STRONG SOLUTION TO STOCHASTIC PENALISED NEMATIC LIQUID CRYSTALS MODEL DRIVEN BY MULTIPLICATIVE GAUSSIAN NOISE

ZDZISŁAW BRZEŻNIK, ERIKA HAUSENBLAS, AND PAUL ANDRÉ RAZAFIMANDIMBY

ABSTRACT. In this paper, we prove the existence of a unique maximal local strong solutions to a stochastic system for both 2D and 3D penalised nematic liquid crystals driven by multiplicative Gaussian noise. In the 2D case, we show that this solution is global. As a by-product of our investigation, but of independent interest, we present a general method based on fixed point arguments to establish the existence and uniqueness of a maximal local solution of an abstract stochastic evolution equations with coefficients satisfying local Lipschitz condition involving the norms of two different Banach spaces.

1. INTRODUCTION

Nematic liquid crystal (NLC) is a liquid crystal phase with has rod-shaped molecules which tend to align along a particular direction denoted by a unit vector \( \mathbf{n} \), called the optical director axis. In addition to \( \mathbf{n} : \mathbb{R}^d \to \mathbb{R}^3 \), the hydrodynamic of an isothermal and incompressible NLC is also described by its pressure \( p : \mathbb{R}^d \to \mathbb{R} \) and velocity \( \mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d \). We refer to [15] and [23] for a comprehensive treatment of the physics of liquid crystals.

Using the Ericksen and Leslie continuum theory for liquid crystals, see [25] and Leslie [43], F. Lin and C. Liu [44] derived the most basic and simplest form of the dynamical system modeling the motion of a nematic liquid crystal (NLC) flowing in \( \mathbb{R}^d (d = 2, 3) \). This system is given by

\[
\begin{align*}
d\mathbf{v} + \left( (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + \nabla p \right) dt &= -\nabla \cdot (\nabla \mathbf{n} \circ \nabla \mathbf{n}) dt + f dt, \\
div \mathbf{v} &= 0, \\
d\mathbf{n} + (\mathbf{v} \cdot \nabla)\mathbf{n} dt &= \left( \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} \right) dt + g dt, \\
|\mathbf{n}|^2 &= 1, 
\end{align*}
\]

where \( f \) and \( g \) are forcings acting on the system. The entries of the matrix \( \nabla \mathbf{n} \circ \nabla \mathbf{n} \) are defined by

\[
[n \circ n]_{ij} = \sum_{k=1}^{3} \frac{\partial n^{(k)}_{i}}{\partial x_{k}} \frac{\partial n^{(k)}_{j}}{\partial x_{k}}, \quad i, j = 1, \ldots, d.
\]

Before proceeding further, we should mention that (1.1)-(1.4) is obtained by neglecting several terms such as the viscous Leslie stress tensor in the equation for \( \mathbf{v} \), the stretching and rotational effects for \( \mathbf{d} \). Thus, it is not known whether the models (1.1)-(1.4) and (1.5)-(1.7) are thermodynamically stable or consistent. However, these models still retain many mathematical and essential features of the dynamics for NLCs. In the recent papers [32] [33], [34], [46] and [49] several thermodynamically consistent and stable models of NLC have been developed and analysed.

This article is part of a project that is currently funded by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 791735 “SELEs”.

1
In this paper, we fix a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, $d = 12, 3$ with smooth boundary and we consider the following stochastic system

$$
dv + \left[ (v \cdot \nabla)v - \Delta v + \nabla p \right] dt = -\nabla \cdot (\nabla n \odot \nabla n) dt + S(v)dW_1, \quad (1.5)$$

$$\text{div } v = 0, \quad (1.6)$$

$$dn + (v \cdot \nabla)n dt = \left[ \Delta n + f(n) \right] dt + (n \times h) \circ dW_2, \quad (1.7)$$

$$v = 0 \text{ and } \frac{\partial n}{\partial \nu} = 0 \text{ on } \partial \mathcal{O}, \quad (1.8)$$

$$v(0) = v_0 \text{ and } n(0) = n_0, \quad (1.9)$$

where $v_0 : \mathcal{O} \rightarrow \mathbb{R}^d$, $n_0 : \mathcal{O} \rightarrow \mathbb{R}^3$ are given mappings, $v$ is the unit outward normal to $\partial \mathcal{O}$, $f$ is a polynomial function satisfying some conditions to be fixed later. Here, $W_1$ and $W_2$ are respectively independent cylindrical Wiener process and standard Brownian motion, $(n \times h) \circ dW_2$ is understood in the Stratonovich sense.

The Fréedericksz transition, which is produced by applying a sufficiently strong external perturbation (e.g. magnetic or electric fields) to an undistorted NLC, and its behaviour under random perturbation have been extensively studied in several physics papers, see [38, 55, 56], all of which neglected the fluid velocity. However, it is pointed out in [23, Chapter 5] that the fluid flow disturbs the alignment and conversely a change in the alignment will induce a flow in the nematic liquid crystal. It is this gap in knowledge that is the motivation for our mathematical study which was initiated in the old unpublished preprints [7] and [8], see also the recent papers [6] and [5].

In this paper, we mainly prove the existence and uniqueness of a maximal local strong solution which is understood in the sense of stochastic calculus and PDEs. This result is a corollary of several abstract results which are proved in Section 5 and are of independent interest. In the case $d = 2$, we show the non-explosion of the maximal solution by an adaptation and combination of Khashminskii test for non-explosions and an idea of Schmalfuß elaborated in [57], see Section 3 for more details. Our novelty is the extension of the Schmalfuß idea, which has been used so far to prove the uniqueness of solutions of Stochastic Navier-Stokes equations and related problems, to the proof of the global existence of a strong solution to the problem (1.5)-(1.9). In particular, we give another proof of the global existence of 2D stochastic Navier-Stokes equations with multiplicative noise and for initial data with finite enstrophy. Thus, our paper can also be seen as a generalization of the results for the existence and uniqueness of maximal local and global solutions of strong solutions of stochastic Navier-Stokes proved in [27], [22] and [51].

We should notice that some of the arguments elaborated in Section 5 have been already used in [1] and [5] which respectively studied the strong solution of some stochastic hydrodynamic equations (NSEs, MHD and 3D Leray $\alpha$-models) driven by Lévy noise, and the existence and uniqueness of a maximal local smooth solution to the stochastic Ericksen-Leslie system (1.1)-(1.4) on the $d$-dimensional torus. We are also strongly convinced that with these general results it is possible, although it has not been done in detail, to prove the existence of strong solution of several stochastic hydrodynamical models such as the NSEs, MHD equations, $\alpha$-models for Navier-Stokes and related problems.

While the deterministic version of (1.5)-(1.7) has been the subject of intensive mathematical studies, see [28, 35, 44, 45, 47, 58, 18] and [14, 36, 37, 39, 63], there are fewer results related to the stochastic system (1.5)-(1.7). The unpublished paper [7] proved the existence and uniqueness of maximal local strong solution to the system (1.5)-(1.7) with a bounded nonlinear term $f(n) = 1_{|n| \leq 1} (1 - |n|^2)n$. The paper [6] deals only with weak (both in PDEs and stochastic calculus
sense) solutions and the maximum principle. Some of the results in [6] and the current paper have already been used in several papers such as [9], [10], [65], [30], [29] and [64]. Very recently we have become aware of a recent paper by Feireisl and Petcu [26], in which they proved the existence of a dissipative martingale, as well as the existence of a local strong solution and weak-strong uniqueness of the solution of the stochastic Navier-Stokes Allen-Cahn Equations. Note that in [26] the second unknown \( n \) is a scalar field, the nonlinear term \( f(\cdot) \) is globally Lipschitz and the derivative of a double-well potential \( F(\cdot) \), and the coefficient of the noise entering the equations for \( n \) is bounded.

The paper [5] is the first paper to deal with the stochastic counterparts of the Ericksen-Leslie equations (1.1)-(1.4). The results in present manuscript is not covered in [5] because in contrast to our framework which considers initial condition (1.1)-(1.3).

To close this introduction, we emphasize that the analysis in the present paper might also be of great interest in the numerical study of stochastic Ericksen-Leslie system. In fact, on the one hand our assumptions on the polynomial \( f(n) \) enable us to consider the typical Ginzburg-Landau function \( f_\epsilon(n) = \frac{1}{\epsilon^2}(1 - |n|^2)n \), see Assumption 2.1 and Remark 2.5. In the numerical context, handling the constraint \( |n| = 1 \) in the Ericksen-Leslie is a rather challenging task and to overcome this difficulty, one usually use the Ginzburg-Landau approximation, see [62]. On the other hand, to get convergence and a rate of convergence of a space discretization for parabolic SPDEs, one often has to consider a regular solution in favour of weak solutions.

2. Preliminary results and notations

2.1. Functional spaces and linear operators. Following [6] we introduce in this section various notations and results that are frequently used in this paper.

For two topological spaces \( X \) and \( Y \) the symbol \( X \hookrightarrow Y \) means that the embedding \( X \) is continuously embedded in \( Y \). Let \( d \in \{2,3\} \) and assume that \( O \subset \mathbb{R}^d \) is a bounded domain with boundary \( \partial O \) of class \( C^\infty \).

For \( p \in [1,\infty) \) and \( k \in \mathbb{N} \) the symbols \( L^p(O) \) or by \( W^{k,p}(O) \) (resp. by \( L^p(O) \) or by \( W^{k,p}(O) \)) respectively denote the Lebesgue and Sobolev spaces of functions \( v : \mathbb{R}^d \to \mathbb{R}^d \) (resp. \( n : \mathbb{R}^d \to \mathbb{R}^3 \)).

For \( p = 2 \) the function spaces \( W^{k,2}(O) \) and \( W^{k,p} \) are respectively denoted by \( H^k \) (\( H^k \)) and their norms are denoted by \( ||\cdot||_k \). The scalar products on \( L^2 \) and \( L^2 \) are denoted by the same symbol \( \langle u, v \rangle \) for \( u, v \in L^2 \) (resp. \( u, v \in L^2 \)) and their associated norms is denoted by \( \|u\|_2, u \in L^2 \) (resp. \( u \in L^2 \)).

By \( H^0_0 \) and \( W^{1,r}_0 \), \( r > 2 \), we mean the spaces of functions in \( H^1 \) and \( W^{1,r} \) that vanish on the boundary on \( O \). It is well known that if \( a = \frac{d}{2} \), then there exists a constants \( c > 0 \) such that

\[
|u|_{L^1} \leq c \begin{cases} |u|^{1-a}_{L^2} |\nabla u|^{a}_{L^2} & \text{if } u \in H^1_0 \\ |u|^{1-a}_{L^2} |u|^{a}_{H^1} & \text{if } u \in H^1, \end{cases} \tag{2.1}
\]

\[
|u|_{L^\infty} \leq c \begin{cases} |u|^{1-a}_{L^4} |\nabla u|^{a}_{L^4} & \text{if } u \in W^{1,4}_0 \\ |u|^{1-a}_{L^4} |u|^{a}_{W^{1,4}} & \text{if } u \in W^{1,4}, \end{cases} \tag{2.2}
\]

and since \( H^2 \hookrightarrow W^{1,4} \), we have

\[
|u|_{L^\infty} \leq \|u\|_1^{1-a} \|u\|_2^{a}, u \in H^2. \tag{2.3}
\]
We now introduce the following spaces
\[ V = \text{closure of } \mathcal{V} \text{ in } H^1_0(O) \]
\[ H = \text{closure of } \mathcal{V} \text{ in } L^2(O). \]
As usual, we endow \( H \) with the scalar product and norm of \( L^2 \), and we equip the space \( V \) with the scalar product \( \langle \nabla u, \nabla v \rangle \) which is equivalent to the \( H^1(O) \)-scalar product on \( V \).

Let \( \Pi : L^2 \to H \) be the Helmholtz-Leray projection from \( L^2 \) onto \( H \). We denote by \( A = -\Delta \Pi \) the Stokes operator with domain \( D(A) = V \cap H^2 \), see for instance [60, Chapter I, Section 2.6]. It is well-known that the spaces \( V_\beta := D(A^\beta), \beta \in \mathbb{R} \), are Hilbert spaces when endowed with the graph inner product and \( V_{-\frac{1}{2}} = V \). It is also well-known that the map \( A^\delta : V_\beta \to V_{\beta-\delta}, \beta, \delta \in \mathbb{R} \), is a linear isomorphism. For all these facts we refer, for instance, to [17].

The Neumann Laplacian acting on \( \mathbb{R}^d \)-valued function will be denoted by \( A_1 \), that is,
\[ D(A_1) := \left\{ u \in H^2 : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial O \right\}, \]
\[ A_1 u := -\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} \quad u \in D(A_1). \]
Notice that the Neumann Laplacian \( A_1 \) can be viewed as a linear map \( A_1 : H^1 \to (H^1)^* \) satisfying
\[ \langle (A_1 u, n)_{H^1} = \langle \nabla u, \nabla n \rangle, \quad \text{for all } u, n \in H^1. \]  
(2.4)

Thanks to [31, Theorem 5.31] one can define and characterize in standard way the spaces \( X_\alpha = D(A_1^\alpha), \alpha \in [0, \infty) \), where \( A_1 = I + A_1 \). Also, it can be shown that \( X_\alpha \hookrightarrow H^{2\alpha} \), for all \( \alpha \geq 0 \) and \( X := X_{\frac{1}{2}} = H^1 \), see, for instance, [61, Sections 4.3.3 & 4.9.2].

Now, let \( h \in L^\infty \) be fixed and define a linear bounded operator \( G \) from \( L^2 \) into itself by
\[ G : L^2 \ni n \mapsto n \times h \in L^2. \]  
(2.5)

It is straightforward to check that there exists a constant \( C > 0 \) such that
\[ \|G(n)\| \leq C\|h\|_{L^\infty} \|n\|_{L^2}, \quad \text{for all } n \in L^2. \]

2.2. The nonlinear terms. Throughout this paper \( B^* \) denotes the dual space of a Banach space \( B \). We also denote by \( \langle \Psi, b \rangle_{B^*, B} \) the value of \( \Psi \in B^* \) on \( b \in B \). Throughout, \( \partial_{x_i} = \frac{\partial}{\partial x_i} \), and \( \phi^{(i)} \) is the \( i \)-th entry of any vector-valued \( \phi \).

Let \( p, q, r \in [1, \infty] \) such that \( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1 \). Then, we define a trilinear form \( b(\cdot, \cdot, \cdot) \) by
\[ b(u, v, w) = \frac{1}{r} \int_\Omega u^{(i)} \frac{\partial v^{(j)}}{\partial x_i} w^{(j)} dx, \quad u \in L^p, v \in W^{1,q}, \text{ and } w \in L^r. \]

Note if \( v \in W^{1,q} \) and \( w \in L^r \), then we have to take the sum over \( j \) from \( j = 1 \) to \( j = 3 \).

It is well known, see [60, Section II.1.2], that there is a bilinear map \( B : V \times V \to V^* \) such that
\[ \langle B(u, v), w \rangle_{V^*, V} = b(u, v, w) \quad \text{for } w \in V, \quad \text{and } u, v \in V. \]  
(2.6)

In a similar way, there is also a bilinear map \( \tilde{B} : V \times H^1 \to (H^1)^* \) such that
\[ \langle \tilde{B}(u, v), w \rangle_{(H^1)^*, H^1} = b(u, v, w) \quad \text{for all } u \in V, \quad v, \quad w \in H^1. \]  
(2.7)
The following lemma was proved in [60, Section II.1.2].

**Lemma 2.1.** The bilinear map $B(\cdot, \cdot)$ maps continuously $V \times H^1$ into $V^*$ and

\[
\langle B(u, v), w \rangle_{V^*, V} = b(u, v, w), \text{ for all } u, v \in H^1, w \in V, \quad (2.8)
\]

\[
\langle B(u, v), w \rangle_{V^*, V} = -b(u, w, v) \text{ for all } u, v \in H^1, w \in V, \quad (2.9)
\]

\[
\langle B(u, v), v \rangle_{V^*, V} = 0 \text{ for all } u, v \in V, \quad (2.10)
\]

\[
|B(u, v)|_{V^*} \leq C_0 \|u\|_{L^2}^{1 - \frac{4}{r}} \|\nabla u\|_{L^2}^{\frac{4}{r}} \|v\|_{L^2}^{1 - \frac{4}{r}} \|\nabla v\|_{L^2}^{\frac{4}{r}}, \text{ for all } u, v \in V, v \in H^1. \quad (2.11)
\]

**Lemma 2.2.** There exists a constant $C_1 > 0$ such that

\[
\|\tilde{B}(v, n)\| \leq C_1 \|v\|_{L^2} \|\nabla v\|_{L^2} \|n\|_{H^2}, \text{ for all } v \in V, n \in H^2, \quad (2.12)
\]

\[
\langle \tilde{B}(v, n), n \rangle = 0, \text{ for any } v \in V, n \in H^2. \quad (2.13)
\]

**Proof.** The estimate (2.12) follows from [6, Lemma 6.2] and the Gagliardo-Nirenberg estimate (2.1). The proof of (2.13) and (2.10) are the same, see [60, Section II.1.2].

Let $r, p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. For $n_1 \in \mathcal{W}^{1,p}$, $n_2 \in \mathcal{W}^{1,q}$ and $u \in \mathcal{W}^{1,r}$ we set

\[
m(n_1, n_2, u) = -\sum_{i,j=1}^{d} \sum_{k=1}^{3} \int_{\Omega} \partial_{x_i} n_1^{(k)} \partial_{x_j} n_2^{(k)} \partial_{x_k} u^{(i)} \, dx. \quad (2.14)
\]

Since $d \leq 4$, the integral in (2.14) is also well defined for $n_1, n_2 \in H^2$ and $u \in V$.

We recall the following proposition which can be found in [6, Proposition 2.2 & Remark 2.3].

**Proposition 2.3.** Let $d \in [1, 4]$. There exists a bilinear map $M : H^2 \times H^2 \to V^*$ such that

\[
\langle M(n_1, n_2, u), v \rangle_{V^*, V} = m(n_1, n_2, u), \quad n_1, n_2 \in H^2, u \in V, \quad (2.15)
\]

\[
\langle M(f, g), v \rangle_{V^*, V} = \langle \Pi[\text{div}(f \circ \nabla g)], v \rangle \text{ for all } f, g \in X_1 \text{ and } v \in H, \quad (2.16)
\]

\[
\langle \tilde{B}(v, n), A_1 n \rangle + \langle M(n, n), v \rangle_{V^*, V} = 0, \quad \text{for all } v \in V, n \in X_1. \quad (2.17)
\]

In some places in this manuscript we use the following shorthand notations:

\[
B(u) := B(u, u) \quad \text{and} \quad M(n) := M(n, n),
\]

for all $u$ and $n$ such that the above quantities are meaningful.

We now fix the standing assumptions on the function $f(\cdot)$.

**Assumption 2.1.** Let $I_d$ be the set defined by

\[
I_d = \begin{cases} \mathbb{N} := \{1, 2, 3, \ldots\} & \text{if } d = 2, \\ \{1\}, & \text{if } d = 3. \end{cases} \quad (2.18)
\]

We fix $N \in I_d$ and $a_k \in \mathbb{R}$, $k = 0, \ldots, N$, with $a_N < 0$. We define a function $\tilde{f} : [0, \infty) \to \mathbb{R}$ by

\[
\tilde{f}(r) = \sum_{k=0}^{N} a_k r^k, \quad \text{for all } r \in \mathbb{R}_+.
\]

We define a map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(n) = \tilde{f}(\|n\|^2) n$ where $\tilde{f}$ is as above.

We now assume that there exists $F : \mathbb{R}^3 \to \mathbb{R}$ a Fréchet differentiable map such that

\[
F'(n)[g] = f(n) \cdot g, \quad n \in \mathbb{R}^d, g \in \mathbb{R}^d.
\]
Remark 2.4. (i) There exists a constant \( \ell_3 > 0 \) such that
\[
|\tilde{f}''(r)| \leq \ell_3(1 + r^{N-2}), \quad r > 0.
\] (2.19)

(ii) From (2.19), we infer that there exist \( c_0, c_1, c_3 > 0 \) such that
\[
|f(n)| \leq c_0(1 + |n|^{2N+1}), \quad |f'(n)| \leq c_1(1 + |n|^{2N}) \quad \text{and} \quad |f''(n)| \leq c_2(1 + |n|^{2N-1}) \quad \text{for all} \quad n \in \mathbb{R}^n.
\]

(iii) Let \( \tilde{q} = 4N + 2 \). It is easy to show that there exists \( C > 0 \) such that for all \( n \in \mathbb{H}^2 \)
\[
\|A_1n\|^2 = \|A_1n + f(n) - f(n)\|^2 \leq 2\|A_1n - f(n)\|^2 + 2\|f(n)\|^2,
\]
\[
\leq 2\|A_1n - f(n)\|^2 + C\|n\|_{L^4}^q + C.
\] (2.20)

(iv) Since the norm \( \|\cdot\|_2 \) is equivalent to \( \|\cdot\| + \|A_1\| \) on \( D(A_1) \), there exists \( C > 0 \) such that
\[
\|n\|^2_2 \leq C(\|A_1n - f(n)\|^2 + \|n\|_{L^4}^q + 1), \quad \text{for all} \quad n \in D(A_1).
\] (2.21)

(v) Since \( \mathbb{H}_1 \hookrightarrow L^{4N+2}, N \in \mathcal{D}_d \), we infer from (2.21) that \( n \in \mathbb{H}^2 \) if \( n \in \mathbb{H}_1 \) and \( A_1n - f(n) \in L^2 \).

Remark 2.5. Let \( \varepsilon > 0 \) and \( \tilde{f}_\varepsilon := \frac{1}{\varepsilon^2}(-r + 1), \quad r \in [0, \infty) \). Examples of maps \( f \) and \( F \) satisfying Assumption 2.1 are the following
\[
f(n) := \hat{f}(|n|^2)n = \frac{1}{\varepsilon^2}(1 - |n|^2)n \quad \text{and} \quad F(n) := \frac{1}{4\varepsilon^2}[\hat{f}(|n|^2)]^2, \quad n \in \mathbb{R}^d.
\]

2.3. The assumption on the coefficients of the noise.

Assumption 2.2. Throughout this paper we are given a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the filtration \( \mathbb{F} = \{\mathcal{F}_t : t \geq 0\} \) satisfying the usual hypothesis, i.e., the filtration is right-continuous and all null sets of \( \mathcal{F} \) are elements of \( \mathcal{F}_0 \).

