GEVREY ANALYTICITY AND DECAY FOR THE COMPRESSIBLE NAVIER-STOKES SYSTEM WITH CAPILLARITY

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Abstract. We are concerned with a system of equations governing the evolution of isothermal, viscous and capillary compressible fluids, that can be used as a phase transition model.

We prove that the global solutions with critical regularity that have been constructed in [11] by the second author and B. Desjardins, are Gevrey analytic, then extend that result to a more general critical $L^p$ framework. As a consequence, we obtain algebraic time-decay estimates in critical Besov spaces (and even exponential decay for the high frequencies) for any derivatives of the solution.

Our approach is partly inspired by the work of Bae, Biswas & Tadmor [2] dedicated to the classical incompressible Navier-Stokes equations, and requires us to establish new bilinear estimates (of independent interest) involving the Gevrey regularity for the product or composition of functions.

To the best of our knowledge, our work is the first one that exhibits Gevrey analyticity for a model of compressible fluids.

1. Introduction

When considering a two-phases liquid mixture, it is generally assumed, as a consequence of the Young-Laplace theory, that the jump in the pressure across the hypersurface separating the phases is proportional to the curvature. In the most common description – the Sharp Interface SI model – the interface between phases corresponds to a discontinuity in the state space. In contrast, in the Diffuse Interface DI model, the change of phase corresponds to a fast but regular transition zone for the density and velocity.

The DI approach has become popular lately as its mathematical and numerical study only requires one set of equations to be solved in a single spatial domain (typically, with a Van der Waals pressure, the phase changes are read through the density values). In contrast, with the SI model one has to solve one system per phase coupled with a free-boundary problem, since the location of the interface is unknown (see e.g. [9, 24] for more details about the modelling of phase transitions).

The DI model we here aim at considering originates from the works of Van der Waals (and, later, Korteweg) more than one century ago. The basic idea is to add to the classical compressible fluids equations a capillary term, that penalizes high variations of the density. In that way, one selects only physically relevant solutions, that is the

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ones with density corresponding to either a gas or a liquid, and such that the length of the phase interfaces is minimal. In fact, if capillary is absent then one can find an infinite number of mathematical solutions, most of them being physically wrong although mathematically correct.

The full derivation of the corresponding equations that we shall name the compressible Navier-Stokes-Korteweg system is due to Dunn and Serrin in [13]. In the barotropic case, it reads:

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - A u + \nabla \Pi &= \text{div} K,
\end{align*}
\]

where \( \Pi \equiv P(\rho) \) is the pressure function, \( A u \equiv \text{div} \left( 2\mu (\rho) D(u) \right) + \nabla (\lambda (\rho) \text{div} u) \) is the diffusion operator, \( D(u) = \frac{1}{2} (\nabla u + \nabla u^T) \) is the symmetric gradient, and the capillarity tensor is given by

\[
K \equiv \rho \text{div} (\kappa(\rho) \nabla \rho) I_{\mathbb{R}^d} + \frac{1}{2} (\kappa(\rho) - \rho \kappa'(\rho)) |\nabla \rho|^2 I_{\mathbb{R}^d} - \kappa(\rho) \nabla \rho \otimes \nabla \rho.
\]

The density-dependent capillarity function \( \kappa \) is assumed to be positive. Note that for smooth enough density and \( \kappa \), we have (see [4])

\[
\text{div} K = \rho \nabla \left( \kappa(\rho) \Delta \rho + \frac{1}{2} \kappa'(\rho) |\nabla \rho|^2 \right).
\]

The coefficients \( \lambda = \lambda(\rho) \) and \( \mu = \mu(\rho) \) designate the bulk and shear viscosities, respectively, and are assumed to satisfy in the neighborhood of some reference constant density \( \bar{\rho} > 0 \) the conditions

\[
\mu > 0 \quad \text{and} \quad \nu \equiv \lambda + 2\mu > 0.
\]

Throughout the paper, we shall assume that the functions \( \lambda, \mu, \kappa \) and \( P \), are real analytic in a neighborhood of \( \bar{\rho} \). Note that this includes the interesting case \( \kappa(\rho) = \frac{1}{\rho} \) that corresponds to the so-called quantum fluids. The reader may for instance refer to the recent paper by B. Haspot in [18] where this case is considered under the ‘shallow water’ assumption for the viscosity coefficients: \( (\mu(\rho), \lambda(\rho)) = (\rho, 0) \).

System (1.1) is supplemented with initial data

\[
(\rho, u) |_{t=0} = (\rho_0, u_0),
\]

and we investigate strong solutions in the whole space \( \mathbb{R}^d \) with \( d \geq 2 \), going to a constant equilibrium \((\bar{\rho}, 0)\) with \( \bar{\rho} > 0 \), at infinity.

The starting point of our paper is the global existence result for System (1.1) in so-called critical Besov spaces that has been established by the second author and B. Desjardins in [11]. Before stating the result, let us introduce the following functional space:

\[
E = \left\{ (a, u) \mid a \in \tilde{C}_b(\mathbb{R}_+; \dot{B}^{d/2-1}_{2,1} \cap \dot{B}^{d/2}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{d/2+1}_{2,1} \cap \dot{B}^{d/2+2}_{2,1});
\right.
\]

\[
\left. u \in \tilde{C}_b(\mathbb{R}_+; \dot{B}^{d/2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{d/2+1}_{2,1}) \right\},
\]

the reader being referred to the appendix for the definition of the Besov spaces coming into play in \( E \).
The following result has been established in [11]:

**Theorem 1.1.** Let \( \bar{\varrho} > 0 \) be such that \( P'(\bar{\varrho}) > 0 \). Suppose that the initial density fluctuation \( \varrho_0 - \bar{\varrho} \) belongs to \( B^{\frac{d}{2} - \frac{1}{2}}_{2,1} \cap B^{\frac{d}{2} - \frac{1}{2}}_{2,1} \) and that the initial velocity \( u_0 \) is in \( B^{\frac{d}{2} - \frac{1}{2}}_{2,1} \).

There exists a constant \( \eta > 0 \) depending only on \( \kappa, \mu, \nu, P'(\bar{\varrho}) \) and \( d \), such that, if

\[
\| \varrho_0 - \bar{\varrho} \|_{B^{\frac{d}{2} - \frac{1}{2}}_{2,1} \cap B^{\frac{d}{2} - \frac{1}{2}}_{2,1}} + \| u_0 \|_{B^{\frac{d}{2} - \frac{1}{2}}_{2,1}} \leq \eta,
\]

then System (1.1) supplemented with (1.4) has a unique global solution \( (\varrho, u) \) such that \( (\varrho - \bar{\varrho}, u) \in E \).

Our first main result states that the solutions constructed in Theorem 1.1 are, in fact, Gevrey analytic.

**Theorem 1.2.** Let the data \( (\rho_0, u_0) \) satisfy the conditions of Theorem 1.1 for some \( \bar{\varrho} > 0 \) such that \( P'(\bar{\varrho}) > 0 \), and assume that the functions \( \kappa, \lambda, \mu \) and \( P \) are analytic. There exist two positive constants \( c_0 \) and \( \eta \) only depending on those functions and on \( d \) such that if we set

\[
F = \left\{ U \in E \mid e^{V_{01}(\Lambda_1)} U \in E \right\},
\]

where \( \Lambda_1 \) stands for the Fourier multiplier with symbol\(^2\) \( |\xi| = \sum_{i=1}^{d} |\xi_i| \), then for any data \( (\varrho_0, u_0) \) satisfying

\[
\| \varrho_0 - \bar{\varrho} \|_{B^{\frac{d}{2} - \frac{1}{2}}_{2,1} \cap B^{\frac{d}{2} - \frac{1}{2}}_{2,1}} + \| u_0 \|_{B^{\frac{d}{2} - \frac{1}{2}}_{2,1}} \leq \eta,
\]

System (1.1)-(1.4) admits a unique solution \( (\varrho, u) \) with \( (\varrho - \bar{\varrho}, u) \in F \).

In Theorem 3.1 below, we shall establish a more general statement that is valid in \( L^p \) type critical Besov spaces. Then, as a by-product, we shall get time-decay estimates for any derivative of the solution (see Theorem 4.1).

Our paper unfolds as follows. The next section is devoted to proving Theorem 1.2. Then, in Section 3, we extend it to the critical \( L^p \) Besov spaces framework: first we extend Theorem 1.1 to the \( L^p \) framework, then we prove that the solutions constructed therein are also Gevrey analytic (see Theorem 3.1). The last section is devoted to proving optimal time-decay estimates. Finally, some basic results and definitions pertaining to Besov spaces are postponed in Appendix.

We end this introductory part by specifying some notations. Throughout the paper, \( C \) stands for a positive harmless “constant”, the meaning of which is clear from the context. Similarly, \( f \lesssim g \) means that \( f \leq Cg \) and \( f \approx g \) means that \( f \lesssim g \) and \( g \lesssim f \). It will be also understood that \( \|(f, g)\|_X \Rightarrow \|f\|_X + \|g\|_X \) for all \( f, g \in X \). Finally, when \( f = (f_1, \ldots, f_d) \) with \( f_i \in X \) for \( i = 1, \ldots, d \), we shall often use, slightly abusively, the notation \( f \in X^d \) instead of \( f \in X^d \).

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\(^1\)Actually, only the case of constant capillarity and viscosity coefficients has been considered therein. The case of smooth coefficients may be treated along the same lines (see also the work by B. Haspot in [15] concerning the general polytropic case).

\(^2\)Also for technical reasons, as observed before in [22], it is more convenient to use the \( \ell^2(\mathbb{R}^d) \) norm instead of the usual \( \ell^1(\mathbb{R}^d) \) norm associated with \( \Lambda = (-\Delta)^{1/2} \).
2. The $L^2$ framework

Proving Theorem 1.2 relies essentially on the classical fixed point theorem in the space $F$. To establish that all the conditions are fulfilled however, we need to prove a couple of a priori estimates for smooth enough solutions. To this end, we first recast the system into a more user-friendly shape, then establish Gevrey type estimates for the corresponding linearized system about the constant reference state $(\bar{\rho}, 0)$, and new nonlinear estimates.

2.1. Renormalization of System (1.1). Throughout the paper, it is convenient to fix some reference viscosity coefficients $\bar{\lambda}$ and $\bar{\mu}$, pressure $\bar{p}$ and capillarity coefficient $\bar{\kappa}$, and to rewrite the diffusion, pressure and capillarity terms as follows:

\[
\begin{align*}
A u &= \bar{\mu} \text{div} \left( 2\mu(\rho) D(u) \right) + \bar{\lambda} \nabla \left( \lambda(\rho) \text{div} u \right), \\
\nabla \Pi &= \bar{p} P'(\rho) \nabla \rho, \\
\n\text{div} K &= \bar{\kappa} \rho \nabla \left( \kappa(\rho) \Delta \rho + \frac{1}{2} \kappa'(\rho) \left| \nabla \rho \right|^2 \right),
\end{align*}
\]

in such a way that $\mu(\bar{\rho}) = \lambda(\bar{\rho}) = \kappa(\bar{\rho}) = P'(\bar{\rho}) = 1$.

If we denote $\nu = 2\bar{\mu} + \bar{\lambda}$, then performing the rescaling:

\[
\begin{align*}
\tilde{\rho}(t,x) &= \frac{1}{\bar{\rho}} \rho \left( \nu t, \frac{\bar{\rho}}{\nu} x \right), \\
\tilde{u}(t,x) &= \frac{1}{\sqrt{\bar{p}}} u \left( \frac{\bar{p}}{\nu} t, \frac{\bar{p}}{\nu} x \right),
\end{align*}
\]

the parameters $(\bar{\rho}, \bar{\mu}, \bar{\lambda}, \bar{p}, \bar{\kappa})$ are changed into $(1, \frac{\bar{\mu}}{\nu}, \frac{\bar{\lambda}}{\nu}, 1, \frac{\bar{\kappa}}{\nu^2})$. We can therefore assume with no loss of generality that

\[
\begin{align*}
\bar{\rho} = 1, \quad \bar{\nu} = 2\bar{\mu} + \bar{\lambda} = 1, \quad \bar{p} = 1, \\
\mu(1) = \lambda(1) = \kappa(1) = P'(1) = 1.
\end{align*}
\]

Then, introducing the density fluctuation $a = \rho - 1$, System (1.1) becomes

\[
\begin{align*}
\partial_t a + \text{div} u &= f, \\
\partial_t u - A u + \nabla a - \kappa \nabla \Delta a &= g,
\end{align*}
\]

with $f = -\text{div}(au)$, and $g = \sum_{j=1}^{5} g_j$, where

\[
\begin{align*}
A u &= \bar{\mu} \text{div} \left( 2\mu(u) D(u) \right) + \bar{\lambda} \nabla \left( \lambda(u) \text{div} u \right), \\
\nabla \Pi &= \bar{p} P'(\rho) \nabla \rho, \\
\text{div} K &= \bar{\kappa} \rho \nabla \left( \kappa(\rho) \Delta \rho + \frac{1}{2} \kappa'(\rho) \left| \nabla \rho \right|^2 \right),
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\mu}(a) &= \mu(1 + a) - 1, \quad \tilde{\lambda}(a) = \lambda(1 + a) - 1, \quad \tilde{\kappa}(a) = \kappa(1 + a) - 1, \\
I(a) &= a_1 \frac{a}{1+a}, \quad J(a) = 1 - \frac{P'(1+a)}{1+a}.
\end{align*}
\]

Let us underline that all those functions are analytic near zero, and vanish at zero.
2.2. The linearized system. The present subsection is devoted to exhibiting the smoothing properties of (2.3), assuming that \( f \) and \( g \) are given. In contrast with the linearized equations for the classical compressible Navier-Stokes system, we shall see that here both the density and the velocity are smoothed out instantaneously. The key to that remarkable property is given by the following lemma where, as in all this subsection, we denote by \( \widehat{\cdot} \) the Fourier transform with respect to the space variable of the function \( z \in C(\mathbb{R}_+; S(\mathbb{R}^d)) \).

**Lemma 2.1.** There exist two positive constants \( c_0 \) and \( C \) depending only on \((\pi, \mu)\) and \( \kappa \), respectively, such that the following inequality holds for all \( \xi \in \mathbb{R}^d \) and \( t \geq 0 \):

\[
|\langle \hat{a}, [\xi] \hat{a} \rangle(t, \xi)| \leq C \left( e^{-c_0|\xi|^2 t} |\langle \hat{a}, [\xi] \hat{a} \rangle(0, \xi)| + \int_0^t e^{-c_0|\xi|^2 (t-\tau)} |\langle \hat{f_\tau}, [\xi] \hat{f_\tau}, \hat{g} \rangle(\tau, \xi)| \, d\tau \right),
\]

**Proof.** It is mainly a matter of adapting to System (2.3) the energy argument of Godunov [14] for partially dissipative first-order symmetric systems (further developed by Kawashima in e.g. [21]).

