JOINT AND DOUBLE COBOUNDARIES
OF COMMUTING CONTRACTIONS

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ABSTRACT. Let $T$ and $S$ be commuting contractions on a Banach space $X$. The elements of $(I-T)(I-S)X$ are called double coboundaries, and the elements of $(I-T)X \cap (I-S)X$ are called joint coboundaries. For $U$ and $V$ the unitary operators induced on $L_2$ by commuting invertible measure preserving transformations which generate an aperiodic $\mathbb{Z}^2$-action, we show that there are joint coboundaries in $L_2$ which are not double coboundaries. We prove that if $\alpha, \beta \in (0,1)$ are irrational, with $T_\alpha$ and $T_\beta$ induced on $L_1(\mathbb{T})$ by the corresponding rotations, then there are joint coboundaries in $C(\mathbb{T})$ which are not measurable double coboundaries (hence not double coboundaries in $L_1(\mathbb{T})$).

1. Introduction

Let $\alpha$ be irrational, and let $\theta_\alpha x = x + \alpha \mod 1$ for $x \in [0,1)$. Then $\theta_\alpha$ preserves Lebesgue’s measure, and the operator $T_\alpha h = h \circ \theta$ defines an invertible isometry on all the spaces $L^p([0,1])$, $1 \leq p \leq \infty$. Motivated by Euler’s formal approach to Fourier series, Wintner [59] studied the existence of solutions $g \in L_1$ of the equation $(I-T_\alpha)g = f$ for a given $f \in L_1$ (or $L_2$). The translation $\theta_\alpha$ corresponds to the rotation $z = e^{2\pi i x} \rightarrow e^{2\pi i (x+\alpha)} = e^{2\pi i \alpha} z$ of the unit circle $\mathbb{T}$. This rotation is minimal (all orbits are dense in $\mathbb{T}$), since $\alpha$ is irrational.

Gottschalk and Hedlund [22, p. 135] proved that if $\theta$ is a minimal homeomorphism of a compact Hausdorff space $K$, then a continuous function $f$ is of the form $f = g - g \circ \theta$ for some continuous $g$ if and only if $\sup_{n} \| \sum_{k=0}^{n-1} f \circ \theta^k \|_{C(K)} < \infty$. Browder [12] proved that if $T$ is a power-bounded operator on a reflexive Banach space $X$, then

\[(1) \quad y \in (I-T)X \quad \text{if and only if} \quad \sup_{n} \| \sum_{k=0}^{n-1} T^k y \| < \infty.\]

Lin and Sine [38, Theorem 7] proved (1) for contractions of $L_1$.

When the equation $(I-T)x = y$ (for $y$ given) has a solution, i.e. $y \in (I-T)X$, $y$ is called a coboundary. Note that when $T$ is induced by a measure preserving transformation and $f \in L_p$, the solution of $(I-T)g = f$ may be in a larger space (e.g. $f \in (I-T)L_1$), or even measurable and non-integrable (and then $f$ is called a measurable coboundary).

Recently, Adams and Rosenblatt [2] studied the following problem: let $(\Omega, \mathbb{F})$ be a standard probability space; given $f \in L_p(\mathbb{F})$, is there some ergodic invertible measure preserving transformation $\theta$ such that $f = g - g \circ \theta$ for some $g$, and what are the integrability properties of $g$?

We refer to the introduction of [13] for additional discussion of developments following the results of Gottschalk-Hedlund and of Browder.


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A two-dimensional extension of Browder’s result was obtained by the present authors in [13, Theorem 3.1]: Let $T$ and $S$ be commuting contractions on a reflexive Banach space $X$. Then

$$(2) \quad y \in (I - T)(I - S)X \quad \text{if and only if} \quad \sup_n \left\| \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^j S^k y \right\| < \infty.$$ 

The elements of $(I - T)(I - S)X$ were called in [13] double coboundaries. Clearly double coboundaries are in $(I - T)X \cap (I - S)X$ (i.e. are joint (common) coboundaries).

The paper deals with the existence of joint coboundaries (of commuting contractions) which are not double coboundaries. We mention that Adams and Rosenblatt [1] studied the existence of joint coboundaries of non-commuting contractions.

Double and joint coboundaries can be interpreted in terms of rates of convergence in mean ergodic theorems. Denote $A_n(T) := \frac{1}{n} \sum_{k=0}^{n-1} T^k$. Then $A_n(T)x \to 0$ if and only if $x \in (I - T)X$, and Browder’s theorem means that the rate is $1/n$ if and only if $x$ is a coboundary. When $X$ is reflexive, $(1/n^2) \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} T^j S^k = A_n(T)A_n(S)$ converges strongly, and $A_n(T)A_n(S)y \to 0$ if and only if $y \in (I - T)X + (I - S)X$ (see Proposition 5.2); (2) means that the rate is $1/n^2$ if and only if $y$ is a double coboundary. The question becomes whether rates of $1/n$ in the convergence to zero of $A_n(T)y$ and $A_n(S)y$ imply the rate of $1/n^2$ for $A_n(T)A_n(S)y \to 0$.

Let $P$ be an ergodic Markov operator on a general state space $(\mathcal{S}, \Sigma)$, with invariant probability $\pi$, and let $(\xi_k)$ be the induced stationary Markov chain on $(\Omega, \mathcal{B}, \mathbb{P}_\pi)$. Gordin and Lifshits [21] proved a central limit theorem for $f(\xi_k)$ when $f$ is a coboundary of $P$ on $L_2(\mathcal{S}, \pi)$. Extensions to central limit theorems for random fields lead to the study of double coboundaries, which play a role in obtaining martingale-coboundary decompositions [20] (see also [58] and references therein).

Let $\mathcal{S}$ be a topological semi-group, and let $\mathbf{R}(s)$ be a bounded representation of $\mathcal{S}$ by linear operators on a Banach space $X$. A cocycle for $\mathbf{R}$ is a function $F : \mathcal{S} \mapsto X$ satisfying

$$F(s_1s_2) = F(s_1) + \mathbf{R}(s_1)F(s_2) \quad \text{for} \quad s_1, s_2 \in \mathcal{S}.$$ 

$F : \mathcal{S} \mapsto X$ is called a coboundary if there exists $x \in X$ such that $F(s) = (I - \mathbf{R}(s))x$ for every $s \in \mathcal{S}$. Parry and Schmidt [48] proved that when $X$ is reflexive and $\mathcal{S}$ is Abelian, a cocycle is a coboundary if and only if it is bounded; when $\mathcal{S} = \mathbb{N}$, we recover (1), since then cocycles are of the form $F(n) = \sum_{k=0}^{n-1} T^k F(1)$.

The following observation by Y. Derriennic (see also [11]) relates our definitions to classical cocycles of representations of $\mathbb{N}^2$.

**Proposition 1.1.** Let $T$ and $S$ be commuting contractions of a Banach space $X$, and let $\mathbf{R}(\bar{u}) := T^{u_1}S^{u_2}$ for $\bar{u} = (u_1, u_2) \in \mathbb{N}^2$.

(i) If $F(\bar{u})$ is a cocycle for $\mathbf{R}$, then $(I - S)F(\bar{e}_1) = (I - T)F(\bar{e}_2)$ is a joint coboundary.

(ii) If $z = (I - T)y = (I - S)x$ is a joint coboundary, then there exists a cocycle $F$ for $\mathbf{R}$ with $F(\bar{e}_1) = x$ and $F(\bar{e}_2) = y$ (the cocycle generated by $x$ and $y$).

(iii) If $F(\bar{u}) = (I - \mathbf{R}(\bar{u}))y$ is a coboundary for $\mathbf{R}$, then $(I - S)F(\bar{e}_1) = (I - T)F(\bar{e}_2)$ is a double coboundary.