Throughout, let \( K_1 \) be a separable Hilbert space, and \( W_1 = (W_1(t))_{t \geq 0} \) and \( W_2 = (W_2(t))_{t \geq 0} \) be independent \( K_1 \)-cylindrical Wiener process and standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). If \( K = K_1 \times \mathbb{R} \) then we can assume that \( W = (W_1(t), W_2(t)) \) is \( K \)-cylindrical Wiener process.

Let \( \tilde{K} \) and \( \tilde{H} \) be a separable Hilbert and Banach spaces. We denote by \( \gamma(\tilde{K}, \tilde{H}) \) the space of \( \gamma \)-radonifying operators which generalises the space of Hilbert-Schmidt operators \( \mathcal{T}_2(\tilde{K}, \tilde{H}) \) if \( \tilde{H} \) is a separable Hilbert space, see [2]. Let \( \mathcal{M}^2(\Omega \times [0, T]; \mathcal{T}_2(\tilde{K}, \tilde{H})) \) the space of all equivalence classes of progressively measurable processes \( \Psi : \Omega \times [0, T] \rightarrow \mathcal{T}_2(\tilde{K}, \tilde{H}) \) satisfying
\[
\mathbb{E}\int_0^T \|\Psi(s)\|^2_{\mathcal{T}_2(\tilde{K}, \tilde{H})} ds < \infty.
\]

For a \( \tilde{K} \)-cylindrical Wiener process \( \tilde{W} \) and \( \Psi \in \mathcal{M}^2(\Omega \times [0, T]; \mathcal{T}_2(\tilde{K}, \tilde{H})) \) the process \( M \) defined by \( M(t) = \int_0^t \Psi(s) d\tilde{W}(s), \quad t \in [0, T], \) is a \( \tilde{H} \)-valued martingale. For more detail on the theory of stochastic integration we refer to [50, Section 26] and [19, Chapter 4].

Let \( G \) be the map defined in (2.5) and \( G^2 = G \circ G \). Then, we have the following relation identity Stratonovich and Itô’s integrals, see [3],
\[
G(n) \circ dW_2 = \frac{1}{2} G^2(n) \, dt + G(n) \, dW_2.
\]

We now introduce the standing set of hypotheses on the function \( S \).
Assumption 2.3. We assume that $S : H \to T_2(K_1, V)$ is a globally Lipschitz map. In particular, there exists $\ell_5 \geq 0$ such that
\[
\|S(u)\|_{T_2(K_1, V)}^2 \leq \ell_5 (1 + |u|_{L^2}^2) , \quad \text{for all } u \in H .
\] (2.22)

Remark 2.6. Notice that the assumption (2.22) implies that there exists a constant $\ell_6 > 0$ such that
\[
\|S(u)\|_{T_2(K_1, V)}^2 \leq \ell_6 (1 + |\nabla u|_{L^2}^2) , \quad \text{for all } u \in H .
\] (2.23)

3. Existence and uniqueness of local and global strong Solution

Using the notations of Section 2, the system (1.5)-(1.9) can be written in the abstract form
\[
dv(t) + \left( Av(t) + B(v(t), v(t)) + M(n(t)) \right) dt = S(v(t))dW_1 ,
\] (3.1)
\[
dn(t) + \left( A_1 n(t) + \tilde{B}(v(t), n(t)) - f(n(t)) - \frac{1}{2} G^2(n(t)) \right) dt = G(n(t))dW_2 ,
\] (3.2)
\[
v(0) = v_0 \text{ and } n(0) = n_0 .
\] (3.3)

In this section we prove the existence and uniqueness of the strong solution to problem (3.1)-(3.3).

3.1. Definition of local solutions. Let $(B_i, \|\cdot\|_{B_i})$, $i = 1, 2$, be two Banach spaces. We endow $B_1 \times B_2$ with the norm $(\|b_1, b_2\| = \sqrt{\|b_1\|_{B_1}^2 + \|b_2\|_{B_2}^2}$). Henceforth, we put
\[
\mathcal{H} = H \times X_2^{1/2} , \quad \mathcal{V} = V \times X_1 \quad \text{and} \quad \mathcal{E} = V_1 \times X_2^{1/2} .
\] (3.4)

Next, we denote by $\{T(t)_{t \geq 0}\}$ and $\{T(t)\}_{t \geq 0}$ the analytic semigroups generated by $-A$ on $H$ and by $A_1$ on $L^2$, respectively. It is well-known that the space $X_2^{1/2}$ is invariant wrt $\{T(t)\}_{t \geq 0}$. The restriction of $\{T(t)\}_{t \geq 0}$ to $X_2^{1/2}$ is also an analytic semigroup which will be denoted by $\{S_2(t)\}_{t \geq 0}$.

The minus infinitesimal generator $\tilde{A}_1$ of $\{S_2(t)\}_{t \geq 0}$ is the part of $A_1$ on $X_2^{1/2}$, that is,
\[
D(\tilde{A}_1) = \{ u \in D(A_1) : A_1 u \in X_2^{1/2} \} , \quad \tilde{A}_1 u = A_1 u \text{ for all } u \in D(\tilde{A}_1) .
\]

Note that $X_2^{1/2} \subset D(\tilde{A}_1)$. With all the above notation, the problem (3.1)-(3.3) can be rewritten as the following stochastic evolution equation in the space $\mathcal{H}$,
\[
dy(t) + Ay(t)dt + F(y(t))dt + L(y(t))dt = G(y(t))dW(t) ,
\] (3.5)
where, for $y = (v, n) \in E$ and $k = (k_1, k_2) \in K$,
\[
Ay = \left( \begin{array}{c} Av \\ A_1 n \end{array} \right) , \quad F(y) = \left( \begin{array}{c} B(v, v) + M(n) \\ \tilde{B}(v, n) - f(n) \end{array} \right) ,
\] (3.6)
\[
L(y) = \left( \begin{array}{c} 0 \\ -\frac{1}{2} G^2(n) \end{array} \right) , \quad G(y)k = \left( \begin{array}{c} S(u)k_1 \\ (G(n)k_2) \end{array} \right) .
\] (3.7)

The operator $-A$ generates an analytic semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{H} = H \times X_2^{1/2}$ defined by
\[
S(t) \left( \begin{array}{c} v \\ n \end{array} \right) = \left( \begin{array}{c} S_1(t)v \\ S_2(t)n \end{array} \right) , \quad (v, n) \in \mathcal{H}.
\]

Important properties of $\{S(t) : t \geq 0\}$ are given in the next two lemma.
Lemma 3.1. Let $T \in (0, \infty)$, $g = \left( \frac{\tilde{g}}{g} \right) \in L^2(0, T; H \times X_1)$ and $\left( v(t) \right) = f_t^T S(t-s) g(s) ds$, $t \geq 0$. Then, there exists $c_1 > 0$ such that

$$\left\| \left( \frac{v}{n} \right) \right\|_{C([0,T];V \times X_1)} + \left\| \left( \frac{v}{n} \right) \right\|_{L^2(0,T;D(A) \times X_2^4)} \leq c_1 \left\| \left( \frac{\tilde{g}}{g} \right) \right\|_{L^2(0,T;H \times X_1^2)}.$$

Proof. This result is well-known and is a special case of [54, Lemma 1.2].

Lemma 3.2. Let $T \in (0, \infty)$, $\zeta = \left( \zeta_1 \zeta_2 \right) \in \mathcal{M}^2(0, T; H \times X_1)$ and $\left( w_1(t), w_2(t) \right) = f_0^T S(t-s) \zeta(s) dW(s)$, $t \geq 0$. Then, there exists $C > 0$ such that

$$\mathbb{E} \left\| \left( w_1 \right) \right\|_{C([0,T];V \times X_1)}^2 + \mathbb{E} \left\| \left( w_1 \right) \right\|_{L^2(0,T;D(A) \times X_2^4)}^2 \leq C \mathbb{E} \left\| \left( \zeta_1 \zeta_2 \right) \right\|_{L^2(0,T;H \times X_1^2)}^2, \quad T \geq 0.$$

Proof. This result is also well-known, see [54, Lemma 1.4].

Let us recall the following notations/definition which are borrowed from [42], see also [3].

Definition 3.3. A random function $\tau : \Omega \to [0, \infty]$ is called a stopping time, see [40, Definition I.2.1], [50, Definition 4.1] and [24, section III.5], iff for each $t \geq 0$, the set $\{ \omega \in \Omega : t < \tau(\omega) \} \in \mathcal{F}_t$ (or equivalently, $\{ \omega \in \Omega : \tau(\omega) \leq t \} \in \mathcal{F}_t$). A stopping time $\tau : \Omega \to [0, \infty]$ is called accessible, see [42, section 2.1, p. 45], iff there exists an increasing sequence$^1$ of stopping times $\tau_n : \Omega \to [0, \infty)$ such that $\mathbb{P}$-a.s. (i) for all $n \in \mathbb{N}$, $\tau_n < \tau$; (ii) and $\lim_{n \to \infty} \tau_n = \tau$. The sequence $(\tau_n)_{n \in \mathbb{N}}$ as above is usually called an announcing sequence for $\tau$.

Remark 3.4. Under the Assumption 2.2 we have the following facts.

(i) It follows from [50, Proposition 6.6 (3)] that a stopping time is accessible if and only if it is predictable. Let us recall, see [50, Definition 4.9], that a stopping time $\tau$ is predictable iff its graph $[\tau] := \{ (t, \omega) \in [0, \infty) \times \Omega : t = \tau(\omega) \}$ is a predictable set. The $\sigma$-field $\mathcal{P}$ of predictable sets is generated by the family $\mathcal{R} := \{ (s, t) \times F : 0 \leq s \leq t, F \in \mathcal{F}_s \} \cup \{ \{0\} \times F : F \in \mathcal{F}_0 \}$, see [50, Theorem 3.3].

(ii) If $(\tau_n)_{n \in \mathbb{N}}$ is a sequence of accessible stopping times, then $\sup_{n \in \mathbb{N}} \tau_n$ is also an accessible stopping times, see [50, Proposition 6.6]. In particular, if $\tau$ and $\sigma$ accessible stopping times with announcing sequences $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$, then $\tau \lor \sigma$ is an accessible stopping time with announcing sequence $(\tau_n \lor \sigma_n)_{n \in \mathbb{N}}$. Furthermore, if $A$ is an arbitrary family of accessible stopping times then a family

$$B := \{ \sup C : C \text{ is a finite subset of } A \}$$

is also a family of accessible stopping times such that $A \subseteq B$ and the supremum of each finite subset of $B$ belongs to $B$. In particular, if $\Delta$ is the family of all accessible stopping times, then the supremum of each finite subset of $\Delta$ belongs to $\Delta$.

Notation. For a stopping time $\tau$ we set

$$\Omega_t(\tau) = \{ \omega \in \Omega : t < \tau(\omega) \},$$

$$[0, t) \times \Omega = \{ (t, \omega) \in [0, \infty) \times \Omega : 0 \leq t < \tau(\omega) \}.$$
Definition 3.5. Assume that $X$ is a topological space. An $X$-valued process local process $\eta : [0, \tau) \times \Omega \to X$ (we will also write $\eta(t), t < \tau$) is admissible if (i) it is adapted, i.e. $\eta|_{\Omega_t(\tau)} : \Omega_t(\tau) \to X$ is $\mathcal{F}_t$ measurable, for all $t \geq 0$; (ii) for almost all $\omega \in \Omega$, the function $[0, \tau(\omega)) \ni t \mapsto \eta(t, \omega)$ in $X$ is continuous.

Two local processes $\eta_i : [0, \tau_i) \times \Omega \to X$, $i = 1, 2$ are called equivalent (and we will write $(\eta_1, \tau_1) \sim (\eta_2, \tau_2)$) iff $\tau_1 = \tau_2$ P-a.s. and, for all $t > 0$, the following condition holds

$$\eta_1(\cdot, \omega) = \eta_2(\cdot, \omega) \text{ on } [0, t]; \text{ for a.e. } \omega \in \Omega_t(\tau_1) \cap \Omega_t(\tau_2).$$

Note that if two local admissible processes $\eta_i : [0, \tau_i) \times \Omega \to X$, $i = 1, 2$ are such that for all $t > 0$ $\eta_1(t)|_{\Omega_t(\tau_1)} = \eta_2(t)|_{\Omega_t(\tau_2)}$ P-a.s., then they are equivalent.

Remark 3.6. Let $\tau$ be an accessible stopping time with an announcing sequence $(\tau_n)_{n \in \mathbb{N}}$ and $\eta : [0, \tau) \times \Omega \to X$ is a local process. Kunita [42, section 2.1, p. 46] defined $\eta$ to be a local adapted process iff the following condition is satisfied for every $n$, the stopped process $(\eta_{t \wedge \tau_n})_{t \geq 0}$ is adapted. We do not know how condition (i) from Definition 3.5 is related to Kunita’s definition.

Definition 3.7. Let $\tau$ be an accessible stopping time with an announcing sequence $(\tau_n)_{n \in \mathbb{N}}$. Motivated by [40, Proposition 2.18], a local process $\eta : [0, \tau) \times \Omega \to X$ is called progressively measurable iff for every $n$, the stopped process $(\eta_{t \wedge \tau_n})_{t \geq 0}$ is progressively measurable.

We now define some concepts of solution to (3.5), see [12, Definition 4.2] or [51, Definition 2.1].

Definition 3.8. Let $y_0 : \Omega \to V$ be $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|y_0\|_V^2 < \infty$. A local solution to problem (3.5) (with the initial time 0) is a pair $(y, \tau)$ such that

1. $\tau$ is an accessible stopping time with an announcing sequence $(\tau_n)_{n \in \mathbb{N}},$
2. $y : [0, \tau) \times \Omega \to V$ is an admissible process,
3. for every $n \in \mathbb{N}$ and $t \in [0, \infty)$, we have

$$\mathbb{E}\left(\sup_{s \in [0, t \wedge \tau_n]} \|y(s)\|_V^2 + \int_0^{t \wedge \tau_n} \|y(s\wedge t)\|_V^2 ds\right) < \infty,$$

and P-a.s.

$$y(t \wedge \tau_n) = S(t \wedge \tau_n) y_0 - \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - s) [F(y(s \wedge \tau_n)) + L(y(s \wedge \tau_n))] ds + I_{\tau_n}(t \wedge \tau_n),$$

where $I_{\tau_n}$ is a continuous $V$-valued process process defined by

$$I_{\tau_n}(t) := \int_0^t 1_{[0, \tau_n)}(s) S(t - s) G(y(s \wedge \tau_n)) dW(s), \quad t \in [0, \infty).$$

(3.10)

Along the lines of the paper [3], we say that a local solution $y(t), t < \tau$ is global iff $\tau = \infty$ P-a.s.

Hereafter, we simply write local solution in place of local mild solution.

Remark 3.9. (i) Since $\tau_n$ is a stopping, the process $1_{[0, \tau_n)}(s), s \in [0, \infty)$ is well-measurable, see [50, Proposition 4.2]. Therefore, since by [50, Theorem 1.6], the $\sigma$-field of well measurable sets is smaller than the $\sigma$-field of progressively measurable sets, it follows that the process $1_{[0, \tau_n)}(s), s \in [0, \infty)$ is progressively measurable. In particular, the integrand in (3.10) is progressively measurable.

(ii) On the other hand, one could use in (3.10) a process $1_{[0, \tau_n)}(s), s \in [0, \infty)$, which according [50, Proposition 4.4], is predictable. However, here we still stick to the processes as in (i).
Remark 3.10. Suppose that \( \tau : \Omega \to [0,\infty) \) is an accessible stopping time and \( \textbf{y} : [0, \tau] \times \Omega \to V \) is an admissible process such that for every \( t \in [0, \infty) \)

\[
\mathbb{E} \left( \sup_{s \in [0,t\land \tau]} \| \textbf{y}(s) \|^2 \right) + \int_0^{t\land \tau} \| \textbf{y}(s) \|^2 ds < \infty, \tag{3.11}
\]

\[
\textbf{y}(t \land \tau) = \mathcal{S}(t \land \tau)\textbf{y}_0 - \int_0^{t\land \tau} \mathcal{S}(t \land \tau - s)[\mathcal{F}(\textbf{y}(s \land \tau)) + \mathcal{L}(\textbf{y}(s \land \tau))] ds + I_\tau(t \land \tau), \quad \mathbb{P}\text{-a.s.,} \tag{3.12}
\]

where \( I_\tau \) is a continuous \( V \)-valued process defined by

\[
I_\tau(t) := \int_0^t 1_{[0,\tau)}(s)\mathcal{S}(t-s)G(\textbf{y}(s \land \tau)) d\mathbb{W}(s), \quad t \in [0, \infty). \tag{3.13}
\]

Let us choose an announcing sequence \( (\tau_n)_{n \in \mathbb{N}} \) for \( \tau \). Then, by using \cite[Lemma A.1]{11}, we can show that for every \( n \), the conditions (3.8) and (3.9) are satisfied with \( I_{\tau_n} \) defined by (3.10). Therefore we infer that the restriction of the process \( \textbf{y} \) to the open stochastic interval \( [0, \tau) \times \Omega \) is a local solution to problem (3.5).

We now introduce the definition of a maximal local solution.

Definition 3.11. Consider a family \( \mathcal{LS} \) of all local solution \( (u, \tau) \) to the problem (3.5). For two elements \( (u, \tau), (v, \sigma) \in \mathcal{LS} \) we write that \( (u, \tau) \preceq (v, \sigma) \) iff \( \tau \leq \sigma \) \( \mathbb{P} \)-a.s. and \( \nu_{[0,\tau) \times \Omega} \sim u \). Note that if \( (u, \tau) \preceq (v, \sigma) \) and \( (v, \sigma) \preceq (u, \tau) \), then \( (u, \tau) \sim (v, \sigma) \). We write \( (u, \tau) < (v, \sigma) \) iff \( (u, \tau) \preceq (v, \sigma) \) and \( (u, \tau) \not\sim (v, \sigma) \). Then, the pair \( (\mathcal{LS}, \preceq) \) is partially ordered. Each maximal element \( (u, \tau) \) in the set \( (\mathcal{LS}, \preceq) \) is called a maximal local solution to the problem (3.5). The existence of an upper bound of every non-empty chain of \( (\mathcal{LS}, \preceq) \) is justified by Amalgamation Lemma 5.3.

If \( (u, \tau) \) is a maximal local solution to equation (3.5), the stopping time \( \tau \) is called its lifetime.

3.2. Existence and uniqueness of a maximal local solution: 2D and 3D cases. By using Theorem 5.15 we will prove in this subsection that the problem (3.5) has a unique maximal local solution. In order to do this, we need to establish several auxiliary results. Throughout this subsection \( d = 2, 3 \) and \( a = \frac{d}{4} \).

Lemma 3.12. There exists \( c_2 > 0 \) such that for all \( \textbf{n}_i \in \mathbb{H}^3, i = 1, 2, \)

\[
|M(\textbf{n}_1) - M(\textbf{n}_2)|_{L^2} \leq c_2 \left( \|\textbf{n}_1 - \textbf{n}_2\|_2 \|\textbf{n}_1\|^{1-a}_2 \|\textbf{n}_1\|_3^2 + \|\textbf{n}_1 - \textbf{n}_2\|^{1-a}_2 \|\textbf{n}_1 - \textbf{n}_2\|_3 \right). \tag{3.14}
\]

Proof. The proof of this result can be found in \cite[Lemma 6.4]{6} \( \square \)

Lemma 3.13. There exist \( c_3 > 0 \) such that for all \( (\textbf{v}_i, \textbf{n}_i) \in \mathcal{E}, i = 1, 2, \)

\[
\|\tilde{B}(\textbf{v}_1, \textbf{n}_1) - \tilde{B}(\textbf{v}_2, \textbf{n}_2)\|_1 \leq c_3 \left( \|\nabla (\textbf{v}_1 - \textbf{v}_2)\| \|\textbf{n}_1\|^{1-a}_2 \|\textbf{n}_1\|_3^2 + \|(\textbf{n}_1 - \textbf{n}_2)\|^{1-a}_2 \|\textbf{n}_1 - \textbf{n}_2\|_3 \right). \tag{3.15}
\]

Proof. Throughout this proof \( C > 0 \) is an universal constant which may change from one term to the other. Let \( (\textbf{v}_i, \textbf{n}_i) \in \mathcal{E}, i = 1, 2, \) and \( (\textbf{w}, \tilde{\textbf{n}}) = (\textbf{v}_1 - \textbf{v}_2, \textbf{n}_1 - \textbf{n}_2) \). Since \( \tilde{B} \) is bilinear, we have

\[
\tilde{B}(\textbf{v}_1, \textbf{n}_1) - \tilde{B}(\textbf{v}_2, \textbf{n}_2) = \tilde{B}(\textbf{w}, \textbf{n}_1) + \tilde{B}(\textbf{v}_2, \tilde{\textbf{n}}) = J_1 + J_2. \tag{3.16}
\]
Proof. The proposition is an easy consequence of Lemmata 3.12-3.14. Thus, we omit its proof.