Note that taking advantage of the Duhamel formula reduces the proof to the case where \( f \equiv 0 \) and \( g \equiv 0 \). Now, applying to the second equation of (2.3) the Leray projector (denoted by \( \mathcal{P} \)) on divergence free vector fields yields

\[
\partial_t \mathcal{P} u - \mu \Delta \mathcal{P} u = 0,
\]

from which we readily get, after taking the (space) Fourier transform,

\[
|\mathcal{P} u(t)| \leq e^{-\mu|\xi|^2 t} |\mathcal{P} u(0)|.
\]

In order to prove the desired inequality for \( a \) and the potential part of the velocity, it is convenient to introduce the function \( v \triangleq \Lambda^{-1} \text{div} u \) (with \( \Lambda^s z \triangleq \mathcal{F}^{-1} (|\xi|^s \mathcal{F} z) \) for \( s \in \mathbb{R} \)). Then, we discover that \((a, v)\) satisfies (recall that \( 2\pi + \lambda = 1 \))

\[
\begin{cases}
\partial_t a + \Lambda v = 0, \\
\partial_t v - \Delta v - \Lambda a - \kappa \Lambda^3 a = 0.
\end{cases}
\]

Hence, taking the Fourier transform of both sides of (2.8) gives

\[
\begin{cases}
\frac{d}{dt} \hat{a} + |\xi| \hat{v} = 0, \\
\frac{d}{dt} \hat{v} + |\xi|^2 \hat{v} - |\xi|(1 + \kappa |\xi|^2) \hat{a} = 0.
\end{cases}
\]

Multiplying the first equation in (2.9) by the conjugate \( \bar{a} \) of \( \hat{a} \), and the second one by \( \bar{\hat{v}} \), we get

\[
\int \left( \frac{d}{dt} |\hat{a}|^2 + |\xi| |\text{Re}(\hat{a} \bar{\hat{v}})| \right) \, dt = 0
\]

and, because \( \text{Re}(\hat{a} \bar{\hat{v}}) = \text{Re}(\bar{a} \hat{v}) \),

\[
\int \left( \frac{d}{dt} |\hat{v}|^2 + |\xi|^2 |\hat{v}|^2 - |\xi|(1 + \kappa |\xi|^2) \text{Re}(\hat{a} \bar{\hat{v}}) \right) \, dt = 0.
\]

Multiplying (2.10) by \( (1 + \kappa |\xi|^2) \), and adding up to (2.11) yields

\[
\int \left( \frac{d}{dt} ((1 + \kappa |\xi|^2) |\hat{a}|^2 + |\hat{v}|^2) + |\xi|^2 |\hat{a}|^2 \right) \, dt = 0.
\]
In order to track the dissipation arising for \( a \), let us multiply the first and second equations of \((2.9)\) by \(-|\xi|\tilde{v} \) and \(-|\xi|\tilde{a} \), respectively. Adding them, we get:

\[
(2.13) \quad \frac{d}{dt} (-|\xi|\text{Re}(\tilde{a} \tilde{v})) = |\xi|^3 \text{Re}(\vec{a} \vec{v}) + |\xi|^2 (1 + \pi |\xi|^2) |\tilde{a}|^2 - |\xi| |\tilde{v}|^2 = 0.
\]

Adding to this \(|\xi|^2(2.10)\) yields

\[
(2.14) \quad \frac{d}{dt} |\xi|^2 = 2|\xi|\text{Re}(\tilde{a} \tilde{v})) + |\xi|^2 (1 + \pi |\xi|^2) |\tilde{a}|^2 - |\xi| |\tilde{v}|^2 = 0.
\]

Therefore, by multiplying \((2.14)\) by a small enough constant \( \beta > 0 \) (to be determined later) and adding it to \((2.12)\), we get

\[
\frac{1}{2} \frac{d}{dt} \mathcal{L}_{|\xi|}^2 (t) + \beta |\xi|^2 (1 + \pi |\xi|^2) |\tilde{a}|^2 + (1 - \beta) |\xi|^2 |\tilde{v}|^2 = 0,
\]

with \( \mathcal{L}_{|\xi|}^2 (t) \triangleq (1 + \pi |\xi|^2) |\tilde{a}|^2 + |\tilde{v}|^2 + \beta (|\xi|^2 |\tilde{v}|^2 - 2|\xi|\text{Re}(\tilde{a} \tilde{v})) \).

Choosing \( \beta = \frac{1}{2} \) we have \( \mathcal{L}_{|\xi|}^2 \approx |(\tilde{a}, |\xi|\tilde{a}, \tilde{v})|^2 \) and using the Cauchy-Schwarz inequality, we deduce that there exists a positive constant \( c_1 \) such that on \( \mathbb{R}_+ \), we have

\[
\frac{d}{dt} \mathcal{L}_{|\xi|}^2 + c_1 |\xi|^2 \mathcal{L}_{|\xi|}^2 \leq 0,
\]

which leads, after time integration, to\(^3\)

\[
(2.15) \quad |(\tilde{a}, |\xi|\tilde{a}, \tilde{v})(t)| \leq Ce^{-c_1|\xi|^2}[|\tilde{a}, |\xi|\tilde{a}, \tilde{v}|(0)].
\]

Putting together with \((2.7)\) completes the proof of the lemma in the case \( f \equiv 0 \) and \( g \equiv 0 \). The general case readily stems from Duhamel formula. \( \square \)

We shall also need the following two results that have been proved in [2].

**Lemma 2.2.** The kernel of operator \( M_1 := e^{-|\sqrt{\tau - \eta} + \sqrt{\eta - \tau}|\Lambda_1} \) with \( 0 < \tau < t \) is integrable, and has a \( L^1 \) norm that may be bounded independently of \( \tau \) and \( t \).

**Lemma 2.3.** The operator \( M_2 := e^{\frac{1}{2}a\sqrt{\tau + \sqrt{\eta}}\Lambda_1} \) is a Fourier multiplier which maps boundedly \( L^p \) to \( L^p \) for all \( 1 < p < \infty \). Furthermore, its operator norm is uniformly bounded with respect to \( a \geq 0 \).

Proving the Gevrey regularity of our solutions will be based on continuity results for the family \( \{B_t\}_{t \geq 0} \) of bilinear operators defined by

\[
B_t(f, g)(x) = \left( e^{\sqrt{\tau}d\Lambda_1} (e^{-\sqrt{\tau}d\Lambda_1} f \cdot e^{-\sqrt{\tau}d\Lambda_1} g) \right)(x)
\]

\[
= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} e^{\sqrt{\tau}d(|\xi+\eta| - |\xi| - |\eta|)} \hat{f}(\xi) \hat{g}(\eta) \, d\xi \, d\eta.
\]

Following [2] and [23], we introduce the following operators acting on functions depending on one real variable:

\[
K_1 f \triangleq \frac{1}{2\pi} \int_{0}^{\infty} e^{ix\xi} \hat{f}(\xi) \, d\xi \quad \text{and} \quad K_{-1} f \triangleq \frac{1}{2\pi} \int_{-\infty}^{0} e^{ix\xi} \hat{f}(\xi) \, d\xi,
\]

and define \( L_{a,1} \) and \( L_{a,-1} \) as follows:

\[
L_{a,1} f \triangleq f \quad \text{and} \quad L_{a,-1} f \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-2\pi a|\xi|} \hat{f}(\xi) \, d\xi.
\]

\(^3\)If one tracks the constants then we get \( c_1 = \frac{3}{2} \min(1, \pi) \) and \( C = \frac{\max(\frac{3}{2}, \pi + 1)}{\min(\frac{3}{2}, \pi)} \).
For $t \geq 0$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \{-1, 1\}^d$, we set
\[
Z_{t,\alpha,\beta} \triangleq K_\beta L_{\sqrt{\alpha_1} \alpha_2} \cdots \otimes K_{\beta_d} L_{\sqrt{\alpha_1} \cdots \sqrt{\alpha_d} \beta_d} \quad \text{and} \quad K_\alpha \triangleq K_{\alpha_1} \otimes \cdots \otimes K_{\alpha_d}.
\]
Then we see that
\[
\tag{2.16}
B_t(f, g) = \sum_{(\alpha, \beta, \gamma) \in \{-1, 1\}^d} K_\alpha(Z_{t,\alpha,\beta} f Z_{t,\alpha,\gamma} g).
\]

Since operators $K_\alpha$ and $Z_{t,\alpha,\beta}$ are linear combinations of smooth homogeneous of degree zero Fourier multipliers, they are bounded on $L^p$ for any $1 < p < \infty$ (but they need not be bounded in $L^1$ and $L^\infty$). Furthermore, they commute with all Fourier multipliers and thus in particular with $\Lambda_1$ and with the Littlewood-Paley cut-off operators $\Delta_j$. We also have the following fundamental result:

**Lemma 2.4.** For any $1 < p, p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have for some constant $C$ independent of $t \geq 0$,
\[
\|B_t(f, g)\|_{L^p} \leq C\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}.
\]

### 2.3. Results of continuity for the paraproduct, remainder and composition

The aim of this section is to establish the nonlinear estimates involving Besov Gevrey regularity that will be needed to bound the right-hand side of (2.3). We shall actually prove more general estimates both because they are of independent interest and since they will be used in the next section, when we shall generalize the statement of Theorem 1.2 to $L^p$ related Besov spaces.

The first part of this subsection will be devoted to product estimates, and will require us to use Bony’s decomposition and to prove new continuity results for the paraproduct and remainder operators.

Recall that, at the formal level, the product of two tempered distributions $f$ and $g$ may be decomposed into
\[
fg = Tf g + T_g f + R(f, g)
\]

with
\[
\tag{2.17} T_f g = \sum_{j \in \mathbb{Z}} S_{j-1} f \Delta_j g \quad \text{and} \quad R(f, g) = \sum_{j \in \mathbb{Z}} \sum_{|j' - j| \leq 1} \Delta_j f \Delta_{j'} g.
\]

The above operators $T$ and $R$ are called “paraproduct” and “remainder,” respectively. The decomposition (2.17) has been first introduced by J.-M. Bony in [5]. The paraproduct and remainder operators possess a lot of continuity properties in Besov spaces (see Chap. 2 in [3]), which motivates their introduction here.

Throughout this section, we fix some real number $c > 0$ and agree that $F(t) \triangleq e^{\sqrt{\pi} \Lambda_1} f$ and $G(t) \triangleq e^{\sqrt{\pi} \Lambda_1} g$ for $t \geq 0$ (dependence on $t$ will be often omitted).

Let us start with paraproduct and remainder estimates in the case where all the Lebesgue indices lie in the range $[1, \infty[$.

**Proposition 2.1.** Let $s \in \mathbb{R}$, $1 < p, p_1, p_2 < \infty$, and $1 \leq r, r_1, r_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$ and $1/r = 1/r_1 + 1/r_2$. There exists a constant $C$ such that for any $f, g$ and $\sigma > 0$ (or $\sigma \geq 0$ if $r_1 = 1$),
\[
\tag{2.18} \|e^{\sqrt{\pi} \Lambda_1} T_f g\|_{\dot{B}^{-\sigma}_{p,r}} \leq C\|F\|_{\dot{B}^{-\sigma}_{p_1,r_1}}\|G\|_{\dot{B}^{-\sigma}_{p_2,r_2}},
\]
and for any $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 > 0$,
\[
\tag{2.19} \|e^{\sqrt{\pi} \Lambda_1} R(f, g)\|_{\dot{B}^{s_1}_{p,r_1}} \leq C\|F\|_{\dot{B}^{s_1}_{p_1,r_1}}\|G\|_{\dot{B}^{s_2}_{p_2,r_2}}.
\]
In order to prove our main results for the Korteweg system, we will need sometimes the estimates corresponding to the case \( p_2 = p \) that are contained in the following statement.

**Proposition 2.2.** Assume that \( 1 < p, q < \infty \) and that \( 1 \leq r, r_1, r_2 \leq \infty \) fulfill \( 1/r = 1/r_1 + 1/r_2 \). There exists a constant \( C \) such that for any \( f, g \) and \( \sigma > 0 \) (or \( \sigma \geq 0 \) if \( r_1 = 1 \)),

\[
\| e^{\sqrt{\alpha}t} T f g \|_{B^{r-\sigma}_{p,r}} \leq C \| F \|_{B^r_{q,\gamma_1}} \| G \|_{B^{r_1}_{p_1,r_1}},
\]

and for any \( s_1, s_2 \in \mathbb{R} \) with \( s_1 + s_2 > 0 \),

\[
\| e^{\sqrt{\alpha}t} R(f, g) \|_{B^{s_1+s_2}_{p,r}} \leq C \| F \|_{B^{s_1}_{p_1,r_1}} \| G \|_{B^{s_2}_{p_2,r_2}}.
\]

**Proof of Proposition 2.1.** By the definition of the paraproduct and of \( B_1 \), we have

\[
e^{\sqrt{\alpha}t} T f g = \sum_{j \in \mathbb{Z}} W_j \quad \text{with} \quad W_j \triangleq B_1(\dot{S}_{j-1} F, \dot{\Delta}_j G).
\]

As no Lebesgue index reaches the endpoints, thanks to Lemma 2.4, we obtain

\[
\| W_j \|_{L^p} \lesssim \| \dot{S}_{j-1} F \|_{L^{p_1}} \| \dot{\Delta}_j G \|_{L^{p_2}} \lesssim \left( \sum_{j' \leq j-2} \| \dot{\Delta}_j F \|_{L^{p_1}} \right) \| \dot{\Delta}_j G \|_{L^{p_2}}.
\]

Therefore, it holds that

\[
2^{j(s-\sigma)} \| W_j \|_{L^p} \lesssim 2^{js} \| \dot{\Delta}_j G \|_{L^{p_2}} \sum_{j' \leq j-2} 2^{\sigma(j'-j)} 2^{-\sigma j'} \| \dot{\Delta}_j F \|_{L^{p_1}}.
\]

As \( \sigma > 0 \), Hölder and Young inequalities for series enable us to obtain

\[
\left( 2^{j(s-\sigma)} \| W_j \|_{L^p} \right)_{L^r} \lesssim \| F \|_{B^{s}_{p_1,r_1}} \| G \|_{B^{s}_{p_2,r_2}},
\]

and one may conclude to (2.18) by using Proposition A.1.

In the case \( \sigma = 0 \), one just has to use the fact that

\[
\| \dot{S}_{j-1} F \|_{L^{p_1}} \lesssim \| F \|_{L^{p_1}} \lesssim \| F \|_{\dot{B}^{0}_{p_1,1}}.
\]

Let us now turn to the remainder: we have for all \( k \in \mathbb{Z} \),

\[
\dot{\Delta}_k e^{\sqrt{\alpha}t} R(f, g) = \sum_{j \geq k-2} \sum_{|j-j'| \leq 1} \dot{\Delta}_k B_1(\dot{\Delta}_j F, \dot{\Delta}_j G).
\]

Taking the \( L^p \) norm with respect to the spatial variable, we deduce by Lemma 2.4 that

\[
\| \dot{\Delta}_k e^{\sqrt{\alpha}t} R(f, g) \|_{L^p} \lesssim \sum_{j \geq k-2} \sum_{|j-j'| \leq 1} \| \dot{\Delta}_j F \|_{L^{p_1}} \| \dot{\Delta}_j G \|_{L^{p_2}}.
\]

Then everything now works as for estimating classical Besov norms:

\[
2^{k(s_1+s_2)} \| \dot{\Delta}_k e^{\sqrt{\alpha}t} R(f, g) \|_{L^p} \lesssim \sum_{j \geq k-2} \sum_{|j-j'| \leq 1} 2^{k(j-j')2s_1+2j_1} \| \dot{\Delta}_j F \|_{L^{p_1}} 2^{(j-j')s_2} 2^{j_1} \| \dot{\Delta}_j G \|_{L^{p_2}},
\]

and Young’s and Hölder inequalities for series allow to get (2.19) as \( s_1 + s_2 > 0 \). \( \square \)
Proof of Proposition 2.2. We argue as in the previous proof, except that one intermediate step is needed for bounding the general term of the paraproduct or remainder. The key point of course is to bound in $L^p$ the general term of $B_t$ in (2.16), while the Lebesgue exponents do not fulfill the conditions of Lemma 2.4.