(iv) If $z = (I - T)(I - S)h$ is a double coboundary, then the cocycle generated by $x = (I - T)h$ and $y = (I - S)h$ is the coboundary $F(\bar{u}) = (I - \mathbf{R}(\bar{u}))h$. 
Proof. Since $\mathbf{R}(\tilde{e}_1) = T$ and $\mathbf{R}(\tilde{e}_2) = S$, (i) follows from

$$F(\tilde{e}_2) + SF(\tilde{e}_1) = F(\tilde{e}_1 + \tilde{e}_2) = F(\tilde{e}_1) + TF(\tilde{e}_2).$$

We define (with empty sum defined as zero)

$$F((n, m)) = F(n\tilde{e}_1 + m\tilde{e}_2) = \sum_{k=0}^{n-1} T^k x + T^n \sum_{j=0}^{m-1} S^j y.$$

Some computations, using $(I - S)x = (I - T)y$, show that $F$ is a cocycle, and (ii) then follows.

Since a coboundary for $\mathbf{R}$ is a cocycle, (iii) follows from (i) and the definition of $F$.

Let $F$ be the cocycle generated by $x$ and $y$, given by (ii). Then

$$F(n\tilde{e}_1 + m\tilde{e}_2) = \sum_{k=0}^{n-1} T^k x + T^n \sum_{j=0}^{m-1} S^j y = (I - T^n)h + T^n(I - S^m)h = (I - T^n S^m)h,$$

which proves (iv). \qed

As mentioned above, in this work we investigate the existence of joint coboundaries for the commuting $T$ and $S$, which are not double coboundaries. In view of Proposition 1.1, the problem is, for actions of $\mathbb{N}^2$ or $\mathbb{Z}^2$, to find cocycles of the representation $\mathbf{R}$ (in different spaces) which are not coboundaries (non-triviality of the first cohomology group). For example, if we have an ergodic action of a countable group on a measure space with an atom, then every cocycle is a (measurable) coboundary [54, Exercise 2.9].

When we have an action of $\mathbb{Z}^2$ generated by commuting homeomorphisms $\theta$ and $\tau$ of a compact metric space $M$, it induces a representation $\mathbf{R}$ on $C(M)$, and any cocycle $F(\tilde{u}) \in C(M)$ is a function on $M$. A special case of interest is that of an Anosov action on a differentiable manifold $M$ (see [32]). In that case the study of cocyles and coboundaries is connected to rigidity properties of the action. Katok and Spatzier [32, Theorem 2.9] proved that every $C^\infty$ (Hölder) cocycle $F(\tilde{u})(t)$ of integral zero is a $C^\infty$ (Hölder) coboundary, and gave some applications. Proposition 1.1 allows us to express this result in terms of joint and double coboundaries. The case of irrational rotations of the circle in Theorem 4.5 below shows that the analogue result of the Katok-Spatzier theorem need not hold for continuous cocycles of commuting (non-hyperbolic) diffeomorphisms.

In Section 2 we study the existence of non-trivial double coboundaries in Banach spaces. We show that if $T \neq I$ and $S$ have the same fixed points, then there exist non-trivial double coboundaries. If in addition $T$ and $S$ are mean ergodic, then the set of double coboundaries is closed if and only if both $T$ and $S$ are uniformly ergodic; if one of the operators is uniformly ergodic, then every joint coboundary is a double one.

In Section 3 we show that if $\theta$ and $\tau$ are commuting invertible measure preserving transformations of a standard probability space which generate an aperiodic $\mathbb{Z}^2$-action, then their induced unitary operators on $L_2$ have a joint coboundary which is not a double coboundary in $L_2$.

In Section 4 we study in detail pairs of irrational rotations of the unit circle $\mathbb{T}$, with induced operators $T_\alpha$ and $T_\beta$ on different function spaces. We show the existence of a joint coboundary $\psi \in (I - T_\alpha)C(\mathbb{T}) \cap (I - T_\beta)C(\mathbb{T})$ which is not even a measurable double coboundary – there is no measurable $h$ such that $(I - T_\alpha)(I - T_\beta)h = \psi$. 


In Section 5 we prove that when $T$ and $S$ are commuting mean ergodic contractions, then $A_n(T)A_n(S)$ converges in operator norm if and only if $(I-T)(I-S)X$ is closed. We prove that if $\theta$ and $\tau$ are commuting ergodic measure preserving transformations of a non-atomic probability space, and $U$ and $V$ are the isometries they induce on $L_p$, $1 \leq p < \infty$, then $(I-U)L_p + (I-V)L_p$ is not closed.

2. On double coboundaries of commuting contractions

In this section we study the existence of non-trivial double coboundaries, and show that when $X$ is reflexive, the operators have the same fixed points, and one of them is uniformly ergodic, then every joint coboundary is a double coboundary.

For a bounded operator $T$ on a Banach space $X$, we denote by $F(T)$ the space of fixed points $\{x \in X : Tx = x\}$. The following "ergodic decomposition" induced by commuting contractions was proved in [13, Theorem 2.4].

**Theorem 2.1.** Let $T_1, T_2, \ldots, T_d$ be commuting mean ergodic contractions of a Banach space $X$. Then

$$X = \sum_{1 \leq j \leq d} F(T_j) \oplus \prod_{1 \leq j \leq d} (I - T_j)X.$$  

**Proposition 2.2.** Let $T$ and $S$ be commuting contractions of a reflexive Banach space $X$. Then

(i) $(I-T)(I-S)z = 0$ if and only if $z \in F(T) + F(S)$.

(ii) If $y \in (I-T)(I-S)X$, then there exists a unique $x \in (I-T)(I-S)X$ with $(I-T)(I-S)x = y$.

**Proof.** (i) By commutativity and continuity, $z \in F(T) + F(S)$ satisfies $(I-T)(I-S)z = 0$. For the converse, let $(I-T)(I-S)z = 0$. Denote $Y := (I-T)(I-S)X$, which is obviously $T$ and $S$ invariant. By the ergodic decomposition (3), $z = z_1 + z_2$ with $z_1 \in F(T) + F(S)$ and $z_2 \in Y$, so $(I-T)(I-S)z_2 = 0$. Hence

$$0 = \frac{1}{N} \sum_{n=1}^{N} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j (I-T)(I-S)z_2 = \frac{1}{N} \sum_{n=1}^{N} (I-T^n)(I-S^n)z_2 =$$

$$z_2 + \frac{1}{N} \sum_{n=1}^{N} (TS)^n z_2 - \frac{1}{N} \sum_{n=1}^{N} T^n z_2 - \frac{1}{N} \sum_{n=1}^{N} S^n z_2.$$

By reflexivity, the three averages converge. Each of the last two limits is in $F(T) + F(S)$ with $z_2 \in Y$, so (3) yields that each of these is zero. The first limit is $TS$-invariant, so by the above $z_2$ is $TS$-invariant, which yields

$$z_2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (TS)^n z_2 = -z_2,$$

which proves $z_2 = 0$.

Proof of (ii): If $(I-T)(I-S)x_1 = (I-T)(I-S)x_2$ for $x_1, x_2 \in Y$, then by (i) $x_1 - x_2 \in F(T) + F(S)$, so $x_1 - x_2 = 0$ by (3). \qed
Remark. The construction of a solution \((I - T)(I - S)x = y\) given in [13, Theorem 3.1] yields the (unique) solution \(x \in Y\), by the invariance of \(Y\) under \(T\) and \(S\).

**Theorem 2.3.** Let \(T\) and \(S\) be commuting contractions of a reflexive Banach space \(X\). Then

\[
(I - T)(I - S)X = (I - T)X \cap (I - S)X = (I - T)X \cap (I - S)X.
\]

**Proof.** Obviously

\[
(I - T)(I - S)X \subset (I - T)X \cap (I - S)X \subset (I - T)X \cap (I - S)X.
\]

We show that \((I - T)(I - S)X = (I - T)X \cap (I - S)X\). If not, by the inclusion above there exists \(z \in (I - T)X \cap (I - S)X\) which is not in \((I - T)(I - S)X\); by the Hahn-Banach theorem there exists \(\phi \in X^*\) with \(\phi(z) \neq 0\) and \(\phi((I - T)(I - S)X) = \{0\}\). We then have \((I - T^*)(I - S^*)\phi = 0\), and since \(X^*\) is reflexive, Proposition 2.2(i) yields that \(\phi \in \overline{F(T^*) + F(S^*)}\). But \(\|\frac{1}{n} \sum_{k=1}^{n} T^k z\| \to 0\) and \(\|\frac{1}{n} \sum_{k=1}^{n} S^k z\| \to 0\), so for \(\psi_1 \in F(T^*)\) and \(\psi_2 \in F(S^*)\) we have

\[
(\psi_1 + \psi_2)(z) = \frac{1}{n} \sum_{k=1}^{n} \psi_1(T^k z) + \frac{1}{n} \sum_{k=1}^{n} \psi_2(S^k z) \to 0.
\]

This yields that \(\phi(z) = 0\), a contradiction, which proves the theorem.