Let \( H \) be a stochastic algebra, we infer that there exists \( c > 0 \) such that

\[
\|\nabla f(n_1) - f(n_2)\|_H \leq c_4 [1 + \|n_1\|_{2N}^2 + \|n_2\|_{2N}^2] \|n_1 - n_2\|_H, \quad \text{for all } n_1, n_2 \in X_1 \cap X_2.
\]

(3.17)

Proof. Let \( N \in I_d, k \in \{0, \ldots, N\} \) and \( f \) be as in Assumption 2.1. By the Young inequality and the fact that \( H^2 \) is an algebra, we infer that there exists \( C > 0 \) such that for all \( n_1, n_2 \in H^2 \)

\[
\|n_1\|_{2N}^2 \|n_2\|_{2N}^2 \leq C\|n_1\|_{2N}^2 \|n_2\|_{2N}^2 \leq C\|n_2\|_H (1 + \|n_1\|_{2N}^2).
\]

(3.18)

Thus, it is enough to establish (3.17) for the leading term \( a_N|n|^{2N}n \). For doing so, we have

\[
|n_1|^{2N}n_1 - |n_2|^{2N}n_2 = |n_1|^{2N}(n_1 - n_2) + n_2(|n_1| - |n_2|)(\sum_{k=0}^{2N-1} |n_1|^{2N-1-k}|n_2|^k),
\]

from which along with (3.18) we infer that (3.17) is true for the leading term \( a_N|n|^{2N}n \).

Lemma 3.14. Let Assumption 2.1 be satisfied. Then, there exists \( c_4 > 0 \) such that

\[
\|f(n_1) - f(n_2)\|_H \leq c_4 [1 + \|n_1\|_{2N}^2 + \|n_2\|_{2N}^2] \|n_1 - n_2\|_H, \quad \text{for all } n_1, n_2 \in X_1 \cap X_2.
\]

(3.17)

Proposition 3.15. Let \( \alpha = \frac{d}{4}, d = 2, 3 \). If Assumption 2.1 is satisfied, then there exists \( C_0 > 0 \) such that for all \( y_i = (v_i, n_i), \quad i = 1, 2 \)

\[
\|F(y_1) - F(y_2)\|_{Y'} \leq C_0 \|y_1 - y_2\|_{Y'}^{1-\alpha} \|y_1\|_{Y'}^\alpha \|y_2\|_{Y'}^\alpha.
\]

\[
+ c_4 \|y_1 - y_2\|_{Y'} \left[ 1 + \|y_1\|_{Y'}^2 + \|y_2\|_{Y'}^2 \right].
\]

Proof. The proposition is an easy consequence of Lemmata 3.12-3.14. Thus, we omit its proof.

Using Theorems 5.14, 5.15 and 5.17 we obtain our first main result.

Theorem 3.16. Let \( d = 2, 3, (v_0, n_0) \in L^2(\Omega; V \times X_1) \) be \( \mathcal{F}_0 \)-measurable, \( h \in H^2 \). If Assumptions 2.1, 2.3 and 2.2 are satisfied, then the problem (3.5) has a unique maximal local solution \((v, n, \tilde{\tau}_\infty)\) satisfying the following properties.

(1) Given \( R > 0 \) and \( \varepsilon > 0 \) there exists \( \tau(\varepsilon, R) > 0 \) such that if \( \mathbb{E}(\|v_0, n_0\|_{V \times X_1}^2) \leq R^2 \), then

\[
\mathbb{P}(\tilde{\tau}_\infty \geq \tau(\varepsilon, R)) \geq 1 - \varepsilon.
\]

(2) We also have

\[
\mathbb{P}\left( \{\tilde{\tau}_\infty < \infty\} \cap \left\{ \sup_{t \in [0, \tau]} (|\nabla v(t)|_{L^2} + \|n(t)\|_2) < \infty \right\} \right) = 0,
\]

(3.19)

\[
\limsup_{t \to \tilde{\tau}_\infty} |\nabla v(t)|_{L^2}^2 + \|n(t)\|_2^2 + \int_0^t (|A v(s)|_{L^2}^2 + \|n(s)\|_b^2) \, ds = \infty \text{ \( \mathbb{P} \)-a.s. on } \{\tilde{\tau}_\infty < \infty\}.
\]

(3.20)
Lemma 3.1-3.2 show that \( \{ \mathcal{S}(t) \}_{t \geq 0} \) on \( \mathcal{H} = H \times X_0 \) satisfies Assumption 5.3. Thanks to Proposition 3.15 we can infer by applying Theorems 5.14, 5.15 and 5.17 that problem (3.5) has a unique maximal local solution satisfying items (1) and (2) of Theorem 3.16.

3.3. Existence and uniqueness of global strong solution: 2D case. By using the Khashminskii test for non-explosions, see [41, Theorem III.4.1], and some arguments from [11], we prove in this section that if \( d = 2 \) then the problem (3.5) has a unique global solution.

For all \( (u, d) \in C([0, T]; H \times H^1) \cap L^2(0, T; V \times H^2) \) and \( t \in [0, T] \) we put
\[
\mathcal{E}[u, d](t) = \frac{1}{2} \left( |u(t)|^2_{L^2} + |d(t)|^2_{L^2} + |\nabla d(t)|^2_{L^2} + \int_{\mathcal{O}} F(d(t, x))dx \right) \quad (3.21)
\]
\[
\mathcal{D}[u, d](t) = |A_1^2 u(t)|^2_{L^2} + |A_1 d(t) - f(d(t))|^2_{L^2}. \quad (3.22)
\]

**Theorem 3.17.** Let \( d = 2, N \in \mathbb{N}, h \in H^2 \) and \( (v_0, n_0) \in L^2(\Omega; V \times X_1) \) such that
\[
\mathbb{E}[\mathcal{E}[v_0, d_0]^{2(4N+2)}] = \mathbb{E} \left( |v_0|^2_{L^2} + |n_0|^2_{L^2} + |\nabla n_0|^2_{L^2} + \int_{\mathcal{O}} F(n_0(x))dx \right)^{2(4N+2)} < \infty. \quad (3.23)
\]
If Assumptions 2.1, 2.2 and 2.3, then the problem (3.5) has a unique global strong solution.

The proof of this theorem is given at the end of this subsection.

**Proposition 3.18.** Let all the assumptions of Theorem 3.17 be satisfied and \( p \in [2, 2(4N + 1)] \). Also, let \( (\tau_k)_{k \in \mathbb{N}} \) be the sequence of stopping times defined by
\[
\tau_k = \inf \{ t \in [0, \infty) : |\nabla v(t)|^2_{L^2} + |n(t)|^2_{L^2} + \int_0^t (|A v(s)|^2_{L^2} + |n(s)|^2_{L^2}) ds > k^2, \quad k \in \mathbb{N}. \quad (3.24)
\]
Then, there exist an increasing function \( \varphi : [0, \infty) \to (0, \infty) \) and \( \kappa_0 = \kappa_0(p, \|h\|_{W^{1, 4}}) > 0 \) such that for all \( k \in \mathbb{N} \)
\[
\mathbb{E} \sup_{t \in [0, T]} (\mathcal{E}[v, n](t \wedge \tau_k))^p + \mathbb{E} \left[ \int_0^{T \wedge \tau_k} \left( \mathcal{D}[v, n](s) - \frac{a_{N+1}}{2} |n(s)|^{2N+2}_{L^{2N+2}} \right) ds \right] \leq \kappa_0 \varphi(T) \left( 1 + \mathbb{E}[\mathcal{E}[v, n](0)]^p \right). \quad (3.25)
\]

**Proof.** The proof of this proposition will be given in Section 4. □

Hereafter, we set
\[
\mathcal{C}_0 = \kappa_0 \varphi(T)(1 + \mathbb{E}[\mathcal{E}[v, n](0)]^{2(4N+2)}) \quad (3.26)
\]

**Corollary 3.19.** Let all the assumptions of Proposition 3.18 be satisfied. Then, there exists \( C > 0 \) such that for all \( k \in \mathbb{N} \)
\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_k} |n(s)|^2_{L^2} \right]^2 \leq C(\mathcal{C}_0 + 1). \quad (3.27)
\]

**Proof.** By (2.21) and \( H^1 \hookrightarrow L^{4N+2} \), which is valid for \( d = 2 \), there exists a constant \( C > 0 \) such that
\[
|n|^2_{L^4} \leq C(|A_1 n - f(n)|^2_{L^2} + |n|^{4N+2}_{L^1} + 1), \quad (3.28)
\]
from which along with (3.25) we conclude the proof of the corollary. □
Let $\Psi_1 : H^2 \to [0, \infty)$, $\Psi_2 : D(A) \to [0, \infty)$ and $\Psi : D(A) \times H^2 \to [0, \infty)$ be defined by
\begin{align}
\Psi_1(d) &= \frac{1}{2} |A_1 d - f(d)|_{L^2}^2, \quad d \in H^2, \\
\Psi_2(u) &= \frac{1}{2} |\nabla u|_{L^2}^2, \quad u \in D(A), \\
\Psi(u, d) &= \Psi_1(d) + \Psi_2(u), \quad (u, d) \in D(A) \times H^2.
\end{align}

(3.29) (3.30) (3.31)

Hereafter, $\Psi_i'$ and $\Psi_i''$, $i = 1, 2$, are the first and second Fréchet derivatives of $\Psi_i$, $i = 1, 2$.

**Lemma 3.20.** There exists $\kappa_1 > 0$ such that for all $d \in H^3$ and $u \in D(A)$ we have
\begin{equation}
-\Psi_1'(d)[u \cdot \nabla d] \leq \kappa_1 \Psi(u, d) \left[ \|d\|_1^2 + 1 \right] \|d\|_{L^2} + \frac{1}{4} |Au|_{L^2}^2 + \frac{1}{6} |\nabla(A_1 d - f(d))|_{L^2}^2.
\end{equation}

(3.32)

**Proof.** In this proof $C > 0$ is an universal constant which may change from one term to the other. Let $d \in H^3 \cap D(A_1)$ and $u \in D(A)$. Observe that
\begin{equation}
\Psi_1'(d)[g] = \langle \Delta d - f(d), A_1 g + f'(d)[g] \rangle \text{ for all } g \in H^2,
\end{equation}

(3.33)

and
\begin{align}
-\Delta (u \cdot \nabla d) &= -(\Delta u \cdot \nabla) d - (u \cdot \nabla) \Delta d - 2\text{tr}(\nabla u \nabla^2) d \\
&= -(\Delta u \cdot \nabla) d - 2\text{tr}(\nabla u \nabla^2) d + (u \cdot \nabla)[\Delta d - f(d)] - f'(d)[u \cdot \nabla d]
\end{align}

Hence, by using the identities (2.13) and (3.33) we obtain
\begin{equation}
-\Psi_1'(d)[u \cdot \nabla d] = \langle A_1 d - f(d), (\Delta u \cdot \nabla) d \rangle + 2 \langle A_1 d - f(d), \text{tr}(\nabla u \nabla^2) d \rangle,
\end{equation}

(3.34)

which along with (2.1) and the Hölder and Young inequalities imply that
\begin{align}
-\Psi_1'(d)[u \cdot \nabla d] &\leq C |A_1 d - f(d)|_{L^4} \|Au|_{L^2} \|
abla d|_{L^4} + |\nabla u|_{L^4} \|\nabla^2 d|_{L^2} \\
&\leq \frac{1}{6} |\nabla(A_1 d - f(d))|_{L^2}^2 + \frac{1}{4} |Au|_{L^2}^2 + C |A_1 d - f(d)|_{L^2}^2 \|\nabla d|_{L^4}^4 + |\nabla^2 d|_{L^2}^2.
\end{align}

(3.35)

Now, (3.32) easily follows from the last line of the the above chain of inequalities.

**Lemma 3.21.** There exists $\kappa_2 > 0$ such that for all $d \in H^3 \cap D(A_1)$ and $u \in D(A)$ we have
\begin{equation}
\langle f'(d)[A_1 d - f(d)], A_1 d - f(d) \rangle \leq \frac{1}{6} |\nabla(A_1 d - f(d))|_{L^2}^2 + \kappa_2 \Psi(u, d)(1 + \|d\|_{1^N}^4).
\end{equation}

(3.36)

**Proof.** Using part (ii) of Remark 2.4, (2.1), the Hölder and Young inequalities, and by $H^1 \hookrightarrow L^{4N}$ (that is valid for $d = 2$) we infer that there exist $C > 0$ and $\kappa_2 > 0$ such that
\begin{align}
\langle f'(d)[A_1 d - f(d)], A_1 d - f(d) \rangle &\leq c_1 \int_{\Omega} (1 + \|d\|_{2^{4N}}^2) |A_1 d - f(d)|^2 dx \\
&\leq C |A_1 d - f(d)|_{L^{4N}}^2 (1 + \|d\|_{2^{4N}}^2)^2 \\
&\leq \frac{1}{6} |\nabla(A_1 d - f(d))|_{L^2}^2 + \kappa_2 |A_1 d - f(d)|_{L^2}^2 (1 + \|d\|_{1^N}^4),
\end{align}

(3.37)

for all $d \in H^3 \cap D(A_1)$ and $u \in D(A)$. This completes the proof of the lemma.

**Lemma 3.22.** Let $h \in H^2$. Then, there exists $\kappa_5 = \kappa_5(\|h\|_2) > 0$ such that
\begin{equation}
|\Psi''_i(d)[d \times h, d \times h]| + |\Psi'_i(d)[(d \times h) \times h]| \leq \Psi_1(d) + \kappa_5(1 + \|d\|_{1^N}^4)\|d\|_{L^2}^2, \text{ for all } d \in D(A_1).
\end{equation}

(3.38)
**Proof.** Let \((u, d) \in D(A) \times D(A_1)\). We firstly recall that

\[
\Psi'_1(d)[g, p] = \langle A_1 d - f(d), -f'(d)[g, p] \rangle + \langle A_1 p - f'(d)[p], A_1 g - f'(d)[g] \rangle, \quad p, g \in D(A_1).
\]

Hence, by recalling that \(G(d) = d \times h\) we have

\[
\begin{align*}
\Psi''_1(d)[G(d), G(d)] &= |A_1(G(d)) - f'(d)[G(d)]|^2 \leq 2 \frac{1}{4} |A_1 d - f(d)|^2_L + |A_1(G(d)) - f'(d)[G(d), G(d)]|^2_L \\
&= I_1 + I_2 + I_3. \tag{3.39}
\end{align*}
\]

Secondly, by the part (ii) of Remark 2.4, \(H^2 \hookrightarrow L^\infty, H^1 \hookrightarrow L^4, L^{4N}\) and the Hölder and Young inequalities we infer that there exists a constant \(C = C(||h||_2) > 0\) such that

\[
I_2 \leq |\Delta(d \times h)|^2 + ||d \times h|^2|f''(d)||^2_L \
\leq 4 \left( |\Delta d \times h|^2 + 2|\nabla d \times \nabla h|^2 + |d \times \Delta h|^2 \right) + C(1 + |d|^{4N})||d \times h|^2 \tag{3.40}
\]

In a similar way, we prove that there exists a constant \(C_7 = C(||h||_2) > 0\) such that

\[
I_3 \leq C||d||^2(1 + ||d||_4^4). \tag{3.41}
\]

Combining this last inequality with (3.40) and (3.41) proves that

\[
|\Psi''_1(d)[G(d), G(d)]| \leq \frac{1}{2} \Psi_1(d) + C_7||d||^2(1 + ||d||_4^4).
\]

In a similar way, we can also show that \(|\Psi'_1(d)[d \times h]| \leq \frac{1}{2} \Psi_1(d) + C_7||d||^2(1 + ||d||_4^4)\). One easily conclude the proof of the lemma from the last two estimates. □

**Lemma 3.23.** Let \(h \in H^2\). Then, there exists \(\kappa_5 = \kappa_5(||h||_2) > 0\) such that for all \(d \in D(A_1)\)

\[
|\Psi'_1(d)[d \times h]| \leq \kappa_6 \left[ 1 + \Psi_1(d) + ||d||_4^{4N+1} + ||d||_1 ||d||_2 \right]. \tag{3.42}
\]

**Proof.** By part (ii) of Remark 2.4, (2.1), (2.2), (2.3), the Hölder and Young inequalities and \(H^1 \hookrightarrow L^{4N+2}, L^4\), we infer that there exists \(\kappa_6 = \kappa_6(||h||_2) > 0\) such that for all \(d \in D(A_1)\)

\[
|\Psi'_1(d)[d \times h]| \leq |A_1 d - f(d)|_L^2 \left( \left| [A_1 d - f(d)] \times h + 2\nabla d \times \nabla h + d \times \Delta h - f(d) \times h \right|_L^2 \right) \\
\leq \kappa_6 \left( \Psi_1(d) + ||\nabla d||_4^2 + ||d||_L^4 + ||d||_L^{4N+2} \right),
\]

This completes the proof of the Lemma 3.23. □

We will also need to the following results.

**Lemma 3.24.** There exists \(\kappa_3 > 0\) such that for all \(u \in D(A)\) and \(d \in D(A_1)\)

\[
-\Psi'_2(u)(B(u, u)) = -\langle B(u, u), Au \rangle \leq \frac{1}{4} |Au|^2 + \kappa_3(|u|^2 \nabla u)^2 \Psi(u, d). \tag{3.43}
\]

**Proof.** Using (2.1), the Hölder and Young inequalities we infer that \(C > 0\) such that

\[
\langle B(u, u), Au \rangle = \langle u \cdot \nabla u, Au \rangle \leq |Au| ||u||_L^4 ||\nabla u||_L^4 \\
\leq |Au|^2 ||u||_L^2 ||\nabla u||_L^2 \leq \frac{1}{4} |Au|^2 + C||u|^2 \nabla u)^2 \Psi_1(u), \tag{3.44}
\]

for all \((u, d) \in D(A) \times D(A_1)\). We easily conclude the proof of Lemma 3.24 from last line. □
Lemma 3.25. There exists $\kappa_4 > 0$ such that
\[ -\Psi_2'(u)(M(d, d)) = -\langle M(d, d), Au \rangle \leq \frac{1}{4}|A|^2 + \frac{1}{6}|\nabla(A_1 d - f(d))|^2 + \kappa_4 \Psi(u, d)||d||_1^2||d||_2^2, \]
for all $d \in D(A_1)$ satisfying $\nabla(A_1 d + f(d)) \in L^2$, and $u \in D(A)$.

Proof. In this proof $C > 0$ is an universal constant. Let $d \in D(A_1)$ be such that $\nabla(A_1 d + f(d)) \in L^2$, and $u \in D(A)$. Firstly, since $\Pi : L^2 \to H$ is self-adjoint, $\nabla F(d) = \nabla f(d)$ and div $Au = 0$, we infer that
\[ \langle M(d, d), Au \rangle = \frac{1}{2}\langle Au, \nabla|\nabla d|^2 \rangle - \langle Au, \nabla d\rangle \]
(3.47)
\[ = -\langle Au \cdot \nabla d, A_1 d + f(d) \rangle + \langle Au, \nabla F(d) \rangle. \]
Secondly, applying the Hölder, the Gagliardo-Nirenberg and the Young inequality yields
\[ \langle M(d, d), Au \rangle = -\langle Au \cdot \nabla d, A_1 d + f(d) \rangle \leq |Au||A_1 d - f(d)||d||_2 \]
(3.48)
\[ \leq \frac{1}{4}|A|^2 + \frac{1}{6}|\nabla(A_1 d - f(d))|^2 + C|A_1 d - f(d)||d||_2. \]
This completes the proof of Lemma 3.25.

Now, let $\kappa_i, i = 1, \ldots, 4$ be the constants from Lemmata 3.20, 3.21, 3.24 and 3.25. For all $t \geq 0$ we set
\[ \Phi(t) = \exp \left( -\int_0^t \left[ (\kappa_1 + \kappa_4)(1 + ||n(s)||_1^2)||n(s)||_2^2 + \kappa_2(1 + ||n(s)||_1^2) + \kappa_3|\nabla n(s)||_1^2 ||\nabla n(s)||_2^2 \right] ds \right). \]
(3.49)
Let $C_0 > 0$ be the constant defined in (3.26) and $C_1 > 0$ the constant defined by
\[ C_1 = \Psi(v_0, n_0) + C_0 + 1. \]
(3.50)

Proposition 3.26. Let $\Psi_1, \Phi, C_0$ and $C_1$ be defined in (3.29), (3.49), (3.26) and (3.50), respectively. Let $(\tau_k)_{k \in N}$ be the sequence of stopping times defined in (3.24). Let $d = 2, N \in N$ and $h \in H^2$.

If all the other assumptions of Theorem 3.17 are satisfied, then there exists an increasing function $\psi : [0, \infty) \to (0, \infty)$ such that for all $k \in N$
\[ \mathbb{E} \sup_{s \in [0, T]} \Phi(s \wedge \tau_k) \left( |\nabla n(s \wedge \tau_k)|^2 + \Psi_1(n(s \wedge \tau_k)) \right) \leq \kappa_4 \psi(T) C_1, \]
(3.51)
\[ \mathbb{E} \int_0^{T \wedge \tau_k} \Phi(s) \left( |\nabla n(s)|^2 + |\nabla f(n(s))|^2 \right) ds \leq \kappa_4 \psi(T) C_1. \]
(3.52)

Proof. The proof of this proposition will be given in Section 4.

Corollary 3.27. Under all the assumptions of Proposition 3.26, there exists a $C > 0$ such that for all $k \in N$
\[ \mathbb{E} \int_0^{T \wedge \tau_k} \Phi(s)||n(s)||_2^3 ds \leq C(C_0 + C_1 + 1). \]
(3.53)

Proof. By part (ii) of Remark 2.4, the Hölder inequality and $H^1 \hookrightarrow L^{2N} \hookrightarrow L^4$ we infer that
\[ |\nabla f(n)|^2 \leq |f'(n)||\nabla n|^2 \leq C((1 + |n|^{2N})||\nabla n||_2^2) \]
\[ \leq |\nabla n||_4^2 + |n||_1^{4N} ||\nabla n||_1^4 \leq C(||n||_2^2 + ||n||_1^{8N+2}). \]
With this at hand we complete the proof by using (3.27), (3.52) and the fact
\[ \|n\|_3^2 \leq \|n\|_2^2 + 2|\nabla(\Delta n + f(n))|_{L^2}^2 + 2|\nabla f(n)|_{L^2}^2. \]

After all these preparations we now proceed to the promised proof of Theorem 3.17.