As an example, let us prove Inequality (2.20) for $\sigma = 0$. We write, combining Hölder and Bernstein inequality (A.7), and the properties of continuity of operators $K_\alpha$ and $Z_{l,\alpha,\beta}$,

$$
\| K_\alpha (Z_{l,\alpha,\beta} \dot{S}_{j-1} F \cdot Z_{l,\alpha,\beta} \dot{\Delta}_j G) \|_{L^p} \lesssim \| Z_{l,\alpha,\beta} \dot{S}_{j-1} F \cdot Z_{l,\alpha,\beta} \dot{\Delta}_j G \|_{L^p} \\
\lesssim \sum_{j' \leq j-2} \| \dot{\Delta}_{j'} Z_{l,\alpha,\beta} F \|_{L^\infty} \| Z_{l,\alpha,\beta} \dot{\Delta}_j G \|_{L^p} \\
\lesssim \sum_{j' \leq j-2} 2^{j'\frac{d}{q}} \| \dot{\Delta}_{j'} Z_{l,\alpha,\beta} F \|_{L^q} \| Z_{l,\alpha,\beta} \dot{\Delta}_j G \|_{L^p} \\
\lesssim \sum_{j' \leq j-2} 2^{j'\frac{d}{q}} \| \dot{\Delta}_{j'} F \|_{L^q} \| \dot{\Delta}_j G \|_{L^p}.
$$

From this, we get

$$
2^{js} \| W_j \|_{L^p} \lesssim \| F \|_{\dot{B}_{q,1}^s} 2^{js} \| \dot{\Delta}_j G \|_{L^p}.
$$

We then obtain (2.20) for $\sigma = 0$ thanks to Proposition A.1. \qed

Combining the above propositions with functional embeddings and Bony’s decomposition, one may deduce the following Gevrey product estimates in Besov spaces that will be of extensive use in what follows:

Proposition 2.3. Let $1 < p < \infty$, $s_1, s_2 \leq d/p$ with $s_1 + s_2 > d \max(0, -1 + 2/p)$. There exists a constant $C$ such that the following estimate holds true:

$$
\| e^{\sqrt{\alpha} \Lambda_1} (f g) \|_{\dot{B}^{s_1 + s_2 - \frac{d}{p}}_{p,1}} \leq C \| F \|_{\dot{B}^s_{p,1}} \| G \|_{\dot{B}^s_{p,1}}.
$$

Proof. In light of decomposition (2.17), we have

$$
e^{\sqrt{\alpha} \Lambda_1} (f g) = e^{\sqrt{\alpha} \Lambda_1} T_j f g + e^{\sqrt{\alpha} \Lambda_1} T_g f + e^{\sqrt{\alpha} \Lambda_1} R(f, g).
$$

Then (2.20) and standard embedding imply that

$$
\| e^{\sqrt{\alpha} \Lambda_1} T_j f g \|_{\dot{B}^{s_1 + s_2 - \frac{d}{p}}_{p,1}} \lesssim \| F \|_{\dot{B}^s_{p,1}} \| G \|_{\dot{B}^s_{p,1}}, \\
\| e^{\sqrt{\alpha} \Lambda_1} T_g f \|_{\dot{B}^{s_1 + s_2 - \frac{d}{p}}_{p,1}} \lesssim \| G \|_{\dot{B}^s_{p,1}} \| F \|_{\dot{B}^s_{p,1}}.
$$

It is easy to deal with the remainder if $p \geq 2$: thanks to embeddings and (2.19), we have

$$
\| e^{\sqrt{\alpha} \Lambda_1} R(f, g) \|_{\dot{B}^{s_1 + s_2 - \frac{d}{p}}_{p,1}} \lesssim \| e^{\sqrt{\alpha} \Lambda_1} R(f, g) \|_{\dot{B}^{s_1 + s_2}_{p/2,1}} \lesssim \| F \|_{\dot{B}^s_{p,1}} \| G \|_{\dot{B}^s_{p,1}}.
$$
If $1 < p < 2$, then we use instead that $\dot{B}^{\sigma + d(\frac{1}{p_0} - \frac{1}{p})}_{p_0,1} \hookrightarrow \dot{B}^\sigma_{p,1}$ for all $1 < p_0 < p$, and Inequality (2.19) thus implies that

$$\left\| e^{\sqrt{d} \Lambda_1} R(f,g) \right\|_{\dot{B}^{s_1 + s_2 - \frac{d}{p}}_{p,1}} \lesssim \left\| e^{\sqrt{d} \Lambda_1} R(f,g) \right\|_{\dot{B}^{s_1 + s_2 - \frac{2d}{p} + \frac{d}{p_0}}_{p_0,1}}$$

$$\lesssim \left\| F \right\|_{\dot{B}^{s_1 - \frac{2d}{p} + \frac{d}{p_0}}_{p_2,1}} \left\| G \right\|_{\dot{B}^{s_2}_{p,1}}$$

$$\lesssim \left\| F \right\|_{\dot{B}^{s_1 - \frac{2d}{p} + d(\frac{1}{p} - \frac{1}{p_0})}_{p,1}} \left\| G \right\|_{\dot{B}^{s_2}_{p,1}} = \left\| F \right\|_{\dot{B}^{s_1}_{p_1,1}} \left\| G \right\|_{\dot{B}^{s_2}_{p_1,1}},$$

whenever $1/p + 1/p_2 = 1/p_0$, and $p_2 \geq p$. Since $p < 2$, it is clear that those two conditions may be satisfied if taking $p_0$ close enough to $1$. \qed

**Remark 2.1.** Proposition 2.3 ensures that the space $\{ f \in \dot{B}^\sigma_{p,1} : e^{\sqrt{d} \Lambda_1} f \in \dot{B}^\sigma_{p,1} \}$ is an algebra whenever $1 < p < \infty$.

The previous estimates can be adapted to the Chemin-Lerner’s spaces $\tilde{L}^q_T(\dot{B}^\sigma_{p,r})$ defined in the Appendix. For example, we have the following result.

**Proposition 2.4.** Let $1 < p < \infty$ and $1 \leq q, q_1, q_2 \leq \infty$ such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. If $\sigma_1, \sigma_2 \leq d/p$ and $\sigma_1 + \sigma_2 > d \max(0, -1 + 2/p)$, then there exists a constant $C > 0$ such that for all $T \geq 0$,

$$\left\| e^{\sqrt{d} \Lambda_1} (f g) \right\|_{\tilde{L}^{q}_{T}(\dot{B}^{\sigma}_{p,1})} \leq C \left\| F \right\|_{\tilde{L}^{q_1}_{T}(\dot{B}^{\sigma_1}_{p_1,1})} \left\| G \right\|_{\tilde{L}^{q_2}_{T}(\dot{B}^{\sigma_2}_{p_2,1})}. \quad (2.26)$$

In order to prove Theorem 1.2, we need not only bilinear estimates involving Gevrey-Besov regularity, but also composition estimates by real analytic functions.

**Lemma 2.5.** Let $\Phi$ be a real analytic function in a neighborhood of 0, such that $\Phi(0) = 0$. Let $1 < p < \infty$ and $-\min(\frac{d}{p}, \frac{d}{p'}) < s \leq \frac{d}{p}$ with $\frac{1}{p'} = 1 - \frac{1}{p}$. There exist two constants $R_0$ and $D$ depending only on $p$, $d$ and $\Phi$ such that if for some $T > 0$,

$$\left\| e^{\sqrt{d} \Lambda_1} z \right\|_{\tilde{L}^{\infty}_{T}(\dot{B}^{s}_{p,1})} \leq R_0, \quad (2.27)$$

then we have for all $q \in [1, \infty]$,

$$\left\| e^{\sqrt{d} \Lambda_1} \Phi(z) \right\|_{\tilde{L}^{q}_{T}(\dot{B}^{s}_{p,1})} \leq D \left\| e^{\sqrt{d} \Lambda_1} z \right\|_{\tilde{L}^{q}_{T}(\dot{B}^{s}_{p,1})}. \quad (2.28)$$

**Remark 2.2.** For proving our main results, we shall use the above lemma with $s = \frac{d}{p}$ or $s = \frac{d}{p} - 1$. Note that the former case requires that $d \geq 2$ and $1 < p < 2d$.

**Proof.** Let us write

$$\Phi(z) = \sum_{n=1}^{+\infty} a_n z^n$$

and denote by $R_\Phi > 0$ the convergence radius of the series. For all $t \geq 0$ (as usual $Z = e^{\sqrt{d} \Lambda_1} z$) we have

$$e^{\sqrt{d} \Lambda_1} \Phi(z) = \sum_{n=1}^{+\infty} a_n e^{\sqrt{d} \Lambda_1} z^n = \sum_{n=1}^{+\infty} a_n e^{\sqrt{d} \Lambda_1} (e^{-\sqrt{d} \Lambda_1} Z)^n, \quad (2.29)$$
which implies from (2.26), by induction and thanks to the condition on s, that
\[
\|e^{\sqrt{\alpha}A_1} \Phi(z)\|_{L^p_T (B^s_{p,1})} \leq C \sum_{n=1}^{+\infty} |a_n| \left( C \|e^{\sqrt{\alpha}A_1} z\|_{L^\infty_T (B^s_{p,1})} \right)^{n-1} \|e^{\sqrt{\alpha}A_1} z\|_{L^p_T (B^s_{p,1})}
\]
\[
\leq \Phi(C \|e^{\sqrt{\alpha}A_1} z\|_{L^\infty_T (B^s_{p,1})}) \|e^{\sqrt{\alpha}A_1} z\|_{L^p_T (B^s_{p,1})},
\]
where we define \( \Phi(z) = \sum_{n=1}^{+\infty} |a_n| z^{n-1} \). So when \( \|e^{\sqrt{\alpha}A_1} z\|_{L^\infty_T (B^s_{p,1})} \leq \frac{R_0}{2C} \) we have (2.28) with \( D = \sup_{z \in \bar{B}(0, \frac{R_0}{2C})} |\Phi(z)| \).

Let us end this section with a variant of the previous result:

**Lemma 2.6.** Let \( \Phi \) be a real analytic function in a neighborhood of 0. Let \( 1 < p < \infty \) and \( -\min\left(\frac{d}{p}, \frac{d}{q}\right) < s \leq \frac{d}{p} \). There exist two constants \( R_0 \) and \( D \) depending only on \( p, d \) and \( \Phi \) such that if for some \( T > 0 \),

\[
\max_{i=1,2} \|e^{\sqrt{\alpha}A_1} z_i\|_{L^p_T (B^s_{p,1})} \leq R_0,
\]
then we have

\[
\|e^{\sqrt{\alpha}A_1} (\Phi(z_2) - \Phi(z_1))\|_{L^p_T (B^s_{p,1})} \leq D \|e^{\sqrt{\alpha}A_1} (z_2 - z_1)\|_{L^p_T (B^s_{p,1})}.
\]

**Proof.** With the same notations as before, Proposition 2.4 with \( (\sigma_1, \sigma_2) = (s, \frac{d}{p}) \) yields:

\[
\|e^{\sqrt{\alpha}A_1} (\Phi(z_2) - \Phi(z_1))\|_{L^p_T (B^s_{p,1})} \leq \sum_{n=1}^{+\infty} |a_n| \|e^{\sqrt{\alpha}A_1} (z_2^n - z_1^n)\|_{L^p_T (B^s_{p,1})}
\]
\[
\leq \sum_{n=1}^{+\infty} |a_n| \|e^{\sqrt{\alpha}A_1} (z_2^n - z_1^n) \sum_{k=0}^{n-1} \bar{\Phi}(\sum_{k=0}^{n-1} z_2^k z_1^{n-1-k})\|_{L^p_T (B^s_{p,1})}
\]
\[
\leq C \sum_{n=1}^{+\infty} |a_n| \|e^{\sqrt{\alpha}A_1} (z_2^n - z_1^n) \sum_{k=0}^{n-1} \|e^{\sqrt{\alpha}A_1} (z_2^k z_1^{n-1-k})\|_{L^p_T (B^s_{p,1})}.
\]

By induction, we get (using \( n \leq 2^{n-1} \))

\[
\|e^{\sqrt{\alpha}A_1} (\Phi(z_2) - \Phi(z_1))\|_{L^p_T (B^s_{p,1})} \leq C \|e^{\sqrt{\alpha}A_1} (z_2 - z_1)\|_{L^p_T (B^s_{p,1})}
\]
\[
\leq C \|e^{\sqrt{\alpha}A_1} (z_2 - z_1)\|_{L^p_T (B^s_{p,1})} \sum_{n=1}^{+\infty} |a_n| \sum_{k=0}^{n-1} C^{n-1} \|e^{\sqrt{\alpha}A_1} z_1\|_{L^p_T (B^s_{p,1})} \|e^{\sqrt{\alpha}A_1} z_2^{n-1-k}\|_{L^p_T (B^s_{p,1})}
\]
\[
\leq C \|e^{\sqrt{\alpha}A_1} (z_2 - z_1)\|_{L^p_T (B^s_{p,1})} \sum_{n=1}^{+\infty} |a_n| \left( 2C \max_{i=1,2} \|e^{\sqrt{\alpha}A_1} z_i\|_{L^p_T (B^s_{p,1})} \right)^{n-1}
\]
\[
\leq C \|e^{\sqrt{\alpha}A_1} (z_2 - z_1)\|_{L^p_T (B^s_{p,1})} \Phi(2C \max_{i=1,2} \|e^{\sqrt{\alpha}A_1} z_i\|_{L^p_T (B^s_{p,1})}).
\]

We conclude as before. \( \square \)
2.4. The proof of Theorem 1.2. One can now come back to the proof of Theorem 1.2. Recall the following estimate that has been shown in [11].

**Lemma 2.7.** Let \((a, u)\) be a solution in \(E\) of System (2.3). There exists a constant \(R_0 > 0\) such that if
\[
\|a\|_{L^\infty(B_{2,1}^d)} \leq R_0
\]
then one has the following a priori estimate:
\[
(2.32) \quad \| (a, u) \|_E \lesssim \| a \|_{B_{2,1}^{d-1}} + \| u_0 \|_{B_{2,1}^{d-1}} + (1 + \| (a, u) \|_E) \| (a, u) \|_E^2.
\]

We want to generalize it in the Gevrey regularity setting, getting the following result:

**Lemma 2.8.** Let \((a, u)\) be the global solution constructed in Theorem 1.1. Denote \(A \equiv e^{\sqrt{c_0} t} a\) and \(U \equiv e^{\sqrt{c_0} t} u\) where \(c_0\) is the constant of Lemma 2.1. There exists a constant \(R_0 > 0\) such that if
\[
(2.33) \quad \| A \|_{L^\infty(B_{2,1}^d)} \leq R_0,
\]
then we have
\[
(2.34) \quad \| (A, U) \|_E \lesssim \| a \|_{B_{2,1}^{d-1}} + \| u_0 \|_{B_{2,1}^{d-1}} + (1 + \| (A, U) \|_E) \| (A, U) \|_E^2.
\]

**Proof.** Apply \(\hat{\Delta}_q\) to (2.6)-(2.8) and repeat the procedure leading to Lemma 2.1. Multiplying by the factor \(e^{\sqrt{c_0} t} \| \xi \|_1\) we end up with
\[
| (\hat{\Delta}_q A, \hat{\Delta}_q \nabla A, \hat{\Delta}_q U)(t, \xi) | \leq C \left( e^{\sqrt{c_0} t} \| \xi \|_1 |(\hat{\Delta}_q a_0, \hat{\Delta}_q \nabla a_0, \hat{\Delta}_q u_0) | + e^{\sqrt{c_0} t} \int_0^t e^{-c_0 |\xi|^2 (t-\tau)} |(\hat{\Delta}_q f, \hat{\Delta}_q \nabla f, \hat{\Delta}_q g)(\tau, \xi) | d\tau \right).
\]

Taking the \(L^2\) norm, thanks to the Fourier-Plancherel theorem, we get for all \(t \geq 0\),
\[
| (\hat{\Delta}_q A, \hat{\Delta}_q \nabla A, \hat{\Delta}_q U)(t) |_{L^2} \leq \| e^{\sqrt{c_0} t} \|_1 |(\hat{\Delta}_q a_0, \hat{\Delta}_q \nabla a_0, \hat{\Delta}_q u_0) |_{L^2} + \int_0^t \| e^{\sqrt{c_0 (t-\tau)} \|_1 |(\hat{\Delta}_q f, \hat{\Delta}_q \nabla f, \hat{\Delta}_q g)(\tau) |_{L^2} d\tau
\]
and thanks to Lemmas 2.2 and 2.3, and to the properties of localization of \(\hat{\Delta}_q\), we obtain, denoting \(c_1 \equiv \frac{n}{32} c_0\),
\[
(2.35) \quad \| (\hat{\Delta}_q A, \hat{\Delta}_q \nabla A, \hat{\Delta}_q U)(t) |_{L^2} \leq C \left( e^{-c_1 t^{2q}} \| (\hat{\Delta}_q a_0, \hat{\Delta}_q \nabla a_0, \hat{\Delta}_q u_0) |_{L^2} + \int_0^t \| e^{-c_1 (t-\tau)^{2q}} \| (\hat{\Delta}_q f, \hat{\Delta}_q \nabla f, \hat{\Delta}_q g)(\tau) |_{L^2} d\tau \right).
\]

Therefore, multiplying by \(2^{n \left( \frac{d}{2} - 1 \right)}\) and summing on \(q \in \mathbb{Z}\), we obtain that for all \(t \geq 0\),
\[
\| (A, U) \|_{E_t} \equiv \| (A, \nabla A, U) \|_{L^\infty_t(B_{2,1}^{d-1})} + \| (A, \nabla A, U) \|_{L^1_t(B_{2,1}^{d-1})} \leq C \left( \| (a_0, \nabla a_0, u_0) \|_{B_{2,1}^{d-1}} + \| (F, \nabla F, G) \|_{L^1_t(B_{2,1}^{d-1})} \right).
\]