**Example.** \(T\) and \(S\) not the identity, with only trivial double coboundaries. Let \(0 \neq E \neq I\) be a (continuous linear) projection on a Banach space \(X\), and put \(T = E\), \(S = I - E\); then \((I - T)(I - S) = 0\). A sightly less trivial example is this: Let \(U\) be a unitary operator on a Hilbert space \(H\), put \(X = H \oplus H = \{(u, v) : u, v \in H\}\), and define \(T(u, v) = (Uu, v)\) and \(S(u, v) = (u, Uv)\). Then \((I - T)(I - S) = 0\).

**Theorem 2.4.** Let \(T\) and \(S\) be commuting contractions of a Banach space \(X\). If \(F(T) \subset F(S) \neq X\), then there exist non-trivial double coboundaries.

**Proof.** Assume all double coboundaries are zero, so for \(x \in X\) we have \((I - T)(I - S)x = 0\). Thus, \((I - S)x \in F(T) \subset F(S)\), so \((I - S)x = \frac{1}{n} \sum_{k=1}^{n} S^k (I - S)x \to 0\), which yields \(Sx = x\). Since this is for any \(x \in X\), \(S = I\), a contradiction.

**Examples.** 1. Let \(T\) and \(S\) be induced by commuting ergodic probability preserving transformations, e.g. irrational rotations of the unit circle. Then the theorem applies.

2. let \(\mu\) and \(\nu \neq \delta_1\) be probabilities on the unit circle \(\mathbb{T}\) such that the closed subgroup of \(\mathbb{T}\) generated by the support of \(\mu\) is all of \(G\), and put \(Tf = \mu \ast f\) and \(Sf = \nu \ast f\) on \(L_p\). The condition on \(\mu\) implies \(F(T) = \{\text{constants}\}\), so the theorem applies.

**Theorem 2.5.** Let \(T\) and \(S\) be commuting mean ergodic contractions on a Banach space \(X\), with \(F(T) = F(S)\). Then \((I - T)(I - S)X\) is closed if and only if both \(T\) and \(S\) are uniformly ergodic.
Proof. By [13, Remark 2.5]; see (3), the assumption $F(T) = F(S)$ implies
\begin{equation}
(I - S)X = (I - T)(I - S)X = (I - T)(I - S)X.
\end{equation}

(a) If $(I - T)(I - S)X$ is closed, then $(I - T)X$ and $(I - S)X$ are closed, and by [37] $T$ and $S$ are uniformly ergodic.

(b) Denote $Y := (I - T)(I - S)X$. If $T$ and $S$ are uniformly ergodic, then $(I - T)X = (I - S)X = Y$ and $I - T$ and $I - S$ are invertible on $Y$. Let $y \in Y$. Then there is $x \in Y$ with $(I - T)x = y$, and then a $z \in Y$ with $(I - S)z = x$. Hence $(I - T)(I - S)z = y$. Thus $(I - T)(I - S)X$ is closed.

\end{proof}

Example. Let $T$ and $S$ be induced on $L^2$ by commuting ergodic probability preserving transformations on a Lebesgue space. Then by [30] $\sigma(T) = \sigma(S) = T$, so neither is uniformly ergodic; hence by the theorem $(I - T)(I - S)X = L^2$ is not closed.

Let $T$ and $S$ be contractions of a Banach space $X$. The elements of $(I - T)X \cap (I - S)X$ are called joint (or common) coboundaries (of $T$ and $S$). When $T$ and $S$ commute, double coboundaries are joint coboundaries, and Theorem 2.4 yields existence of non-trivial joint coboundaries. Theorem 2.3 shows that in reflexive spaces, every joint coboundary can be approximated by double coboundaries. We want to address the question of existence of joint coboundaries which are not double coboundaries. Adams and Rosenblatt [1] studied existence of non-trivial joint coboundaries in the non-commutative case.

Corollary 2.6. Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$, with $F(T) = F(S)$. Then the following are equivalent:

(i) The space of double coboundaries $(I - T)(I - S)X$ is closed.

(ii) The space of joint coboundaries $(I - T)X \cap (I - S)X$ is closed.

(iii) Both $T$ and $S$ are uniformly ergodic.

Proof. By (4), (i) implies

\begin{equation}
(I - T)X = (I - T)(I - S)X \subset (I - T)X \cap (I - S)X \subset (I - T)X,
\end{equation}

which yields (ii).

By (4), and (ii), we have

\begin{equation}
(I - T)X = (I - T)X \cap (I - S)X \subset (I - T)X,
\end{equation}

which yields that $(I - T)X$ is closed, so $T$ is uniformly ergodic, and similarly for $S$.

\end{proof}

Lemma 2.7. Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$, with $F(T) = F(S)$, and assume that $T$ is uniformly ergodic. Then:

(i) Every coboundary of $S$, in particular every joint coboundary, is a double coboundary.

(ii) $\lim_{\min(n,m) \to \infty} \| \frac{1}{nm} \sum_{k=0}^{n} \sum_{j=0}^{m} T^k S^j - E \| \to 0$, where $E$ is the projection on $F(T)$ with null space $(I - T)X$.

Proof. The assumption $F(T) = F(S)$ implies $(I - S)X = (I - T)(I - S)X$, by [13, Remark 2.5].

(i) Let $x = (I - S)z$ be a coboundary of $S$. By mean ergodicity of $S$, we can take (uniquely) $z \in (I - S)X$. By uniform ergodicity of $T$, $Y := (I - T)X$ is closed (and $T$
invariant), and $I - T$ is invertible on $Y$. Then by the above $z \in (I - T)X$, which yields that
\[ x = (I - S)z = (I - S)(I - T)(I - T|_Y)^{-1}z \]
is a double coboundary.

(ii) Put $M_n(T) = \frac{1}{n} \sum_{k=0}^n T^k$, and $M_m(S) = \frac{1}{m} \sum_{j=0}^m S^j$. By uniform ergodicity,
\[ \|M_n(T) - E\| \to 0. \]
Since $T$ and $S$ commute and have the same ergodic decomposition, $SE = E$. Hence
\[ \|M_m(S)M_n(T) - E\| = \|M_m(S)(M_n(T) - E)\| \leq \|M_n(T) - E\| \to 0 \quad \text{as } \min(n,m) \to \infty. \]
\[ \square \]

**Remark.** The unitary operator $T$ induced on the complex $L_2$ by an ergodic invertible measure preserving transformation of a non-atomic probability space (with $T^k \neq I$ for $k \in \mathbb{N}$) is not uniformly ergodic, since its spectrum is $\mathbb{T}$ [30] (this is immediate for an irrational rotation of the unit circle, since the eigenvalues are dense in $\mathbb{T}$).

**Theorem 2.8.** Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$ with $F(T) = F(S)$. Then $T$ is uniformly ergodic if and only if every coboundary of $S$ is a double coboundary.

**Proof.** If $T$ is uniformly ergodic, we apply Lemma 2.7(i).

Assume that every coboundary of $S$ is a double coboundary. Fix $z \in \overline{(I - T)X} = \overline{(I - S)X}$ and put $x = (I - S)z$. By assumption, there is $y \in X$ with $x = (I - T)(I - S)y$. Hence $(I - S)[z - (I - T)y] = 0$, so $z - (I - T)y \in F(T)$, which yields $z = (I - T)y$, since $z \in \overline{(I - T)X}$. Thus $(I - T)X$ is closed, so $T$ is uniformly ergodic by [37]. \[ \square \]

**Remark.** Lemma 2.7(i) yields that when $F(T) = F(S)$, a necessary condition for the existence of a joint coboundary which is not a double coboundary is that neither $T$ nor $S$ be uniformly ergodic.