**Proof of Theorem 3.17.** By Theorem 3.16 the problem (3.5) has a unique maximal local solution \((\mathbf{v}, n; \tau_\infty)\). We shall prove that \(\mathbb{P}\left(\tau_\infty < \infty\right) = 0\). For this aim, let \(\{\tau_k; k \in \mathbb{N}\}\) be the sequence of stopping times defined in (3.24). We first establish the following chain of inequalities
\[ \mathbb{P}(\tau_k < t) \leq \mathbb{E}\left(1_{\{\tau_k < t\}} \Phi(t \wedge \tau_k) \left(\|\nabla\mathbf{v}(t \wedge \tau_k)\|_{L^2}^2 + \|\mathbf{n}(t \wedge \tau_k)\|_2^2\right)\right) + \mathbb{E}\left(1_{\{\tau_k < t\}} \int_{t \wedge \tau_k}^{t \wedge \tau_k} 1_{\{\tau \leq k^2\}} e^{-\int_{0}^{\tau} \phi(s)ds} d\tau\right) + \mathbb{E}\left(1_{\{\tau_k < t\}} \int_{t \wedge \tau_k}^{t \wedge \tau_k} 1_{\{\tau \geq 2 \log k\}} e^{-\int_{0}^{\tau} \phi(s)ds} d\tau\right) =: I + II. \]

Now, we estimate I and II separately. Firstly, from the definition of \(\tau_k\) and \(\Phi\) we have
\[ I \leq \frac{1}{k^2} \mathbb{E}\left[1_{\{\tau_k < t\}} \Phi(t \wedge \tau_k) \left(\|\nabla\mathbf{v}(t \wedge \tau_k)\|_{L^2}^2 + \|\mathbf{n}(t \wedge \tau_k)\|_2^2\right)\right]. \]

Secondly, we estimate II as follows
\[ II = \mathbb{E}\left(1_{\{\tau_k < t\}} \int_{t \wedge \tau_k}^{t \wedge \tau_k} \phi(r) dr\right) \leq \int_{\{\tau_k < t\}} \int_{t \wedge \tau_k}^{t \wedge \tau_k} \phi(r) dr d\mathbb{P} \leq \frac{1}{2 \log k} \int_{\{\tau_k < t\}} \int_{0}^{t \wedge \tau_k} \phi(r) dr d\mathbb{P}. \]

Now, from (3.25) and (3.27) we infer that there exists a constant \(C > 0\) such that for all \(k \in \mathbb{N}\)
\[ I \leq \frac{1}{k^2} C(\mathcal{C}_0 + \mathcal{C}_1 + 1). \]

Thus, there exists a constant \(C > 0\) such that for all \(k \in \mathbb{N}\)
\[ II \leq \frac{1}{2 \log k} \mathbb{E}\left(\int_{0}^{t \wedge \tau_k} \phi(s) ds\right) \leq \frac{C(\mathcal{C}_0 + 1)}{\log k}. \]
Collecting the information about I and II together, we infer that
\[
\lim_{k \to \infty} \mathbb{P}(\tau_k < t) \leq \lim_{k \to \infty} C(\mathcal{C}_0 + \mathcal{C}_1 + 1) \left[ \frac{1}{k^2} + \frac{1}{\log k} \right] = 0.
\]
Now, we easily infer from the last estimate and part (2) of Theorem 3.16 that \( \mathbb{P}(\tilde{\tau}_\infty < \infty) = 0. \) This completes the proof of Theorem 3.17.

4. Basic estimates for the solution \((v, n)\)

Throughout this section, \(((v, n); \tilde{\tau}_\infty)\) is the maximal local solution to problem (3.5) from Theorem 3.16 and \( \{\tau_k : k \in \mathbb{N}\} \) is the sequence defined in (3.24). The first and second subsections are devoted to the proofs of Proposition 3.18 and Proposition 3.26, respectively.

4.1. Proof of Proposition 3.18. Before proceeding to the actual proof of Proposition 3.18 we state and prove the following result.

**Lemma 4.1.** Let \( h \in W^{1,4} \). Then, there exists \( C = C(\|h\|_{W^{1,4}}) > 0 \) such that for all \( d \in H^1 \)
\[
|\nabla G(d)|^2_{L^2} + \langle \nabla d, \nabla G^2(d) \rangle \leq C\|d\|^2_{L^4}.
\]

**Proof.** Let \( h \in W^{1,4} \). Using the Hölder inequality, the embeddings \( H^1 \hookrightarrow L^4 \) and \( W^{1,4} \hookrightarrow L^\infty \), and
\[
a \cdot ((b \times c) \times d) = -(a \times d) \cdot (b \times c), \quad \forall \ a, b, c, d \in \mathbb{R}^3,
\]
we infer that for all \( d \in H^1 \) and \( i \in \{1, 2\} \)
\[
|\partial_i(d \times h)|^2_{L^2} + \langle \partial_i d, \partial_i ((d \times h) \times h) \rangle = |d \times \partial_i h|^2_{L^2} + \langle \partial_i d \times h, d \times \partial_i h \rangle + \langle \partial_i d, (d \times h) \times \partial_i h \rangle 
\]
\[
\leq |\partial_i d|^2_{L^4} |\partial_i h|^2_{L^4} + |\partial_i d|^2_{L^2} |d|_{L^4} |h|_{L^\infty} |\partial_i h|_{L^4} \leq C\|d\|^2_{L^4} \|h\|^2_{W^{1,4}}.
\]
Hence, summing over \( i \) from 1 to 2 imply the desired inequality (4.1).

We now give the promised of the proposition.

**Proof of Proposition 3.18.** Without loss of generality (Wlog) we only give a proof for \( p = 2(4N + 1) \). Let \( h \in W^{1,4} \). Throughout this proof \( C = C(\|h\|_{W^{1,4}}) > 0 \) is a constant which may change from one term to the next one. Let \( k \in \mathbb{N} \) be fixed and \( \tau_k \) be defined by (3.24).

Firstly, let \( \Lambda : H^1 \to [0, \infty) \) be the map defined by
\[
\Lambda(d) = \frac{1}{2} |d|^2_{L^2} + |\nabla d|^2_{L^2} + \frac{1}{2} \int_{\mathcal{O}} F(|d(x)|^2) dx, \quad d \in H^1.
\]

By Assumption 2.1 and [12, Lemma 8.10] the map \( \Lambda(\cdot) \) is twice Fréchet differentiable. Moreover, elementary calculations and (4.2) imply
\[
\Lambda'(d)[G(d)] = \langle \nabla d, d \times \nabla h \rangle,
\]
\[
\frac{1}{2} \Lambda''(G^2(d)) + \frac{1}{2} \Lambda''(G(d), G(d)) = \frac{1}{2} |\nabla G(d)|^2_{L^2} + \frac{1}{2} \langle \nabla d, \nabla G^2(d) \rangle.
\]

We also observe that if \( u \in V \) such that \( \text{div} \ u = 0 \), then
\[
\langle u \cdot \nabla d, f(d) \rangle = \frac{1}{2} \int_{\mathcal{O}} u(x) \cdot \nabla F(d(x)) dx = 0.
\]

(4.6)
Secondly, applying the Itô formula to $\frac{1}{2} |v(t \wedge \tau_k)|_{L^2}^2 + \Lambda(n(t \wedge \tau_k))$ and using (2.13), (2.10), (2.17), (4.4), (4.5) and (4.6) yield

\[
\mathcal{E}[v, n](t \wedge \tau_k) - \mathcal{E}[v, n](0) + \int_0^{t \wedge \tau_k} \mathcal{D}[v, n](s) \, ds - \int_0^{t \wedge \tau_k} \langle n(s), f(n(s)) \rangle \, ds - \frac{1}{2} \int_0^{t \wedge \tau_k} |\nabla G(n(s))|^2 \, ds \, ds
\]

\[
= \frac{1}{2} \int_0^{t \wedge \tau_k} \langle \nabla n(s), \nabla G^2(n(s)) \rangle \, ds + \int_0^{t \wedge \tau_k} \langle v(s), S(v(s))dW_1(s) \rangle + \int_0^{t \wedge \tau_k} \langle \nabla n(s), n(s) \times \nabla h \rangle dW_2.
\]

Before proceeding further, we should observe that thanks to Assumption 2.1 and [12, Lemma 8.7] we infer that there exists $c > 0$ independent of $k$ such that

\[
-\frac{a_{N+1}}{2} \int_\mathcal{O} |n(x)|^{2N+2} dx - c \int_\mathcal{O} |n(x)|^2 dx \leq (-f(n), n).
\]

Hence, plugging this inequality and (4.1) into (4.7) implies

\[
\mathcal{E}[v, n](t \wedge \tau_k) - \mathcal{E}[v, n](0) + \int_0^{t \wedge \tau_k} \mathcal{D}[v, n](s) \, ds - \frac{a_{N+1}}{2} \int_0^{t \wedge \tau_k} |n(s)|^{2N+2} \, ds
\]

\[
\leq \int_0^{t \wedge \tau_k} \langle v(s), S(v(s))dW_1(s) \rangle + \int_0^{t \wedge \tau_k} \langle \nabla n(s), n(s) \times \nabla h \rangle dW_2 + \frac{1}{2} \int_0^{t \wedge \tau_k} \mathcal{E}[v, n](s) \, ds.
\]

Thirdly, by taking the supremum over $s \in [0, t]$, raising to the power $p$, taking the mathematical expectation to both sides of the above inequality and applying the Hölder inequality we obtain

\[
\mathbb{E} \sup_{s \in [0, t]} |\mathcal{E}[v, n](s \wedge \tau_k)|^p + \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \left( |\mathcal{D}[v, n](s)| - \frac{a_{N+1}}{2} |n(s)|^{2N+2} \right) \, ds \right]^p
\]

\[
\leq C\mathbb{E}|\mathcal{E}[v, n](0)|^p + Ct^{p-1}\mathbb{E} \int_0^{t \wedge \tau_k} |\mathcal{E}[v, n](s)| \, ds + \mathbb{E} \sup_{s \in [0, t]} \left[ \int_0^{s \wedge \tau_k} \langle v(s), S(v(s))dW_1(s) \rangle \right]^p
\]

\[
+ C\mathbb{E} \sup_{s \in [0, t]} \left[ \int_0^{s \wedge \tau_k} \langle \nabla n(s), n(s) \times h \rangle dW_2(s) \right]^p.
\]

For the time being let us assume that there exists $C = C(p, \|h\|_{W^{1,4}}) > 0$ such that

\[
\mathcal{M}(t \wedge \tau_k) := C\mathbb{E} \sup_{s \in [0, t]} \left| \int_0^{s \wedge \tau_k} \langle v(s), S(v(s))dW_1(s) \rangle \right|^p + \mathbb{E} \sup_{s \in [0, t]} \left| \int_0^{s \wedge \tau_k} \langle \nabla n(s), n(s) \times h \rangle dW_2(s) \right|^p
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} |\mathcal{E}[v, n](s \wedge \tau_k)|^p + C_1 t^{p \frac{2}{3}} + C_1 t^{p \frac{p-2}{2}} \mathbb{E} \int_0^{t \wedge \tau_k} |\mathcal{E}[v, n](s)| \, ds,
\]

from which along with (4.8) we infer that there exists $C = C(p, \|h\|_{W^{1,4}}) > 0$ such that

\[
\mathbb{E} \sup_{s \in [0, t]} |\mathcal{E}[v, n](s \wedge \tau_k)|^p + 2\mathbb{E} \left[ \int_0^{t \wedge \tau_k} \left( |\mathcal{D}[v, n](s)| - \frac{a_{N+1}}{2} |n(s)|^{2N+2} \right) \, ds \right]^p
\]

\[
\leq C\mathbb{E}|\mathcal{E}[v, n](0)|^p + Ct^{p \frac{2}{3}} + C(t^{p-1} + t^{p-2}) \mathbb{E} \int_0^{t \wedge \tau_k} |\mathcal{E}[v, n](s)| \, ds.
\]

We then apply the Gronwall lemma and obtain the desired (3.25).

Thus, it remains to prove (4.9). For this purpose, by applying the Burkholder-Davis-Gundy (BDG),
Let $\mathcal{M}(t \wedge \tau_k) \leq C_4 t^{\frac{p-2}{2}} \mathbb{E} \left[ \int_0^{t \wedge \tau_k} |\mathbf{v}(s)|^2 V_{1,2} |S(\mathbf{v}(s))|^2 s |n(s) \times \nabla h_{1,2}^p \right]^\frac{p}{2} \\
\leq C_4 t^{\frac{p-2}{2}} \mathbb{E} \left[ \int_0^{t \wedge \tau_k} |\mathbf{v}(s)|^2 V_{1,2} |S(\mathbf{v}(s))|^2 s |n(s) \times \nabla h_{1,2}^p \right]^\frac{p}{2} ds \\
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0,T]} |\mathcal{E}[\mathbf{v},n](s \wedge \tau_k)|^p + C_{5} \mathbb{E} \left[ \int_0^{t \wedge \tau_k} (1 + |\mathbf{v}(s)|^2 V_{1,2} |n(s)|^2 V_{1,2} |\nabla h_{1,2}^p|^2) ds \right].$

The last line and $\mathbf{H}^1 \hookrightarrow \mathbf{L}^4$ imply (4.9). This completes the proof of Proposition 3.18.

### 4.2. Proof of Proposition 3.26

Before giving the promised proof we firstly state and prove two important lemmata. The first one is inspired by [13, Lemma 6.3].

**Lemma 4.2.** Let $V_1, H_1, \tilde{V}_1$ be two separable Hilbert spaces such that the embeddings $V_1 \hookrightarrow H_1 \hookrightarrow \tilde{V}_1$ are dense and continuous. Let $A : V_1 \rightarrow \tilde{V}_1$ be a bounded linear map and $\mathbf{f} : [0,T] \rightarrow H_1$ and $g : [0,T] \rightarrow V_1$ measurable and progressively measurable respectively such that

$$\mathbb{E} \int_0^T \left[ |\mathbf{f}(t)|^2 V_1 + |g(t)|^2 H_1 \right] dt < \infty. \quad (4.11)$$

Let $x : [0,T] \times \Omega \rightarrow V_1$ be a progressively measurable and $H_1$-continuous process such that

$$\mathbb{E} \int_0^T |x(s)|^2 V_1 |d{s}| < \infty, \quad (4.12)$$

$$x(t) = x(0) + \int_0^t A x(s) |d{s}| + \int_0^t \mathbf{f}(s) |d{s}| + \int_0^t g(s) |dW_2(s)| \quad \text{for all } t \ \mathbb{P}\text{-a.s..} \quad (4.13)$$

Now, let $V_2, H_2$ be two separable Hilbert spaces and $V_2^*$ the dual of $V_2$. We identify $H_2$ with its dual and we assume that the embeddings $V_2 \hookrightarrow H_2 \leftrightarrow V_2^*$ are continuous and dense. Let $B : V_2 \rightarrow V_2^*$ be a bounded linear map. Let $L : \tilde{V}_1 \rightarrow V_2^*$ be a twice Fréchet differentiable map such that:

1. $L(V_1) \subset V_2$ and $L(H_1) \subset H_2$,

2. there exists $\mathcal{H} : V_1 \rightarrow H_2$ such that for every $z \in V_1$

$$\mathcal{L}(z)[A z] = B L(z) + \mathcal{H}(z). \quad (4.14)$$

3. The map $L''$ is bounded on balls.

Then for every $t \in [0,T]$, $\mathbb{P}$-a.s. the following identity holds in $V_2^*$

$$L(x(t)) = L(x(0)) + \int_0^t B L(x(s)) |d{s}| + \int_0^t \left( L'(x(s)) |\mathbf{f}(s)| + \mathcal{H}(x(s)) \right) |d{s}| \quad (4.15)$$

$$\quad + \frac{1}{2} \int_0^t L''(x(s)) |g(s), g(s)| |d{s}| + \int_0^t L'(x(s)) |g(s), g(s)| |dW_2(s)|.$$
Using (4.14) yields
\[
L\varphi(x(t)) = L\varphi(x(0)) + \int_0^t \int \langle B(x(s)), \varphi \rangle V_2 \, ds + \int_0^t \int \langle L'(x(s))[f(s)] + H(x(s)), \varphi \rangle V_2 \, ds
+ \frac{1}{2} \int_0^t \int_2 \langle L''(x(s))[g(s), g(s)], \varphi \rangle \, ds + \int_0^t \int_2 \langle L'(x(s))[g(s)], \varphi \rangle \, dW_2(s).
\]
This completes the proof of (4.15). □

**Lemma 4.3.** Let \( h \in H^2 \) and \( \{y(t) : t \in [0, \bar{\tau}_x)\} \) be the local process defined by
\[
y(t) := A_1 n(t) - f(n(t)), \quad t \in [0, \bar{\tau}_x).
\]
Then, for all \( t \in [0,T], \ k \in \mathbb{N}, \ \mathbb{P} \text{-a.s.} \) the following equation holds in \((H^1)^*\)
\[
y(t \wedge \tau_k) + \int_0^{t \wedge \tau_k} \left( A_1 y(r) + (A_1 - f'(n(r))) [\nu(r) \cdot \nabla n(r)] \right) \, dr - \int_0^{t \wedge \tau_k} (A_1 - f'(n(r)))[G(n(r))] \, dW_2
= y(0) + \frac{1}{2} \int_0^{t \wedge \tau_k} \left( 2f'(n(r))y(r) + (A_1 - f'(n(r)))[G^2(n(r))] - f''(n(r))[G(n(r)), G(n(r))] \right) \, dr.
\]

**Proof of Lemma 4.3.** Let \( h \in H^2 \). Let us put \( V_1 = D((I + A_1)\frac{3}{2}), \ H_1 = D(A_1), \ \tilde{V}_1 = H^1 \) and \( A = B = -A_1 \). We also set \( V_2 = H^1, \ H_2 = L^2, \ V_2^* = (H^1)^* \). The map \( L : H^1 \ni z \mapsto L(z) := A_1 z - f(z) \in (H^1)^* \) satisfies the assumptions of Lemma 4.2. In particular, if we set \( \mathcal{H}(z) = A_1 f(z) - f'(z)[A_1 z], \ z \in V_1, \) then
\[
L'(z)[A_1 z] = A_1[A_1 z - f(z)] + A_1 f(z) - f'(z)[A_1 z] = A_1 L(z) + \mathcal{H}(z) \in (H^1)^*.
\]
Now, let \( k \in \mathbb{N} \) and
\[
f = 1_{[0, \tau_k]} \left(-\nu \cdot \nabla n + f(n) + \frac{1}{2} G^2(n)\right), \quad (4.18)
g = 1_{[0, \tau_k]} G(n).
\]
By Lemmata 3.13 and 3.14, and the definition of \( \tau_k \) we infer that there exists \( C > 0 \) such that
\[
\mathbb{E} \int_0^t |f(r)|^2_{H^1} \leq \mathbb{E} \int_0^{t \wedge \tau_k} \left( |\nu(r) \cdot \nabla n(r)|^2_{H^1} + |f(n(r))|^2_{H^1} + \frac{1}{2} (n(r) \times h) \times h^2_{H^1} \right) \, dr \leq C, \quad (4.20)
\]
\[
\mathbb{E} \int_0^{t \wedge \tau_k} |G(n(r))|^2_{H^2} \, dr = \mathbb{E} \int_0^{t \wedge \tau_k} |n(r) \times h^2_{H^2} \, dr \leq C.
\]
These mean that \( f \) and \( g \) satisfy (4.11). Because the local strong solution \( y = (\nu, n) \) of (3.5) satisfies (3.8), the process \( x(t) = n(t \wedge \tau_k) \) satisfies (4.12) and (4.13) with \( f \) and \( g \) as defined above. By setting \( y(t \wedge \tau_k) = L(n(t \wedge \tau_k)), \ t \geq 0 \), and applying Lemma 4.2 we obtain
\[
y(t \wedge \tau_k) = y(0) + \int_0^{t \wedge \tau_k} \left( -A_1 y(r) + L'(n(r))[f(r)] + \mathcal{H}(\nu(r)) \right) \, dr + \frac{1}{2} \int_0^{t \wedge \tau_k} L''(n(r))[g(r), g(r)] \, dr + \int_0^{t \wedge \tau_k} L'(n(r))[g(r)] \, dW_2(r).
\]
We complete the proof of the lemma by taking into account the last line and the following identity
\[
L'(z) = A_1 - f'(z) \quad \text{and} \quad L''(z) = -f''(z) \quad \text{for every} \ z \in H^1.
\]
We now give the promised proof of Proposition 3.26.