We are left with estimating the external force terms as in the classical Besov case, but using the laws suited to Gevrey regularity.
Regarding \( F \), we have thanks to Proposition 2.3,
\[
\int_0^t \| F(\tau) \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq \int_0^t \| e^{\sqrt{t} \Lambda_1}(au) \|_{B^\frac{d}{2},1}_{2,1} d\tau \\
\leq C \int_0^t \| A \|_{B^\frac{d}{2},1}_{2,1} \| U \|_{B^\frac{d}{2},1}_{2,1} d\tau \\
\leq C \| A \|_{L^2_t(B^\frac{d}{2},1)} \| U \|_{L^2_t(B^\frac{d}{2},1)}.
\]

Estimating \( \nabla F \) is also based on Proposition 2.3, after using that \( f = -u \cdot \nabla a - \nabla (a \div u) \).
Then one may write that
\[
\int_0^t \| \nabla F(\tau) \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq \int_0^t \| e^{\sqrt{t} \Lambda_1}(u \cdot \nabla a + a \div u) \|_{B^\frac{d}{2},1}_{2,1} d\tau \\
\leq C \int_0^t \left( \| U \|_{B^\frac{d}{2},1}_{2,1} \| \nabla A \|_{B^\frac{d}{2},1}_{2,1} + \| A \|_{B^\frac{d}{2},1}_{2,1} \| \div U \|_{B^\frac{d}{2},1}_{2,1} \right) d\tau \\
\leq C \left( \| U \|_{L^2_t(B^\frac{d}{2},1)} \| A \|_{L^\infty_t(B^\frac{d}{2},1)} + \| A \|_{L^\infty_t(B^\frac{d}{2},1)} \| U \|_{L^1_t(B^\frac{d}{2},1)} \right).
\]

One can now turn to \( g \): using Proposition 2.3 with \( (s_1, s_2) = (\frac{d}{2} - 1, \frac{d}{2}) \) yields
\[
\int_0^t \| e^{\sqrt{t} \Lambda_1} g_1 \|_{B^\frac{d}{2},1}_{2,1} d\tau = \int_0^t \| e^{\sqrt{t} \Lambda_1}(u \cdot \nabla u) \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq C \| U \|^2_{L^2_t(B^\frac{d}{2},1)}.
\]

Using the same product law together with Lemma 2.5, and under the following condition that depends on the convergence radii of the analytic functions appearing in \( g \):
\[
\| e^{\sqrt{t} \Lambda_1} a \|_{L^\infty(B^\frac{d}{2},1)} \leq \frac{1}{2C} \min(R_I, R_{\bar{r}}, R_\lambda, R_{\bar{\lambda}}, R_J),
\]
we get that
\[
\int_0^t \| e^{\sqrt{t} \Lambda_1} g_3 \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq C \| A \|_{L^\infty_t(B^\frac{d}{2},1)} \| U \|_{L^1_t(B^\frac{d}{2},1)}.
\]

Similarly, we obtain:
\[
\begin{cases}
\int_0^t \| e^{\sqrt{t} \Lambda_1} g_2 \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq C(1 + \| A \|_{L^\infty_t(B^\frac{d}{2},1)}) \| A \|_{L^\infty_t(B^\frac{d}{2},1)} \| U \|_{L^1_t(B^\frac{d}{2},1)}, \\
\int_0^t \| e^{\sqrt{t} \Lambda_1} g_4 \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq C \| A \|^2_{L^2_t(B^\frac{d}{2},1)} , \\
\int_0^t \| e^{\sqrt{t} \Lambda_1} \nabla(\tilde{\kappa}(a) \Delta a) \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq C \| A \|_{L^\infty_t(B^\frac{d}{2},1)} \| A \|_{L^1_t(B^\frac{d}{2},1)}.
\end{cases}
\]

We have to be careful with the second part of \( g_5 \): as Lemma 2.5 requires the regularity index to be less than \( \frac{d}{2} \), we have to rewrite the term into:
\[
\int_0^t \| e^{\sqrt{t} \Lambda_1} \nabla(\tilde{\kappa}(a) \cdot \nabla a) \|_{B^\frac{d}{2},1}_{2,1} d\tau \leq \int_0^t \| e^{\sqrt{t} \Lambda_1} (\tilde{\kappa}'(a) \nabla a \cdot \nabla a) \|_{B^\frac{d}{2},1}_{2,1} d\tau \\
\leq C(1 + \| A \|_{L^\infty_t(B^\frac{d}{2},1)}) \| A \|^2_{L^2_t(B^\frac{d}{2},1)}.
\]

Putting all the above estimates together, we conclude the proof of Lemma 2.8. \( \square \)
Now we are able to complete the proof of Theorem 1.2 by means of the fixed point theorem. Let \( W(t) \) be the semi-group associated to the left-hand side of (2.3). According to the standard Duhamel formula, one has

\[
\begin{pmatrix}
  a(t)
  \\
  u(t)
\end{pmatrix} = \begin{pmatrix}
  a_L \\
  u_L
\end{pmatrix} + \int_0^t W(t-\tau) \begin{pmatrix}
  f(\tau) \\
  g(\tau)
\end{pmatrix} \, d\tau \quad \text{with} \quad \begin{pmatrix}
  a_L \\
  u_L
\end{pmatrix} \triangleq W(t) \begin{pmatrix}
  a_0 \\
  u_0
\end{pmatrix}.
\]

Define the functional \( \Psi_{(a_L,u_L)} \) in a neighborhood of zero in the space \( F \) by

\[
\Psi_{(a_L,u_L)}(\bar{a}, \bar{u}) = \int_0^t W(t-\tau) \begin{pmatrix}
  f(a_L + \bar{a}, u_L + \bar{u}) \\
  g(a_L + \bar{a}, u_L + \bar{u})
\end{pmatrix} \, d\tau.
\]

To get the existence part of the theorem, it suffices to show that \( \Psi_{(a_L,u_L)} \) has a fixed point in \( F \). Our procedure is divided into two steps: stability of some closed ball \( B(0, r) \) of \( F \) by \( \Psi_{(a_L,u_L)} \), then contraction in that ball. As those two properties have been established for the space \( E \) in [11], we shall concentrate on proving suitable bounds for \( e^{\sqrt{\sigma}A_1} \Psi_{(a_L,u_L)}(\bar{a}, \bar{u}) \).

**Step 1:** Stability of some ball \( B(0, r) \). We prove that the ball \( B(0, r) \) of \( F \) is stable under \( \Psi_{(a_L,u_L)} \), provided the radius \( r \) is small enough. Let \( a = a_L + \bar{a} \) and \( u = u_L + \bar{u} \).

- If the data fulfill (1.5), then from Lemmas 2.7, 2.8 and the definition of the space \( F \), we get

\[
\|e^{\sqrt{\sigma}A_1}(a_L, u_L)\|_E \leq C(\|a_0\|_{B_{\frac{d}{4}}^1} \cap B_{\frac{d}{4}}^1 + \|u_0\|_{B_{\frac{d}{4}}^1}) \leq C\eta,
\]

and

\[
\|e^{\sqrt{\sigma}A_1}\Psi_{(a_L,u_L)}(\bar{a}, \bar{u})\|_E \leq C\|e^{\sqrt{\sigma}A_1}(f, \nabla f, g)\|_{L_1(\frac{d}{2}+1)}.
\]

Assuming that \( r \) is so small that:

\[
\|e^{\sqrt{\sigma}A_1} a\|_{L^\infty(\frac{d}{2}+1)} \leq \|(a, u)\|_F \leq r \leq \frac{1}{2C} \min(R_I, R_{\bar{\mu}}, R_{\bar{\lambda}}, R_{\bar{\kappa}}, R_J),
\]

and also that \( 2C\eta \leq r \), we get

\[
\|\Psi_{(a_L,u_L)}(\bar{a}, \bar{u})\|_F \leq C\|e^{\sqrt{\sigma}A_1}(a_L + \bar{a}, u_L + \bar{u})\|^2 \left(1 + \|(a_L + \bar{a}, u_L + \bar{u})\|_F\right)
\]

\[
\leq C(C\eta + r)^2(1 + C\eta + r) \leq C\frac{9}{4}r^2 \left(1 + \frac{3}{2}r\right).
\]

Finally, choosing \((r, \eta)\) such that

\[
r \leq \min \left(1, \frac{8}{45C}, \frac{1}{2C} \min(R_I, R_{\bar{\mu}}, R_{\bar{\lambda}}, R_{\bar{\kappa}}, R_J)\right) \quad \text{and} \quad \eta \leq \frac{r}{2C},
\]

assumption (2.43) is satisfied. Hence, it follows from (2.44) that

\[
\Psi_{(a_L,u_L)}(B(0, r)) \subset B(0, r).
\]

**Step 2:** The contraction property. Let \((\bar{a}_1, \bar{u}_1)\) and \((\bar{a}_2, \bar{u}_2)\) be in \( B(0, r) \). Denote \( a_i = a_L + \bar{a}_i \) and \( u_i = u_L + \bar{u}_i \) for \( i = 1, 2 \). According to (2.40) and Lemmas 2.7, 2.8, we have

\[
\|\Psi_{(a_L,u_L)}(\bar{a}_2, \bar{u}_2) - \Psi_{(a_L,u_L)}(\bar{a}_1, \bar{u}_1)\|_F
\]

\[
= \|e^{\sqrt{\sigma}A_1} f(a_2, u_2) - f(a_1, u_1), \nabla f(a_2, u_2) - \nabla f(a_1, u_1), g(a_2, u_2) - g(a_1, u_1)\|_{L^1(\frac{d}{2}+1)}
\]
where $f$ and $g$ are defined in (2.4). All terms are estimated exactly as in the previous step except that we use in addition Lemma 2.6. Let us for example give details for $g_2 = (1 - I(a)) \text{div} (\tilde{\mu}(a) \cdot \nabla a))$ (assume that $\lambda = 0$ for conciseness):

$$
(2.45) \quad \|g_2(a_2, u_2) - g_2(a_1, u_1)\|_{L^1(B_{d,1}^{\frac{d}{2} - 1})} \leq \| (I(a_2) - I(a_1)) \text{div} (\tilde{\mu}(a_2) \cdot \nabla a_2)\|_{L^1(B_{d,1}^{\frac{d}{2} - 1})} \\
+ \| (1 - I(a_1)) \text{div} ((\tilde{\mu}(a_2) - \tilde{\mu}(a_1)) \nabla u_2 + \tilde{\mu}(a_1) \cdot \nabla (u_2 - u_1))\|_{L^1(B_{d,1}^{\frac{d}{2} - 1})}.
$$

Following the previous computations (together with Lemma 2.6 for the first and second terms), we obtain, if $r$ and $\eta$ are small enough,

$$
\|\Psi(\bar{a}_2, \bar{u}_2) - \Psi(\bar{a}_1, \bar{u}_1)\|_F \\
\leq C\left( \|(a_1, u_1)\|_F + \|(a_2, u_2)\|_F \right) \left( 1 + \|(a_1, u_1)\|_F + \|(a_2, u_2)\|_F \right) \|(\bar{a}_2 - \bar{a}_1, \bar{u}_2 - \bar{u}_1)\|_F \\
\leq 4C(r + C\eta)(1 + r + C\eta)(\|(\bar{a}_2 - \bar{a}_1, \bar{u}_2 - \bar{u}_1)\|_F \\
\leq \frac{1}{4} \|(\bar{a}_2 - \bar{a}_1, \bar{u}_2 - \bar{u}_1)\|_F.
$$

Hence, combining the two steps completes the proof of Theorem 1.2. \hfill \Box

3. The $L^p$ framework

Our aim here is to extend Theorem 1.2 to more general critical Besov spaces. Recall that for the classical compressible Navier-Stokes equations, the first two authors [6] and Chen-Miao-Zhang [8] established a global existence result for small data in $L^p$ type critical Besov spaces. The proofs therein are based on the study of the parabolized system combined with a Lagrangian change of coordinates. A more elementary method has been proposed afterward by B. Haspot in [17]. It relies on the introduction of some suitable effective velocity that, somehow, allows to uncouple the velocity equation from the mass equation.

In the present section, by combining Haspot’s approach with estimates in the same spirit as in the previous section, we shall not only extend the critical regularity result in $L^p$ spaces to the capillary case, but also obtain Gevrey analytic regularity:

**Theorem 3.1.** Assume that the functions $\kappa$, $\lambda$, $\mu$ and $P$ are real analytic and that the condition $P'(\overline{\gamma}) > 0$ is fulfilled. Let $p \in [2, \min(4, 2d/(d-2))]$ with, additionally, $p \neq 4$ if $d = 2$. There exists an integer $k_0 \in \mathbb{N}$ and a real number $\eta > 0$ depending only on the functions $\kappa$, $\lambda$, $\mu$, and $P$, and on $p$ and $d$, such that if one defines the threshold between low and high frequencies as in (A.11), if $a_0 \in \tilde{B}_{p,1}^{\frac{d}{2}}$ and $u_0 \in \tilde{B}_{p,1}^{\frac{d}{2} - 1}$ with, besides, $(a^\ell, u^\ell_0)$ in $\tilde{B}_{d,1}^{\frac{d}{2} - 1}$ satisfy

$$
X_{p,0} \triangleq \|(a_0, u_0)\|_{\tilde{B}_{d,1}^{\frac{d}{2} - 1}} + \|a_0\|_{\tilde{B}_{p,1}^{\frac{d}{2}}} + \|u_0\|_{\tilde{B}_{p,1}^{\frac{d}{2} - 1}} \leq \eta,
$$

then (2.3) has a unique global-in-time solution $(a, u)$ in the space $X_p$ defined by

$$
X_p \triangleq \{(a, u) | (a, u)^\ell \in \tilde{C}_b(\mathbb{R}_+; \tilde{B}_{d,1}^{\frac{d}{2} - 1}) \cap L^1(\mathbb{R}_+; \tilde{B}_{d,1}^{\frac{d}{2} + 1}), a^h \in \tilde{C}_b(\mathbb{R}_+; \tilde{B}_{p,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+; \tilde{B}_{d,1}^{\frac{d}{2} + 2}), u^h \in \tilde{C}_b(\mathbb{R}_+; \tilde{B}_{p,1}^{\frac{d}{2} - 1}) \cap L^1(\mathbb{R}_+; \tilde{B}_{d,1}^{\frac{d}{2} + 1}) \}.
$$

Furthermore, there exists a constant $c_0$ so that $(a, u)$ belongs to the space

$$
Y_p \triangleq \{(a, u) \in X_p | e^{\sqrt{\mu} t}A_1(a, u) \in X_p \}.
$$
Remark 3.1. In the physical dimensions $d = 2, 3$, Condition (3.1) allows us to consider the case $p > d$, and the velocity regularity exponent $d/p - 1$ thus becomes negative. Therefore, our result applies to large highly oscillating initial velocities (see e.g. [6] for more explanations).

3.1. Global estimates in $X_p$ for (2.3). As in Section 2, the proof of Theorem 3.1 is based on the fixed point theorem in complete metric spaces. Another important ingredient is the following endpoint maximal regularity property of the heat equation with complex diffusion coefficient.

Lemma 3.1. Let $T > 0$, $s \in \mathbb{R}$ and $1 \leq \rho_2, p, r \leq \infty$. Let $u$ satisfy

$$
\begin{cases}
\partial_t u - \beta \Delta u = f, \\
u|_{t=0} = u_0(x),
\end{cases}
$$

where $\beta \in \mathbb{C}$ is a complex number with $\text{Re} \beta > 0$. Then, there exists a constant $C$ depending only on $d$ and such that for all $\rho_1 \in [\rho_2, \infty]$, one has

$$
\|e^{\beta t \Delta} \Delta_0 z\|_{L^p} \leq Ce^{-\text{Re} \beta |t|^{2j}}\|\hat{\Delta}_j z\|_{L^p}, \quad t \geq 0, \quad j \in \mathbb{Z}.
$$

Indeed, using a suitable rescaling, it suffices to prove (3.4) for $j = 0$. Now, if we fix some smooth function $\tilde{\varphi}$ compactly supported away from 0 and with value 1 on the support of $\varphi$ ($\varphi$ is the function we used in the Littlewood-Paley decomposition, see Appendix), then we may write

$$
e^{\beta \Delta} \hat{\Delta}_0 z = \mathcal{F}^{-1}\left(\tilde{\varphi}e^{-\beta|\xi|^2}\hat{\Delta}_0 z\right)
$$

$$
= g_{\beta t} \ast \Delta_0 z \quad \text{with} \quad g_{\beta t}(x) \triangleq (2\pi)^{-d} \int e^{ix \cdot \xi}(\text{Id} - \Delta\xi)^d(\tilde{\varphi}(\xi)e^{-\beta|\xi|^2})d\xi.
$$

Then, integrating by parts, we discover that for all $x \in \mathbb{R}^d$, we have

$$
g_{\beta t}(x) = (1 + |x|^2)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi}(\text{Id} - \Delta\xi)^d(\tilde{\varphi}(\xi)e^{-\beta|\xi|^2})d\xi.
$$

Expanding the last term and using the fact that integration may be performed on some annulus, we get for some positive constants $c, C$ and $C'$,

$$
\|g_{\beta t}\|_{L^1} \leq C\|(1 + |\cdot|^2)^d g_{\beta t}\|_{L^\infty} \leq C'e^{-c\text{Re} \beta}.
$$

Then, using the convolution inequality $L^1 \ast L^p \to L^p$ yields (3.4). From it, we get

$$
\|\Delta_j u(t)\|_{L^p} \leq C\left(e^{-c\text{Re} \beta 2j\tau}\|\hat{\Delta}_j u_0\|_{L^p} + \int_0^t e^{-c\text{Re} \beta 2j(t-\tau)}\|\hat{\Delta}_j f(\tau)\|_{L^p} d\tau\right).
$$

Then, (3.3) follows from exactly the same calculations as in [3].