**Corollary 2.9.** Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$ with $F(T) = F(S)$, and assume $T$ is not uniformly ergodic. If $(I - S)X \subset (I - T)X$, then there exists a joint coboundary which is not a double coboundary.

**Proof.** Since $T$ is not uniformly ergodic, by Theorem 2.8 there exists $y \in (I - S)X$ which is not a double coboundary. Then $y$ is a joint coboundary, since $(I - S)X \subset (I - T)X$. \[ \square \]

**Remark.** If $T$ and $S$ are induced by invertible ergodic probability preserving transformations, then by Kornfeld [36, Theorem 2], the assumption $(I - S)X \subset (I - T)X$ implies that $S = T^k$ for some $k \in \mathbb{Z}$. Therefore, if $T \neq S^k$ and $S \neq T^k$, then there are coboundaries of $T$ and of $S$ which are not joint coboundaries.

**Theorem 2.10.** Let $R$ be a mean ergodic contraction, which is not uniformly ergodic, on a Banach space $X$, and let $T = R^k$ and $S = R^j$ be mean ergodic (e.g. $X$ is reflexive), with $F(T) = F(S) = F(R)$. Then $T$ and $S$ have a joint coboundary which is not a double coboundary.

**Proof.** Since $R$ is not uniformly ergodic, neither are $T$ nor $S$. By Theorem 2.9, there is a joint coboundary $u$ for $R$ and $S$ which is not a double coboundary for them. We put
is not a double coboundary in $L^2$. Joint coboundaries of commuting measure-preserving transformations $S$ and $T$ are uniformly ergodic if and only if $(I - S) \neq 0$, so we obtain

$$(I - R)x = u = [u - (I - R)(I - S)z] + (I - R)(I - S)z.$$  

Uniqueness in the ergodic decomposition (with respect to $R$) yields $u = (I - R)(I - S)z$, which means that $u$ is a double coboundary of $R$ and $S$, contradicting the choice of $u$. \hfill $\Box$

**Remark.** If $T$ is a mean ergodic contraction, taking $S = T$ we obtain that $T$ is uniformly ergodic if and only if $(I - T)X = (I - T)^2 X$.

3. JOINT COBOUNDARIES OF COMMUTING MEASURE-PRESERVING TRANSFORMATIONS

In this section we show that for commuting invertible measure-preserving transformations $\theta$ and $\tau$ of a standard probability space $(\Omega, \mathcal{B}, \mathbb{P})$ which generate an aperiodic $\mathbb{Z}^2$-action, their induced unitary operators on $L_2(\Omega, \mathbb{P})$ have a joint coboundary in $L_2$ which is not a double coboundary in $L_2$.

A bounded operator $T$ on a Banach space is called *aperiodic* if $T^n \neq I$ for any $n \in \mathbb{N}$ (see [19]). If $T^k = I$, then $T$ is power-bounded and uniformly ergodic. Hence for any commuting mean ergodic contractions with $F(T) = F(S)$, to have joint coboundaries which are not double coboundaries, it is necessary that both operators be aperiodic (by Lemma 2.7). A probability preserving transformation $\theta$ is called *aperiodic* if $\theta^n \neq id$ for any $n \geq 1$, i.e. its induced operator on $L_2$ is aperiodic (a more restrictive definition is given in [30]). Ergodic probability preserving transformations of a standard probability space are aperiodic.

If $\theta$ and $\tau$ are invertible probability preserving transformations on $(\Omega, \mathbb{P})$, we say that the $\mathbb{Z}^2$-action they generate is *aperiodic* (see [14], [34]) if for $j, k \in \mathbb{Z}$ which are not both zero, $\mathbb{P} \{ x \in \Omega : \theta^j \tau^k x = x \} = 0$; in that case, the induced unitary operators $U$ and $V$ satisfy $U^j V^k \neq I$ whenever $j$ and $k$ are not both zero.

Let $U$ and $V$ be commuting unitary operators on a complex Hilbert space $H$. They generate a unitary representation of $\mathbb{Z}^2$, to which we apply the general Stone spectral theorem (e.g. [3]) to obtain: There exists a (unique) projection valued spectral measure $E(\cdot)$ on the Borel sets of $\mathbb{T}^2 = \overline{\mathbb{Z}^2}$ such that (in the strong operator topology)

$$U^n V^m = \int_{\mathbb{T}^2} z_1^n z_2^m dE(z_1, z_2), \quad n, m \in \mathbb{Z}.$$  

Hence $P(U, V) = \int_{\mathbb{T}^2} P(z_1, z_2) dE(z_1, z_2)$ for every polynomial $P$ in two commuting variables. We denote by $\sigma_f(\cdot) := \langle E(\cdot), f \rangle$ the spectral measure of $f \in H$, and obtain that $\|P(U, V)f\|^2 = \int_{\mathbb{T}^2} |P(z_1, z_2)|^2 d\sigma_f(z_1, z_2).$
Assume convergence of the integral in (ii); it implies that 
\[ \sigma \]

Proof. (i) is a well-known consequence of Browder’s theorem [12]. The details of the proof will be clear from the proof of (ii) below.

It was proved in [13, Theorem 3.1] that \( f \in (I - U)(I - V)H \) if and only if

\[
\sup_n \left\| \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} U^{k}V^{j}f \right\| < \infty.
\]

Assume convergence of the integral in (ii); it implies that \( \sigma_f \{ \{1 \} \times \mathcal{T} \} = \sigma_f (\mathcal{T} \times \{1 \}) = 0 \), so

\[
\left\| \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} U^{k}V^{j}f \right\|^2 = \int_{\mathcal{T}^{2}} | \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} z_{1}^{k}z_{2}^{j} |^{2}d\sigma_f(z_{1}, z_{2}) = \int_{\mathcal{T}^{2}} \frac{|z_{1}^{n} - 1|^{2}|z_{2}^{n} - 1|^{2}}{|z_{1} - 1|^{2}|z_{2} - 1|^{2}}d\sigma_f(z_{1}, z_{2}).
\]

This yields

\[
\sup_n \left\| \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} U^{k}V^{j}f \right\|^2 \leq 16 \int_{\mathcal{T}^{2}} \frac{d\sigma_f(z_{1}, z_{2})}{|z_{1} - 1|^{2}|z_{2} - 1|^{2}} < \infty,
\]

which proves that \( f \in (I - U)(I - V)H \).

Assume now that \( f \in (I - U)(I - V)H \). Then

\[
\sup_N \left\| \frac{1}{N^{2}} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} U^{k}V^{j}f \right\|^2 = M < \infty.
\]

Since the spectral measure of \( U \) is \( E_{U}(B) = E(B \times \mathcal{T}) \), we have \( E(\{1\} \times \mathcal{T})g = E_{U}(\{1\})g = 0 \) when \( g \in (I - U)H \). Hence \( f \in (I - U)(I - V)H \) yields \( \sigma_f(\{1\} \times \mathcal{T}) = \sigma_f (\mathcal{T} \times \{1 \}) = 0 \), so in particular \( \sigma_f(\{(1, 1)\}) = 0 \).

\[
\left\| \frac{1}{N^{2}} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} U^{k}V^{j}f \right\|^2 = \int_{\mathcal{T}^{2}} \left| \frac{1}{N} \sum_{m=1}^{N} (z_{1}^{m} - 1) \right| \left| \frac{1}{N} \sum_{n=1}^{N} (z_{2}^{n} - 1) \right|^{2} \frac{1}{|z_{1} - 1|^{2}|z_{2} - 1|^{2}}d\sigma_f(z_{1}, z_{2}).
\]

The limit \( \lim_{N \to \infty} \left| \left( \frac{1}{N} \sum_{m=1}^{N} (z_{1}^{m} - 1) \right) \left( \frac{1}{N} \sum_{n=1}^{N} (z_{2}^{n} - 1) \right) \right|^{2} \) exists for every \( (z_{1}, z_{2}) \in \mathcal{T} \); it is 0 when \( z_{1} = 1 \) or \( z_{2} = 1 \), and 1 otherwise; hence the limit is 1 \( \sigma_f \) a.e. By Fatou’s lemma we have

\[
\int_{\mathcal{T}^{2}} \frac{d\sigma_f(z_{1}, z_{2})}{|z_{1} - 1|^{2}|z_{2} - 1|^{2}} = \]
\[
\int_{T^2} \lim_{N \to \infty} \left| \left( \frac{1}{N} \sum_{m=1}^{N} (z_m^n - 1) \right) \left( \frac{1}{N} \sum_{n=1}^{N} \left( z_n^2 - 1 \right) \right) \right|^2 \frac{1}{|z_1 - 1|^2 |z_2 - 1|^2} d\sigma_f(z_1, z_2) \leq \\
\lim \inf_{N \to \infty} \int_{T^2} \left| \left( \frac{1}{N} \sum_{m=1}^{N} (z_m^n - 1) \right) \left( \frac{1}{N} \sum_{n=1}^{N} \left( z_n^2 - 1 \right) \right) \right|^2 \frac{1}{|z_1 - 1|^2 |z_2 - 1|^2} d\sigma_f(z_1, z_2) \leq M,
\]
which proves (ii).

