C(\rho_1, T). \quad (4.18)$$

Proof. The lower bound of density is easily obtained, based on (4.4) and the following fact that

$$\rho(x, t) \geq \inf_{x \in \mathbb{R}^3} \rho_0(x) \exp \left\{ - \int_0^t \|\text{div} u\|_{L^\infty} ds \right\} \geq c(\rho_1, T).$$

In order to estimate \(\|\nabla u\|_{L^3}\), we first make use of (4.16) to rewrite (2.1) as

$$u_t^j + u \cdot \nabla u + R \partial_j \theta + R \rho^{-1} \partial_j \rho \rho^{-1} (\mu \Delta u^j + (\mu + \lambda) \partial_j \text{div} u) + \partial_j f,$$
which, differentiated with respect to $x_i$, multiplied by $|\nabla u|\partial_i u^j$, and then integrated by parts over $\mathbb{R}^3$, gives

$$
\text{L.H.S.} \triangleq \frac{1}{3} \frac{d}{dt} \|\nabla u\|^3_{L^3} + \mu \int \rho^{-1} (|\nabla u||\nabla^2 u|^2 + |\nabla u||\nabla u||\nabla u|^2) \, dx \\
+ (\mu + \lambda) \int \rho^{-1} (|\nabla u||\nabla \text{div} u|^2 + \partial_i \text{div} u \partial_i u^j \partial_j |\nabla u|) \, dx \\
\leq C \int |\nabla \rho||\nabla u|^2 |\nabla^2 u| \, dx + C \int |\nabla \theta||\nabla u||\nabla^2 u| \, dx \\
+ C \int |\theta||\nabla \rho||\nabla u||\nabla^2 u| \, dx + C \int |u||\nabla u|^2 |\nabla^2 u| \, dx \\
+ C \int |\nabla f||\nabla u||\nabla^2 u| \, dx \triangleq \sum_{i=1}^5 I_i.
$$

(4.19)

It is easily derived from direct calculations that

$$
\mu |\nabla u||\nabla|\nabla u||^2 + (\mu + \lambda) |\nabla u||\nabla \text{div} u|^2 + (\mu + \lambda) \partial_i \text{div} u \partial_i u^j \partial_j |\nabla u| \\
\geq (\mu + \lambda) |\nabla u| \left( |\nabla \text{div} u| - \frac{|\nabla|\nabla u||}{2} \right)^2 + \frac{3\mu - \lambda}{4} |\nabla u||\nabla|\nabla u||^2 \\
\geq \frac{3\mu - \lambda}{4} |\nabla u||\nabla|\nabla u||^2.
$$

Hence, if $7\mu > \lambda$, then there exists a positive number $\bar{\mu} \in (0, \min\{\mu, (7\mu - \lambda)/4\})$ such that

$$
\mu |\nabla u| \left( |\nabla^2 u|^2 + |\nabla|\nabla u||^2 \right) + (\mu + \lambda) \left( |\nabla u||\nabla \text{div} u|^2 + \partial_i \text{div} u \partial_i u^j \partial_j |\nabla u| \right) \\
\geq \bar{\mu} |\nabla u||\nabla^2 u|^2 + (\mu - \bar{\mu}) |\nabla u||\nabla^2 u|^2 + \frac{3\mu - \lambda}{4} |\nabla u||\nabla|\nabla u||^2 \\
\geq \bar{\mu} |\nabla u||\nabla^2 u|^2 + \frac{7\mu - \lambda - 4\bar{\mu}}{4} |\nabla u||\nabla|\nabla u||^2 \\
\geq \bar{\mu} |\nabla u||\nabla^2 u|^2,
$$

which, combined with (4.16), yields that there exists a positive number $\mu_0 > 0$, depending on $\mu$, $\lambda$, $\inf \rho_0$ and $T$, such that

$$
\text{L.H.S.} \geq \frac{1}{3} \frac{d}{dt} \|\nabla u\|^3_{L^3} + \mu_0 \int |\nabla u||\nabla^2 u|^2 \, dx.
$$

(4.20)

Next, we estimate the right-hand side of (4.19) term by term, using Proposition 3.1, Lemmas 4.1–4.3 and (4.16). First, using Lemma 2.1 and (5.4), we find

$$
I_1 \leq C \|\nabla \rho\|_{L^p} \|\nabla u\|^\frac{3}{2} \|\nabla u\|^\frac{3(p-3)}{2p} \|\nabla u\|^\frac{1}{2} \|\nabla^2 u\|_{L^2} \\
\leq C \|\nabla u\|^\frac{2}{L^p} \|\nabla u\|^\frac{2}{L^p} \|\nabla u\|^\frac{1}{2} \|\nabla^2 u\|_{L^2} \\
\leq C \|\nabla u\|^\frac{3}{L^3} \|\nabla u\|^\frac{3}{L^3} \|\nabla^2 u\|^\frac{1}{2} \|\nabla^2 u\|_{L^2} \\
\leq C \|\nabla u\|^\frac{3}{L^3} \|\nabla u\|^\frac{1}{2} \|\nabla^2 u\|^\frac{2}{L^2} + C(\delta) \|\nabla u\|^3_{L^3}.
$$
In a similar manner, we can estimate the other terms as follows:

\[ I_2 \leq C \left\| \nabla u \right\|_{L^\infty} \left( \left\| \nabla \theta \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| \nabla^2 u \right\|_{L^2} \right) \]

\[ \leq \delta \left( \left\| \nabla u \right\|_{L^2}^2 + C(\delta) \left\| \nabla u \right\|_{L^\infty} \right) \]

\[ I_3 \leq C \left( \left\| \theta - 1 \right\|_{L^6} \left\| \nabla \rho \right\|_{L^3} + \left\| \nabla \rho \right\|_{L^2} \right) \left\| \nabla u \right\|_{L^\infty} \left\| \nabla u \right\|_{L^2} \left\| \nabla^2 u \right\|_{L^2} \]

\[ \leq \delta \left( \left\| \nabla u \right\|_{L^2}^2 \left\| \nabla^2 u \right\|_{L^2} + C(\delta) \left\| \nabla u \right\|_{L^\infty} \right) \]

\[ I_4 \leq C \left\| \nabla u \right\|_{L^6} \left\| \nabla u \right\|_{L^{9/2}} \left\| \nabla u \right\|_{L^2} \left\| \nabla^2 u \right\|_{L^2} \]

\[ \leq C \left( \left\| \nabla u \right\|_{L^3}^2 + \left\| \nabla u \right\|_{L^3} \left\| \nabla^2 u \right\|_{L^2} \right) \]

\[ \leq \delta \left( \left\| \nabla u \right\|_{L^2}^2 + C(\delta) \left\| \nabla u \right\|_{L^3} \right) \]

Thus, inserting (4.20) and the estimates of \( I_i \) \((i = 1, \ldots, 5)\) into (4.19) and choosing \( \delta > 0 \) small enough, we arrive at

\[ \frac{d}{dt} \left( \left\| \nabla u \right\|_{L^3}^2 + \left\| \nabla u \right\|_{L^2}^2 \right) \leq C \left( 1 + \left\| \nabla u \right\|_{L^\infty} \right) \left( 1 + \left\| \nabla u \right\|_{L^3}^3 + \left\| \nabla \theta \right\|_{L^2}^2 \right) \]  \quad (4.21)

To estimate \( \left\| \nabla \theta \right\|_{L^2} \), we obtain after choosing \( m = 0 \) in (3.63) that

\[ \frac{\kappa}{2} \frac{d}{dt} \left\| \nabla \theta \right\|_{L^2}^2 + c_V \sqrt{\rho} \left\| \nabla \theta \right\|_{L^2}^2 = -\kappa \int \nabla \theta \cdot \nabla (u \cdot \nabla \theta) \, dx + \lambda \int (\nabla u)^2 \nabla \theta \, dx \]

\[ + 2\mu \int \mathcal{D}(u)^2 \nabla \theta \, dx - R \int \rho \theta (\nabla u)^2 \nabla \theta \, dx \triangleq \sum_{i=1}^4 J_i. \]  \quad (4.22)

Analogously to the derivation of (3.47), by (4.16) we deduce from (2.1) that

\[ \left\| \nabla^2 \theta \right\|_{L^2} \leq C \left( \left\| \nabla \theta \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} + \left( \theta - 1 \right) \nabla u \right\|_{L^2} \left\| \nabla u \right\|_{L^4} \right) \]

\[ \leq C \left( 1 + \left\| \nabla \theta \right\|_{L^2} + \left\| \nabla \theta \right\|_{L^2} \left\| \nabla u \right\|_{L^3} + \left\| \nabla u \right\|_{L^3} \left\| \nabla u \right\|_{L^6} \right) \]  \quad (4.23)

and hence, we obtain after integrating by parts that

\[ J_1 \leq C \left\| \nabla u \right\|_{L^3} \left\| \nabla \theta \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| \nabla^2 \theta \right\|_{L^2} \]

\[ \leq \eta \left\| \nabla \theta \right\|_{L^2}^2 + C(\eta) \left( 1 + \left\| \nabla u \right\|_{L^3}^2 \left\| \nabla \theta \right\|_{L^2}^2 + \left\| \nabla u \right\|_{L^2}^2 \left\| \nabla \theta \right\|_{H^1} \right) \]

\[ \leq \eta \left\| \nabla \theta \right\|_{L^2}^2 + C(\eta) \left( 1 + \left\| \nabla u \right\|_{L^3}^2 \right) \left( \left\| \nabla \theta \right\|_{L^2}^2 + \left\| \nabla u \right\|_{H^1}^2 \right) \]