Combining Lemma 3.1 with the low frequency estimates of the previous section and introducing some suitable effective velocity will enable us to get the following result.
Theorem 3.2. There exists some constant \( C \) such that for all \( t \geq 0 \),
\[
\mathcal{X}_p(t) \leq C(\mathcal{X}_{p,0} + \mathcal{X}_p^2(t) + \mathcal{X}_p^3(t)),
\]
where
\[
\mathcal{X}_p(t) \triangleq \| (a, u) \|_{L^\infty_t(B^\frac{d-1}2_{x^1})} + \| (a, u) \|_{L^1_t(B^\frac{d+1}2_{x^1})} + \| a \|_{L^h_t(B^\frac{d-1}2_{x^1} \cap L^2_t(B^\frac{d+2}2_{x^1}) - 1)} \| p \|_{L^h_t(B^\frac{d-1}2_{x^1} \cap L^2_t(B^\frac{d+2}2_{x^1}) - 2)}.\]

Proof. We start from the linearized system (2.3):
\[
\begin{cases}
\partial_t a + div u = f, \\
\partial_t u - \mathcal{A} u + \nabla a - \mathcal{A} \nabla a = g.
\end{cases}
\]
The incompressible part of the velocity fulfills the heat equation
\[
\partial_t u - \mathcal{A} u = \mathcal{P} g.
\]
Hence, using the notation \( z_j := \Delta_j z \) for \( z \) in \( S' \), we see that there exists a constant \( c > 0 \) such that we have for all \( j \in \mathbb{Z}^* \),
\[
\| \mathcal{P} u_j(t) \|_{L^p} \leq Ce^{-c2^j t} \left( \| \mathcal{P} u_j(0) \|_{L^p} + \int_0^t e^{c2^j \tau} \| \mathcal{P} g_j \|_{L^p} d\tau \right),
\]
which leads for all \( T > 0 \), after summation on \( j \geq k_0 \), to
\[
\| \mathcal{P} u \|_{L^p_t(B^\frac{d-1}2_{x^1})} + \| \mathcal{P} u \|_{L^2_t(B^\frac{d+1}2_{x^1})} \leq \| \mathcal{P} u_0 \|_{L^p_t(B^\frac{d-1}2_{x^1})} + \| \mathcal{P} g \|_{L^h_t(B^\frac{d+2}2_{x^1})}.
\]

Let \( Q := I_d - \mathcal{P} = -(\Delta)^{-1} \nabla \text{div} \). To estimate \( a \) and \( Qu \), following Haspot in [17], we introduce the modified velocity
\[
v \triangleq Qu + (\Delta)^{-1} \nabla a
\]
so that \( \text{div} v = \text{div} u - a \), and discover that, since \( \bar{\lambda} + 2\mu = 1 \),
\[
\begin{cases}
\partial_t a + \nabla a + \Delta v = \nabla f, \\
\partial_t v - \Delta v - \pi \Delta \nabla a = Qg + (\Delta)^{-1} \nabla f + v - (\Delta)^{-1} \nabla a.
\end{cases}
\]
In the Fourier space, the eigenvalues of the associated matrix read (with the convention that \( \sqrt{\mathcal{T}} := i\sqrt{\mathcal{T}} \) if \( \mathcal{T} < 0 \)):
\[
\lambda^\pm(\xi) = \frac{1}{2} \left( 1 + |\xi|^2 \pm \sqrt{(1 - 4\pi)|\xi|^4 - 2|\xi|^2 + 1} \right).
\]
Therefore, in the high frequency regime, we expect that for any \( \pi > 0 \), the system has a parabolic behavior. This may be easily justified by considering suitable linear combinations of \( v \) and \( \nabla a \). Indeed, for all \( \alpha \in \mathbb{C} \), we have
\[
\partial_t (v + \alpha \nabla a) - (1 - \alpha) \Delta v - \pi \Delta \nabla a + \alpha \nabla a = \alpha \nabla f + Qg + (\Delta)^{-1} \nabla f + v - (\Delta)^{-1} \nabla a.
\]
Therefore, if we set
\[
w \triangleq v + \alpha \nabla a \quad \text{with} \quad \alpha \text{ satisfying } \alpha = \frac{\pi}{1 - \alpha},
\]
and remember that \( Qg = -(\Delta)^{-1} \nabla \text{div} g \), then we get
\[
\partial_t w - (1 - \alpha) \Delta w = -\alpha \nabla a + \alpha \nabla f + (\Delta)^{-1} \nabla (f - \text{div} g) + v - (\Delta)^{-1} \nabla a.
\]
A possible choice is \( \alpha = \frac{1}{2} \left( 1 + \sqrt{1 - 4\pi} \right) \). so that \( 1 - \alpha = \frac{1}{2} \left( 1 - \sqrt{1 - 4\pi} \right) \).

Obviously, the real part of \( 1 - \alpha \) is positive for any value of \( \pi \). Hence one can take advantage of (3.3) and get

\[
(\text{3.11}) \quad \|w\|_{L^p_t(B^d_{p,1})}^h + \|w\|_{L^1_t(B^d_{p,1})}^h \lesssim \|w\|_{L^1_t(B^d_{p,1})}^h + \|\alpha \nabla f + (-\Delta)^{-1} \nabla (f - \text{div} \, g)\|_{L^1_t(B^d_{p,1})}^h + \|v - \alpha \nabla a - (-\Delta)^{-1} \nabla a\|_{L^1_t(B^d_{p,1})}^h.
\]

Because \( \nabla (-\Delta)^{-1} \) is an homogeneous Fourier multiplier of degree \(-1\), we have

\[
\|\alpha \nabla f + (-\Delta)^{-1} \nabla (f - \text{div} \, g)\|_{L^1_t(B^d_{p,1})}^h \lesssim \|f\|_{L^1_t(B^d_{p,1})}^h + \|f - \text{div} \, g\|_{L^1_t(B^d_{p,1})}^h \lesssim \|f\|_{L^1_t(B^d_{p,1})}^h + \|f\|_{L^1_t(B^d_{p,1})}^h + \|g\|_{L^1_t(B^d_{p,1})}^h.
\]

Next, let us observe that, owing to the high frequency cut-off, we have for some universal constant \( C \),

\[
\|\alpha \nabla a\|_{L^1_t(B^d_{p,1})}^h \leq C 2^{-2k_0} \|a\|_{L^1_t(B^d_{p,1})}^h, \quad \|\nabla a\|_{L^1_t(B^d_{p,1})}^h \leq C 2^{-2k_0} \|v\|_{L^1_t(B^d_{p,1})}^h,
\]

\[
\|f\|_{L^1_t(B^d_{p,1})}^h \leq C 2^{-2k_0} \|f\|_{L^1_t(B^d_{p,1})}^h \quad \text{and} \quad \|(-\Delta)^{-1} \nabla a\|_{L^1_t(B^d_{p,1})}^h \leq C 2^{-4k_0} \|a\|_{L^1_t(B^d_{p,1})}^h.
\]

Consequently, it follows that

\[
(\text{3.12}) \quad \|w\|_{L^p_t(B^d_{p,1})}^h + \|w\|_{L^1_t(B^d_{p,1})}^h \lesssim \|w\|_{B^d_{p,1}}^h + \|1 + 2^{-k_0} \|f\|_{L^1_t(B^d_{p,1})}^h + \|g\|_{L^1_t(B^d_{p,1})}^h + 2^{-2k_0} \|a\|_{B^d_{p,1}}^h + 2^{-2k_0} \|\nabla a\|_{L^1_t(B^d_{p,1})}^h + 2^{-4k_0} \|a\|_{L^1_t(B^d_{p,1})}^h.
\]

Now, in order to estimate \( v \), we use the fact that

\[
(\text{3.13}) \quad \nabla a = \frac{w - v}{\alpha}
\]

so that the equation for \( v \) rewrites

\[
\partial_t v - \frac{\alpha - \pi}{\alpha} \Delta v = \frac{\pi}{\alpha} \Delta w + \nabla (-\Delta)^{-1} (f - \text{div} \, g) + v - (-\Delta)^{-1} \nabla a.
\]

The important observation is that

\[
\frac{\alpha - \pi}{\alpha} = \frac{\pi}{1 - \alpha}.
\]

Hence one can again take advantage of (3.3), and get

\[
(\text{3.13}) \quad \|v\|_{L^p_t(B^d_{p,1})}^h + \|v\|_{L^1_t(B^d_{p,1})}^h \lesssim \|w\|_{B^d_{p,1}}^h + \|\nabla (-\Delta)^{-1} (f - \text{div} \, g)\|_{L^1_t(B^d_{p,1})}^h + \|\frac{\pi}{\alpha} \Delta w + v - (-\Delta)^{-1} \nabla a\|_{L^1_t(B^d_{p,1})}^h.
\]
whence, arguing as for proving (3.12),

\begin{equation}
\left\| v \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| v \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h \lesssim \left\| v_0 \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| g \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| f \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h \\
+ \left\| \nabla u \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + 2^{-2k_0} \left\| v \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + 2^{-4k_0} \left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h.
\end{equation}

Plugging (3.12) in (3.14) and taking $k_0$ large enough, we arrive at

\begin{equation}
\left\| v \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| v \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h \\
\lesssim \left\| v_0 \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| u_0 \right\|_{B_{p,1}^{\frac{d}{p}+1}}^h + \left\| f \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| g \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| a \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h.
\end{equation}

Then, inserting that latter inequality in (3.12) and using (13.3), we get

\begin{equation}
\left\| (\nabla a, v) \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| (\nabla a, v) \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h \\
\lesssim \left\| (\nabla a_0, v_0) \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| f \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| g \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h.
\end{equation}

Finally, keeping in mind that $u = v - (-\Delta)^{-1} \nabla a + \mathcal{P} u$, we conclude that

\begin{equation}
\left\| (\nabla a, u) \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| (\nabla a, u) \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h \\
\lesssim \left\| (\nabla a_0, u_0) \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| f \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h + \left\| g \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h.
\end{equation}

Let us next turn to estimates for the nonlinear terms. For the high frequencies of $f$, we just write that, since $f = -a \div \nabla u - u \cdot \nabla a$ and the space $B_{p,1}^{\frac{d}{p}}$ is a Banach algebra,

\begin{equation}
\left\| f \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^h \lesssim \left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^h \left\| u \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^h + \left\| u \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^h \left\| a \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^h.
\end{equation}

From the triangle and the Bernstein inequalities, and from the definition of $\mathcal{X}_p$, we get

\begin{equation}
\left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^\ell \lesssim \left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^h + \left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^h \\
\lesssim \left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^h.
\end{equation}

and

\begin{equation}
\left\| u \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1}}^h \lesssim \left\| u \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1}}^h + \left\| u \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1}}^h.
\end{equation}

and similar inequalities for the last term of (3.16). Hence we get for some constant $C$ that may depend on $k_0$,

\begin{equation}
\left\| f \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}})}^h \lesssim C \mathcal{X}_p^2(T).
\end{equation}

All terms in $g$, but $\nabla (\tilde{\kappa}(a) \Delta a)$ and $\nabla (\tilde{\kappa}'(a) |\nabla a|^2)$ have been treated in e.g. [17] for the classical compressible Navier-Stokes equations; they are bounded by the right-hand side of (3.6). Now, regarding the high frequencies of these two capillary terms, one can use the fact that the space $B_{p,1}^{\frac{d}{p}}$ is stable by product and composition, and get

\begin{equation}
\left\| \nabla (\tilde{\kappa}(a) \Delta a) \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h \lesssim \left\| \tilde{\kappa}(a) \Delta a \right\|_{L^1_t(B_{p,1}^{\frac{d}{p}+1})}^h \lesssim \left\| a \right\|_{L^\infty_t(B_{p,1}^{\frac{d}{p}})}^h \left\| \Delta a \right\|_{L^2_t(B_{p,1}^{\frac{d}{p}})}^h.
\end{equation}
Similarly,

$$
\| \nabla (\tilde{\kappa}'(a) | \nabla a|)^{\frac{h}{2}} \|_{L^2(T)} \lesssim \| \tilde{\kappa}'(1) + (\tilde{\kappa}'(a) - \tilde{\kappa}'(1)) | \nabla a| \|^2_{L^2(T)} \lesssim (1 + \| a \|^2_{L^2(T)}) \| \nabla a\|^2_{L^2(T)}.
$$

(3.18)

To handle the low frequencies, one can use the fact that, owing to Lemma 2.1,

$$
\| \nabla \Delta a \|_{L^2(B^{d-1}_{p,1})} \lesssim \| \tilde{\kappa}(a) \Delta a \|_{L^2(B^{d-1}_{p,1})}.
$$

(3.19)

Again, taking advantage of prior works on the compressible Navier-Stokes equations, we just have to check that the capillary terms satisfy (3.6). Now, we have

$$
\| \nabla (\tilde{\kappa}(a) \Delta a) \|_{L^2(B^{d-1}_{p,1})} \lesssim \| \tilde{\kappa}(a) \Delta a \|_{L^2(B^{d-1}_{p,1})}.
$$

In order to estimate the r.h.s., we use the following Bony decomposition:

$$
\tilde{\kappa}(a) \Delta a = T\tilde{\kappa}(a) \Delta a + T\Delta a \tilde{\kappa}(a) + R(\tilde{\kappa}(a), \Delta a).
$$

Recall that $T : \dot{B}_{p,1}^{\frac{d}{p}-1} \times \dot{B}_{p,1}^{\frac{d}{p}} \rightarrow \dot{B}_{p,1}^{\frac{d}{p}-1}$ for $2 \leq p \leq \min(4, \frac{2d}{d-2})$. Hence we have, using the second part of Proposition A.2,

$$
\| T\tilde{\kappa}(a) \Delta a + T\Delta a \tilde{\kappa}(a) \|_{\dot{B}_{p,1}^{\frac{d}{p}}} \lesssim \| T\tilde{\kappa}(a) \Delta a + T\Delta a \tilde{\kappa}(a) \|_{\dot{B}_{p,1}^{\frac{d}{p}-1}} \\
\lesssim \| \tilde{\kappa}(a) \|_{\dot{B}_{p,1}^{\frac{d}{p}-1}} \| \Delta a \|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \| \Delta a \|_{\dot{B}_{p,1}^{\frac{d}{p}-1}} \| \tilde{\kappa}(a) \|_{\dot{B}_{p,1}^{\frac{d}{p}}} \\
\lesssim (1 + \| a \|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \| a \|_{\dot{B}_{p,1}^{\frac{d}{p}-1}} + \| a \|_{\dot{B}_{p,1}^{\frac{d}{p}+1}} \| a \|_{\dot{B}_{p,1}^{\frac{d}{p}}}.
$$

For the remainder term, one can use that $R : \dot{B}_{p,1}^{\frac{d}{p}} \times \dot{B}_{p,1}^{\frac{d}{p}} \rightarrow \dot{B}_{p,1}^{\frac{d}{p}}$ if $2 \leq p \leq 4$. Hence we eventually get, if $p \leq \min(4, \frac{2d}{d-2})$

$$
\| \nabla (\tilde{\kappa}(a) \Delta a) \|_{L^2(B^{d-1}_{p,1})} \lesssim (1 + \| a \|_{L^2(T)})(\| a \|_{L^2(T)} + \| a \|_{\dot{B}_{p,1}^{\frac{d}{p}+1}} + \| a \|_{\dot{B}_{p,1}^{\frac{d}{p}+2}}).
$$