It follows from Proposition 3.1(i) that \( f \) is a joint coboundary if and only if

\[\int_{T^2} \frac{|z_1 - 1|^2 + |z_2 - 1|^2}{|z_1 - 1|^2 |z_2 - 1|^2} d\sigma_f(z_1, z_2) < \infty.\]

The problem of finding a joint coboundary for \( U \) and \( V \) which is not a double coboundary is therefore the problem of finding \( 0 \neq f \in H \) such that \( \sigma_f \) satisfies (5) and the integral in Proposition 3.1(ii) diverges. This requires a deeper study of spectral measures, summarized below.

Abstract (orthogonal projection valued) spectral measures (called spectral families in [3]), defined on a measurable space \((S, \Sigma)\) (and not necessarily connected to any unitary representation or any operator), were studied in the books of Halmos [23] and Nadkarni [46]. We fix a complex Hilbert space \( H \) and a spectral measure \( E(\cdot) \) with values in \( B(H) \), and define as before the spectral measure of \( f \in H \) by \( \sigma_f(\cdot) := \langle E(\cdot) f, f \rangle \), which is a positive finite measure. For \( f \in H \) we define the cyclic subspace \( Z(f) \) generated by \( f \) as the closed linear manifold generated by \( \{ E(A) f : A \in \Sigma \} \). The orthogonal projection on \( Z(f) \) commutes with \( E(\cdot) \) [23, p. 91]; hence also \( Z(f)^\perp \) is invariant under \( E(\cdot) \), and \( Z(g) \perp Z(f) \) for \( g \perp Z(f) \).

**Theorem 3.2.** Let \( H \) be a separable complex Hilbert space. Then there exists a vector \( \psi \in H \) such that \( \sigma_\psi(A) = 0 \) if and only if \( E(A) = 0 \); hence \( \sigma_g << \sigma_\psi \) for every \( g \in H \).

The proof is given in [46, p. 11-12]. The equivalence class of \( \sigma_\psi \) is called the maximal spectral type of \( E \), and is often denoted by \( \sigma_\psi \) instead of \( [\sigma_\psi] \). More detailed information is given by the Hahn-Hellinger theorem [46].

**Theorem 3.3.** [23, p. 104]. Let \( \nu \) be a finite measure on \((S, \Sigma)\). If \( \nu << \sigma_f \) for some \( f \in H \), then there exists \( g \in Z(f) \) such that \( \nu = \sigma_g \). When \( \nu \sim \sigma_f \) we have \( Z(g) = Z(f) \).

The following proposition and its proof are inspired by the one-dimensional result for unitary operators in [30, p. 290]. We use the Harte joint spectrum of \( d \) commuting operators in a complex Banach space \( X \), with respect to the Banach algebra \( B(X) \) of all bounded linear operators on \( X \) [25]. For completeness we repeat the definition.

**Definition.** The joint spectrum of the operators \( T_1, \ldots, T_d \in B(X) \), denoted by \( \sigma(T_1, \ldots, T_d) \) or \( \sigma(\{T_j\}) \), is the set of \( (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \) such that one of the equations

\[\sum_{j=1}^{d} A_j(\lambda_j I - T_j) = I \quad A_1, \ldots, A_d \in B(X) \]

\[\sum_{j=1}^{d} (\lambda_j I - T_j) B_j = I \quad B_1, \ldots, B_d \in B(X) \]
has no solution. The joint spectrum is closed [25, p. 872], and the following properties follow directly from the definition:

\[(8) \quad \sigma(T_1, \ldots, T_d) \subset \sigma(T_1) \times \cdots \times \sigma(T_d).\]

\[(9) \quad \inf_{\|x\|=1} \sum_{j=1}^{d} \| (\lambda_j I - T_j)x \| = 0 \implies (\lambda_1, \ldots, \lambda_d) \in \sigma(T_1, \ldots, T_d),\]

Since equation (6) cannot have a solution. The set of points satisfying (9) is the approximate point spectrum \(\sigma_\pi(T_1, \ldots, T_d)\).

If \(X = X_1 \oplus X_2\) with \(X_1\) and \(X_2\) each invariant under all the \(T_j\), then the restrictions \(T_j^{(k)}\) of \(T_j\) to \(X_k\) satisfy

\[(10) \quad \sigma(T_1, \ldots, T_d) \subset \sigma(T_1^{(1)}, \ldots, T_d^{(1)}) \cup \sigma(T_1^{(2)}, \ldots, T_d^{(2)}).\]

For additional information when the operators act in a Hilbert space, see [40, Chapter 5]. Note that by [57, Proposition 2.10], [9, p. 30], the joint spectrum of operators on \(H\) is a subset of the Taylor spectrum [55] \(\sigma_T(T_1, \ldots, T_d)\), while for normal operators on \(H\), these spectra are equal, by [43, pp. 30-31].

**Proposition 3.4.** Let \(U\) and \(V\) be commuting isometries on a complex Hilbert space \(H\).

If for every \(n > 2\) there exists a vector \(v_n \neq 0\) such that the vectors \(\{U^kV^n v_n : 0 \leq k \leq n, \ 0 \leq j \leq n\}\) are orthogonal, then \(T \times T \subset \sigma(U, V)\).

If \(U\) and \(V\) are invertible (unitary operators), then \(T \times T = \sigma(U, V)\).

**Proof.** We fix \(n > 2\) and denote \(v = v_n\). Let \(|\lambda| = |\nu| = 1\), and define

\[x_n := \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \lambda^{n-k} U^k \nu^{n-j} V^j v.\]

By the assumed orthogonality, \(\|x_n\|^2 = n^2 \|v\|^2 \neq 0\). We compute

\[(\lambda I - U)x_n = \nu^{n-j} V^j [(\lambda I - U) \sum_{k=0}^{n-1} \lambda^{n-k} U^k v] = \]

\[\sum_{j=0}^{n-1} \nu^{n-j} V^j [\sum_{k=0}^{n-1} (\lambda^{n+1-k} U^k v - \lambda^{n-k} U^{k+1} v)] = \sum_{j=0}^{n-1} \nu^{n-j} V^j (\lambda^{n+1} v - \lambda U^n v).\]

By the orthogonality assumption we obtain \(\|(\lambda I - U)x_n\|^2 = 2n \|v\|^2 = \frac{2}{n} \|x_n\|^2\), and similarly \(\|(\nu I - V)x_n\|^2 = \frac{2}{n} \|x_n\|^2\), so

\[(11) \quad \|(\lambda I - U)x_n\| + \|(\nu I - V)x_n\| = \frac{2\sqrt{2}}{\sqrt{n}} \|x_n\|.\]

Thus, (11) shows that \((\lambda, \nu)\) is in the ”approximate point spectrum” \(\sigma_\pi(U, V)\).

Hence \(T \times T \subset \sigma_\pi(U, V) \subset \sigma(U, V)\), the last inclusion by (9).

When \(U\) and \(V\) are unitary, \(\sigma(U, V) \subset \sigma(U) \times \sigma(V) \subset T \times T\) by (8), which completes the proof. \(\square\)
Corollary 3.5. Let θ and τ be commuting measure preserving transformations of a non-atomic probability space \((Ω, B, P)\) which generate a free \(\mathbb{N}_0^2\) action. Then for \(1 \leq p < \infty\), the joint spectrum of the isometric operators \(U\) and \(V\) induced on the complex \(L_p\) by \(θ\) and \(τ\) contains \(T \times T\). If θ and τ are invertible, then \(σ(U, V) = T \times T\).