Similarly, we also have

\[ J_2 + J_3 \leq C \left\| \nabla \theta \right\|_{L^2} \left\| \nabla u \right\|_{L^3}^2 \left\| \nabla u \right\|_{L^6} \left\| \nabla u \right\|_{L^3} \left\| \nabla u \right\|_{L^6} \]

\[ \leq \eta \left\| \nabla \theta \right\|_{L^2}^2 + C(\eta) \left( \left\| \nabla u \right\|_{H^1}^2 \left\| \nabla u \right\|_{L^3}^2 \right) \]

and

\[ J_4 \leq C \left( \left\| \theta - 1 \right\|_{L^6} \left\| \nabla u \right\|_{L^3} + \left\| \nabla u \right\|_{L^2} \right) \left\| \nabla \theta \right\|_{L^2} \]
\[ \leq \eta \|\dot{\theta}\|_{L^2}^2 + C(\eta) \left( 1 + \|\nabla \theta\|_{L^2}^2 \right) \left( 1 + \|\nabla u\|_{L^3}^3 \right). \]

Now, choosing \( \eta > 0 \) suitably small and putting the estimates of \( J_i \) \( (i = 1, \ldots, 4) \) into (4.22), we get
\[
\frac{d}{dt} \|\nabla \theta\|^2_{L^2} + \|\dot{\theta}\|^2_{L^2} \leq C \left( 1 + \|\nabla \theta\|^2_{L^2} + \|\nabla u\|^2_{H^1} \right) \left( 1 + \|\nabla u\|_{L^3}^3 \right),
\]
which, combined with (4.21), gives
\[
\frac{d}{dt} \left( \|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} \right) + \left( \|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} \right) \leq C \left( 1 + \|\nabla u\|_{L^\infty} + \|\nabla \theta\|^2_{L^2} + \|\nabla u\|^2_{H^1} \right) \left( 1 + \|\nabla u\|_{L^3}^3 + \|\nabla \theta\|^2_{L^2} \right). \tag{4.24}
\]

Thus, by virtue of (3.35), (4.4), (4.13) and the Gronwall’s inequality, we immediately deduce from (4.24) that
\[
\sup_{0 \leq t \leq T} \left( \|\nabla u\|^3_{L^3} + \|\nabla \theta\|^2_{L^2} \right) + \int_0^T \left( \|\nabla u\|^2_{L^2} + \|\dot{\theta}\|^2_{L^2} \right) dt \leq C(T),
\]
which, together with (3.35), (4.13) and (4.23), leads to the desired estimate of (4.18).

The following lemma, concerning the \( t \)-weighted estimate of \( \dot{\theta} \), also plays an important role in the analysis of uniqueness.

**Lemma 4.5** Let the conditions of Theorem 1.2 be in force. Then for any \( T > 0 \), there exists a positive constant \( C(T) \), depending on \( T \), such that
\[
\sup_{0 \leq t \leq T} \left( t \|\dot{\theta}(t)\|^2_{L^2} \right) + \int_0^T t \|\nabla \dot{\theta}\|^2_{L^2} dt \leq C(T). \tag{4.25}
\]

**Proof.** Applying the operator \( \partial_i + \text{div}(u \cdot) \) to both sides of (2.13) and multiplying it by \( \dot{\theta} \) in \( L^2 \), by tedious but direct calculations we find
\[
\frac{c_V}{2} \frac{d}{dt} \|\sqrt{\rho} \dot{\theta}\|^2_{L^2} + \kappa \|\nabla \dot{\theta}\|^2_{L^2} = \left\langle \kappa \left( \text{div} u \Delta \theta - \partial_i (\partial_i u \cdot \nabla \theta) - \partial_i u \cdot \nabla \partial_i \theta \right), \dot{\theta} \right\rangle \\
- \left\langle R_\rho \left( \partial_i \text{div} u + \theta \partial_i \dot{u}, - \theta \partial_i u \right), \dot{\theta} \right\rangle \\
+ \left\langle \text{div} \left( \lambda (\text{div} u)^2 + 2\mu |\mathbf{D}(u)|^2 \right), \dot{\theta} \right\rangle + \left\langle 2\lambda \text{div} u \left( \text{div} \dot{u} - \partial_i u \partial_i u \right), \dot{\theta} \right\rangle \\
+ \left\langle \mu (\partial_i u^i + \partial_j u^j) \left( \partial_i \dot{u}^i + \partial_j \dot{u}^j - \partial_i u^k \partial_k \dot{u}^i - \partial_j u^k \partial_k \dot{u}^j \right), \dot{\theta} \right\rangle \equiv \sum_{i=1}^5 N_i,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the standard \( L^2 \)-inner product. The right-hand side of (4.26) can be estimated term by term as follows, using Lemma 2.1 and integration by parts.