Proof of Proposition 3.26. Throughout, \( L(z) = A_1z - f(z) \) and \( \mathcal{H}(z) = A_1f(z) - f'(z)[A_1z] \) be defined as in the proof of Lemma 4.3. Keeping in mind the notations of 4.3, in particular (4.18), (4.19) and (4.23), we set

\[
v = -\mathbf{1}_{[0, \tau_k]}A_1L(n) + L'(n)[f] + \mathbf{1}_{[0, \tau_k]}\mathcal{H}(n) + \frac{1}{2}L''(n)[g, g], \quad k \in \mathbb{N}.
\] (4.24)

Then, for all \( k \in \mathbb{N} \) and \( F \in L^2(\Omega \times [0, t]; \mathbb{L}^2) \), \( t \geq 0 \),

\[
\mathbb{E} \int_0^{t \wedge \tau_k} |f'(n(r))[F(r)]|_{L_2}^2 \, dr \leq c_1^2(1 + k^2) \mathbb{E} \int_0^{t \wedge \tau_k} |f'(n(r, x))[F(r, x)]|^2 \, dx \, dr
\]

\[
\leq c_1^2 \mathbb{E} \int_0^{t \wedge \tau_k} (1 + |n(r, x)|^{4N}) |F(r, x)|^2 \, dx \, dr
\]

\[
\leq c_1^2 \mathbb{E} \left[ 1 + \sup_{t \in [0, T]} |n(t \wedge \tau_k)|^{4N} |H_r|^2 \right] \int_0^{t \wedge \tau_k} |F(r)|_{L_2}^2 \, dr
\]

\[
\leq c_1^2(1 + k^{2N}) \mathbb{E} \int_0^{t \wedge \tau_k} |F(r)|_{L_2}^2 \, dr.
\] (4.25)

In fact, from Remark 2.4(ii), the embedding \( \mathbb{H}^2 \hookrightarrow \mathbb{L}^\infty \) and (4.20) we infer that

\[
\mathbb{E} \int_0^{t \wedge \tau_k} |f''(n(r))[g(r), g(r)]|_{L_2}^2 \leq c_2^2 \mathbb{E} \int_0^{t \wedge \tau_k} |h|_{L_2}^2(1 + k^{2N}) \mathbb{E} \int_0^{t \wedge \tau_k} |n(r)|_{L_2}^2 \, dr.
\] (4.26)

From the continuity of the linear map \( A_1 : \mathbb{H}^1 \rightarrow (\mathbb{H}^1)^* \), the embedding \( \mathbb{H}^1 \hookrightarrow \mathbb{L}^2 \), (4.25), (4.26) along with (4.20) and (4.21) we infer that there exits \( C = C(k) > 0 \) such that

\[
\mathbb{E} \int_0^t |v(r)|_{L^2}^2 \leq CE \int_0^{t \wedge \tau_k} \left( |L(n(r))|_{L_2}^2 + |f(r)|_{H_1}^2 + |f(n(r))|_{H_1}^2 + |f''(n(r))[g(r), g(r)]|_{L_2}^2 \right) \, dr < \infty.
\] (4.27)

Next, let \( \{y(t) : t \in [0, \bar{\tau}_\infty)\} \) be the process defined in (4.16). The process \( \{y(t \wedge \tau_k) : t \geq 0\} \) is an \( \mathbb{L}^2 \)-valued process and satisfies the equivalent equations (4.17) and (4.22). Hence, by (4.27) we can apply the Itô formula to \( \frac{1}{2}y(t \wedge \tau_k)|_{L_2}^2 = \frac{1}{2}A_1d - f(d)|_{L_2}^2 = \Psi_1(d) \) (see [54, Theorem 3.2]), and use the fact

\[
-(A_1 - f'(n))[v \cdot \nabla n] + (f'(n)y - (A_1 - f'(n))|\frac{1}{2}G^2(n)| - \frac{1}{2}f''(n)[G(n), G(n)] \in \mathbb{L}^2,
\]

to infer that for all \( k \in \mathbb{N} \) and \( t \geq 0 \)

\[
\Psi_1(n(t \wedge \tau_k)) + \int_0^{t \wedge \tau_k} |\nabla(A_1n(r) - f(n(r)))|_{L_2}^2 \, dr + \int_0^{t \wedge \tau_k} \Psi'_1(n(r))[v(r) \cdot \nabla n(r)] \, dr
\]

\[
= \Psi_1(n_0) + \int_0^{t \wedge \tau_k} \Psi'_1(n(r))[\frac{1}{2}G^2(n(r))] \, dr + \frac{1}{2} \int_0^{t \wedge \tau_k} \Psi''_1(n(r))[G(n(r)), G(n(r))] \, dr
\]

\[
- \int_0^{t \wedge \tau_k} (A_1n(r) - f(n(r)), f'(n(r))[A_1n(r) - f(n(r))]) \, dr + \int_0^{t \wedge \tau_k} \Psi_1(n(r))[G(n(r))] \, dW_2(r).
\] (4.28)
In preparation of our next step, let us set
\[
M(t \wedge \tau_k) = \int_0^{t \wedge \tau_k} \Phi(s) \Psi_2(v(s)) \circ S(v(s)) dW_1(s) + \int_0^{t \wedge \tau_k} \Phi(s) \Psi_1(n(s))[G(n(s))] dW_2(s), \quad k \in \mathbb{N}, t \geq 0.
\]

With (4.28) and the definitions of \( \Psi_2, \Psi \) and \( \Phi \) (see (3.30), (3.31) and (3.49)) in mind, we apply the Itô formula to \( Y(t \wedge \tau_k) = \Phi(t \wedge \tau_k)\Psi(u, n)(t \wedge \tau_k) \) and obtain that for all \( k \in \mathbb{N} \) and \( t \geq 0 \)
\[
Y(t \wedge \tau_k) - Y(0) = M(t \wedge \tau_k) + \int_0^{t \wedge \tau_k} \Phi(s) \Psi_1'(v(s)) [-B(v(s), v(s)) - M(n(s), n(s))] ds
\]
\[
+ \int_0^{t \wedge \tau_k} \Phi(s) \Psi_1'(n(s)) [-v(s) \cdot \nabla n(s) + \frac{1}{2} G^2(n(s))] ds
\]
\[
+ \frac{1}{2} \int_0^{t \wedge \tau_k} \Phi(s) \Psi_1''(n(s)) [G(n(s)), G(n(s))] ds + \int_0^{t \wedge \tau_k} d \frac{ds}{ds} \Phi(s) \Psi(v(s), n(s))) ds
\]
\[
- \int_0^{t \wedge \tau_k} \Phi(s) (A_1 n(s) - f(n(s)), f'(n(s))) [A_1 n(s) - f(n(s))] ds
\]
\[
- \int_0^{t \wedge \tau_k} \Phi(s) \left( |\nabla(A_1 n(s) - f(n(s)))|^2_{L^2} + |Av(s)|^2 - |S(v(s))|^2_{\mathcal{T}_2(\mathbf{K}_1, V)} \right) ds.
\]

By Assumption 2.3, (2.23), Lemma 3.20-3.25 and the facts \(|\Phi| \leq 1\) and
\[
\frac{d}{ds} \Phi(s) \Psi(v(s), n(s))) = - \left[ ((\kappa_1 + \kappa_4) ||n(s)||^2 + \kappa_2)(1 + ||n(s)||^2_1) + \kappa_3 |v(s)|^2_{L^2} |\nabla v(s)|^2_{L^2} \right]
\]
\[
\times \Phi(s) \Psi(v(s), n(s)),
\]
we infer that there exists \( \kappa_7 > 0 \) such that for all \( k \in \mathbb{N} \) and \( t \geq 0 \)
\[
\mathbb{E} \sup_{s \in [0, t]} Y(s \wedge \tau_k) + \int_0^{t \wedge \tau_k} \Phi(s) \left[ \frac{1}{4} |Av(s)|^2_{L^2} + \frac{1}{2} |\nabla(A_1 n(s) + f(n(s)))|^2_{L^2} \right] ds
\]
\[
\leq Y(0) + \kappa_7 T + \kappa_7 \mathbb{E} \int_0^{t \wedge \tau_k} Y(s) ds + \kappa_5 \mathbb{E} \int_0^{t \wedge \tau_k} (1 + ||n(s)||^2_{L^2}) ||n(s)||^2_{L^2} ds + \mathbb{E} \sup_{s \in [0, t]} |M(s \wedge \tau_k)|.
\]

Next, by applying the BDG inequality, taking into account Assumption 2.3, (3.43) and the fact \(|\Phi| \leq 1\) we infer that there exists \( \kappa_8 > 0 \) such that for all \( k \in \mathbb{N} \) and \( t \geq 0 \)
\[
\mathbb{E} \sup_{s \in [0, t]} |M(s \wedge \tau_k)|
\]
\[
\leq \kappa_8 \mathbb{E} \left( \int_0^{t \wedge \tau_k} \Phi(s) \Psi_2(2)[1 + \Psi_2(s)] \Phi(s) ds \right)^{\frac{1}{2}} + \kappa_8 \mathbb{E} \left( \int_0^{t \wedge \tau_k} \Phi(s) \Psi_1'(n(s))[G(n(s))]^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{8} \mathbb{E} \sup_{s \in [0, t]} \Phi(s) \Psi_2(v(s)) + \kappa_8 T + \kappa_8 \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \Phi(s) \Psi_1(n(s))^2 ds \right]^{\frac{1}{2}}
\]
\[
+ \kappa_8 \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \left( 1 + ||n(s)||^8_{L^1} + ||n(s)||_1 ||n(s)||^2_{L^2} \right) ds \right]^{\frac{1}{2}} + \kappa_8 \mathbb{E} \int_0^{t \wedge \tau_k} \Phi(s) \Psi_2(v(s)) ds
\]
\[
\leq \frac{1}{4} \mathbb{E} \sup_{s \in [0, t]} Y(s) + \kappa_8 T + \kappa_8 \mathbb{E} \int_0^{t \wedge \tau_k} Y(s) ds + \kappa_8 \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \left( 1 + ||n(s)||^8_{L^1} + ||n(s)||_1 ||n(s)||^2_{L^2} \right) ds \right]^{\frac{1}{2}}.
\]
Using the last inequality and absorbing the term $\frac{1}{4} \mathbb{E} \sup_{s \in [0, t]} \mathcal{Y}(s)$ in the LHS of (4.29), and applying Gronwall inequality imply that there exist an increasing function $\psi : [0, \infty) \to (0, \infty)$ and a constant $\kappa_9 > 0$ such that for all $k \in \mathbb{N}$ and $T \geq 0$

$$
\mathbb{E} \sup_{s \in [0, T]} \mathcal{Y}(s \wedge \tau_k) + \mathbb{E} \int_0^{T \wedge \tau_k} \Phi(s) \left[ \frac{1}{4} |A\mathbf{v}(s)|^2_{L^2} + \frac{1}{2} |\nabla(A_1 \mathbf{n}(s) + f(\mathbf{n}(s)))|^2_{L^2} \right] ds \\
\leq \psi(T) \left( 1 + \mathcal{Y}(0) + \mathbb{E} \int_0^{T \wedge \tau_k} \left( 1 + \|\mathbf{n}(s)\|^8_{H} + \|\mathbf{n}(s)\|_{V}^2 \right) ds \right)^{\frac{1}{2}} \\
+ \mathbb{E} \int_0^{T \wedge \tau_k} \left( 1 + \|\mathbf{n}(s)\|^{8N}_{V} \right) \|\mathbf{n}(s)\|_{H}^2 ds
$$

(4.30)

Using the estimate (3.25) we easily conclude that there exists a constant $\tilde{\kappa}_0 > 0$ such that

$$
\mathbb{E} \left[ \int_0^{t \wedge \tau_k} \left( 1 + \|\mathbf{n}(s)\|^{8N}_{H} + \|\mathbf{n}(s)\|_{V}^2 \right) ds \right]^{\frac{1}{2}} + \mathbb{E} \int_0^{t \wedge \tau_k} \left( 1 + \|\mathbf{n}(s)\|^{8N}_{V} \right) \|\mathbf{n}(s)\|_{H}^2 ds \\
\leq \tilde{\kappa}_0 (C_0 + 1),
$$

which along with (4.30) complete the proof of Proposition 3.26

\[ \square \]

5. Strong solution for an abstract stochastic equation

By a fixed point method we prove in this section general results about the existence and uniqueness of maximal local solution to stochastic evolution equations (SEE) with Lipschitz coefficients.

5.1. Notations and Preliminary. Let $V$, $E$ and $H$ be separable Banach spaces such that $E \hookrightarrow V$. We denote the norm in $V$ by $\| \cdot \|$ and for $a, b \in [0, \infty)$ with $a < b$ we put

$$
X_{a,b} := C([a, b]; V) \cap L^2(a, b; E)
$$

(5.1)

with the norm $\| \cdot \|_{X_{a,b}}$ defined by

$$
\| u \|_{X_{a,b}}^2 := \sup_{s \in [a, b]} \| u(s) \|^2 + \int_a^b \| u(s) \|_{H}^2 ds.
$$

(5.2)

If $a = 0$ we simply write $X_{a,b} = X_b$. If $a = b$ then the space $X_{a,a}$ is isomorphic to $V$.

Suppose that $\delta \in [0, T]$ and $a \in X_\delta$. Define

$$
\tilde{X}_{\delta;T,a} := \{ u \in X_T : u|_{[0, \delta]} = a \}.
$$

(5.3)

Obviously, $\tilde{X}_{\delta;T,a}$ is a closed subspace $X_T$ and hence a Banach space with the norm of $\| \cdot \|_{X_T}$. Note that if $\delta = 0$, $a \in X_0$ can be identified with $a = a(0) \in V$ and so $\tilde{X}_{0,T,a} = \{ u \in X_T : u(0) = a(0) \}$.

Let $F$ and $G$ be two nonlinear mappings satisfying the following sets of conditions.

Assumption 5.1. Suppose that $F : E \to H$ is such that $F(0) = 0$ and there exist $N \in \mathbb{N}$ and $p_i \geq 1$, $\alpha_i \in (0, 1)$, $i = 1, \cdots, N$, and $C > 0$ such that for all $x, y \in E$.

$$
|F(y) - F(x)|_H \leq C \sum_{i=1}^N \left[ |y - x||y||p_i - \alpha_i||y||E_{p_i}^{\alpha_i} + |y - x||y||E_{p_i}^{\alpha_i} \right].
$$

(5.4)
Assumption 5.2. Assume that $G : E \to V$ such that $G(0) = 0$ and there exists $k \geq 1$, $\beta \in [0, 1)$ and $C_G > 0$ such that

$$\|G(y) - G(x)\| \leq C_G \left[\|y - x\|^{k - \beta}\|y\|^{\beta} + \|y - x\|^{\beta} \|y - x\|^{1 - \beta}\|x\|^{k}\right],$$

for all $x, y \in E$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual hypothesis. By $\mathcal{M}^2(X_T)$ we denote the Banach space of all $E$-valued processes $u$ that are progressively measurable and with trajectories belonging to $X_T$ $\mathbb{P}$-a.s., with the norm

$$|u|_{\mathcal{M}^2(X_T)} = \left(\mathbb{E}\left[\sup_{s \in [0,T]} \|u(s)\|^2 + \int_0^T |u(s)|_E^2 \, ds\right]\right)^{\frac{1}{2}}$$

Let us also formulate the following assumptions.

Assumption 5.3. Suppose that $E \ni V \ni H$. Consider (for simplicity) a one-dimensional Wiener process $W(t)$.

Assume that $S(t), t \in [0, \infty)$, is a family of bounded linear operators on the space $H$ such that: the following properties are satisfied.

(i) For every $T > 0$, the linear map

$$L^2(0, T; H) \ni f \mapsto \{S * f(t) = \int_0^t S(t - r)f(r) \, dr; t \in [0, T]\} \in X_T,$$

is continuous.

(ii) For every $T > 0$, the linear map

$$\mathcal{M}^2(0, T; V) \ni \xi \mapsto \{S \diamond \xi(t) := \int_0^t S(t - r) \xi(r) \, dW(r); t \in [0, T]\} \in \mathcal{M}^2(X_T),$$

is continuous.

(iii) For every $T > 0$, the linear map

$$V \ni u_0 \mapsto \{[0, T] \ni t \mapsto Su_0(t) := S(t)u_0\} \in X_T,$$

is continuous.

Now let us consider a semigroup $S(t), t \in [0, \infty)$ as above and the abstract SEE

$$u(t) = S(t)u_0 + \int_0^t S(t - s)F(u(s)) \, ds + \int_0^t S(t - s)G(u(s))dW(s), \text{ for all } t > 0$$

which is a mild version of the problem

$$\begin{cases}
  du(t) = Au(t) \, dt + F(u(t)) \, dt + G(u(t))dW(t), & t > 0, \\
  u(0) = u_0.
\end{cases}$$

Definition 5.1. Assume that a $V$-valued $\mathcal{F}_0$ measurable random variable $u_0$ is given. A local solution to problem (5.8) (with the initial time 0) is a pair $(u, \tau)$ such that

1. $\tau$ is an accessible stopping time,
2. $u : [0, \tau) \times \Omega \to V$ is an admissible\textsuperscript{2} process,

\textsuperscript{2}This also follows from condition (3) below.
(3) there exists a sequence \((\tau_m)_{m \in \mathbb{N}}\) of finite stopping times such that \(\tau_m \not\sim \tau\) \(\mathbb{P}\)-a.s. and, for every \(m \in \mathbb{N}\) and \(t \geq 0\), we have

\[
\mathbb{E}\left( \sup_{s \in [0, t \wedge \tau_m]} \|u(s)\|^2 + \int_0^{t \wedge \tau_m} |u(s)|^2 ds \right) < \infty, \tag{5.9}
\]

\[
u(t \wedge \tau_m) = S(t \wedge \tau_m)u_0 + \int_0^{t \wedge \tau_m} S(t \wedge \tau_m - s)F(u(s)) \, ds + \int_0^t 1_{[0, \tau_m]}(s)G(u(s \wedge \tau_m)) \, dW(s). \tag{5.10}
\]

Along the lines of [3], we said that a local solution \(u(t), t < \tau\) is called global iff \(\tau = \infty\) \(\mathbb{P}\)-a.s.

Let us first formulate the following useful result.

**Proposition 5.2.** Assume that a pair \((u, \tau)\) is a local solution to problem (5.8). Then for every finite stopping time \(\sigma\), a pair \((u|_{[0, \tau \wedge \sigma]} \times \Omega, \tau \wedge \sigma)\) is also a local mild solution to problem (5.8).

Secondly, we state the following lemma result which is a generalisation of [24, Lemmata III 6A and 6B].

**Lemma 5.3. (The Amalgamation Lemma)**

1. Let \(\Delta\) be a family of accessible stopping times taking values in \([0, \infty)\). Then a supremum of \(\Delta\), i.e., \(\tau := \sup \Delta\), is an accessible stopping time with values in \([0, \infty]\) and there exists an \(\Delta\)-valued increasing sequence \(\{\alpha_n\}_{n=1}^{\infty}\) such that \(\tau(\omega) = \lim_{n \to \infty} \alpha_n(\omega)\), for all \(\omega \in \Omega\).
2. Assume also that for each \(\alpha \in \Delta\), \(I_{\alpha} : [0, \alpha) \times \Omega \to V\) is an admissible process such that for all \(\alpha, \beta \in \Delta\) and every \(t > 0\),

\[
I_{\alpha}(t) = I_{\beta}(t) \quad \mathbb{P}\text{-a.s. on } \Omega_{t}(\alpha \wedge \beta). \tag{5.11}
\]

Then, there exists an admissible process \(I : [0, \tau) \times \Omega \to V\), such that every \(t > 0\),

\[
I(t) = I_{\alpha}(t) \quad \mathbb{P}\text{-a.s. on } \Omega_{t}(\alpha). \tag{5.12}
\]

3. Moreover, if \(\tilde{I} : [0, \tau) \times \Omega \to X\) is any process satisfying (5.12) then the process \(\tilde{I}\) is a version of the process \(I\), i.e. for all \(t \in [0, \infty)\)

\[
\mathbb{P}\left( \{\omega \in \Omega : t < \tau(\omega), I(t, \omega) \neq \tilde{I}(t, \omega)\} \right) = 0. \tag{5.13}
\]

In particular, if in addition \(\tilde{I}\) is an admissible process, then

\[
I = \tilde{I}. \tag{5.14}
\]

**Remark 5.4.** Let us note that because both processes \(I : [0, \tau) \times \Omega \to V\) and \(I_{\alpha} : [0, \alpha) \times \Omega \to V\) are admissible (and hence with almost sure continuous trajectories), and since \(\alpha \leq \tau\), condition (5.12) is equivalent to the following one:

\[
I_{|[0, \alpha)} \times \Omega = I_{\alpha}. \tag{5.15}
\]

Similarly, condition (5.11) is equivalent to the following one

\[
I_{\alpha|[0, \alpha \wedge \beta)} \times \Omega = I_{\beta|[0, \alpha \wedge \beta)} \times \Omega. \tag{5.16}
\]

**Proof of Lemma 5.3.** Let \(\Delta\) be the family of accessible stopping times with values in \([0, \infty]\). This set satisfies the assumptions of Lemma [24, Lemma III.6A], where the set \(\Delta\) is denoted by \(A\). Indeed, by Remark 3.4, the supremum of every finite subset of \(\Delta\) belongs to \(\Delta\). Therefore, there exists an \(\mathcal{F}\)-measurable function \(\tau : \Omega \to [0, \infty]\) such that
(i) if \( \sigma \in \Delta \), then \( \tau \geq \sigma \), \( \mathbb{P}\)-a.s.;
(ii) if a random variable \( \eta : \Omega \to [0, \infty] \) satisfies \( \tau \geq \sigma \), \( \mathbb{P}\)-a.s., for all \( \sigma \in \Delta \), then \( \eta \geq \tau \), \( \mathbb{P}\)-a.s.
(iii) there exists a sequence \((\alpha_n)_{n\in\mathbb{N}}\) of elements of \( \Delta \) such that for all \( \omega \in \Omega \), \( \alpha_n(\omega) \leq \alpha_{n+1}(\omega) \leq \tau(\omega) \) for all \( n \in \mathbb{N} \) and \( \tau(\omega) = \lim_{n\to\infty} \alpha_n(\omega) = \sup_{n\in\mathbb{N}} \alpha_n(\omega) \).

Moreover, \( \tau \) is unique in the sense that if \( \hat{\tau} \) satisfies the above conditions (i) and (ii), then \( \hat{\tau} \geq \tau \), \( \mathbb{P}\)-a.s.. Hence, since for every \( n \in \mathbb{N} \), \( \alpha_n \), is an accessible stopping time, by [24, Proposition III.5B], Remark 3.4 and [50, Proposition 4.11] we infer that \( \tau \) an accessible stopping time. This proves part (1) of Lemma 5.3.

The proof of parts (2) and (3) is the same as the proof of [24, Lemma III 6 B], so we omit it. \( \square \)

**Definition 5.5.** Consider a family \( \mathcal{LS} \) of all local solution \( (u, \tau) \) to the problem (5.8). For two elements \( (u, \tau), (v, \sigma) \in \mathcal{LS} \) we write that \( (u, \tau) \preceq (v, \sigma) \) iff \( \tau \leq \sigma \) \( \mathbb{P}\)-a.s. and \( v_{[0,\tau} \times \Omega} = u \). Note that if \( (u, \tau) \preceq (v, \sigma) \) and \( (v, \sigma) \preceq (u, \tau) \), then \( (u, \tau) = (v, \sigma) \). We write \( (u, \tau) \prec (v, \sigma) \) iff \( (u, \tau) \preceq (v, \sigma) \) and \( (u, \tau) \not\preceq (v, \sigma) \). Then, the pair \((\mathcal{LS}, \preceq)\) is partially ordered. Each maximal element \((u, \tau)\) in the set \((\mathcal{LS}, \preceq)\) is called a maximal local solution to the problem (5.8). The existence of an upper bound of every non-empty chain of \((\mathcal{LS}, \preceq)\) is justified by Amalgamation Lemma 5.3.