**Proof.** The Rokhlin-Kakutani lemma for free \(\mathbb{N}_0^d\) actions, proved by Avila and Candela [6] (for non-periodic \(\mathbb{Z}^d\) actions see Conze [14, Lemme 3.1] or Katznelson and Weiss [34, Theorem 1]), yields that for every \(n\) there is a measurable set \(E_n\) such that the sets \(\{θ^{-k}τ^{-j}E_n : 0 \leq k \leq n, \ 0 \leq j \leq n\}\) are disjoint. For \(p = 2\) the vector \(v_n = 1_{E_n}\) satisfies the assumptions of Proposition 3.4.

When \(p \neq 2\), note that for \(v_n = 1_{E_n}\) the functions \(\{U^kV^jv_n : 0 \leq k \leq n, \ 0 \leq j \leq n\}\) have disjoint supports. Replacing orthogonality by disjointness of supports and taking norms to power \(p\) instead of squares in the proof of Proposition 3.4, that proof yields that also in \(L_p\) we have \(T \times T \subset σ(U, V)\), with equality when \(θ\) and \(τ\) are invertible.

**Remarks.** 1. The one-dimensional result for invertible transformations and \(p = 2\) was obtained in [30, Corollary 1]. It could have been proved there (even for \(p \neq 2\)) by using the original Rokhlin’s lemma and Remark 5 in [30, p. 290]. Another proof (for \(p = 2\)) was given in [19, Corollaire 1].

2. If \(θ\) and \(τ\) are invertible and generate an aperiodic \(\mathbb{Z}^2\)-action, then the \(\mathbb{N}_0^2\)-action is free.

We now return to the problem of joint coboundaries which are not double coboundaries.

Theorem 3.6. Let \(θ\) and \(τ\) be commuting invertible measure preserving transformations of a standard probability space \((Ω, B, P)\) which generate an aperiodic \(\mathbb{Z}^2\) action, and let \(U\) and \(V\) be their corresponding unitary operators on \(L_2(Ω, B, P)\). Then there exists a function \(g ∈ (I − U)L_2 ∩ (I − V)L_2\) which is not in \((I − U)(I − V)L_2\).

**Proof.** Let \(E(\cdot)\) be the spectral measure of the pair \((U, V)\) given by the general Stone spectral theorem. Since (the complex) \(L_2(\mathbb{C})\) is separable (standard probability space), by Theorem 3.2, there exists \(ψ ∈ H\) such that \(σ_ψ\) is the maximal spectral type of \(E\).

For \((1, 1) \neq (z_1, z_2) ∈ \mathbb{T}^2\) define

\[
φ(z_1, z_2) := \frac{|z_1 - 1|^2|z_2 - 1|^2}{|z_1 - 1|^2 + |z_2 - 1|^2},
\]

and put \(φ(1, 1) = 0\). Then \(φ(z_1, z_2) ≤ |z_2 - 1|^2 ≤ 4\), so \(φ\) is bounded, and vanishes on \(A := \{(z_1, z_2) : φ(z_1, z_2) = 0\} = \{(1) × \mathbb{T}\} ∪ (\mathbb{T} × \{1\})\). Since \(U \neq I\) and \(V \neq I\), \(E(\{1\} × \mathbb{T}) = E_U(\{1\}) \neq I\) and \(E(\mathbb{T} × \{1\}) = E_V(\{1\}) \neq I\); hence \(σ_ψ(A^c) := σ_ψ(\mathbb{T}^2 − A) > 0\).

For any finite measure \(μ << σ_ψ\) we define \(ν_μ\) by \(dν_μ/dμ := φ\). Then \(ν_μ\) is a finite measure with \(ν_μ(A) = 0, ν_μ \neq 0\) when \(μ(A^c) > 0\), and \(ν_μ\) satisfies

\[
∫_{\mathbb{T}^2} \frac{|z_1 - 1|^2 + |z_2 - 1|^2}{|z_1 - 1|^2 + |z_2 - 1|^2} dν_μ = ∫_{A^c} \frac{|z_1 - 1|^2 + |z_2 - 1|^2}{|z_1 - 1|^2 + |z_2 - 1|^2} φ(z_1, z_2) dμ = μ(A^c) < ∞.
\]

Hence \(ν_μ\) satisfies (5). We now show that for some \(μ\) we have \(∫_{\mathbb{T}^2} \frac{dν_μ(z_1, z_2)}{|z_1 - 1|^2|z_2 - 1|^2} = ∞\).

Assume not; then for every \(μ << σ_ψ\) with \(μ(A^c) > 0\) we have

\[
∫_{A^c} \frac{dμ}{|z_1 - 1|^2 + |z_2 - 1|^2} = ∫_{A^c} \frac{φ(z_1, z_2) dμ}{|z_1 - 1|^2|z_2 - 1|^2} = ∫_{\mathbb{T}^2} \frac{dν_μ(z_1, z_2)}{|z_1 - 1|^2|z_2 - 1|^2} < ∞.
\]
By a well-known lemma (Lemma 3.7 below) we conclude that \(|z_1 - 1|^2 + |z_2 - 1|^2|^{-1} \in L^{\infty}(A', \sigma_{\psi})
.

By a theorem of Hastings [27, Theorem 3], the support of \(E(\cdot)\) (which is the support of \(\sigma_{\psi}\)), equals the joint spectrum \(\sigma''(U, V)\) with respect to the double commutant of \((U, V)\); but this equals the Harte spectrum \(\sigma(U, V)\) by [43, Proposition 4 and Example (A)]. See also [49, Theorem 2.2]. By our aperiodicity assumption on the transformations, Corollary 3.5 yields \(\sigma(U, V) = \mathbb{T}^2\), so the support of \(\sigma_{\psi}\) is all of \(\mathbb{T}^2\). Hence \((|z_1 - 1|^2 + |z_2 - 1|^2)^{-1}\), which is unbounded in any neighborhood of \((1, 1)\), cannot be in \(L^{\infty}(A', \sigma_{\psi})\). This contradiction shows that for some \(\mu_0 << \sigma_{\psi}\) (with \(\mu_0(A') > 0\)), we

have \(\int_{\mathbb{T}^2} \frac{d\nu_{\mu_0}(z_1, z_2)}{|z_1 - 1|^2|z_2 - 1|^2} = \infty\). By the construction, \(0 \neq \nu_{\mu_0} << \sigma_{\psi}\), and by Theorem 3.3 there exists \(g \in Z(f) \subset L_2\) such that \(\sigma_g = \nu_{\mu_0}\). By what we saw, \(\sigma_g\) satisfies (5), and the integral with respect to \(\sigma_g\) in Proposition 3.1(ii) diverges. Hence \(g\) is a joint coboundary which is not a double coboundary. It can be shown (using the spectral theorem, [23, Theorem 1, p. 95] and \(L_2\)-density of trigonometric polynomials on \(\mathbb{T}^2\)) that \(Z(\psi)\) is the closed linear manifold generated by the \(Z^2\)-orbit \(\{U^nV^m\psi : m, n \in \mathbb{Z}\}\).

The above proof yields \(g\) in the complex \(L_2\). Let \(L_2^{(\mathbb{R})}\) be the real \(L_2\), which is invariant under \(U\) and \(V\). If \(g = (I - U)h\), then \(\Re g = (I - U)\Re h\), etc. Hence \(\Re g\) and \(\Im g\) are both joint coboundaries. If there are \(h_1, h_2 \in L_2^{(\mathbb{R})}\) such that \((I - U)(I - V)h_1 = \Re g\) and \((I - U)(I - V)h_2 = \Im g\), then \((I - U)(I - V)(h_1 + ih_2) = g\), contradicting the choice of \(g\). Hence \(\Re g\) or \(\Im g\) is a joint coboundary in \(L_2^{(\mathbb{R})}\) which is not a double coboundary. \(\square\)

**Lemma 3.7.** Let \(h \geq 0\) be a measurable function on a finite measure space \((S, \Sigma, \sigma)\). If \(\int hf\,d\sigma < \infty\) for every \(0 \leq f \in L_1(\sigma)\), then \(h \in L_\infty(\sigma)\).