At this stage, one observes that

$$
\| a \|_{L^2(T)} \lesssim \| a \|_{L^2(T)}^{\frac{d}{p}} + \| a \|_{L^2(T)}^{\frac{d}{p}} \lesssim \mathcal{X}_p(T)
$$

and

$$
\| a \|_{\dot{B}_{p,1}^{\frac{d}{p}+1}} \lesssim \| a \|_{L^2(T)}^{\frac{d}{p}} + \| a \|_{L^2(T)}^{\frac{d}{p}+1} \lesssim \mathcal{X}_p(T).
$$

In order to estimate the other capillary terms, we simply use that $\tilde{\kappa}'(a) \nabla a = \nabla (\tilde{\kappa}(a))$, with $\tilde{\kappa}(0) = 0$. Now, thanks to Bony’s decomposition:

$$
\nabla a \cdot \nabla (\tilde{\kappa}(a)) = T\nabla \nabla (\tilde{\kappa}(a)) + T\nabla (\tilde{\kappa}(a)) \nabla a + R(\nabla (\tilde{\kappa}(a)), \nabla a).
$$

and to

$$
\| T\nabla \nabla (\tilde{\kappa}(a)) + T\nabla (\tilde{\kappa}(a)) \nabla a \|_{\dot{B}_{p,1}^{\frac{d}{p}}} \lesssim \| \nabla \nabla (\tilde{\kappa}(a)) \|_{\dot{B}_{p,1}^{\frac{d}{p}}} + \| \nabla (\tilde{\kappa}(a)) \|_{\dot{B}_{p,1}^{\frac{d}{p}}} \| \nabla a \|_{\dot{B}_{p,1}^{\frac{d}{p}}}.
$$

(3.18)
and
\[ \| R(\nabla(\tilde{\kappa}(a)), \nabla a) \|_{B^s_{p,1}} \lesssim \| \nabla(\tilde{\kappa}(a)) \|_{B^{p-\frac{d}{4} + \frac{s}{p}}_{p,1}} \| \nabla a \|_{B^{p-\frac{d}{4}}_{p,1}} \lesssim (1 + \| a \|_{B^{p}_{p,1}}) \| a \|_{B^{p}_{p,1}} \| \nabla a \|_{B^{p}_{p,1}}, \]
we end up with
\[ \| \nabla(\kappa'(a)|\nabla a|^2) \|_{L^2(\mathbb{R}_+^d)} \lesssim (1 + \| a \|_{L^p(\mathbb{R}_+^d)}) \| a \|_{L^p(\mathbb{R}_+^d \cap \mathbb{R}^d_{p,1})} \| \nabla a \|_{L^p(\mathbb{R}_+^d)} \]
Since, from Bernstein inequality then interpolation, we have
\[ \| a \|_{L^2(B^s_{p,1} \cap B^{p-\frac{d}{4}}_{p,1})} \lesssim \| a \|_{L^2(B^s_{p,1})} + \| a \|_{L^2(B^{p-\frac{d}{4} + \frac{s}{p}}_{p,1})} \]
\[ \lesssim \sqrt{\| a \|_{L^2(B^s_{p,1})}^2 + \| a \|_{L^2(B^{p-\frac{d}{4} + \frac{s}{p}}_{p,1})}^2} \]
\[ \lesssim \mathcal{X}_p(T), \]
one may conclude that
\[ \| \nabla(\kappa'(a)|\nabla a|^2) \|_{L^2(\mathbb{R}_+^d)} \lesssim (1 + \mathcal{X}_p(T))\mathcal{X}_p(T). \]
Combining with the already proved estimates for the other nonlinear terms (see [10]), we conclude that (3.6) is fulfilled. From this, it is not difficult to work out a fixed point argument as in the previous section, and to prove the first part of Theorem 3.1. \qed

3.2. More paraproduct, remainder and product estimates. In order to investigate the Gevrey regularity of solutions in the $L^p$ framework, resorting only to Propositions 2.1-2.2 does not allow to get suitable bounds for the low frequency part of some nonlinear terms. The goal of this short subsection is to establish more estimates for the paraproduct, remainder operators in $L^2$ based Besov spaces, when the two functions under consideration belong to some $L^p$ type Besov space.

**Proposition 3.1.** Assume that $2 \leq p \leq \min(4, \frac{2d}{d-2})$ and $s \in \mathbb{R}$. There exists a constant $C > 0$ such that
\[ \| e^{\sqrt{\alpha} A_t} T f g \|_{B^{s}_{p,1}} \leq C \| F \|_{B^{s-\frac{d}{4} + \frac{s}{p}}_{p,1}} \| G \|_{B^{s+\frac{d}{4} + \frac{s}{p}}_{p,1}} \]
with $F \triangleq e^{\sqrt{\alpha} A_t} f$ and $G \triangleq e^{\sqrt{\alpha} A_t} g$.

**Proof.** If $p > 2$, then we define $p^*$ by the relation $\frac{1}{2} = \frac{1}{p} + \frac{1}{p^*}$. Then applying inequality (2.18) with exponents $(s, \sigma, p, p_1, p_2, r, r_1, r_2) = (s + 1 - \frac{d}{p}, 1 - \frac{d}{p^*}, 2, p^*, p, 1, 1, \infty)$ which is possible since $p^* \geq p$ (that is $2 \leq p \leq 4$) and $-\sigma \triangleq \frac{d}{p^*} - 1 \leq 0$ (or, equivalently, $p \leq \frac{2d}{d-2}$), we get
\[ \| e^{\sqrt{\alpha} A_t} T f g \|_{B^{s}_{p,1}} \leq C \| F \|_{B^{s-\frac{d}{4} + \frac{s}{p}}_{p,1}} \| G \|_{B^{s+\frac{d}{4} + \frac{s}{p}}_{p,1}}. \]
Then using the embedding $\dot{B}^{d}_{p,1} \hookrightarrow \dot{B}^{s}_{p,1}$ (note that $p^* \geq p$) and $\dot{B}^{s+\frac{d}{4} + \frac{s}{p}}_{p,1} \hookrightarrow \dot{B}^{d}_{p,1}$ gives the desired inequality.

The endpoint case $p = 2$ stems from (2.20) with the exponents $(s, \sigma, p, q, r, r_1, r_2) = (s + 1, 1, 2, 2, 1, 1, \infty)$.

As a consequence of Proposition 2.1 and of the embedding $\dot{B}^{s+\frac{d}{2} - \frac{d}{p}}_{p,2} \hookrightarrow \dot{B}^{s}_{p,1}$ for any $2 \leq p \leq 4$ and $s \in \mathbb{R}$, we readily get:
**Proposition 3.2.** Let $d \geq 2$ and $2 \leq p \leq 4$. If $s_1 + s_2 > d\left(\frac{1}{2} - \frac{1}{p}\right)$ then there exists a constant $C > 0$ such that

\begin{equation}
\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} R(f,g)\|_{B^{s_1 + s_2}_{p,1}} \leq C\|F\|_{B^{s_1 + d\left(\frac{1}{2} - \frac{1}{p}\right)}_{p,1}} \|G\|_{B^{s_2}_{p,1}}.
\end{equation}

**Proposition 3.3.** Assume that $2 \leq p \leq \min\{4, \frac{2d}{d+2}\}$ and $p < 2d$. There exists a constant $C > 0$ such that:

\begin{equation}
\begin{cases}
\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (f,g)\|_{B^{s}_{2,1}}^{\ell} \lesssim \|F\|_{B^{\frac{d}{p}-1}_{p,1}} \|G\|_{B^{\frac{d}{p}+1}_{p,1}} + \|F\|_{B^{\frac{d}{p}+1}_{p,1}} \|G\|_{B^{\frac{d}{p}-1}_{p,1}}, \\
\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (f,g)\|_{B^{s}_{2,1}}^{d-1} \lesssim \|F\|_{B^{d\frac{d}{p}-1}_{p,1}} \|G\|_{B^{d\frac{d}{p}+1}_{p,1}} + \|F\|_{B^{d\frac{d}{p}+1}_{p,1}} \|G\|_{B^{d\frac{d}{p}-1}_{p,1}}, \\
\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (f,g)\|_{B^{s}_{2,1}}^{d-1} \lesssim \|F\|_{B^{d\frac{d}{p}-1}_{p,1}} \cap B^{d\frac{d}{p}+1}_{p,1} \|G\|_{B^{d\frac{d}{p}-1}_{p,1}}.
\end{cases}
\end{equation}

**Proof.** From Bony’s decomposition, we have

\[\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (f,g)\|_{B^{s}_{2,1}}^{\ell} = \|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (Tfg + T_g f + R(f,g))\|_{B^{s}_{2,1}}^{\ell}.\]

Thanks to Propositions 3.1 and 3.2 (with $(s, s_1, s_2) = \left(\frac{d}{2}, \frac{d}{2} - \frac{1}{p} - 1, \frac{d}{p} + 1\right)$), we get that:

\[\begin{cases}
\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (Tfg)\|_{B^{s}_{2,1}}^{\ell} + \|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (R(f,g))\|_{B^{s}_{2,1}}^{\ell} \lesssim \|F\|_{B^{d\frac{d}{p}-1}_{p,1}} \|G\|_{B^{d\frac{d}{p}+1}_{p,1}}, \\
\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (T_g f)\|_{B^{s}_{2,1}}^{d-1} \lesssim \|G\|_{B^{d\frac{d}{p}+1}_{p,1}} \|F\|_{B^{d\frac{d}{p}-1}_{p,1}}.
\end{cases}\]

The second estimate is proved the same way but with $(s, s_1, s_2) = \left(\frac{d}{2} - 1, \frac{d}{2} - \frac{1}{p} - 1, \frac{d}{p}\right)$. For the last estimate, we write that, taking advantage of the low frequency cut-off,

\[\|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (f,g)\|_{B^{s}_{2,1}}^{d-1} \lesssim \|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} (Tfg + T_g f)\|_{B^{s}_{2,1}}^{d-2} + \|e^{\sqrt{\frac{\gamma}{\Lambda_1}}} R(f,g))\|_{B^{s}_{2,1}}^{d-1}.\]

The last term may be bounded as before, and for the first two terms, we apply Proposition 3.1 with $s = \frac{d}{2} - 2$. \qed

### 3.3. A priori estimates for Gevrey regularity

This paragraph is devoted to proving estimates for Gevrey regularity in the $L^p$ Besov framework. This will be based on the following lemma.

**Lemma 3.3.** Let $(a, u)$ satisfy (2.3). If $\|A\|_{L^\infty(B^{\frac{d}{p}}_{p,1})}$ is small enough, then the following a priori estimate holds true

\begin{equation}
Y_p(t) \leq C(X_{p,0} + Y^2_p(t)) \quad \text{for all} \quad t \geq 0,
\end{equation}

with

\[Y_p(t) \triangleq \|(A, U)\|_{L^\infty(B^{d-1}_{p,1})} + \|(A, U)\|_{L^1_{1/2}(B_{p,1}^{d+1})} \]

\[+ \|A\|_{L^\infty(B_{p,1}^{d})} + \|U\|_{L^\infty(B_{p,1}^{d})} \cap L^1_{1/2}(B_{p,1}^{d+1}) + \|U\|_{L^\infty(B_{p,1}^{d})} \cap L^1_{1/2}(B_{p,1}^{d+1}).\]

**Proof.** Summing up inequality (2.35) for $j \leq k_0$, we get for all $t \geq 0$,

\begin{equation}
\|(A, U)\|_{L^\infty(B^{d-1}_{p,1})} + \|(A, U)\|_{L^1_{1/2}(B_{p,1}^{d+1})} \lesssim \|(a_0, u_0)\|_{B^{d-1}_{p,1}} + \|F\|_{L^1_{1/2}(B_{p,1}^{d+1})} + \|G\|_{L^1_{1/2}(B_{p,1}^{d+1})}.\end{equation}
Regarding the high frequency estimates, we plan to repeat the computations of the previous section after introducing \(e^{\sqrt{\alpha}A_1}\) everywhere. Now, using again the auxiliary functions
\[
v \triangleq Q u + (-\Delta)^{-1} \nabla a \quad \text{and} \quad w \triangleq v + \alpha \nabla a \quad \text{with} \quad \alpha = \frac{1}{2} (1 + \sqrt{1 - 4\bar{\alpha}}),
\]
and setting \(\alpha \triangleq 1 - \alpha\) and \(\bar{g} \triangleq (-\Delta)^{-1} \nabla (f - \text{div} g) + v - (-\Delta)^{-1} \nabla a\), we discover that
\[
w(t) = e^{\tilde{\alpha} \Delta} w_0 + \int_0^t e^{\tilde{\alpha}(t-\tau)}(-\alpha \nabla a + \alpha \nabla f + \bar{g})(\tau) \, d\tau.
\]
Hence \(W(t) \triangleq e^{\sqrt{\alpha}A_1}w(t)\) fulfills (with obvious notation):
\[
W(t) = e^{\sqrt{\alpha}A_1 + \tilde{\alpha} \Delta} w_0 + \int_0^t e^{(\sqrt{\alpha} - \sqrt{\alpha}A_1 + \tilde{\alpha}(t-\tau))\Delta}(-\alpha \nabla A + \alpha \nabla F + \bar{G})(\tau) \, d\tau.
\]
Arguing as in the proof of Lemma 3.2, it follows from Lemmas 2.2-2.3 that for the same threshold \(k_0\) as in (3.11) and (3.12), we have
\[
\|W\|_{L^\infty_t(B^{\frac{d}{p} - 1}_{p,1})} + \|W\|_{L^1_t(B^{\frac{d}{p}}_{p,1})} \lesssim \|w_0\|_{B^{\frac{d}{p} - 1}_{p,1}}^h + \|A\|_{L^1_t(B^{\frac{d}{p}}_{p,1})}^h + \|F\|_{L^1_t(B^{\frac{d}{p}}_{p,1})}^h + \|\bar{G}\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h \\
\lesssim \|w_0\|_{B^{\frac{d}{p} - 1}_{p,1}}^h + 2^{-2k_0}\|A\|_{L^1_t(B^{\frac{d}{p}}_{p,1})}^h + 2^{-2k_0}\|V\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h \\
+ (1 + 2^{-2k_0})\|F\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h + \|G\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h.
\]
Then one can revert to \(v\) as in (3.13), applying \(e^{\sqrt{\alpha}A_1}\) to:
\[
\partial_t v - \frac{\bar{\alpha}}{1 - \alpha} \Delta v = \frac{\bar{\alpha}}{\alpha} \Delta w + \bar{g}.
\]
Denoting \(V \triangleq e^{\sqrt{\alpha}A_1} v\) and following the procedure leading to (3.14), one gets
\[
\|V\|_{L^\infty_t(B^{\frac{d}{p} - 1}_{p,1})} + \|V\|_{L^1_t(B^{\frac{d}{p}}_{p,1})} \lesssim \|W\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h + \|\bar{G}\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h \\
\lesssim \|w_0\|_{B^{\frac{d}{p} - 1}_{p,1}}^h + \|W\|_{L^1_t(B^{\frac{d}{p}}_{p,1})}^h + 2^{-2k_0}\|V\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h \\
+ 2^{-4k_0}\|A\|_{L^1_t(B^{\frac{d}{p}}_{p,1})}^h + (1 + 2^{-2k_0})\|F\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h + \|G\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h.
\]
For the incompressible part of velocity, applying \(e^{\sqrt{\alpha}A_1}\) to (3.8) yields
\[
\|PU\|_{L^\infty_t(B^{\frac{d}{p} - 1}_{p,1})} + \|PU\|_{L^1_t(B^{\frac{d}{p}}_{p,1})} \lesssim \|Pu_0\|_{B^{\frac{d}{p} - 1}_{p,1}}^h + \|G\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h.
\]
Therefore, taking the same large enough \(k_0\) as in the previous section, and using (3.13), we deduce that
\[
\|((\nabla A, U))\|_{L^\infty_t(B^{\frac{d}{p} - 1}_{p,1})} + \|((\nabla A, U))\|_{L^1_t(B^{\frac{d}{p} + 1}_{p,1})} \lesssim \|((\nabla a_0, u_0))_{B^{\frac{d}{p} - 1}_{p,1}} + \|F\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h + \|G\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h.
\]
Putting together with (3.23), we end up with
\[
\mathcal{V}_p(t) \lesssim X_{p,0} + \|F\|_{L^1_t(B^{\frac{d}{p} + 1}_{p,1})}^h + \|G\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h + \|F\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h + \|G\|_{L^1_t(B^{\frac{d}{p} - 1}_{p,1})}^h.
\]
All that remains to do is to bound $F$ and $G$ in terms of $\mathcal{Y}_p$. As regards the low frequencies, it will be strongly based on Proposition 3.3.