\[
N_1 \leq C \left( \|\nabla u\|_{L^3} \|\nabla^2 \theta\|_{L^2} \|\dot{\theta}\|_{L^6} + \|\nabla u\|_{L^3} \|\nabla \theta\|_{L^6} \|\nabla \dot{\theta}\|_{L^2} \right) \\
\leq \gamma \|\nabla \dot{\theta}\|_{L^2}^2 + C(\gamma) \|\nabla^2 \theta\|_{L^2}^2,
\]
\[
N_2 \leq C \|\ddot{\theta}\|_{L^2} \|\dot{\theta}\|_{L^6} \|\nabla u\|_{L^3} + C \|\ddot{\theta}\|_{L^2} \left( \|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla u\|_{L^6} \right) \\
+ C \|\theta - 1\|_{L^3} \|\dot{\theta}\|_{L^6} \left( \|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla u\|_{L^6} \right) \\
\leq \gamma \|\nabla \dot{\theta}\|_{L^2}^2 + C(\gamma) \left( \|\ddot{\theta}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \right),
\]
\[
N_3 \leq C \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^3} \|\dot{\theta}\|_{L^6} \leq \gamma \|\nabla \dot{\theta}\|_{L^2}^2 + C(\gamma) \|\nabla u\|_{H^1}^2,
\]

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and
\[ N_4 + N_5 \leq C \| \nabla u \|_{L^1} \| \nabla \dot{u} \|_{L^2} \| \dot{\theta} \|_{L^6} + C \| \nabla u \|_{L^6} \| \nabla u \|_{L^4}^2 \| \dot{\theta} \|_{L^6} \]
\[ \leq \gamma \| \nabla \dot{\theta} \|_{L^2}^2 + C(\gamma) \left( \| \nabla \dot{u} \|_{L^2}^2 + \| \nabla u \|_{H^1}^2 \right). \]

Thus, choosing \( \gamma > 0 \) small enough and inserting the estimates of \( N_i \) with \( i = 1, \ldots, 5 \) into (4.26), we immediately arrive at the desired estimates stated in (4.25), using Lemmas 3.7, 4.3, 4.4 and the Gronwall’s inequality.

\[ \square \]

5 Proofs of Theorems 1.1 and 1.2

5.1 Proof of Theorem 1.1

Based on the global a priori estimates established in Section 3 and the standard local existence result (cf. [21, Theorem 5.2]), we can prove Theorem 1.1 in a similar manner as that in [14]. For completeness, we sketch the proofs below.

Step I. Construction of Approximate Solutions

Assume that the conditions of Theorem 1.1 hold and \( C_0 \) satisfies (1.15) with \( \varepsilon = \varepsilon_0/2 \), where \( \varepsilon_0 \) is the one given in (3.8). For sufficiently small \( \delta, \eta \in (0, \bar{\rho} - \sup \rho_0(x)) \), let \( j_\delta(x) \) be the standard mollifier. Define
\[ \rho^\delta_\eta \triangleq \frac{j_\delta * \rho_0 + \eta}{1 + \eta}, \quad u^\delta_\eta \triangleq j_\delta * u_0, \quad \theta^\delta_\eta \triangleq \frac{j_\delta * (\rho_0 \theta_0) + \eta}{j_\delta * \rho_0 + \eta}, \quad f^\delta_\eta \triangleq j_\delta * f. \] (5.1)