**Proof.** If \(h\) is not in \(L_\infty\), for every \(n\) there exists \(f_n \in L_1\) with \(\|f_n\|_1 = 1\) and \(\int hf_n\,d\sigma > 2^n\). Then \(\int hf_n\,d\sigma > 2^n\), and \(f = \sum_{n=1}^{\infty} 2^{-n}|f_n|\) satisfies \(\|f\|_1 = 1\). But \(\int hf\,d\sigma = \sum_{n=1}^{\infty} 2^n \int hf_n\,d\sigma = \infty\), a contradiction. \(\square\)

**Remark.** The condition that the \(Z^2\)-action be free is not necessary. If \(\theta\) is aperiodic on a standard probability space, then \(\sigma(U) = \mathbb{T}\), so \(U\) is not uniformly ergodic, and taking \(\tau = \theta\) the theorem holds, since \((I - U)H \neq (I - U)^2H\), by the remark to Theorem 2.10.

**Example.** Commuting transformations with a joint non-double coboundary in any \(L_p\)

The measure space is \(\mathbb{N}^2\) with the counting measure \(m\). The transformations are \(\theta(j, k) = (j + 1, k)\) and \(\tau(j, k) = (j, k + 1)\) for \(j \geq 1, k \geq 1\). The counting measure is not invariant, but subinvariant for \(\theta\) and \(\tau\). Fix \(1 \leq p < \infty\), and put \(X := \ell_p(\mathbb{N}^2) := \{f_{j,k} : \sum_{j,k=1}^{\infty} |f_{j,k}|^p < \infty\}\). Define \(Uf = f \circ \theta\) and \(Vf = f \circ \tau\), so \((Uf)_{j,k} = f_{j+1,k}\) and \((Vf)_{j,k} = f_{j,k+1}\). Then \(U\) and \(V\) are commuting contractions of \(X\), satisfying \(U^n \to 0\) and \(V^n \to 0\) strongly.

Fix \(p > 1\) and put \(a = p + 1\). Define \(h \in X\) by \(h_{j,k} = 1/(j + k)^{a/p}\) for \(j, k \geq 1\). Then \(h \in X\). By definition \(Uh = Vh = (1/(j + k + 1)^{a/p})\), so \(f := (I - U)h\) is a joint coboundary. Suppose \(f = (I - U)(I - V)q\) for some \(q \in X\). Then \((I - U)[h - (I - V)q] = 0\), and since \(U\) has no fixed points, \(h = (I - V)q\), which yields \(q = \sum_{n=0}^{\infty} V^n h\). Hence

\[
q_{j,k} = \sum_{n=0}^{\infty} h_{j,k+n} = \sum_{n=0}^{\infty} \frac{1}{(j+k+n)^{a/p}} = \sum_{n=j+k}^{\infty} \frac{1}{n^{a/p}} \geq \frac{c}{(j+k)^{(a-p)/p}}.
\]
Since \( a - p = 1 \), we have \( \sum_{k=1}^{\infty} |q_{j,k}|^p = \infty \), so \( q \notin X \), a contradiction; hence \( f \) is not a double coboundary. Note that the solution \( q \) is in the larger space \( \ell_r(\mathbb{N}^2) \) for \( r > 2p \).

Now let \( p = 1 \), and define \( h \) by \( h_{j,k} = 1/[(j + k)^2[\log(j + k)]^2] \). Then \( h \in X \), since

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} h_{j,k} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j + k)^2[\log(j + k)]^2} = \sum_{j=1}^{\infty} \frac{1}{[\log(j + 1)]^2} \sum_{k=j+1}^{\infty} \frac{k^2}{[\log(j + 1)]^2} \leq \sum_{j=1}^{\infty} \frac{1}{j[\log(j + 1)]^2} < \infty.
\]

As before, \( Vh = Uh \) and \( f = (I - U)(I - V)q \), then, as before, \( q = \sum_{n=0}^{\infty} V^nh \), so

\[
q_{j,k} = \sum_{n=0}^{\infty} h_{j,k+n} = \sum_{n=j+k}^{\infty} n^2[\log n]^2 \geq \sum_{n=j+k}^{\infty} \frac{1}{n^{5/2}} \geq \frac{c}{(j + k)^{3/2}}
\]

when \( j > J_0 \). Then \( \sum_{k=1}^{\infty} q_{j,k} \geq c \sum_{k=j+1}^{\infty} k^{-3/2} \geq c' j^{-1/2} \) for \( j > J_0 \), so \( q \notin X \). This contradiction shows that \( f \) is not a double coboundary. Since \( q_{j,k} \leq \sum_{n=j+k}^{\infty} n^{-2} \leq (j + k - 1)^{-1} \), we obtain that \( q \in \ell_r(\mathbb{N}^2) \) for \( r > 2 \).

4. Joint coboundaries of irrational rotations of the circle

In this section we look at two irrational rotations of the unit circle \( T \). In this case we refine Theorem 3.6, showing first the existence of a joint coboundary in \( C(T) \) which is not a double coboundary, not even in \( L_1(T) \), and then exhibit such a joint coboundary which is not even a measurable double coboundary.

Let \( \alpha \) be a real number. We denote \( e(x) := e^{2\pi ix} \). For \( f \in L_1(T) \) we define \( T_\alpha f(z) = f(e(\alpha)z) \), which preserves Lebesgue’s measure on \( T \). The operator \( T_\alpha \) is a contraction of all the \( L_p(T) \) spaces, \( 1 \leq p \leq \infty \), and is mean ergodic for \( 1 \leq p < \infty \). It is also a mean ergodic contraction of \( C(T) \). If \( \beta \) is rational, then for some \( k \) we have \( T_\beta^k = I \), so \( T_\beta \) is uniformly ergodic, and by Lemma 2.7 every joint coboundary of \( T_\beta \) and any \( T_\alpha \) in \( C(T) \) or in \( L_p \), \( 1 \leq p < \infty \), is a double coboundary.

Let \( \alpha \) and \( \beta \) be irrational numbers. Clearly, if \( \alpha \equiv \beta \mod 1 \) (i.e. \( \alpha - \beta \in \mathbb{Z} \)), then \( T_\alpha = T_\beta \). Each \( T_\alpha \) is invertible, with \( T_\alpha^{-1} = T_{-\alpha} \), and for any integer \( n \), \( T_{n\alpha} = T_\alpha^n \).

Since \( I - T_\beta = T_\beta(T_\beta^{-1} - I) \), the double or joint coboundaries of \( T_\alpha, T_\beta \) are the same as the respective ones of \( T_\alpha, T_\beta^{-1} \).

Let \( \phi \) be a centered trigonometric polynomial \( \sum_{|k|=1}^{n} a_k e(kx) \). Then for any irrational \( \alpha \) we have \( (I - T_\alpha) \sum_{|k|=1}^{n} \frac{a_k}{1 - e(k\alpha)} e(kx) = \phi \). Hence \( \phi \) is a double coboundary in \( C(T) \) for any two irrationals \( \alpha \) and \( \beta \).

The question for "how many" irrational rotations an \( L_1(T) \) function is a coboundary in \( L_1(T) \) was studied by Baggett et al. [8].