Let us start with $F$. Then, thanks to (3.21)$_1$ and Besov injections (as $p \geq 2$), we get

$$
\|F\|_{L^1_t(B^{\frac{d}{2}-1} \_2)} \lesssim \|A\|_{L^\infty_t(B^{\frac{d}{2}} \_p, B^{\frac{d}{2}+1} \_p)} \|U\|_{L^1_t(B^{\frac{d}{2}+1} \_p)} + \|U\|_{L^\infty_t(B^{\frac{d}{2}} \_p)} \|A\|_{L^1_t(B^{\frac{d}{2}+1} \_p)}
$$

Similarly, we estimate

$$
\|G\|_{L^1_t(B^{\frac{d}{2}} \_2)} \lesssim \left( \|A\|_{L^\infty_t(B^{\frac{d}{2}-1} \_p)} + \|A\|_{L^\infty_t(B^{\frac{d}{2}} \_p)} \right) \left( \|U\|_{L^1_t(B^{\frac{d}{2}+1} \_p)} + \|U\|_{L^\infty_t(B^{\frac{d}{2}} \_p)} \right)
$$

Next, we bound the norm $\|G\|_{L^1_t(B^{\frac{d}{2}} \_2)}$. Using (3.21)$_2$ for $u$ and $\nabla u$, we obtain

$$
\|G\|_{L^1_t(B^{\frac{d}{2}} \_2)} = \|\sqrt{c_0A_1}(u \cdot \nabla u)\|_{L^1_t(B^{\frac{d}{2}} \_2)}
$$

Using (3.21)$_3$ and Lemma 2.5:

$$
\|G\|_{L^1_t(B^{\frac{d}{2}} \_2)} \lesssim \|U\|_{L^\infty_t(B^{\frac{d}{2}-1} \_p)} \|U\|_{L^1_t(B^{\frac{d}{2}+1} \_p)} + \|U\|_{L^\infty_t(B^{\frac{d}{2}} \_p)} \lesssim \mathcal{Y}_p^2(t).
$$

Let us now turn to $G_3 = -\sqrt{c_0A_1}(I(a)\tilde{A}u)$. Thanks to (3.21)$_3$ and Lemma 2.5:

$$
\|G_3\|_{L^1_t(B^{\frac{d}{2}} \_2)} \lesssim \int_0^t \|\sqrt{c_0A_1}(I(a))\|_{L^1_t(B^{\frac{d}{2}+1} \_p)} d\tau
$$

Similarly, we estimate $G_4 = \sqrt{c_0A_1}(J(a)\nabla u)$ using (3.21)$_2$, Lemma 2.5 and interpolation as follows:

$$
\|G_4\|_{L^1_t(B^{\frac{d}{2}} \_2)} \lesssim \int_0^t \left( \|\sqrt{c_0A_1}(J(a))\|_{B^{\frac{d}{2}+1} \_p} \|\nabla A\|_{B^{\frac{d}{2}} \_p} + \|\sqrt{c_0A_1}(J(a))\|_{B^{\frac{d}{2}} \_p} \|\nabla A\|_{B^{\frac{d}{2}+1} \_p} \right) d\tau
$$

In order to bound the term corresponding to $g_2$, it suffices to consider $\tilde{G}_2 = \sqrt{c_0A_1}(1 - I(a)) \nabla (\tilde{\mu}(a)\nabla u)$, the other term being similar. Now, we have:

$$
\|\tilde{G}_2\|_{L^1_t(B^{\frac{d}{2}} \_2)} \leq \left\| \sqrt{c_0A_1}(\tilde{\mu}(a)\nabla u) \right\|_{L^1_t(B^{\frac{d}{2}} \_2)} + \|\sqrt{c_0A_1}(I(a)\nabla (\tilde{\mu}(a)\nabla u))\|_{L^1_t(B^{\frac{d}{2}} \_2)}.
$$
The first term may be bounded (taking once again advantage of the low frequencies cut-off) according to (3.21) and Proposition 2.5 as follows:

\[
I \lesssim \| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a) \nabla u) \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})}^\ell \\
\lesssim \int_0^t \left( \| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a)) \|_{B_{p,1}^{\frac{d}{q} - 1}}^\ell \| U \|_{B_{p,1}^{\frac{d}{q} + 1}} + \| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a)) \|_{B_{p,1}^{\frac{d}{q}}} \| U \|_{B_{p,1}^{\frac{d}{q}}}) d\tau.
\]

The second term is bounded using (3.21) and Proposition 2.3 as follows:

\[
II \lesssim \int_0^t \| e^{\sqrt{\gamma} A_1} I(a) \|_{B_{p,1}^{\frac{d}{q} - 1} \cap B_{p,1}^{\frac{d}{q}}} \| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a) \nabla u) \|_{B_{p,1}^{\frac{d}{q} - 1}} d\tau \\
\lesssim \int_0^t \| A \|_{B_{p,1}^{\frac{d}{q} - 1} \cap B_{p,1}^{\frac{d}{q}}} \| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a)) \|_{B_{p,1}^{\frac{d}{q}}} \| U \|_{B_{p,1}^{\frac{d}{q} + 1}} d\tau.
\]

Now, using the fact that \( \| A \|_{L^\infty(B_{p,1}^{\frac{d}{q}})} \) is small, Lemma 2.5 allows to write that

\[
\| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a)) \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \lesssim \| A \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \quad \text{and} \quad \| e^{\sqrt{\gamma} A_1} (\bar{\mu}(a)) \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1})} \lesssim \| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1})}
\]

and we finally obtain after splitting \( a \) and \( u \) into low and high frequencies that

\[
\| G_2 \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \lesssim \mathcal{Y}^2(t).
\]

To bound the capillary terms, we use (3.21), writing that

\[
\| e^{\sqrt{\gamma} A_1} \nabla (\bar{K}(a) \Delta a) \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \lesssim \| e^{\sqrt{\gamma} A_1} (\bar{K}(a) \Delta a) \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \\
\lesssim \| e^{\sqrt{\gamma} A_1} \bar{K}(a) \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1}} \| \Delta A \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \\
+ \| \Delta A \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \| e^{\sqrt{\gamma} A_1} \bar{K}(a) \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1}} \\
\lesssim \| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1})} \| A \|_{L^1_t(B_{p,1}^{\frac{d}{q} + 1})} + \| A \|_{L^1_t(B_{p,1}^{\frac{d}{q} + 1})} \| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1})}.
\]

As we just have to bound the low frequencies, one gets thanks to (3.21),

\[
\| e^{\sqrt{\gamma} A_1} \nabla \left( \frac{1}{2} \nabla \bar{K}(a) \cdot \nabla a \right) \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \lesssim \| e^{\sqrt{\gamma} A_1} (\nabla \bar{K}(a) \cdot \nabla a) \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} \\
\lesssim \int_0^t \left( \| e^{\sqrt{\gamma} A_1} (\nabla \bar{K}(a)) \|_{B_{p,1}^{\frac{d}{q} - 1}} \| \nabla A \|_{B_{p,1}^{\frac{d}{q}}} \\
+ \| e^{\sqrt{\gamma} A_1} (\nabla \bar{K}(a)) \|_{B_{p,1}^{\frac{d}{q}}} \| \nabla A \|_{B_{p,1}^{\frac{d}{q} + 1}} \right) d\tau.
\]

(3.27)

Thanks to Lemma 2.5 we see that the first term of the right-hand side of (3.27) is bounded by:

\[
\| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1})} \| A \|_{L^1_t(B_{p,1}^{\frac{d}{q} + 1})} \\
\lesssim \left( \| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} - 1})} + \| A \|_{L^\infty_t(B_{p,1}^{\frac{d}{q} + 1})} \right) \left( \| A \|_{L^1_t(B_{p,1}^{\frac{d}{q} - 1})} + \| A \|_{L^1_t(B_{p,1}^{\frac{d}{q} + 1})} \right).
\]
We have to be careful for the last term of (3.27) as \( \frac{d}{p} + 1 \) is not in the range of Lemma 2.5. However, we have \( \nabla \tilde{\kappa}(a) = \tilde{\kappa}'(a) \nabla a \) and thus,
\[
\int_0^t \| e^{\sqrt{\varphi} t A_1} (\tilde{\kappa}'(a) \nabla a) \|_{B^d_{p,1}} \| A \|_{B^d_{p,1}} d\tau \\
\lesssim \int_0^t \left( \| e^{\sqrt{\varphi} t A_1} (\tilde{\kappa}'(a) - \tilde{\kappa}'(0)) \|_{B^d_{p,1}} + \| \tilde{\kappa}'(0) \| \| \nabla A \|_{B^d_{p,1}} \| A \|_{B^d_{p,1}} \right) d\tau \\
\lesssim (1 + \| A \|_{L^\infty(B^d_{p,1})}) \| A \|_{L^\infty(B^d_{p,1})} \| A \|_{L^1(B^d_{p,1})},
\]
which enables us to obtain, since \( \| A \|_{L^\infty(B^d_{p,1})} \) is small:
\[
\| e^{\sqrt{\varphi} t A_1} g_\delta(t) \|_{L^1(B^d_{p,1})} \lesssim \mathcal{Y}_p^2(t).
\]
To complete the proof, we need to bound the high frequencies of \( F \) and \( G \). This turns out to be rather straightforward, as we only need Proposition 2.4 and Lemma 2.5. More precisely, we get
\[
\| F \|_{L^1(B^d_{p,1})} \lesssim \| e^{\sqrt{\varphi} t A_1} (\operatorname{div} u + u \cdot \nabla a) \|_{L^1(B^d_{p,1})}
\]
and
\[
\| G \|_{L^1(B^d_{p,1})} \lesssim \| U \|_{L^\infty(B^d_{p,1})} \| U \|_{L^1(B^d_{p,1})} + \| A \|_{L^\infty(B^d_{p,1})} \| A \|_{L^1(B^d_{p,1})}
\]
\[
+ \| A \|_{L^\infty(B^d_{p,1})} \| U \|_{L^1(B^d_{p,1})} + \| A \|_{L^\infty(B^d_{p,1})} \| A \|_{L^1(B^d_{p,1})} + \| A \|_{L^\infty(B^d_{p,1})} \| A \|_{L^1(B^d_{p,1})} + (1 + \| A \|_{L^\infty(B^d_{p,1})}) \| A \|_{L^1(B^d_{p,1})}^2.
\]
Since \( \| A \|_{L^\infty(B^d_{p,1})} \) is small, one can conclude that
\[
\| F \|_{L^1(B^d_{p,1})} + \| G \|_{L^1(B^d_{p,1})} \lesssim \mathcal{Y}_p^2(t).
\]
Hence, putting all the above estimates together ends the proof of Lemma 3.3. \( \square \)

Finally, as in the previous section, using a suitable contracting mapping argument enables us to complete the proof of Theorem 3.1. The details are left to the reader. As for uniqueness, it stems from [11, Thm. 5].

4. The time-decay of solutions in Besov spaces

We here aim at exhibiting the time-decay properties of the solutions that have been constructed in Theorems 1.2 and 3.1. They will come up as a consequence of the following lemma.

**Lemma 4.1.** There exists a universal constant \( c > 0 \) such that for all \( s \in \mathbb{R} \), there exists a constant \( C_s \) such that for any tempered distribution \( u \) and real number \( \delta > 0 \), the following inequality holds true for all \( p \in [1, \infty] \) and \( j \in \mathbb{Z} \):
\[
\| \Lambda^s e^{-\delta \Lambda} \Delta_j u \|_{L^p} \leq C s 2^{js} e^{-c 2^j} \| \Delta_j u \|_{L^p}.
\]
Proof. The starting point is the fact that by definition of operator $e^{-\delta \Delta_1}$, we have for all $v \in \mathcal{S}'(\mathbb{R}^d)$,

$$e^{-\delta \Delta_1} v = h_\delta \ast v \quad \text{with} \quad h_\delta = \mathcal{F}^{-1}(e^{-\delta |\xi|}).$$

Now, we notice that $h_\delta$ is nonnegative, since

$$\int_{\mathbb{R}} e^{-|\eta|} e^{ix\eta} d\eta = \frac{2}{1 + x^2}$$

and, owing to the definition of $|\xi|_1$, we have

$$\mathcal{F}^{-1}(e^{-\delta |\xi|_1})(x) = \frac{1}{(2\pi)^d} \prod_{j=1}^d \left( \int_{\mathbb{R}} e^{-\delta |\xi_j|} e^{ix_j \xi_j} d\xi_j \right).$$

Therefore

$$\|h_\delta\|_{L^1} = \int_{\mathbb{R}^d} h_\delta(x) dx = \mathcal{F}(\mathcal{F}^{-1}(e^{-\sqrt{3}|\xi|}))(0) = 1.$$}

From this, we deduce by Young inequality that for all $\delta \geq 0$,

$$\|e^{-\delta \Delta_1} v\|_{L^p} \leq \|v\|_{L^p}. \quad (4.2)$$

In order to get (4.1) for $s = 0$, one has to refine the argument. First, performing a suitable rescaling reduces the proof to the case $j = 0$. Then, we introduce a family $(\phi_k)_{1 \leq k \leq d}$ of smooth functions on $\mathbb{R}^d$ such that

1. $\operatorname{Supp} \phi_k \subset \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \text{ and } \frac{3}{4\sqrt{d}} \leq |\xi_k| \}$;
2. $\sum_{k=1}^d \phi_k \equiv 1$ on $\operatorname{Supp} \varphi$, where $\varphi$ is the function used in the definition of the Littlewood-Paley decomposition.

As we obviously have

$$e^{-\delta |\xi|} \mathcal{F}(\hat{\Delta}_0 u)(\xi) = \sum_{k=1}^d (e^{-\delta |\xi|} \phi_k(\xi)) \mathcal{F}(\hat{\Delta}_0 u)(\xi),$$

one may write

$$e^{-\delta \Delta_1} \Delta_0 u = \sum_{k=1}^d h_k \ast \Delta_0 u \quad \text{with} \quad h_k \equiv \mathcal{F}^{-1}(e^{-\delta |\xi|} \phi_k).$$

If we prove that for some $c > 0$ and $C > 0$ independent of $\delta$, we have

$$\|h_k\|_{L^1} \leq C \left( \frac{1 + \delta}{\delta} \right)^d e^{-c\delta}, \quad (4.3)$$

then, combining with (4.2) will complete the proof of the lemma for $s = 0$.