Also let \( \rho^\delta_s \triangleq \rho^\delta_s(x) \) be the unique solution of (1.11) with \( f \) being replaced by \( f^\delta_\eta \). Then, similarly to that in [14], we have
\[ \left( \rho^\delta_\eta - 1, u^\delta_\eta, \theta^\delta_\eta - 1, f^\delta_\eta \right) \in H^\infty, \] (5.2)
\[ \frac{\eta}{1 + \eta} \leq \rho^\delta_\eta \leq \frac{\bar{\rho} + \eta}{1 + \eta} < \bar{\rho}, \quad \frac{\eta}{\bar{\rho} + \eta} \leq \theta^\delta_\eta \leq \bar{\theta}, \quad \| \nabla u^\delta_\eta \|_{L^2} \leq M, \] (5.3)
\[ \lim_{\eta \to 0} \lim_{\delta \to 0} \left( \| \rho^\delta_\eta - \rho_0 \|_{L^2} + \| u^\delta_\eta \|_{H^1} + \| \rho^\delta_\eta \theta^\delta_\eta - \rho_0 \theta_0 \|_{L^2} \right) = 0, \] (5.4)
\[ \lim_{\eta \to 0} \lim_{\delta \to 0} \left( \| f^\delta_\eta - f \|_{H^2 \cap W^{1,\infty}} + \| \rho^\delta_s - \rho_s \|_{H^2 \cap W^{1,\infty}} \right) = 0, \] (5.5)
and moreover,
\[ \lim_{\eta \to 0} \lim_{\delta \to 0} C^\delta_\eta = C_0, \] (5.6)
where the initial energy \( C^\delta_\eta \) is defined as the one in (1.13) with \( (\rho_0, u_0, \theta_0) \) and \( \rho_s \) being replaced by \( (\rho^\delta_\eta, u^\delta_\eta, \theta^\delta_\eta) \) and \( \rho^\delta_s \), respectively. It follows from (5.6) and the fact that \( C_0 \leq \varepsilon = \varepsilon_0/2 \) that there exists an \( \tilde{\eta} > 0 \) such that for any \( \eta \in (0, \tilde{\eta}) \), there exists some \( \tilde{\delta}(\eta) > 0 \) such that
\[ C^\delta_\eta \leq C_0 + \varepsilon_0 / 2 \leq \varepsilon_0, \] (5.7)
provided
\[ 0 < \eta \leq \tilde{\eta}, \quad 0 < \delta \leq \tilde{\delta}(\eta). \] (5.8)
Then, it follows from the standard local existence results (cf. [21, Theorem 5.2]) that there exist a small time $T_0$ and a unique smooth solution $(\rho^{\delta, \eta}, u^{\delta, \eta}, \theta^{\delta, \eta})$ to the Cauchy problem (2.1)–(2.3) with mollified initial data $(\rho_0^{\delta, \eta}, u_0^{\delta, \eta}, \theta_0^{\delta, \eta})$ on $\mathbb{R}^3 \times [0, T_0]$, such that

$$
\inf_{x \in \mathbb{R}^3, t \in [0, T_0]} \rho^{\delta, \eta}(x, t) \geq \frac{1}{2} \inf_{x \in \mathbb{R}^3, t \in [0, T_0]} \rho_i^{\delta, \eta}(x), \quad \inf_{x \in \mathbb{R}^3, t \in [0, T_0]} \theta^{\delta, \eta}(x, t) > 0,
$$

\begin{align*}
& (\rho^{\delta, \eta} - 1, u^{\delta, \eta}, \theta^{\delta, \eta} - 1) \in C([0, T_0]; H^3), \quad \rho_t^{\delta, \eta} \in C([0, T_0]; H^2), \\
& (u_t^{\delta, \eta}, \theta_t^{\delta, \eta}) \in C([0, T_0]; H^1), \quad (u^{\delta, \eta}, \theta^{\delta, \eta} - 1) \in L^2(0, T_0; H^4),
\end{align*}

where $T_0 > 0$ may depend on $\delta, \eta$.

**Step II. Global Existence of Approximate Solutions**

Let $\eta, \delta$ satisfy (5.8). We shall show that the local-in-time solution $(\rho^{\delta, \eta}, u^{\delta, \eta}, \theta^{\delta, \eta})$ is indeed a global smooth one, provided $(\rho_0^{\delta, \eta}, u_0^{\delta, \eta}, \theta_0^{\delta, \eta})$ satisfies the small energy condition (5.7). To this end, let $A_i(T)$ be the ones defined in (3.1)–(3.4) with $(\rho, u, \theta)$ being replaced by $(\rho^{\delta, \eta}, u^{\delta, \eta}, \theta^{\delta, \eta})$.

Since (3.6) holds for $t = 0$, it follows from the continuity arguments that there exists a positive time $T_1 \in (0, T_0]$, such that (3.6) holds for $T = T_1$. Let

$$
T_* \doteq \sup \{T \leq T^* \mid (3.6) \text{ holds}\}.
$$

Clearly, $T_* \geq T_1 > 0$. In fact, it holds that

$$
T_* = \infty.
$$

Otherwise, $T_* < \infty$. Then, Proposition 3.1 implies that (3.7) holds for all $0 < T < T_*$. Moreover, since $(\rho_0^{\delta, \eta} - 1, u_0^{\delta, \eta}, \theta_0^{\delta, \eta} - 1) \in H^3$ and $\inf \rho_0^{\delta, \eta} > 0$, it is easily seen that

$$
(u_t^{\delta, \eta}, \theta_t^{\delta, \eta}) |_{t=0} \in H^1 \quad \text{are well defined.}
$$

So, similarly to the derivations of (3.35), (3.68), (3.78), (3.90), (3.96) and (4.4), by (5.9) we deduce that there exists a positive constant $\tilde{C}$, which may depend on $\delta, \eta$ and $T_*$, such that for any $0 < T < T_*,

$$
\sup_{0 \leq t \leq T} \left( \|\rho^{\delta, \eta} - 1\|_{H^1} + \|(u^{\delta, \eta}, \theta^{\delta, \eta} - 1)\|_{H^2} + \|(u_t^{\delta, \eta}, \theta_t^{\delta, \eta})\|_{L^2} \right) \leq \tilde{C}.
$$