If \((u, \tau)\) is a maximal local solution to equation (5.8), the stopping time \( \tau \) is called its lifetime.

A priori, there may be many maximal elements in \((\mathcal{LS}, \preceq)\) and hence many maximal local solutions to the problem (5.8). However, if the uniqueness of local solutions holds, then the uniqueness of the maximal local solution will follow.

**Definition 5.6.** A local solution \((u, \tau)\) to problem (5.8) is unique iff for all other local solution \((v, \sigma)\) to (5.8) the restricted processes \( u_{[0,\tau] \times \Omega} = v_{[0,\tau \wedge \sigma] \times \Omega} \) are equivalent.

**Proposition 5.7.** Suppose that \( u_0 \) is a \( V \)-valued random variable and \( \mathcal{F}_0 \)-measurable. Assume that the following two conditions are satisfied:

(i) there exist at least one local solution \((u^0, \tau^0)\) to problem (5.8)
(ii) if \((u^1, \tau^1)\) and \((u^2, \tau^2)\) are local solutions, then for every \( t > 0 \),
\[
u^1(t) = u^2(t) \quad \text{\( \mathbb{P}\)-a.s. on } \Omega_t(\tau^1 \land \tau^2) . \tag{5.17}\]

Then, problem problem (5.8) has a unique maximal local solution \((\hat{u}, \hat{\tau})\) satisfying \((u^0, \tau^0) \preceq (\hat{u}, \hat{\tau})\).

**Remark 5.8.** Let us note that similarly to Remark 5.4, because both the local solutions \( u^1 \) and \( u^2 \) are admissible (hence with almost sure continuous trajectories), condition (5.17) is equivalent to
\[
u^1_{[0,\tau^1 \land \tau^2]} = u^2_{[0,\tau^1 \land \tau^2]} \times \Omega . \tag{5.18}\]

**Proof of Proposition 5.7.** Let us choose and fix a local solution \((u^0, \tau^0)\) to problem (5.8) and let us consider the family \( \mathcal{LS} \) of all local solution \((u, \tau)\) to the problem (5.8) such that \((u^0, \tau^0) \preceq (u, \tau)\).

By assumptions this set is non-empty. Due to the assumptions (i) and (ii) of Proposition 5.7, by the Amalgamation Lemma 5.3 we infer that there exists an accessible stopping time
\[
\hat{\tau} := \sup \{ \tau : (u, \tau) \in \mathcal{LS} \}
\]

and an admissible process \( \hat{u} : [0, \hat{\tau}] \times \Omega \to V \), such that for all \((u, \tau) \in \mathcal{LS}\) and for \( t > 0 \),
\[
\hat{u}(t) = u(t) \quad \text{\( \mathbb{P}\)-a.s. on } \Omega_t(\tau) . \tag{5.19}\]

Moreover, there exists an increasing sequence \((\tau_n)\) of accessible stopping times such that \( \tau(\omega) = \lim_{n\to\infty} \tau_n(\omega) \), for all \( \omega \in \Omega \).
In order to complete the proof of the existence of a maximal local solution, we shall prove that \((\hat{u}, \hat{\tau}) \in \mathcal{LS}\). For this aim, we closely follow the proof of [3, Theorem 2.26]. Let us define an auxiliary process \(\tilde{\eta} = (\tilde{\eta}(t)), t \in [0, \tau)\), such that for each \(n \in \mathbb{N}\) and \(t \geq 0\), the following equality holds \(\mathbb{P}\)-a.s.

\[
\tilde{\eta}(t \wedge \tau_n) = S(t \wedge \tau_n)u_0 + \int_0^{t \wedge \tau_n} S(t - s)F(\hat{u}(s \wedge \tau_n))\, ds + I_{\tau_n}(t \wedge \tau_n),
\]

where \(I_{\tau_n}\) is a continuous \(V\)-valued process process defined by

\[
I_{\tau_n}(t) := \int_0^t 1_{[0,\tau_n)}(s)S(t - s)G(\hat{u}(s \wedge \tau_n))\, dW(s), \ t \geq 0.
\]

Assume that \((u, \tau) \in \mathcal{LS}\). Define a process \(\eta = (\eta(t)), t \in [0, \tau)\) by the above formulae (5.20)-(5.21) with \(\hat{u}\) replaced by \(u\) and the announcing sequence \(\tau_n\) of the accessible stopping time \(\hat{\tau}\) replaced by announcing sequence of the accessible stopping time \(\tau\). Because \((u, \tau)\) is a local solution, we infer that the process \(\eta(t), t \in [0, \tau)\) is a version of the process \(u(t), t \in [0, \tau)\). Since \(\hat{u}\) satisfies (5.19) and assumption (ii) of Proposition 5.7 is satisfied, we infer that

\[
\hat{\eta}(t) = u(t) \ \mathbb{P}\text{-a.s. on } \Omega(\tau).
\]

Hence, by the part (3) of Lemma 5.3, we infer that the process \(\hat{\eta}(t), t \in [0, \hat{\tau}),\) is a version of the process \(\check{u}(t), t \in [0, \hat{\tau})\) and therefore we can replace \(\hat{\eta}\) by \(\hat{u}\) on the LHS of (5.20). Therefore, we deduce that \((\hat{u}, \hat{\tau}) \in \mathcal{LS}\). This completes the existence of a local maximal solution.

As a byproduct of the above proof of the existence of a local maximal solution \((\hat{u}, \hat{\tau})\) we showed that \((\check{u}, \check{\tau}) \in \mathcal{LS}\). This, in conjunction with the definition of \(\mathcal{LS}\) implies that \((u^0, \tau^0) \preccurlyeq (\hat{u}, \hat{\tau})\).

It remains to prove the uniqueness of the local maximal solutions. For this aim let us suppose that \((u^1, \tau^1)\) and \((u^2, \tau^2)\) are two local maximal solutions. Let us put \(\check{\tau} = \tau^1 \vee \tau^2\). Then, by part (ii) of Remark 3.4 \(\tilde{\tau}\) is an accessible stopping time with announcing sequence \((\tau_n := \tau^1_n \vee \tau^2_n)_{n \in \mathbb{N}}\), where \((\tau^i_n)_{n \in \mathbb{N}}, \ i = 1, 2\) is an announcing sequence of \(\tau^i\). By the uniqueness assumption (ii), we infer that

\[
(u^1_{\leq \tau^1, \tau^2}, \tau^1 \wedge \tau^2) \sim (u^2_{\leq \tau^1, \tau^2}, \tau^1 \wedge \tau^2).
\]

We shall now prove that \(\tau^1 = \tau^2\) \(\mathbb{P}\text{-a.s.}\). Suppose by contradiction that \(\mathbb{P}\{\tau^1 \neq \tau^2\} > 0\). Let \(\Omega_1 := \{\tau^1 \geq \tau^2\}\) and \(\Omega_2 := \{\tau^2 > \tau^1\}\). We define a process \((\check{u}, \check{\tau})\) by the following formula

\[
\check{u}(t, \omega) = \begin{cases} u^1(t, \omega) & \text{if } \omega \in \Omega_1 \text{ and } t \in [0, \tau^1(\omega)) \\ u^2(t, \omega) & \text{if } \omega \in \Omega_2 \text{ and } t \in [0, \tau^2(\omega)). \end{cases}
\]

We now claim that the process \((\check{u}, \check{\tau})\) is a local solution to Problem (5.8). Let us fix \(n \in \mathbb{N}\) and \(t \geq 0\). By symmetry, we can assume that \(\tau^1_n(\omega) \leq \tau^2_n(\omega)\) for all \(\omega \in \Omega\) and \(n \in \mathbb{N}\). Firstly, the proof of the admissibility of \(\check{u}(t), t \in [0, \check{\tau})\) is very similar to the proof in [3, Corollary 2.28]. Secondly, let us also observe that on \(\Omega_1\) we have \(\check{\tau}_n < \tau^1 \wedge \tau^2\). Hence, we deduce from (5.24) and (5.23) that

\[
\check{u}(t \wedge \tau_n) = u^1(t \wedge \tau^2_n) = u^2(t \wedge \tau^2_n) = S(t \wedge \tau^2_n)u_0 + \int_0^{t \wedge \tau^2_n} S(t \wedge \tau^2_n - s)F(u^2(s))\, ds + I^2_{\tau^2_n}(t \wedge \tau^2_n) \ t \geq 0,
\]

where

\[
I^2_{\tau^2_n}(t) := \int_0^t 1_{(0,\tau^2_n)}(s)S(t - s)G(u^2(s \wedge \tau^2_n))\, dW(s), \ t \geq 0.
\]
The last equality follows from the fact that \((u^2, \tau^2)\) is a local solution. Since \(\tau_n = \tilde{\tau}_n\), by using (5.23) and [3, Proposition 2.10] we deduce that

\[
\tilde{u}(t \wedge \tilde{\tau}_n) = S(t \wedge \tilde{\tau}_n)u_0 + \int_0^{t \wedge \tilde{\tau}_n} S(t \wedge \tilde{\tau}_n - s)F(\tilde{u}(s)) \, ds + \tilde{I}_{\tilde{\tau}_n}(t \wedge \tilde{\tau}_n), \quad t \geq 0,
\]

where

\[
\tilde{I}_{\tilde{\tau}_n}(t) := \int_0^t 1_{[0,\tilde{\tau}_n)} S(t - s)G(\tilde{u}(s \wedge \tilde{\tau}_n))dW(s), \quad t \geq 0.
\]

Hence, \((\tilde{u}, \tilde{\tau})\) satisfies equation (5.10) on \(\Omega_1\). In a similar way, we can also show that \(\hat{\tilde{u}}, \hat{\tilde{\tau}}\) satisfies (5.10) on \(\Omega_2\). Hence, \((\hat{\tilde{u}}, \hat{\tilde{\tau}}) \in \mathcal{L}S\).

Now, by construction we have \((u^i, \tau^i) \preceq (\tilde{u}, \tilde{\tau})\) for \(i \in \{1, 2\}\) and there exists \(i_0 \in \{1, 2\}\) such that \((u^{i_0}, \tau^{i_0}) \preceq (\hat{\tilde{u}}, \hat{\tilde{\tau}})\). This contradicts the maximality of \((u^{i_0}, \tau^{i_0})\) and completes the proof of Proposition 5.7.

As a byproduct of the proof of the above Proposition 5.7 we deduce the following general result.

**Corollary 5.9.** Let \((x, \sigma)\) and \((y, \tau)\) be two local solution to problem (5.8) such that for every \(t > 0\),

\[
y(t) = x(t) \, \mathbb{P}\text{-}a.s. \text{ on } \Omega_t(\tau \wedge \sigma).
\]

Then the process \((z, \sigma \vee \tau)\) defined by the following formula

\[
z(t, \omega) = \begin{cases} x(t, \omega), & \text{if } \sigma(\omega) \geq \tau(\omega) \text{ and } t \in [0, \sigma(\omega)), \\ y(t, \omega), & \text{if } \sigma(\omega) < \tau(\omega) \text{ and } t \in [0, \tau(\omega)), \end{cases}
\]

is local solution to problem (5.8). The process \((z, \sigma \vee \tau)\) is called supremum of \((x, \sigma)\) and \((y, \tau)\).

### 5.2. An abstract result.

In this subsection we prove by a fixed point method some results about the existence and uniqueness of maximal local mild solution to (5.7).

Let \(\theta : \mathbb{R}^+ \to [0, 1]\) be a \(C^\infty\) non increasing function such that

\[
\inf_{x \in \mathbb{R}^+} \theta'(x) \geq -1, \quad \theta(x) = 1 \text{ iff } x \in [0, 1] \quad \text{and } \theta(x) = 0 \text{ iff } x \in [2, \infty).
\]

(5.27)

and for \(n \geq 1\) set \(\theta_n(\cdot) = \theta(\frac{\cdot}{n})\). Note that if \(h : \mathbb{R}^+ \to \mathbb{R}^+\) is a non decreasing function, then

\[
\theta_n(x)h(x) \leq h(2n), \quad |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x - y|, \text{ for every } x, y \in \mathbb{R}.
\]

(5.28)

**Proposition 5.10.** Let \(F\) be a mapping satisfying Assumption 5.1. Assume that \(\delta \in [0, T], \, a \in X_\delta\). Then the map

\[
\Phi^n_{\delta,T,a} : X_{\delta,T,a} \ni u \mapsto \theta_n(||u||_X)F(u) \in L^2(0,T;H).
\]

is globally Lipschitz and moreover, for all \(u_1, u_2 \in X_{\delta,T,a}\),

\[
|\Phi^n_{\delta,T,a}(u_1) - \Phi^n_{\delta,T,a}(u_2)|_{L^2(0,T;H)} \leq C(C + 1) \sum_{i=1}^N (2n)^{p_i+2}(T - \delta)^{(1-\alpha_i)/2}|u_1 - u_2|_{X_T}.
\]

(5.29)

In particular, the Lipschitz constant of \(\Phi^n_{\delta,T,a}\) is independent of \(a\).

The proof is based on a proof from [12] which in turn was based on a proof from [20, 21]. For simplicity of notation, below we will write \(\Phi_T\) instead of \(\Phi^n_{\delta,T,a}\).
Proof of Proposition 5.10. Wlog we can assume that $N = 1$ and we will use notation $\alpha = \alpha_1$ and $p = p_1$. In this case, the inequality (5.29) takes the following form. For every $n \in \mathbb{N}$ there exists $C(n) > 0$ such that for all $T > \delta \geq 0$, all $a \in X_{\delta}$ and all $u_1, u_2 \in \hat{X}_{\delta,T,a}$,

$$|\Phi^n_{\delta,T,a}(u_1) - \Phi^n_{\delta,T,a}(u_2)|_{L^2(0,T;H)} \leq C(n)(T - \delta)^{(1-\alpha)/2}|u_1 - u_2|_{X_T}. \quad (5.30)$$

In what follows we will prove (5.30). Let us fix $n \in \mathbb{N}$, $T > \delta \geq 0$, $a \in X_{\delta}$ and $u_1, u_2 \in \hat{X}_{\delta,T,a}$.

Note that $\Phi_T(0) = 0$. Assume that $u_1, u_2 \in X_T$. Denote, for $i = 1, 2$,

$$\tau_i = \inf \{t \in [0,T] : |u_i|_{X_T} \geq 2n \}.$$

Note that if the set on the RHS above is empty, i.e. $|u_i|_{X_T} < 2n$ for all $t \in [0,T]$, then $\tau_i = T$.

Wlog we can assume that $\tau_1 \leq \tau_2$. Because for $i = 1, 2$, $\theta_n(|u_i|_{X_T}) = 0$ for $t \geq \tau_2$, we have

$$|\Phi_T(u_1) - \Phi_T(u_2)|_{L^2(0,T;H)} = \left[ \int_0^{\tau_2} |\theta_n(|u_1|_{X_T})F(u_1(t)) - \theta_n(|u_2|_{X_T})F(u_2(t))|^2_H dt \right]^{1/2} \leq \left[ \int_0^{\tau_2} |\theta_n(|u_1|_{X_T}) - \theta_n(|u_2|_{X_T})|^2_H dt \right]^{1/2} + \left[ \int_0^{\tau_2} \theta_n(|u_1|_{X_T})F(u_1(t)) - F(u_2(t))|^2_H dt \right]^{1/2} =: A + B.$$

Next, since $\theta_n$ is Lipschitz with Lipschitz constant 2$\alpha$ and $u_1_{[0,\delta]} = u_2_{[0,\delta]} = a$ we have

$$A^2 = \int_0^{\delta \wedge \tau_2} \big| \theta_n(|u_1|_{X_T}) - \theta_n(|u_2|_{X_T}) \big|^2_H dt + \int_{\delta \wedge \tau_2}^{\tau_2} \big| \theta_n(|u_1|_{X_T}) - \theta_n(|u_2|_{X_T}) \big|^2_H dt \leq 4n^2 C^2 \int_{\delta \wedge \tau_2}^{\tau_2} \big||u_1|_{X_T} - |u_2|_{X_T} \big|^2_H dt \leq 4n^2 C^2 \big||u_1|_{X_T} - |u_2|_{X_T} \big|^2_H dt.$$

Next, by assumptions and some elementary calculations

$$\int_{\delta \wedge \tau_2}^{\tau_2} |F(u_2(t))|^2_H dt \leq C^2 \int_{\delta \wedge \tau_2}^{\tau_2} \|u_2(t)\|^{2p+2-2\alpha} \|u_2(t)\|^{2\alpha} dt \leq C^2 \sup|t \in [\delta \wedge \tau_2, \tau_2]| \big|u_2(t)\|^2 H \big| \big( \int_{\delta \wedge \tau_2}^{\tau_2} |u_2(t)|^{2\alpha} dt \big)^{\alpha - 2} (\tau_2 - \delta \wedge \tau_2)^{1-\alpha} \leq C^2 (T - \delta)^{1-\alpha} |u_2|^{2p+2}_{X_{\delta \wedge \tau_2, \tau_2}} \leq C^2 (T - \delta)^{1-\alpha} (2n)^{2p+2}.$$
This term can be estimated as follows
\[
\tilde{B}_T \leq C \sup_{t \in [0,T]} \|u_1(t) - u_2(t)\| \sup_{t \in [\delta, T \wedge \tau I]} \|u_1(t)\|^{p-\alpha} \left[ \int_{\delta \wedge \tau I}^{T \wedge \tau I} \left| u_1(t) \right|_{E^2} dt \right]^{\alpha/2} \left( \tau I - \delta \wedge \tau I \right)^{(1-\alpha)/2}
\]
\[
+ C \sup_{t \in [0,T]} \|u_1(t) - u_2(t)\|^{1-\alpha} \sup_{t \in [\delta \wedge \tau I]} \|u_2(t)\|^{p} \left[ \int_{\delta \wedge \tau I}^{T \wedge \tau I} \left| u_1(t) - u_2(t) \right|_{E^2} dt \right]^{\alpha/2} \left( \tau I - \delta \wedge \tau I \right)^{(1-\alpha)/2}
\]
\[
\leq C \|u_1 - u_2\|_{X_T^1} \|u_1\|_{X_{T1}}^{p} (T - \delta)^{(1-\alpha)/2} + C \|u_1 - u_2\|_{X_T^1} \|u_2\|_{X_{T1}}^{p} (T - \delta)^{(1-\alpha)/2}
\]
\[
\leq C(T - \delta)^{(1-\alpha)/2} \|u_1 - u_2\|_{X_{T}^1} \left[ |u_1|_{X_{T1}} + |u_2|_{X_{T1}} \right] \leq C(2n)^{p+1} (T - \delta)^{(1-\alpha)/2} \|u_1 - u_2\|_{X_{T}^1}.
\]

Summing up, we proved the following inequality
\[
|\Phi_T(u_1) - \Phi_T(u_2)|_{L^2(0,T;H)} \leq C(2nC + 1)(2n)^{p+1} \|u_1 - u_2\|_{X_{T}^1},
\]
which competes the proof of the proposition. \qed

The following result is a special case of Proposition 5.10 with \( H = V \).

**Corollary 5.11.** Let \( G \) be a nonlinear mapping satisfying Assumption 5.2. Assume that \( n \in \mathbb{N} \), \( T > 0 \), \( \delta \in [0,T] \) and \( a \in X_{0,\delta} \). Define a map \( \tilde{\Phi}_{T,a}^{n} \) by
\[
\tilde{\Phi}_{T,a}^{n} : X_{T,\delta} \ni u \mapsto \theta_n(|u|_{X_T})G(u) \in L^2(0,T;V).
\]

Then \( \tilde{\Phi}_{T,a}^{n} \) is globally Lipschitz and moreover, for all \( u_1, u_2 \in X_{T,\delta} \),
\[
|\tilde{\Phi}_{T,a}^{n}(u_1) - \tilde{\Phi}_{T,a}^{n}(u_2)|_{L^2(0,T;V)} \leq (2n)^{k+2} C_G (C_G + 1) (T - \delta)^{(1-\beta)/2} \|u_1 - u_2\|_{X_{T}^1}.
\]

In particular, the Lipschitz constant of \( \tilde{\Phi}_{T,a}^{n} \) is independent of \( a \).

**Proposition 5.12.** Assume that Assumptions 5.1 and 5.3 hold. Assume that \( n \in \mathbb{N} \), \( T > 0 \), \( \delta \in [0,T] \) and \( a \in \mathcal{M}(X_{0,\delta}) \). Then the map \( \Psi_{T,a}^{n} \) defined by
\[
\Psi_{T,a}^{n} : \mathcal{M}(X_{T,\delta}) \ni u \mapsto \left| S(\cdot)(a(0)) + S \ast \Phi_{T,a}^{n}(u) + S \ast \tilde{\Phi}_{T,a}^{n}(u) \right| \in \mathcal{M}(X_T),
\]
is globally Lipschitz and moreover, for all \( u_1, u_2 \in \mathcal{M}(X_{T,\delta}) \),
\[
|\Psi_{T,a}^{n}(u_1) - \Psi_{T,a}^{n}(u_2)|_{\mathcal{M}(X_T)} \leq \tilde{C}(n) \left[ \max_{1 \leq i \leq N} (T - \delta)^{1-\alpha_i} \vee (T - \delta)^{1-\beta_i} \right]^{1/2} \|u_1 - u_2\|_{\mathcal{M}(X_T)},
\]
where \( \tilde{C}(n) \) is dependent only on \( n \) and is given by, for some \( D > 1 \),
\[
\tilde{C}(n) = C_1 C_F (C_F + 1) \sum_{i=1}^{N} (2n)^{p_i+2} + C_2 C_G (2n)^{k+2} (C_G + 1) \leq C_3 n^D.
\]
In particular, the Lipschitz constant of \( \Psi_{T,a}^{n} \) is independent of \( a \).