Let us prove (4.3) for $k = 1$ (the other cases being similar). Then we introduce the notation $\xi = (\xi_1, \xi')$ and $x = (x_1, x')$. Since $(\delta^2 + x_1^2)e^{ix \xi} = (\delta^2 - \partial_{\xi_1}^2)(e^{ix \xi})$, integrating by parts with respect to the variable $\xi_1$ in the integral defining $h_1$ yields:

$$(\delta^2 + x_1^2)h_1(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \xi} e^{-\delta |\xi'|_1}(\delta^2 - \partial_{\xi_1}^2)(\phi_1(\xi)e^{-\delta |\xi|_1}) d\xi.$$}

Now, let us observe that

$$(e^{-\delta |\xi|})' = -\delta e^{-\delta |\xi|} \operatorname{sgn} r \quad \text{and} \quad \delta^2 e^{-\delta |\xi|} - (e^{-\delta |\xi|})'' = 2\delta \delta_0.$$}

Therefore,

$$(\delta^2 - \partial_{\xi_1}^2)(\phi_1(\xi)e^{-\delta |\xi|_1}) = 2\delta \phi_1(0, \xi')\delta_{\xi_1=0} + e^{-\delta |\xi|_1}(2\delta \operatorname{sgn}(\xi_1)\partial_{\xi_1} \phi_1(\xi) - \partial_{\xi_1}^2 \phi_1(\xi)),$$
and thus (taking advantage of the fact that $\phi_1(0, \xi') = 0$)

$$(\delta^2 + x_1^2)h_1(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-\delta|\xi|_1} (2\delta \sgn(\xi_1)\partial_1 \phi_1(\xi) - \partial^2_1 \phi_1(\xi)) d\xi.$$ 

Multiplying by $(\delta^2 + x_1^2)$, the same arguments lead to (denoting $\xi'_2 = (\xi_1, 0, \xi_2, \ldots, \xi_d)$ and $\phi^2_1(\xi) = 2\delta \sgn(\xi_1)\partial_1 \phi_1(\xi) - \partial^2_1 \phi_1(\xi) )$

$$(\delta^2 + x_1^2)(\delta^2 + x_2^2)h_1(x) = \frac{1}{(2\pi)^d} \left( 2\delta \int_{\mathbb{R}^{d-1}} e^{ix_2 \cdot \xi_2} e^{-\delta|\xi_2|_1} \phi^2_1(\xi_2) d\xi_2' + \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-\delta|\xi|_1} (2\delta \sgn(\xi_2)\partial_2 \phi^2_1(\xi) - \partial^2_2 \phi^2_1(\xi)) d\xi \right).$$

Multiplying the above equality by $(\delta^2 + x_3^2) \cdots (\delta^2 + x_d^2)$, repeating the above computation, and using the fact that,

$$\forall \xi \in \text{Supp} \phi_1, e^{-\delta|\xi|_1} \leq e^{-\delta|\xi|_1} \leq e^{-\frac{3\delta}{4\sqrt{d}}}.$$

we end up with

$$\prod_{\ell=1}^d (\delta^2 + x_\ell^2)h_1(x) \leq C(\delta + 1)^d e^{-\frac{3\delta}{4\sqrt{d}}} \|

which implies (4.3), and thus the lemma for $s = 0$.

Proving the general case $s \geq 0$ follows from the case $s = 0$: indeed, Inequality (A.8) ensures that

$$\|\Lambda^s e^{-\delta \Lambda^1 \hat{\Delta}} u\|_{L^p} \leq C_s 2^{js} \|e^{-\delta \Lambda^1 \hat{\Delta}} u\|_{L^p},$$

and bounding the right-hand side according to (4.1) thus yields the desired inequality. \(\square\)

One can now state our main decay estimates.

**Theorem 4.1.** Let $(\rho, u)$ be the solution constructed in Theorem 3.1. Then, for any $s \geq 0$, there exists a constant $C_s$ such that for all $t > 0$, it holds that

$$
\|\rho(t) - \bar{\rho}\|_{\dot{B}^{0}_{2,1}-1+s} \leq C_s \|X_{p,0} t^{-\frac{s}{2}}
\|u(t)\|_{\dot{B}^{0}_{2,1}-1+s} \leq C_s \|X_{p,0} t^{-\frac{s}{2}},
\|\rho(t) - \bar{\rho}\|_{\dot{B}^{0}_{p,1} + s} \leq C_s \|X_{p,0} t^{-\frac{s}{2}} e^{-c\sqrt{t}},
\|u(t)\|_{\dot{B}^{0}_{p,1} + s} \leq C_s \|X_{p,0} t^{-\frac{s}{2}} e^{-c\sqrt{t}}.
$$

**Proof.** Recall that the solution constructed in Theorem 3.1 fulfills

$$\|(\rho - \bar{\rho}, u)\|_{Y_p} \leq C X_{p,0}.$$ 

Now, Inequality (A.8) implies that

$$
\|u(t)\|_{\dot{B}^{0}_{2,1}-1+s} \leq C_s \|\Lambda^s X_{p,0} u(t)\|_{\dot{B}^{0}_{2,1}}.
$$
Then we write, denoting $U = e^{\sqrt{\alpha} t \Lambda_1} u$ and using the previous lemma, that
\[ t^2 \| \Lambda^s u \|_{B^d_{2,1}}^{t, \frac{d}{2} - 1} = \sum_{j \leq k_0} t^2 2^j 2^{j(\frac{d}{2} - 1)} \| \Lambda^s e^{-\sqrt{\alpha} t \Lambda_1} \hat{\Delta}_j U(t) \|_{L^2} \leq C_s \sum_{j \leq k_0} (\sqrt{t} 2^j)^s e^{-c \sqrt{\alpha} t 2^j} 2^j 2^{j(\frac{d}{2} - 1)} \| \hat{\Delta}_j U(t) \|_{L^2} \leq C_s \| U(t) \|_{B^d_{2,1}}^{t, \frac{d}{2} - 1} \leq C_s X_{p,0}.
\]
Similarly, we have
\[ t^2 \| u(t) \|_{B^m_{p,1}}^{h, \frac{d}{2} + m} \leq C_s \| \Lambda^s u(t) \|_{B^m_{p,1}}^{h, \frac{d}{2} + m} \leq C_s \sum_{j \geq k_0} 2^j 2^{j(\frac{d}{2} - 1)} t^2 2^j \| e^{-\sqrt{\alpha} t \Lambda_1} \Lambda^s \hat{\Delta}_j U(t) \|_{L^p} \leq C_s \sum_{j \geq k_0} e^{-\frac{1}{2} \sqrt{\alpha} t 2^j} 2^j 2^{j(\frac{d}{2} - 1)} (\sqrt{t} 2^j)^s e^{-\frac{1}{2} \sqrt{\alpha} t 2^j} \| \hat{\Delta}_j U(t) \|_{L^p} \leq C_s e^{-\frac{1}{2} \sqrt{\alpha} t 2^{k_0}} \| |U(t)|_{B^m_{p,1}}^{h, \frac{d}{2} + m} \leq C_s e^{-\frac{1}{2} \sqrt{\alpha} t 2^{k_0}} X_{p,0}.
\]
Proving the inequalities for $\rho$ is totally similar. \(\square\)

**Remark 4.1.** Decay in Theorem 4.1 is faster than that of the standard compressible Navier-Stokes pointed out in e.g. [20] or, more recently, in [12]. This, somehow, reflects the fact that internal capillarity smooths out the density in compressible flows.

**Appendix A. Littlewood-Paley Decomposition and Besov Spaces**

Here we recall a few basic results concerning the Littlewood-Paley decomposition and Besov spaces. More details may be found in e.g. [3, Chap. 2].

To build the Littlewood-Paley decomposition, one need a smooth radial function $\chi$ supported in the ball $B(0, \frac{3}{2})$ and with value 1 on $B(0, \frac{3}{4})$. Let $\varphi(\xi) \triangleq \chi(\xi/2) - \chi(\xi)$. Then, $\varphi$ is compactly supported in the annulus $\{ \xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \}$ and fulfills
\[ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}) = 1 \text{ in } \mathbb{R}^d \setminus \{0\}.
\]
Define the dyadic blocks $(\hat{\Delta}_j)_{j \in \mathbb{Z}}$ by $\hat{\Delta}_j = \varphi(2^{-j} D)$ (that is, $\hat{\Delta}_j f := \varphi(2^{-j} \xi) \hat{f}(\xi)$ for all tempered distribution $f$). The (formal) homogeneous Littlewood-Paley decomposition of $f$ reads
\[ f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f.
\]
That equality holds true in the set $S'_h$ of tempered distributions whenever $f$ belongs to
\[ S'_h \triangleq \{ f \in S', \lim_{j \to -\infty} \| \hat{S}_j f \|_{L^\infty} = 0 \},
\]
where $\hat{S}_j$ stands for the low frequency cut-off defined by $\hat{S}_j = \chi(2^{-j} D)$. 


Definition A.1. For \( \sigma \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), we set
\[
\|f\|_{\dot{B}^\sigma_{p,r}} = \left\|2^{j\sigma} \|\Delta_j f\|_{L^p(\mathbb{R}^d)}\right\|_{l^r(\mathbb{Z})}.
\]
We then define the homogeneous Besov space \( \dot{B}^\sigma_{p,r} \) to be the subset of distributions \( f \in S'_h \) such that \( \|f\|_{\dot{B}^\sigma_{p,r}} < \infty \).

Homogeneous Besov spaces on \( \mathbb{R}^d \) possess the following scaling invariance for any \( \sigma \in \mathbb{R} \) and \( (p, r) \in [1, +\infty]^2 \):
\[
C^{-1}\lambda^{\sigma-\frac{d}{p}}\|f\|_{\dot{B}^\sigma_{p,r}} \leq \|f(\lambda\cdot)\|_{\dot{B}^\sigma_{p,r}} \leq C\lambda^{\sigma-\frac{d}{r}}\|f\|_{\dot{B}^\sigma_{p,r}}, \quad \lambda > 0,
\]
where the constant \( C \) depends only on \( \sigma, p \) and on the dimension \( d \).

The following properties have been used repeatedly in the paper:

- The space \( \dot{B}^\sigma_{p,r} \) is complete whenever \( s < d/p \), or \( s \leq d/p \) and \( r = 1 \).
- For any \( p \in [1, \infty] \), we have the continuous embedding \( \dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty} \).
- If \( \sigma \in \mathbb{R}, \, 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq r_1 \leq r_2 \leq \infty \), then \( \dot{B}^\sigma_{p_1,r_1} \hookrightarrow \dot{B}^\sigma_{p_2,r_2} \hookrightarrow \dot{B}^{\sigma-d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}_{p_2,r_2} \).
- The space \( \dot{B}^\sigma_{p,1} \) is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if \( p < \infty \)).
- If \( K \) is a smooth homogeneous of degree \( m \) function on \( \mathbb{R}^d \setminus \{0\} \) that maps \( S'_h \) to itself, then
\[
K(D) : \dot{B}^\sigma_{p,r} \rightarrow \dot{B}^{\sigma-m}_{p,r}.
\]

In particular, the gradient operator maps \( \dot{B}^\sigma_{p,r} \) to \( \dot{B}^{\sigma-1}_{p,r} \).

Let us also mention the following interpolation inequality that is satisfied whenever \( 1 \leq p, r_1, r_2, s \leq \infty, \sigma_1 \neq \sigma_2 \) and \( \theta \in (0, 1) \):
\[
\|f\|_{\dot{B}^{\sigma_2+(1-\theta)\sigma_1}_{p,r_1}} \lesssim \|f\|_{\dot{B}^{\sigma_1}_{p,r_1}}^{1-\theta} \|f\|_{\dot{B}^{\sigma_2}_{p,r_2}}^\theta.
\]

The following proposition has been used in this paper.

Proposition A.1. Let \( \sigma \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \). Let \( (f_j)_{j \in \mathbb{Z}} \) be a sequence of \( L^p \) functions such that \( \sum_{j \in \mathbb{Z}} f_j \) converges to some distribution \( f \) in \( S'_h \) and
\[
\left\|2^{j\sigma}\|f_j\|_{L^p(\mathbb{R}^d)}\right\|_{l^r(\mathbb{Z})} < \infty.
\]
If \( \text{Supp}\hat{f}_j \subset C(0, 2^j R_1, 2^j R_2) \) for some \( 0 < R_1 < R_2 \), then \( f \) belongs to \( \dot{B}^\sigma_{p,r} \) and there exists a constant \( C \) such that
\[
\|f\|_{\dot{B}^\sigma_{p,r}} \leq C\left\|2^{j\sigma}\|f_j\|_{L^p(\mathbb{R}^d)}\right\|_{l^r(\mathbb{Z})}.
\]

The following result was used to bound the terms of System (1.1) involving compositions of functions:

Proposition A.2. Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be smooth with \( F(0) = 0 \). For all \( 1 \leq p, r \leq \infty \) and \( \sigma > 0 \) we have \( F(f) \in \dot{B}^\sigma_{p,r} \cap L^\infty \) for \( f \in \dot{B}^\sigma_{p,r} \cap L^\infty \), and
\[
\|F(f)\|_{\dot{B}^\sigma_{p,r}} \leq C\|f\|_{\dot{B}^\sigma_{p,r}},
\]
with \( C \) depending only on \( \|f\|_{L^\infty}, F' \) (and higher derivatives), \( \sigma, p \) and \( d \).
If $\sigma > -\min(\frac{d}{p}, \frac{d}{p'})$, then $f \in \dot{B}^\sigma_{p,r} \cap \dot{B}^{\frac{d}{p}}_{p,1}$ implies that $F(f) \in \dot{B}^\sigma_{p,r} \cap \dot{B}^{\frac{d}{p}}_{p,1}$, and

\[(A.6) \quad \|F(f)\|_{\dot{B}^\sigma_{p,r}} \leq C(1 + \|f\|_{\dot{B}^\sigma_{p,r} \cap \dot{B}^{\frac{d}{p}}_{p,1}})\|f\|_{\dot{B}^\sigma_{p,r}}.\]

Let us finally recall the following classical Bernstein inequality:

\[(A.7) \quad \|D^k f\|_{L^q} \leq C^{1+k} \lambda^{k+d(\frac{1}{q} - \frac{1}{2})}\|f\|_{L^2}\]

that holds for all function $f$ such that $\text{Supp} \mathcal{F}f \subset \{\xi \in \mathbb{R}^d : |\xi| \leq R\}$ for some $R > 0$ and $\lambda > 0$, if $k \in \mathbb{N}$ and $1 \leq a \leq b \leq \infty$.

Let us also recall that, as a consequence of [3, Lemma 2.2], we have for all $s \in \mathbb{R}$ if $\text{Supp} \mathcal{F}f \subset \{\xi \in \mathbb{R}^d : r \lambda \leq |\xi| \leq R\lambda\}$ for some $0 < r < R$,

\[(A.8) \quad \|\Lambda^s f\|_{L^q} \approx \lambda^s \|f\|_{L^q} \quad \text{with} \quad \Lambda^s \triangleq (-\Delta)^s.\]

When localizing PDE’s by means of Littlewood-Paley decomposition, one ends up with bounds for each dyadic block in spaces of type $L_t^q(L^p) \triangleq L^q(0,T;L^p(\mathbb{R}^d))$. To get a Besov type information, we then have to perform a summation on $k^s(\mathbb{Z})$, which motivates the following definition that has been first introduced by J.-Y. Chemin in [7] for $0 \leq T \leq +\infty$, $\sigma \in \mathbb{R}$ and $1 \leq p, q, r < \infty$:

\[\|f\| \lesssim_{\mathcal{F}} \|f\|_{L^q_t(L^p_x)} \triangleq \left\|\left(2^{ks} |\hat{\Delta}_j f| L^q_t(L^p_x)\right)\right\|_{L^r(\mathbb{Z})}.
\]

For notational simplicity, index $T$ is omitted if $T = +\infty$.

We also used the following functional space:

\[(A.9) \quad \tilde{C}_b(\mathbb{R}^+; \dot{B}^\sigma_{p,r}) \triangleq \left\{f \in C(\mathbb{R}^+; \dot{B}^\sigma_{p,r}) \text{ s.t. } \|f\|_{L^\infty(\dot{B}^\sigma_{p,r})} < \infty\right\}.
\]

The above norms may be compared with those of the more standard Lebesgue-Besov inequality:

\[(A.10) \quad \|f\| \lesssim_{\mathcal{F}} \|f\|_{L^q_t(\dot{B}^\sigma_{p,r})} \quad \text{if} \quad r \geq q, \quad \|f\| \lesssim_{\mathcal{F}} \|f\|_{L^q_t(\dot{B}^\sigma_{p,r})} \quad \text{if} \quad r \leq q.
\]

Restricting the above norms to the low or high frequencies parts of distributions is fundamental in our approach. For some fixed integer $k_0$ (the value of which follows from the proof of the main theorem), we put $z^\ell \triangleq \mathcal{S}_{k_0} z$ and $z^h \triangleq z - z^\ell$, and

\[(A.11) \quad \|z\|_{L^\infty_t(\dot{B}^\sigma_{p,1})} \triangleq \sum_{k \leq k_0} 2^{k\sigma} \|\dot{\Delta}_k z\|_{L^p}, \quad \|z\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \triangleq \sum_{k \geq k_0 - 1} 2^{k\sigma} \|\dot{\Delta}_k z\|_{L^p}, \quad \|z\|_{L^\infty_t(\dot{B}^\sigma_{p,1})} \triangleq \sum_{k \geq k_0} 2^{k\sigma} \|\dot{\Delta}_k z\|_{L^\infty_t(L^p)} \quad \text{and} \quad \|z\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} \triangleq \sum_{k \geq k_0 - 1} 2^{k\sigma} \|\dot{\Delta}_k z\|_{L^\infty_t(L^p)}.
\]

REFERENCES


\textsuperscript{4}For technical reasons, we need a small overlap between low and high frequencies.