With the help of (5.12) and (5.13), one can proceed to show that for any $0 < T < T_*$,

$$
\sup_{0 \leq t \leq T} \|(\rho^{\delta, \eta} - 1, u^{\delta, \eta}, \theta^{\delta, \eta} - 1)(t)\|_{H^3} \leq \tilde{C},
$$

whose proofs are routine and omitted here for simplicity (see, for example, [21, 14]). Thus, combining (5.14) with the local existence result (cf. [21, Theorem 5.2]), by continuity arguments we conclude from (3.7) that there exists some $T** > T_*$ such that (3.6) holds for $T = T**$, which contradicts (5.10). As a result, (5.11) holds, and $(\rho^{\delta, \eta}, u^{\delta, \eta}, \theta^{\delta, \eta})$ is indeed a global smooth solution of (2.1)–(2.3) with mollified initial data $(\rho_0^{\delta, \eta}, u_0^{\delta, \eta}, \theta_0^{\delta, \eta})$ on $\mathbb{R}^3 \times [0, T]$ for all $T > 0$.

**Step III. Passing to the Limits as $\delta, \eta \to 0$**

Since for any $T > 0$, the approximate solutions $(\rho^{\delta, \eta}, u^{\delta, \eta}, \theta^{\delta, \eta})$ satisfy the $(\delta, \eta)$-independent estimates (3.6), (3.9) and (3.96), passing to the limit first $\delta \to 0$ and then $\eta \to 0$ (up to a subsequence if necessary), by virtue of the Aubin-Lions lemma and [19, Lemma C.1] one infers from standard compactness arguments that the limit $(\rho, u, \theta)$ of $(\rho^{\delta, \eta}, u^{\delta, \eta}, \theta^{\delta, \eta})$ is a weak solution
of (1.1)–(1.6) in the sense of Definition 1.1 and satisfies (1.16) and (1.18). For more details, we refer to [6, 13, 14].

**Step IV. Large-time Behavior**

Similarly to the derivation of (3.76), by formal calculations, which can be made rigorously by mollifying techniques (see [20, 14]), we find

\[
\left| \frac{d}{dt} \left\| \rho - \rho_s \right\|_{L^4} \right| \leq C \left( \left\| \rho - \rho_s \right\|_{L^4}^4 + \left\| \nabla u \right\|_{L^2}^2 + \left\| \rho(\theta - 1) \right\|_{L^4}^4 + \left\| F \right\|_{L^4}^4 \right),
\]

which, together with (3.35), (3.74), (3.75) and (3.77), gives

\[
\int_1^\infty \left| \frac{d}{dt} \left\| \rho - \rho_s \right\|_{L^4} \right| dt \leq C,
\]

and hence,

\[
\lim_{t \to \infty} \left\| (\rho - \rho_s)(t) \right\|_{L^4} = 0.
\]

This, combined with (3.9) and the fact that \(0 \leq \rho \leq 2\tilde{\rho}\), gives

\[
\lim_{t \to \infty} \left\| (\rho - \rho_s)(t) \right\|_{L^p} = 0, \quad \forall \ p \in (2, \infty).
\]  

(5.15)

In view of Lemma 2.2, (3.9), (3.35), (3.39) and (3.96), we have

\[
\int_1^\infty \left( \left\| u \right\|_{L^4}^4 + \left\| \nabla \theta \right\|_{L^2}^2 \right) dt \leq C + C \int_1^\infty \left\| u \right\|_{L^2} \left\| \nabla u \right\|_{L^2}^3 dt
\]

\[
\leq C + C \int_1^\infty \left( \left\| \sqrt{\rho} u \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} \right) \left\| \nabla u \right\|_{L^2}^3 dt \leq C
\]

and

\[
\int_1^\infty \left( \frac{d}{dt} \left\| u \right\|_{L^4}^4 \right) + \left( \frac{d}{dt} \left\| \nabla \theta \right\|_{L^2}^2 \right) dt
\]

\[
\leq C \int_1^\infty \left( \int \left| u \right|^3 |u_t| dx \right) + \left( \int \left| \nabla \theta \right| |\nabla \theta_t| dx \right) dt
\]

\[
\leq C \int_1^\infty \left( \left\| \nabla u \right\|_{L^2}^2 \left\| u_t \right\|_{L^2} + \left\| \nabla \theta \right\|_{L^2} \left\| \nabla \theta_t \right\|_{L^2} \right) dt \leq C,
\]

so that

\[
\lim_{t \to \infty} \left( \left\| u \right\|_{L^4} + \left\| \nabla \theta \right\|_{L^2} \right) = 0,
\]

which, together with Lemma 2.1 and (3.96), shows

\[
\lim_{t \to \infty} \left( \left\| u \right\|_{L^p \cap L^\infty} + \left\| \nabla \theta \right\|_{L^r} \right) = 0, \quad \forall \ p \in (2, \infty), \ r \in [2, 6). \quad (5.16)
\]

Thus, collecting (5.15) and (5.16) together yields (1.17). The proof of Theorem 1.1 is therefore complete.