**Proof of Proposition 5.12.** For simplicity of notation we will write \( \Psi_T \) instead of \( \Psi_{T,a}^{n} \). We will also write \( \Phi_T \) (resp. \( \Phi_G \)) instead of \( \Phi_{T,a}^{n} \) (resp. \( \Phi_{T,a}^{n} \)). Obviously in view of Assumption 5.3 the map \( \Psi_T \) is well defined. Let us fix \( u_1, u_2 \in \mathcal{M}(X_{T,\delta}) \). Then by the Fubini Theorem, Assumption
5.3. Proposition 5.10 and Corollary 5.11 we infer that
\[
|\Psi_T(u_1) - \Psi_T(u_2)|_{\mathcal{H}^2(X_T)} \leq |S \ast \Phi_F(u_1) - S \ast \Phi_F(u_2)|_{\mathcal{H}^2(X_T)} + |S \circ \Phi_G(u_1) - S \circ \Phi_G(u_2)|_{\mathcal{H}^2(X_T)} \\
\leq C_1|\Phi_F(u_1) - \Phi_F(u_2)|_{\mathcal{H}^2(0,T;H)} + C_2|\Phi_G(u_1) - \Phi_G(u_2)|_{\mathcal{H}^2(0,T;H)} \\
\leq \tilde{C}(n) \left[ \max_{1 \leq i \leq N} (T - \delta)^{1-\alpha_i} \vee (T - \delta)^{1-\beta} \right] |u_1 - u_2|_{\mathcal{H}^2(X_T)}.
\]

The proof is complete. □

Since our method is based on finding fixed points of $\Psi^n_{S,T,a}$, the following auxiliary result is useful.

**Lemma 5.13.** Assume that $n \in \mathbb{N}$ and $T > S > 0$ and $x \in V$. Assume that $a \in \mathcal{M}^2(\hat{X}_{0,S,X})$ is a fixed point of $\Psi^n_{S,T,a}$. Then $\Psi^n_{S,T,a}$ maps $\mathcal{M}^2(\hat{X}_{S,T,a})$ into itself.

**Proof of Lemma 5.13.** Let us choose and fix $n \in \mathbb{N}$, $T > S > 0$, $x \in V$ and $a \in \mathcal{M}^2(\hat{X}_{0,S,X})$, a fixed point of $\Psi^n_{0,S,X}$. We will show that $\Psi^n_{S,T,a}$ maps $\mathcal{M}^2(\hat{X}_{S,T,a})$ into itself. Take an arbitrary $u \in \hat{X}_{S,T,a}$. Since by Proposition 5.12, $v := [\Psi^n_{S,T,a}](u) \in \mathcal{M}^2(X_T)$ we only need to show that $v|_{[0,S]} = a$. For this aim let us observe that by Definition 5.33, we have for $t \in [0,S]$,

\[
v(t) = S(t)(a(0)) + \int_0^t S(t - r)\theta_n(|u|_{X_\tau})F(u(r))dr + \int_0^t S(t - r)\theta_n(|u|_{X_\tau})G(u(r))dW(r).
\]

Because $u|_{[0,S]} = a$, $a(0) = x$ and, by assumptions $\Psi^n_{0,S,X}(a) = a$ in $\mathcal{M}^2(\hat{X}_{0,S,X})$, we infer that

\[
v(t) = S(t)x + \int_0^t S(t - r)\theta_n(|a|_{X_\tau})F(a(r))dr + \int_0^t S(t - r)\theta_n(|a|_{X_\tau})G(a(r))dW(r)
\]

\[
= [\Psi^n_{0,S,X}(a)](t) = a(t), \quad t \in [0,S].
\]

This completes the proof of Lemma 5.13. □

The first two main results of this subsection are given in the following two theorems.

**Theorem 5.14.** Suppose that Assumption 5.1-Assumption 5.3 hold. Let $u_0$ be a $\mathcal{F}_0$-measurable $V$-valued square integrable random variable $u_0$ and $(u, \tau)$ and $(v, \sigma)$ two local solutions of (5.7).

Then,

\[
(u_{|_{[0,\alpha(r)]} \times \Omega}, \sigma \wedge \tau) \sim (v_{|_{[0,\alpha(r)]} \times \Omega}, \sigma \wedge \tau).
\]

**Proof of Theorem 5.14.** Wlog we can assume in Assumptions 5.1 and 5.2 that $N = 1$ and $p_i = k$ and will use the notations $\alpha_1 = \alpha$, $p_1 = p = k$. Let $(u, \tau)$ and $(v, \sigma)$ be two local solutions of (5.7). Let $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ be the announcing sequences of $\tau$ and $\sigma$, respectively. By [50, Propositions 4.3 & 4.11 and Theorem 6.6] the stopping time $\varrho := \tau \wedge \sigma$ is accessible and it is easy to show that $(\varrho_n)_{n \in \mathbb{N}} := (\tau_n \wedge \sigma_n)_{n \in \mathbb{N}}$ is an announcing sequence of $\varrho$.

Hereafter we fix $n \in \mathbb{N}$. Since $(v, \sigma)$ is a local solution to (5.8) and $\varrho_n \leq \sigma_n$, by Corollary A.2 we infer that for all $t \geq 0$ $\mathbb{P}$-a.s.

\[
v(t \wedge \varrho_n) = S_{t \wedge \varrho_n}u_0 + \int_0^{t \wedge \varrho_n} S_{t \wedge \varrho_n-r}F(v(r))dr + I_{\sigma_n}(t \wedge \varrho_n)
\]

\[
= S_{t \wedge \varrho_n}u_0 + \int_0^{t \wedge \varrho_n} S_{t \wedge \varrho_n-r}F(v(r))dr + I_{\varrho_n}(t \wedge \varrho_n),
\]

where

\[
I_{\sigma}(t) := \int_0^t 1_{[0,\varrho_n]}(r)S_{t-r}G(v(r))dW(r), \quad t \geq 0.
\]
The identity (5.35) proves that \((v, \sigma \wedge \tau)\) is a local solution to (5.8). In a similar way, we prove that \((u, \sigma \wedge \tau)\) is a local solution to (5.8) as well.

Thirdly, for \(k \in \mathbb{N}\) we put \(\varrho_{n,k} = \tau_n \wedge \sigma_n \wedge \tilde{\tau}_k \wedge \tilde{\sigma}_k\), where

\[
\tilde{\tau}_k = \inf\{t \in [0, T] : |v|_{X_t} \geq k\} \wedge \tau \quad \text{and} \quad \tilde{\sigma}_k = \inf\{t \in [0, T] : |v|_{X_t} \geq k\} \wedge \sigma, \quad k \in \mathbb{N}.
\]

We observe that for all \(k \in \mathbb{N}\), \(\varrho_{n,k} \leq \varrho_{n,k+1} \leq \sigma_n \wedge \tau_n\) \(\mathbb{P}\)-a.s. and \(\varrho_{n,k} \nearrow \sigma_n \wedge \tau_n\) \(\mathbb{P}\)-a.s. if \(k \to \infty\).

Let us now fix \(k \in \mathbb{N}\). Arguing as in the proof of (5.35) we show that for all \(t \geq 0\), \(\mathbb{P}\)-a.s.

\[
v(t \wedge \varrho_{n,k}) = S_{t \wedge \varrho_{n,k}} u_0 + \int_0^{t \wedge \varrho_{n,k}} S_{t \wedge \varrho_{n,k} - r} F(v(r)) dr + I_{\varrho_{n,k}}(t \wedge \varrho_{n,k}).
\]

(5.36)

In a similar way, we prove that the same identity holds with \(v\) replaced by \(u\). Hence, setting \(w = u - v\) we infer that for all \(t \geq 0\), \(\mathbb{P}\)-a.s.

\[
w(t \wedge \varrho_{n,k}) = \int_0^{t \wedge \varrho_{n,k}} S_{t \wedge \varrho_{n,k} - r} [F(u(r)) - F(v(r))] dr + \tilde{I}_{\varrho_{n,k}}(t \wedge \varrho_{n,k}).
\]

(5.37)

where

\[
\tilde{I}_{\varrho_{n,k}}(t) := \int_0^t 1_{[0, \varrho_{n,k}]}(r) S_{t \wedge \varrho_{n,k} - r} [G(u(r)) - G(v(r))] dW(r), \quad t \geq 0.
\]

Hereafter, \(c > 0\) denotes an universal constant (independent of \(n\) and \(k\)) which may change from one term to the other. Following the lines of [4, Proof of Lemma 3.8] or [11, Page 134] and using Assumptions 5.3, 5.1 and 5.2 we infer that for all \(t \geq 0\), \(\mathbb{P}\)-a.s.

\[
\mathbb{E}[w]_{X_t \wedge \varrho_{n,k}}^2 \leq c \mathbb{E} \int_0^{t \wedge \varrho_{n,k}} \left[ |F(u(r)) - F(v(r))|^2_H + \|G(u(r)) - G(v(r))\|^2 \right] dr,
\]

\[
\leq c \mathbb{E} \int_0^{t \wedge \varrho_{n,k}} \left( \|w(s)\|^2 \|u(s)\|^{2(p-\alpha)} \|u(s)\|_{\mathbb{E}}^{\frac{2}{\alpha}} \right) ds + c \mathbb{E} \int_0^{t \wedge \varrho_{n,k}} \left( \|w(s)\|_{\mathbb{E}}^2 \|w(s)\|^{2(1-\alpha)} \|v(s)\|^{2p} \right) ds,
\]

\[
=: c \mathbb{E} I_1 + c \mathbb{E} I_2.
\]

(5.38)

The Hölder inequality and the definition of the stopping time \(\varrho_{n,k}\) imply that for all \(t \geq 0\), \(\mathbb{P}\)-a.s.

\[
I_1 \leq c \left( \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|^{\frac{2p}{p-\alpha}} \|u(s)\|^{\frac{2p(1-\alpha)}{p-\alpha}} ds \right)^{\frac{p}{p-\alpha}} \left( \int_0^{t \wedge \varrho_{n,k}} \|u(s)\|_{\mathbb{E}}^{\frac{2}{1-\alpha}} ds \right)^{1-\alpha},
\]

\[
(5.39)
\]

\[
\leq c R^{2\alpha} \sup_{s \in [0, t \wedge \varrho_{n,k}]} \|u(s)\|^{2(p-\alpha)} \left( \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|^{2(1-\alpha)} ds \right)^{1-\alpha},
\]

\[
(5.40)
\]

\[
\leq c R^{2p} \|w\|_{X_t \wedge \varrho_{n,k}}^{2p} \left( \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|^2 ds \right)^{1-\alpha} \leq \frac{1}{4} \|w\|_{X_t \wedge \varrho_{n,k}}^2 + c R^{\frac{2p}{1-\alpha}} \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|^2 ds.
\]

(5.41)

In a similar way one can prove that for all \(t \geq 0\), \(\mathbb{P}\)-a.s.

\[
I_2 \leq c \left( \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|^2 \|v(s)\|^{2p(1-\alpha)} ds \right)^{1-\alpha} \left( \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|_{\mathbb{E}}^{2p} ds \right)^{\frac{p}{p-\alpha}},
\]

\[
(5.42)
\]

\[
\leq \frac{1}{4} \|w\|_{X_t \wedge \varrho_{n,k}}^2 + c R^{\frac{2p}{1-\alpha}} \int_0^{t \wedge \varrho_{n,k}} \|w(s)\|^2 ds.
\]

(5.43)

Hence, plugging (5.41) and (5.43) in (5.38) and using \(\|w(t \wedge \varrho_{n,k})\|^2 \leq \|w\|_{X_t \wedge \varrho_{n,k}}^2\), we infer that

\[
\mathbb{E}[w(t \wedge \varrho_{n,k})] \leq 2 c R^{\frac{2p}{1-\alpha}} \int_0^{t \wedge \varrho_{n,k}} \|w(s \wedge \varrho_{n,k})\|^2 ds, \quad \forall t \geq 0.
\]

(5.44)
This along the Gronwall lemma implies that for all $t \geq 0$, $\mathbb{E}\|w(t \wedge \tau_{n,k})\|^2 = 0$. Hence, by letting $k \to \infty$ we infer that for all $t \geq 0$, $\mathbb{E}\|w(t \wedge \tau_n \wedge \sigma)\|^2 = 0$, which along with the continuity of $w$ completes the proof of the theorem.

\textbf{Theorem 5.15.} Suppose that Assumptions 5.1-5.3 are satisfied. Then

(I) for every $\mathcal{F}_0$-measurable $V$-valued square integrable random variable $u_0$ there exits a local process $u = (u(t), t \in [0, T_1])$ which is the unique local solution to problem (5.8),

(II) if $R > 0$ and $\varepsilon > 0$ then there exists a number $T^*(\varepsilon, R) > 0$, such that for every set $\Omega_1 \in \mathcal{F}_0$ and every $\mathcal{F}_0$-measurable $V$-valued random variable $u_0$ such that

$$\|u_0\| \leq R \, \mathbb{P}\text{-a.s. on } \Omega_1,$$

one has

$$\mathbb{P}(\{T_1 \geq T^*(\varepsilon, R)\} \cap \Omega_1) \geq (1 - \varepsilon)\mathbb{P}(\Omega_1).$$

\textbf{Proof of Theorem 5.15.} Wlog we can assume that $N = 1$ and we will use notation $\alpha = \alpha_1$ and $p = p_1$. Let $u_0 \in L^2(\Omega, \mathbb{P}; V)$. We also fix a natural number $n \in \mathbb{N}$ in Steps 1-5. In the first part consisting of Steps 1-7 we will prove the part (I) of the Theorem, i.e. the existence and uniqueness of a local solution to problem (5.8). Part (II) of the Theorem will be proven in Steps 8-9.

\textbf{Proof of part (I)}

\textbf{Step 1.} Let us fix $n \in \mathbb{N}$ and $T > 0$. Let $\Psi^n_{0,T,u_0} : \mathcal{M}^2(\hat{X}_{0,T,u_0}) \to \mathcal{M}^2(\hat{X}_{0,T,u_0})$. By Proposition 5.12 the map $\Psi^n_{0,T,u_0}$ is well defined and for sufficiently small $T = \delta_n$, and all $a_0$, it is an $1/2$-contraction. Thus, by the Banach Fixed Point Theorem, there exists a unique

$$u^{[n,1]} = \Psi^n_{0,\delta_n,u_0}(u^{[n,1]}).$$

We fix $u^{[n,1]}$ for the rest of the proof. We also put $M := \frac{T}{\delta_n} \in \mathbb{N}$.

\textbf{Step 2.} By Lemma 5.13 $\Psi^n_{\delta_n,2\delta_n,u^{[n,1]}}$ maps $\mathcal{M}^2(\hat{X}_{\delta_n,2\delta_n,u^{[n,1]}})$ into itself and by Proposition 5.12 and inequality (5.34) it is an $1/2$-contraction. Therefore, we can find a unique $u^{[n,2]} \in \mathcal{M}^2(\hat{X}_{\delta_n,2\delta_n,u^{[n,1]}})$, which we fix for the rest of the proof, such that

$$u^{[n,2]} = \Psi^n_{\delta_n,2\delta_n,u^{[n,1]}},(u^{[n,2]}) \in \mathcal{M}^2(\hat{X}_{\delta_n,2\delta_n,u^{[n,1]}}).$$

\textbf{Step 3.} By induction we can construct a sequence $(u^{[n,k]})_{k=1}^{\infty}$ such that

$$u^{[n,k]} = \Psi^n_{(k-1)\delta_n,k\delta_n,u^{[n,k-1]}},(u^{[n,k]}) \in \mathcal{M}^2(\hat{X}_{(k-1)\delta_n,k\delta_n,u^{[n,k-1]}}, k = 2, \ldots$$

Note that by construction, the restriction of $u^{[n,k]}$ to interval $[0, (k-1)\delta_n]$ is equal to $u^{[n,k-1]}$.

\textbf{Step 4.} By \textbf{Step 3} we can define a process $u^n \in \mathcal{M}^2(X_T)$ by $u^n(t) = u^{[n,k]}(t)$, if $t \in [0, k\delta_n]$. Moreover, for every $t \in [0, T]$, $\mathbb{P}$-a.s.,

$$u^n(t) = S_t u_0 + \int_0^t S_{t-s}[\theta_n(|u^n|_{X_r})F(u^n(r))]dr + \int_0^t S_{t-s}[\theta_n(|u^n|_{X_r})G(u^n(r))]dW(r).$$

\textbf{Step 5.} Let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of stopping times defined by

$$\tau_n = \inf\{t \in [0, \infty) : |u^n|_{X_t} \geq n\}.$$  

Let us fix $n \in \mathbb{N}$. By [11, Lemma A.1], we infer from (5.46) that for every $t \in [0, T]$, $\mathbb{P}$-a.s.

$$u^n(t \wedge \tau_n) = S_{t \wedge \tau_n} u_0 + \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n - s}[\theta_n(|u^n|_{X_r})F(u^n(r))]dr + \tilde{I}^n_{t \wedge \tau_n}(t \wedge \tau_n),$$

where $\tilde{I}^n_{t \wedge \tau_n}(t \wedge \tau_n)$ is a $\mathcal{F}_{t \wedge \tau_n}$-adapted $\mathbb{R}$-valued random process of finite $\mathbb{P}$-almost surely bounded variations. Note that $u^n(t \wedge \tau_n)$ is $\mathcal{F}_{t \wedge \tau_n}$-measurable for every $t \in [0, T]$. Hence, we have constructed a sequence $\{u^{[n,k]}\}_{k=1}^{\infty}$ of local processes, which is a Cauchy sequence in $\mathcal{M}^2(X_T)$, and thus converges to a unique $\mathcal{F}_T$-measurable $V$-valued process $u$.
where \( \tilde{I}^n_{\tau_n} \) is a continuous \( V \)-valued
\[
\tilde{I}^n_{\tau_n}(t) := \int_0^t \mathbf{1}_{(0,\tau_n)}(s) S_{t-s-r} \theta_n([u^n]_{X_\tau}) G(u^n(r)) dW(r), \quad t \in [0, \infty).
\]
By the definition of the function \( \theta_n \) we infer that \( \theta_n([u^n]_{X_\tau}) = 1 \) for \( r \in [0, t \wedge \tau_n] \). Hence
\[
\theta_n([u^n]_{X_\tau}) F(u^n(r)) = F(u^n(r)), \quad r \in [0, t \wedge \tau_n), \quad t \in [0, \infty).
\]
Therefore, we deduce that for every \( t \in [0, \infty) \), \( \mathbb{P} \)-a.s.
\[
\tilde{I}^n_{\tau_n}(t) = \int_0^t \mathbf{1}_{(0,\tau_n)}(s) S_{t-s-r} G(u^n(r)) dW(r) =: I^n_{\tau_n}(t).
\]
Thus, we infer that \( u^n \) satisfies, for every \( t \in [0, \infty) \), \( \mathbb{P} \)-a.s.
\[
u^n(t \wedge \tau_n) = S_{t \wedge \tau_n} u_0 + \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n-r} [F(u^n(r))] \, dr + I^n_{\tau_n}(t \wedge \tau_n). \tag{5.49}
\]

**Step 6.** Arguing as in the proof of proof of [12, Lemma 5.1] we can show that for every \( n \in \mathbb{N} \),
\[
\tau_n < \tau_{n+1} \quad \mathbb{P} \text{-a.s.} \tag{5.50}
\]
and
\[
u^n(t) = u^{n+1}(t) \quad \text{if } t \in [0, \tau_n) \text{ and } n \in \mathbb{N}, \quad \mathbb{P} \text{-a.s.} \tag{5.51}
\]
By taking appropriate modifications we can assume that (5.50) is satisfied on the whole space \( \Omega \).
Hence, the following limit exists
\[
\tau_{\infty}(\omega) = \lim_{n \to \infty} \tau_n(\omega), \quad \omega \in \Omega. \tag{5.52}
\]
Since our probability basis satisfies the usual hypothesis, \( \tau_{\infty} \) is an accessible stopping time, see [40, Proposition 2.3 and Lemma 2.11] with \( (\tau_n) \) being the announcing sequence for \( \tau_{\infty} \). The two claims made at the beginning of **Step 6** enable us to define a local process \((u, \tau_{\infty})\) in the following way
\[
u(t, \omega) = u^n(t, \omega) \quad \text{if } t < \tau_n(\omega), \omega \in \Omega. \tag{5.53}
\]

**Step 7.** We claim that \((u, \tau_{\infty})\) is a local solution to problem (5.8).
Indeed, arguing as in **Step 5**, in particular using [11, Lemma A.1], we can show that \( u^n \) satisfies, for every \( t \in [0, \infty) \), \( \mathbb{P} \)-a.s.
\[
u(t \wedge \tau_n) = S_{t \wedge \tau_n} u_0 + \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n-r} [F(u(r))] \, dr + I_{\tau_n}(t \wedge \tau_n). \tag{5.54}
\]
where \( I_{\tau_n} \) is an \( V \)-valued continuous process defined by
\[
I_{\tau_n}(t) = \int_0^t \mathbf{1}_{(0,\tau_n)}(s) S_{t-s-r} G(u(r)) dW(r), \quad t \in [0, \infty).
\]
Since, as observed above, \( \tau_{\infty} \) is an accessible stopping time with the announcing sequence \((\tau_n)\), by definition 3.8 we infer that the local process \((u, \tau_{\infty})\) is a local solution to problem (5.8).
This also ends the proof of the first part of Theorem 5.15.

**Proof of part (III)**
Let us recall that \( \Omega_1 \in \mathcal{F}_0 \). Let \( i : \Omega_1 \to \Omega \) be the natural embedding and \( W^1 \) a process given by
\[
W^1(t) := W(t) \circ i, \quad t \geq 0.
\]
We define \( \mathcal{F}_t^1 \) by
\[
\mathcal{F}_t^1 := \{ A \cap \Omega_1 = i^{-1}(A) : A \in \mathcal{F}_t \}, \quad t \geq 0.
\]
Similarly we define $\mathcal{F}^1$. We put $\mathbb{P}^1 = (\mathcal{F}^1_t)_{t \geq 0}$.