5.2 **Proof of Theorem 1.2**

The existence part of Theorem 1.2 can be shown in a similar manner as that used for the proof of Theorem 1.1, and thus, the details are omitted for simplicity. It remains to prove the uniqueness
of solutions belonging to the class of functions (1.26). To do this, let \((\rho_1, u_1, \theta_1)\) and \((\rho_2, u_2, \theta_2)\), satisfying (1.26), be two solutions of the problem (1.1)–(1.6) on \(\mathbb{R}^3 \times [0, T]\). Define
\[
\varrho \triangleq \rho_1 - \rho_2, \quad \nu \triangleq u_1 - u_2, \quad \vartheta \triangleq \theta_1 - \theta_2.
\]

It is easy to check that \(\varrho = \varrho(x, t)\) satisfies
\[
\varrho_t + u_2 \cdot \nabla \varrho + \varrho \text{div} u_2 + \rho_1 \text{div} v + v \cdot \nabla \rho_1 = 0,
\]
which, multiplied by \(\varrho\) in \(L^2\) and integrated by parts, gives
\[
\frac{d}{dt} \|\varrho\|_{L^2}^2 \leq C\|\text{div} u_2\|_{L^\infty} \|\varrho\|_{L^2}^2 + C (\|\nabla v\|_{L^2} + \|\rho_1\|_{L^3}) \|\varrho\|_{L^2}^2
\]
\[
\leq C\|\text{div} u_2\|_{L^\infty} \|\varrho\|_{L^2}^2 + C\|\nabla v\|_{L^2} \|\varrho\|_{L^2}.
\]

Due to (1.26), \(\|\text{div} u_2\|_{L^\infty} \in L^1(0, T)\). We thus infer from (5.17) and the Gronwall’s inequality that
\[
\|\varrho(t)\|_{L^2}^2 \leq C \int_0^t \|\nabla v\|_{L^2} ds \leq C \sqrt{t} \left( \int_0^t \|\nabla v\|_{L^2}^2 ds \right)^{\frac{1}{2}}, \quad \forall \ t \in [0, T].
\]

It follows from (2.1) that \(v\) satisfies
\[
\rho_1 v_t + \rho_1 u_1 \cdot \nabla v - \mu \Delta v - (\mu + \lambda) \nabla \text{div} v
\]
\[
= -c_v \rho_2 v - \rho_1 v \cdot \nabla u_2 - \nabla (P(\rho_1, \theta_1) - P(\rho_2, \theta_2)) + \varrho \text{div} f,
\]
which, multiplied by \(v\) in \(L^2\) and integrated by parts, yields
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho_1} v\|_{L^2}^2 + \mu \|\nabla v\|_{L^2}^2 + (\mu + \lambda) \|\text{div} v\|_{L^2}^2
\]
\[
\leq C\|v\|_{L^2} \|\varrho_2\|_{L^3} \|v\|_{L^6} + C\|v\|_{L^2} \|\nabla u_2\|_{L^3} \|v\|_{L^6}
\]
\[
+ C (\|\theta_2\|_{L^\infty} \|v\|_{L^2} + \|\vartheta\|_{L^2}) \|\nabla v\|_{L^2} + C \|\varrho\|_{L^2} \|\nabla f\|_{L^3} \|v\|_{L^6}
\]
\[
= \frac{\mu}{2} \|\nabla v\|_{L^2}^2 + C (1 + \|\varrho_2\|_{L^3}^2 + \|\theta_2 - 1\|_{L^\infty}^2) \|\varrho\|_{L^2}^2 + C (\|\vartheta\|_{L^2}^2 + \|v\|_{L^2}^2),
\]

so that
\[
\frac{d}{dt} \|\sqrt{\rho_1} v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \leq C (1 + \|\varrho_2\|_{L^3}^2 + \|\theta_2 - 1\|_{L^\infty}^2) \|\varrho\|_{L^2}^2
\]
\[
+ C (\|\vartheta\|_{L^2}^2 + \|v\|_{L^2}^2).
\]

Next, noting that
\[
c_v (\rho_1 u_t + \rho_1 u_1 \cdot \nabla \vartheta) - \kappa \Delta \vartheta
\]
\[
= -c_v \rho_2 \vartheta - c_v \rho_1 v \cdot \nabla \theta_2 - (P(\rho_1, \theta_1) \text{div} u_1 - P(\rho_2, \theta_2) \text{div} u_2)
\]
\[
+ 2 \mu (|\mathcal{D}(u_1)|^2 - |\mathcal{D}(u_2)|^2) + \lambda (|\text{div} u_1|^2 - |\text{div} u_2|^2),
\]
where \(\vartheta_2 = \theta_2 t + u_2 \cdot \nabla \theta_2\), we obtain in a similar manner as the derivation of (5.19) that
\[
\frac{d}{dt} \|\sqrt{\rho_1} \vartheta\|_{L^2}^2 + \|\nabla \vartheta\|_{L^2}^2
\]
\[
\leq C\|\varrho\|_{L^2} \|\vartheta_2\|_{L^3} \|\vartheta\|_{L^6} + C\|v\|_{L^2} \|\nabla \theta_2\|_{L^3} \|\vartheta\|_{L^6}
\]
\[
+ C\|\varrho\|_{L^2} \|\theta_2 - 1\|_{L^\infty} \|\nabla u_2\|_{L^3} \|\vartheta\|_{L^6} + C\|\varrho\|_{L^2} \|\nabla u_2\|_{L^3} \|\vartheta\|_{L^6}.
\]
and hence,

\[
\frac{d}{dt} (\sqrt{\rho_1^2} v^2_{L^2} + \sqrt{\theta^2} \nabla v^2_{L^2} + \nabla \theta^2_{L^2} + \nabla v^2_{L^2} + \nabla \theta^2_{L^2}) 
\leq C \left( 1 + \|\dot{\theta}^2_{L^2} + \|\theta_2 - 1\|_{L^\infty}^2 \right) \|v\|_{L^2}^2 
+ C \left( 1 + \|\nabla \theta_2^2_{L^2} \right) \left( \|v\|_{L^2}^2 + ||v||_{L^2}^2 \right).
\]  

(5.20)

Here, we have used the following simple facts that

\[ |P(\rho_1, \theta_1) \text{div} u_1 - P(\rho_2, \theta_2) \text{div} u_2| \leq C (|\rho_2| \|\nabla u_2\| + |\nabla u_2| + |\theta_1| \|\nabla v\|) \]
\[ \leq C (|\rho_2| \|u_2\| + |\nabla u_2| + |\theta_1| \|u_2\| + |\theta_1 - 1| \|v\| + |\nabla v|) \]

and

\[ |\Omega(u_1)|^2 - |\Omega(u_2)|^2 + |(\text{div} u_1)^2 - (\text{div} u_2)^2| \leq C \left( \|\nabla u_1\| + \|\nabla u_2\| \right) \|v\|.
\]

Thus, in view of (5.19) with (5.20), we infer from (5.18) that

\[
\frac{d}{dt} \left( \sqrt{\rho_1^2} v^2_{L^2} + \sqrt{\theta^2} \nabla v^2_{L^2} + \nabla \theta^2_{L^2} \right) 
\leq C t \left( 1 + \|\dot{\theta}^2_{L^2} + \|\theta_2 - 1\|_{L^\infty}^2 \right) \|v\|_{L^2}^2 
+ C \left( 1 + \|\nabla \theta_2^2_{L^2} \right) \left( \|v\|_{L^2}^2 + \|v\|_{L^2}^2 \right).
\]  

(5.21)

Let

\[ U(t) \triangleq \left( \sqrt{\rho_1^2} v^2_{L^2} + \sqrt{\theta^2} \nabla v^2_{L^2} \right) + \int_0^t \left( \|\nabla v\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) ds.
\]

Then, due to the fact that \( \rho_1 \) has a positive lower bound, one infers from (5.21) that

\[ U'(t) \leq \Gamma(t) U(t) \quad \text{with} \quad U(0) = 0,
\]  

(5.22)

where \( \Gamma(t) \) is defined and bounded as follows, using (1.26) and the Sobolev’s inequalities,

\[ \Gamma(t) \triangleq Ct \left( 1 + \|\dot{\theta}^2_{L^2} + \|\theta_2 - 1\|_{L^\infty}^2 \right) + C \left( 1 + \|\nabla \theta_2^2_{L^2} \right) \]
\[ \leq Ct \left( 1 + \|\dot{\theta}^2_{L^2} + \|\nabla \theta_2^2_{L^2} + \|\nabla \theta_2^2_{L^2} \right) + C \left( 1 + \|\nabla \theta_2^2_{L^2} \right) \in L^1(0, T).
\]

As a result, one deduces from (5.22) and the Gronwall’s inequality that \( U(t) = 0, \) i.e.,

\[ \left( \|v(t)\|_{L^2}^2 + ||v(t)||_{L^2}^2 \right) + \int_0^t \left( \|\nabla v\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) ds = 0, \quad \forall \ t \in [0, T],
\]

from which it follows that \( v(x, t) = 0 \) and \( \theta(x, t) = 0 \) a.e. on \( \mathbb{R}^3 \times [0, T]. \) This, combined with (5.18), also yields \( \varphi(x, t) = 0 \) a.e. on \( \mathbb{R}^3 \times [0, T]. \) The proof of the uniqueness is therefore complete. \( \square \)
References