We also define a measure

$$\mathbb{P}^1 : \mathcal{F}^1 \ni A \mapsto \frac{\mathbb{P}(A \cap \Omega_1)}{\mathbb{P}(\Omega_1)} \in [0, 1].$$

It is easy to check that $(\Omega_1, \mathcal{F}^1, \mathbb{P}^1, \mathbb{F}^1)$ is a filtered probability space satisfying the usual conditions. Moreover, since the Wiener process $W$ is independent of $\mathcal{F}_0$, $W^1$ is a Wiener process on $(\Omega_1, \mathcal{F}^1, \mathbb{P}^1, \mathbb{F}^1)$.

Hence, it is sufficient to prove our result in the case when $\Omega_1 = \Omega$. Indeed, the following holds true.

**Lemma 5.16.** If a local process $u = (u(t), t \in [0, \tau])$ is a local solution to problem (5.8) on the original probability basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, then the process $u^1 = (u(t), t \in [0, \tau^1])$ defined by

$$u^1(t, \omega) := u(t, i(\omega)), \quad t \in [0, \tau(\omega)), \quad \omega \in \Omega_1,$$

and $\tau^1 := \tau \circ i$, is a local solution to problem (5.8) on the new probability basis $(\Omega_1, \mathcal{F}^1, \mathbb{P}^1, \mathbb{F}^1)$.

Thus we assume that $u_0$ is a $\mathcal{F}_0$-measurable $V$-valued random variable such that

$$\|u_0\| \leq R \mathbb{P} \text{-a.s. on } \Omega.$$

Our proof follows the lines of [12, Theorem 5.3]. In order to simplify the notation we will write $\Psi^\alpha_T$ instead of $\Psi^\alpha_{0,T,u_0}$. We will also write $\Phi^\alpha_F$ (resp. $\hat{\Phi}^\alpha_F$) instead of $\Phi^\alpha_{\delta,T,u}$ (resp. $\hat{\Phi}^\alpha_{\delta,T,u}$).

We modify the initial data $u_0$ by replacing it by $\tilde{u}_0 = u_01\Omega_1$. Then $\|\tilde{u}_0\| \leq R$ on $\Omega$.

**Step 8.** Let us fix $\varepsilon > 0$ and choose $M$ such that $M \geq (C_0 + 1)\varepsilon^{-\frac{1}{2}}$, where $C_0$ is the constant appearing in inequality (5.55) below. Thanks to Propositions 5.10 and 5.12, Corollary 5.11 and Assumption 5.3 we can find $D_i(n) > 0$, $i = 1, 2$, $n \in \mathbb{N}$ such that for all $u \in \mathcal{M}^2(X_{0,T,u_0})$ we have

$$|S * \Phi^\alpha_F(u)|_{\mathcal{M}^2(X_T)} \leq \tilde{D}_1(n)\tilde{C}_2(2n\tilde{C}_2 + 1)(2n)^{p+2}T^{(1-\alpha)/2},$$

$$|S * \hat{\Phi}^\alpha_F(u)|_{\mathcal{M}^2(X_T)} \leq \tilde{D}_2(n)\tilde{C}_4(2n)^{k+2}[2n\tilde{C}_4 + 1]T^{(1-\beta)/2}.$$

Hence, since $1 - \alpha, 1 - \beta > 0$ we infer that there exists a sequence $(K_n(T))_n$ of numerical functions such that for all $n$, $\lim_{T \to 0} K_n(T) = 0$ and

$$|S * \Phi^\alpha_F(u) + S * \hat{\Phi}^\alpha_F(u)|_{\mathcal{M}^2(X_T)} \leq K_n(T), \quad u \in \mathcal{M}^2(X_T).$$

Let us put $n = MR^2$ and choose $\delta_1(\varepsilon, R) > 0$ such that $K_n(\delta_1(\varepsilon, R)) \leq R$. Let $\Psi_T^\alpha$ be the mapping defined by (5.33). Since $\mathbb{E}[\|\tilde{u}_0\|^2] \leq R^2$, we infer by the Assumption 5.3 that

$$|\Psi_T^\alpha(u)|_{\mathcal{M}^2(X_T)} \leq C_0R + K_n(T) \leq (C_0 + 1)R,$$

for all $T \leq \delta_1(\varepsilon, R)$. Furthermore, Propositions 5.12 implies that there exists $C > 0$ such that

$$|\Psi_T^\alpha(u_1) - \Psi_T^\alpha(u_2)|_{\mathcal{M}^2(X_T)} \leq C_3M^D R^D \left[T^{(1-\alpha)/2} \vee T^{(1-\beta)/2}\right] |u_1 - u_2|_{\mathcal{M}^2(X_T)},$$

for all $u_1, u_2 \in \mathcal{M}^2(X_T)$.

Since $M \geq (C_0 + 1)\varepsilon^{-\frac{1}{2}}$ we infer that

$$|\Psi_T^\alpha(u_1) - \Psi_T^\alpha(u_2)|_{\mathcal{M}^2(X_T)} \leq C_3M^D R^D \left[T^{(1-\alpha)/2} \vee T^{(1-\beta)/2}\right] |u_1 - u_2|_{\mathcal{M}^2(X_T)}.$$

Hence we can find $\delta_2(\varepsilon, R) > 0$ such that $\Psi_T^\alpha$ is a strict contraction for all $T \leq \delta_2(\varepsilon, R)$. Thus if one puts $T^*(\varepsilon, R) = \delta_1(\varepsilon, R) \wedge \delta_2(\varepsilon, R)$, the mapping $\Psi_T^\alpha$ has a unique fixed point $\hat{u}^\alpha$ which satisfies

$$\mathbb{E}[\|\hat{u}^\alpha\|_{X_{T^*(\varepsilon, R)}}^2] \leq (C_0 + 1)^2 R^2.$$ (5.56)
Similarly to (5.47) we can define a new stopping time $\hat{\tau}_n$ by

$$\hat{\tau}_n := \inf\{ t \in [0, \infty) : |\hat{u}^n| \geq n \}.$$  

Arguing as in Step 5 we can show that $\hat{u}^n$ satisfies, for every $t \in [0, T], \mathbb{P}$-a.s.

$$\hat{u}^n(t \wedge \tau_n) = S_{t \wedge \tau_n} u_0 + \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n - r} [F(\hat{u}^n(r))] dr + \hat{I}^n_{\tau_n}(t \wedge \hat{\tau}_n).$$  

(5.57)

where $\hat{I}^n_{\tau_n}$ is a continuous $\bar{V}$-valued process defined by

$$\hat{I}^n_{\tau_n}(t) := \int_0^t 1_{[0, \tau_n)}(s) S_{t - r} [G(\hat{u}^n(r))] dW(r), \quad t \in [0, T].$$

By the definition of the stopping time $\hat{\tau}_n$, $\{\hat{\tau}_n \leq T^*(\varepsilon, R)\} \subset \{|u^n|_{X_{T^*(\varepsilon, R)}} \geq n\}$. Therefore, by the Chebyshev inequality and inequality (5.56) we infer that

$$\mathbb{P}(\hat{\tau}_n \leq T^*(\varepsilon, R)) \leq \mathbb{P}(|u^n|_{X_{T^*(\varepsilon, R)}} \geq n) \leq \frac{1}{n} \mathbb{E}|u^n|_{X_{T^*(\varepsilon, R)}}^2 \leq \frac{1}{n} (C_0 + 1)^2 R^2.$$  

Since $n = NR^2$ and $N \geq (C_0 + 1)^2 \varepsilon^{-\frac{1}{2}}$ we get

$$\mathbb{P}(\hat{\tau}_n \leq T^*(\varepsilon, R)) \leq (C_0 + 1)^2 N^{-2} \leq \varepsilon.$$

Hence, we have prove (5.45).

**Step 9.** To conclude the proof, we observe that in view of Remark 3.10, it follows from equality (5.57) that the process $\hat{u}^n$ restricted to the open random interval $[0, \hat{\tau}_n) \times \Omega$ is a local solution to problem (5.8). On the other hand, in Step 7 we proved that also $(u, \tau_\infty)$ is a local solution to problem (5.8). Because local uniqueness holds for problem (5.8), see Theorem 5.14, we infer by applying Corollary 5.9 that the supremum of $(u, \tau_\infty)$ and $(\hat{u}^n, \hat{\tau}_n)$ is another local solution to problem (5.8). Hence, the stopping time $T_1 = \tau_\infty \vee \hat{\tau}_n$ satisfies the requirements of the theorem.

This concludes the proof of part (III) of Theorem 5.15 and thus of the whole theorem.

The next result is about the existence and uniqueness of a maximal solution and the characterization of its lifespan.

**Theorem 5.17.** Let $u_0 \in L^2(\Omega, \mathcal{F}_0, V)$. Then, problem (5.8) has a unique maximal local solution $(\hat{u}, \hat{\tau})$. Moreover, $(u, \tau_\infty) \sim (\hat{u}, \hat{\tau})$ and

$$\lim_{t \searrow \hat{\tau}} |\hat{u}|_{\bar{X}_t} = \infty \quad \mathbb{P} \text{-a.s. on } \{\hat{\tau} < \infty\},$$  

(5.58)

$$\mathbb{P}\left\{\omega \in \Omega : \hat{\tau}(\omega) < \infty \text{ and } \sup_{t \in [0, \hat{\tau}(\omega)]} |\hat{u}(t)(\omega)|_V < \infty\right\} = 0.$$  

(5.59)

**Proof.** Let us choose and fix $u_0 \in L^2(\Omega, \mathcal{F}_0, V)$. Wlog we can assume that $\mathbb{P}\{\{\hat{\tau} < \infty\}\} > 0$.

Firstly, we observe that it follows from the proof of Theorem 5.15 that the local process $(u, \tau_\infty)$ defined in (5.53) is a local solution to (5.8). In particular, the set of local solutions to problem (5.8) is non-empty. Since by Theorem 5.14 the local uniqueness holds for problem (5.8), we infer by applying Proposition 5.7 that there exists a unique maximal local solution $(\hat{u}, \hat{\tau})$ to (5.8). Moreover, $(\hat{u}, \hat{\tau})$ satisfies the following

$$\hat{\tau} \geq \tau_\infty \mathbb{P} \text{-a.s. and } \hat{u}|_{[0, \tau_\infty) \times \Omega} = u.$$  

(5.60)

Secondly, suppose that $\mathbb{P}(\hat{\tau} > \tau_\infty) > 0$. Let $(\hat{\tau}_n)$ be the announcing sequence of $\hat{\tau}$. Since $\hat{\tau}_n \uparrow \hat{\tau}$ $\mathbb{P}$-a.s., we infer that there exists $n \in \mathbb{N}$ such that $\mathbb{P}(\hat{\tau} > \hat{\tau}_n > \tau_\infty) > 0$. Thus we infer that there
exists \( t > 0 \) such that \( \mathbb{P}\left(t > \hat{\tau} > \hat{\tau}_n > \tau_\infty\right) > 0. \)

On the other hand, by the definition (5.47) of the announcing sequence \((\tau_n)_{n \in \mathbb{N}}\) for \(\tau_\infty\) and the definition, see (5.53), of the local process \((u, \tau)\) that the sequence \(|u|_{X_{\tau_n}}\) converges to \(\infty\) on \(\{\tau_\infty < \infty\}\). Hence we infer that \(|u|_{X_{\tau_\infty}} = \infty\) a.s. on \(\{t > \hat{\tau} > \hat{\tau}_n > \tau_\infty\}\). Since the probability of the last set is \(> 0\), this contradicts condition (5.9) of Definition 5.1 of a local solution. This contradiction implies that \(\mathbb{P}(\hat{\tau} > \tau_\infty) = 0\), and in view of (5.60), we also have \((u, \tau_\infty) \sim (\hat{u}, \hat{\tau})\).

Thirdly, we infer from the definition (5.47) of the announcing sequence \((\tau_n)_{n \in \mathbb{N}}\) for \(\tau_\infty\) and the definition, see (5.53), of the local process \((u, \tau_\infty)\) that the sequence \(|u|_{X_{\tau_n}}\) converges to \(\infty\) on \(\{\tau_\infty < \infty\}\). Since the function \(t \mapsto |u|_{X_t}\) is increasing, by (5.60), we infer that

\[
\lim_{t \nearrow \tau_\infty} |u|_{X_t} = \lim_{n \nearrow \infty} |u|_{X_{\tau_n}} = \infty \text{ a.s. on } \{\hat{\tau} < \infty\}. \tag{5.61}
\]

This proves (5.58).

Fourthly, we shall prove (5.59). By contradiction, we assume that there exists \(\varepsilon > 0\) such that

\[
\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \{\sup_{t \in [0, \tau_\infty]} |\hat{u}(t)|_V < \infty\}\right) = 4\varepsilon > 0.
\]

Hence we can easily deduce that there exists \(R > 0\) such that

\[
\mathbb{P}\left(\{|u(t)|_V < R \text{ for all } t \in [0, \tau_\infty]\}\right) \geq 3\varepsilon.
\]

Let us now choose \(\alpha\) such that \(\alpha = \frac{1}{2}T^*(\varepsilon, R)\), where the number \(T^*(\varepsilon, R) > 0\) depending only on \(\varepsilon\) and \(R\) comes from part (II) of Theorem 5.15. By the definition of an announcing sequence \((\tau_n)\) of \(\tau_\infty\) we infer that for arbitrary \(\delta > 0\) there exists \(n_0 > 0\) such that \(\mathbb{P}(\Omega_0) \geq (1 - \delta)\mathbb{P}(\hat{\Omega})\), where \(\Omega_0 := \{\omega \in \hat{\Omega} : |u(t)|_V < R \text{ for all } t \in [0, \hat{\tau}] \text{ and } \hat{\tau} - \tau_{n_0} < \alpha\}\). Choosing \(\delta = \frac{1}{3}\) we get \(\mathbb{P}(\Omega_0) \geq 2\varepsilon\).

Let \(y_0 = u(\tau_{n_0})\). Note that \(y_0\) is \(\mathcal{F}_{\tau_{n_0}}\)-measurable and \(|y_0|_V \leq R\) on \(\Omega_0\). With the previously chosen \(R, \varepsilon\) and \(T^*(\varepsilon, R) > 0\), by applying part (II) of Theorem 5.15 we find a local solution \(y(t), t \in [\tau_{n_0}, \tau_{n_0} + T_1]\) to problem (5.7) with the initial condition (starting at \(\tau_{n_0}\)) \(y(\tau_{n_0}) = y_0\) such that \(\mathbb{P}(T_1 \geq T^*(\varepsilon, R)) > 1 - \varepsilon\). Also, let \(\Omega_1 := \Omega_0 \cap \{T_1 \geq T^*(\varepsilon, R)\}\). Since \(\mathbb{P}(\hat{\tau} - T_0 < \frac{1}{2}T^*(\varepsilon, R)) \geq 2\varepsilon\), we infer that

\[
\mathbb{P}(\Omega_1) \geq \varepsilon > 0.
\]

By a generalization of [3, Corollary 2.28] to the case of SPDEs we infer that a local stochastic process \(v(t), t \in [0, \tau_{n_0}(\omega) + T_1]\) defined by

\[
v(t, \omega) = \begin{cases} u(t, \omega) & \text{if } t \in [0, \tau_{n_0}(\omega)], \\ y(t, \omega) & \text{if } t \in [\tau_{n_0}(\omega), \tau_{n_0}(\omega) + T_1]. \end{cases}
\]

is a local solution to problem (5.7) with the initial data \(v(0) = u_0\). However, on the \(\Omega_1\), we have

\[\tau_{n_0} + T_1 - \tau_\infty = T_1 - (\tau_\infty - \tau_{n_0}) \geq 2\alpha - \alpha = \alpha > 0,\]

what contradicts the maximality of the solution \((u, \tau_\infty)\) proved in the earlier part of the theorem. This completes the proof of (5.59) as well as the theorem.

\[\square\]

**Acknowledgments.** This article is part of a project that is currently funded by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 791735 “SELEs”. The authors are also grateful for the support they received from the FWF-Austrian Science through the Stand-Alone project P28010. Z. Brzeźniak presented a lecture based on a preliminary version of this paper at the RIMS Symposium on Mathematical Analysis of Incompressible Flow held at Kyoto in February 2013. He would like to thank Professor
Toshiaki Hishida for the kind invitation. Razafimandimby is also very grateful to the organizers
of the conference “Nonlinear PDEs in Micromagnetism: Analysis, Numerics and Applications”,
which was held in ICMS Edinburgh, UK, for their invitation to present a talk based on preliminary
result of this paper at this meeting. He is very grateful for the financial support he received from
the International Centre for Mathematical Sciences (ICMS) Edinburgh. Last, but not the least, the
authors wish to thank Professor Guoli Zhou for pointing out some gaps in the previous version of
this paper [8].

APPENDIX A. ON STOPPED STOCHASTIC CONVOLUTION PROCESSES

Let $K$ and $E$ be two separable Hilbert spaces. Suppose that $W$ is a canonical cylindrical Wiener
process on $K$ and that $\xi : [0, \infty) \rightarrow \gamma(K, E)$ is a progressively measurable process such that
\[
\int_0^t \|\xi(s)\|^2_{\gamma(K, E)} ds < \infty, \text{ for all } t \geq 0, \mathbb{P}\text{-almost surely. (A.1)}
\]
Assume that $S = (S_t)_{t \geq 0}$ is a $C_0$ semigroup on $E$. Let us define a process $I$ by
\[
I(t) := \int_0^t S_{t-s} \xi(s) dW(s), \ t \geq 0.
\] (A.2)
Assume that $\tau$ is a finite stopping time. Let us define a process $I_\tau$ by
\[
I_\tau(t) := \int_0^t 1_{[0,\tau)}(s) S_{t-s} \xi(s) dW(s), \ t \geq 0.
\] (A.3)
Let us observe that since $\tau$ is a stopping, the stochastic process $1_{[0,\tau)}(s)$, $s \in [0, \infty)$ is well-
measurable, see [50, Proposition 4.2]. Therefore, since by [50, Theorem 1.6], the $\sigma$-field of well
measurable sets is smaller than the $\sigma$-field of progressively measurable sets, it follows that the
stochastic process $1_{[0,\tau)}$, is progressively measurable. In particular, the integrand in (3.10) is pro-
gressively measurable.

If both processes $I$ and $I_\tau$ have continuous paths $\mathbb{P}$-a.s, then the next lemma was proved in [11].

Lemma A.1.

\[
S_{t-\land \tau}^\land I(t^\land) = I_\tau(t) \text{ for all } t \geq 0, \mathbb{P}\text{-a.s.},
\] (A.4)
and
\[
I(t^\land \tau) = I_\tau(t^\land \tau) \text{ for all } t \geq 0, \mathbb{P}\text{-a.s.}
\] (A.5)

Let us observe that the process $I_\tau$ is well defined even if the integrand $\xi$ is only defined on the
random interval $[0, \tau) \times \Omega$ and that it satisfies the following modification of the condition (A.1), i.e.
\[
\int_0^{t^\land} \|\xi(s)\|^2_{\gamma(K, E)} ds < \infty, \text{ for all } t \geq 0, \mathbb{P}\text{-a.s. (A.6)}
\]
In particular, if $\xi$ is defined on the random closed interval $[0, \tau] \times \Omega$ and that it satisfies (A.1), i.e.
\[
\int_0^{\tau} \|\xi(s)\|^2_{\gamma(K, E)} ds < \infty, \mathbb{P}\text{-a.s. (A.7)}
\]
Let us now formulate a useful corollary of the above result.

Corollary A.2. Under the above assumptions, if $\xi$ is a progressively measurable process defined on
$[0, \tau) \times \Omega$ and $\xi$ satisfies condition (A.6) and $\sigma$ is another stopping time such that $\sigma \leq \tau$, then
\[
I_\tau(t^\land \sigma) = I_\sigma(t^\land \sigma) \text{ for all } t \geq 0, \mathbb{P}\text{-a.s. (A.8)}
\]
Proof of Corollary A.2. Let us define a new process \( \eta = 1_{[0,\tau)}(s) \xi \). Obviously, the process \( \eta \) satisfies the assumptions of Lemma A.1. In particular, if we define continuous processes \( J \) and \( J_\sigma \) by formulae (A.2)-(A.3) with \( \tau \) replaced by \( \sigma \) and \( \xi \) replaced by \( \eta \), i.e.

\[
J(t) := \int_0^t S_{t-s} \eta(s) \, dW(s) \quad \text{and} \quad J_\sigma(t) := \int_0^t 1_{[0,\sigma)}(s) S_{t-s} \eta(s) \, dW(s) \quad t \geq 0,
\]

then by Lemma A.1 we have

\[
J(t \wedge \sigma) = J_\sigma(t \wedge \sigma) \quad \text{for all} \quad t \geq 0, \quad \mathbb{P} \text{-a.s.} \quad (A.10)
\]

On the other hand, we trivially have the following identities,

\[
J_\sigma(t) = I_\sigma(t) \quad \text{for all} \quad t \geq 0, \quad \mathbb{P} \text{-a.s.,} \quad (A.11)
\]

\[
J(t) = I_\tau(t) \quad \text{for all} \quad t \geq 0, \quad \mathbb{P} \text{-a.s.} \quad (A.12)
\]

Therefore, by (A.11), (A.9) and (A.12) (in that order), we infer that

\[
I_\sigma(t \wedge \sigma) = J_\sigma(t \wedge \sigma) = J(t \wedge \sigma) = I_\tau(t \wedge \sigma) \quad \text{for all} \quad t \geq 0, \quad \mathbb{P} \text{-a.s.}
\]

what proves equality (A.8) and the corollary. \( \square \)

Remark A.3. The approach from [11] we follow here was used implicitly in several papers in particular, in the paper [4], but it seems to have been discussed explicitly for the first time only in [16], section 4.3 (in a way different from the one presented above).

References


Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
E-mail address: zdzislaw.brzezniak@york.ac.uk

Department of Mathematics and Information Technology, Montanuniversity of Leoben, Franz Josef Strasse 18, 8700 Leoben, Austria
E-mail address: erika.hausenblas@unileoben.ac.at

Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
E-mail address: paul.razafimandimby@york.ac.uk