Theorem 4.1. Let \( \alpha \) and \( \beta \) be positive irrational numbers such that \( \{\alpha, \beta, 1\} \) are linearly dependent over \( \mathbb{Z} \) (i.e., there exist \( m, n, p \in \mathbb{Z} \) with \( m \neq 0 \) and \( ma + nb + p = 0 \)). If the g.c.d. of \( (|m|, |n|) \) is 1, then, in \( C(T) \) and in any \( L_p(T) \), the rotations \( T_\alpha \) and \( T_\beta \) have a joint coboundary which is not a double coboundary.
Proof. Since \( \alpha \) and \( \beta \) are irrational, also \( n \neq 0 \), and we may assume \( n > 0 \). Since the g.c.d. of \( |m| \) and \( |n| \) is 1, there are integers \( j, k \) such that \( 1 = jm + kn \). Hence
\[
0 = m\alpha + n\beta + (jm + kn)p = m(\alpha + j\beta) + n(\beta + kp).
\]

Put \( \gamma := \frac{\alpha + j\beta}{n} = \frac{-\beta + kp}{m} \). Then \( T_\alpha = T_\gamma^n \) and \( T_\beta^{-1} = T_{-\gamma}^m \). If \( m > 0 \), by Theorem 2.10 \( T_\alpha \) and \( T_\beta^{-1} \) (and therefore \( T_\alpha \) and \( T_\beta \)) have a joint coboundary which is not a double coboundary; if \( m < 0 \), then \( T_\beta = T_\gamma^{-m} \), and then \( T_\alpha \) and \( T_\beta \) have a joint coboundary which is not a double coboundary. \( \square \)

Remarks. 1. The assumptions of the theorem are equivalent to assuming that for some \( \alpha' \equiv \alpha \) and \( \beta' \equiv \beta \) we have \( \alpha'/\beta' \) rational. Indeed, if \( (\alpha + j)/(\beta + k) \) is a rational \( n/m \), then \( m\alpha - n\beta + (jm - kn) = 0 \), and g.c.d. of \( |m| \) and \( |n| \) divides \( p := jm - kn \). The converse implication is shown in the proof of the theorem.

2. Theorem 4.1 is a special case of Theorem 4.5, but its proof follows from general principles.

Theorem 4.1 shows that for every irrational \( \alpha \in (0,1) \) there are countably infinitely many \( \beta \), necessarily irrational (of the form \( \beta = r(\alpha + j) + k \), \( r \) rational and \( j, k \) integers), such that \( T_\alpha \) and \( T_\beta \) have a joint coboundary in \( L_p \) \((1 \leq p < \infty) \) which is not a double coboundary.

Lemma 4.2. Let \( T \) and \( S \) be induced by commuting ergodic measure preserving transformations of a probability space \((\Omega, \mathcal{F})\). If \( h \) is measurable such that \( (I - T)(I - S)h = 0 \), then \( h \) is a constant a.e.

Proof. Since \( T \) is ergodic, \((I - S)h\) is a constant, hence integrable. By Anosov [4, Theorem 1] \( \int_\Omega (I - S)h \, d\Omega = 0 \), so \((I - S)h = 0 \); by ergodicity of \( S \), \( h \) is constant a.e. \( \square \)

For rotations, Lemma 4.2 is more general than (and independent of) Proposition 2.2(i).

Proposition 4.3. Let \( \alpha \) and \( \beta \) be irrational numbers. Then there exist a continuous \( \phi \in (I - T_\alpha)C(\mathbb{T}) \) and a measurable non-integrable \( h \) such that \( (I - T_\alpha)(I - T_\beta)h = \phi \), and every measurable \( u \) satisfying \( (I - T_\alpha)(I - T_\beta)u = \phi \) is non-integrable.

Proof. By Anosov [4, Theorem 2] (see also [35], [8, Theorem 3]), there exist \( f \in C(\mathbb{T}) \) and \( h \) measurable non-integrable with \((I - T_\beta)h = f \). We put
\[
\phi = (I - T_\alpha)f = (I - T_\alpha)(I - T_\beta)h.
\]

Let \( u \) be measurable and satisfy \((I - T_\alpha)(I - T_\beta)u = \phi \). Then \( u - h = \text{const} \) by Lemma 4.2, so \( u = h + \text{const} \) is not integrable, since \( h \) is not. \( \square \)

The following corollary strengthens a particular case of Theorem 4.1.

Corollary 4.4. Let \( \alpha \) and \( \beta \) be irrational numbers, such that \( T_\alpha = T_\beta^k \) for some \( k \in \mathbb{Z} \). Then there exist continuous functions \( f \) and \( g \) such that \((I - T_\alpha)f = (I - T_\beta)g \), but there is no integrable \( h \) with \((I - T_\alpha)(I - T_\beta)h = (I - T_\alpha)f \).

Proof. We prove for \( k > 0 \). Let \( h \) and \( f \) be as in the proof of Proposition 4.3. Then \( \phi = (I - T_\alpha)f = (I - T_\beta)g \) with \( g = \sum_{j=0}^{k-1} T_\beta^j f \in C(\mathbb{T}) \). Proposition 4.3 shows that any \( u \) with \((I - T_\alpha)(I - T_\beta)u = (I - T_\alpha)f \) is non-integrable. \( \square \)
Remark. Proposition 4.3 and Corollary 4.4 hold also when we replace $T_\alpha$ and $T_\beta$ by $T$ and $S$ induced by commuting uniquely ergodic homeomorphisms of a compact metric space; the proofs are similar, with [4, Theorem 2] replaced by its extension by Kornfeld [35].

Our main purpose now is to study the existence of a joint coboundary for $T_\alpha$ and $T_\beta$ which is not a double coboundary, when $\alpha$ and $\beta$ are irrationals (which are not rationally dependent). Wintner [59] seems to have been the first to study coboundaries of rotation operators $T_\alpha$ for $\alpha$ irrational (using Fourier series methods).

As usual, we denote by $\{\cdot\}$ the fractional part. For a real number $\alpha$ we denote by $\|\alpha\|$ its distance from the nearest integer, so $\|\alpha\| = \min\{\{\alpha\}, 1 - \{\alpha\}\}$.

Theorem 4.5. Let $\alpha$ and $\beta$ be irrational numbers. Then:

(i) There exists a joint coboundary for $T_\alpha$ and $T_\beta$ acting in $C(\mathbb{T})$ which is not a double coboundary, not even in $L_1(\mathbb{T})$.

(ii) For fixed $p \in [1, \infty)$, there exists a joint coboundary for $T_\alpha$ and $T_\beta$ acting in $L_p(\mathbb{T})$ which is not a double coboundary in $L_1(\mathbb{T})$.

Proof. Since $C(\mathbb{T}) \subset L_p(\mathbb{T}) \subset L_1(\mathbb{T})$ for $1 < p < \infty$, both parts of the theorem will follow if we produce continuous functions $f$ and $g$ with $(I - T_\alpha)f = (I - T_\beta)g$ (a joint coboundary in $C(\mathbb{T})$), such that there is no $h \in L_1(\mathbb{T})$ satisfying $(I - T_\alpha)f = (I - T_\alpha)(I - T_\beta)h$. Remember that if $\sum_{n \in \mathbb{Z}} |a_n| < \infty$, then $f(z) := \sum_{n} a_n z^n$ is continuous on $\mathbb{T}$.

By the two-dimensional Dirichlet theorem [24, Theorem 200], there are infinitely many positive integers $q$ such that $\max\{\|q\alpha\|, \|q\beta\|\} < \frac{1}{\sqrt{q}}$. We take an increasing subsequence $(q_k)$ of these $q$, such that $\sum_k \frac{1}{\sqrt{q_k}}$ converges.

Joint coboundaries of $T_\alpha$ and $T_\beta$ in $C(\mathbb{T})$ are given by continuous functions $f, g$ for which $(I - T_\alpha)f = (I - T_\beta)g$. This equality implies the following relations between the Fourier coefficients of $f$ and $g$:

\[ (1 - e^{2\pi i n \alpha}) \hat{f}_n = (1 - e^{2\pi i n \beta}) \hat{g}_n \quad \text{for every } n \in \mathbb{Z}. \]

We define $\hat{f}_n = \hat{g}_n = 0$ for $n \not\in (q_k)$, and put $\hat{f}_{q_k} = \|q_k \beta\|$. By the choice of $(q_k)$, we have $f \in C(\mathbb{T})$. We then define $\hat{g}_{q_k}$ by the relation (12).

Since $\frac{\sin(\pi x)}{\pi x}$ is positive and decreases on $(0, 1/2)$ and tends to 1 as $x \to 0^+$, we have $\frac{2}{\pi} \leq \frac{\sin(\pi x)}{\pi x} \leq 1$ for $0 < x \leq \frac{1}{2}$. We then obtain,

\[ \sum_n |\hat{g}_n| = \sum_k |\hat{f}_{q_k}| \frac{|1 - e^{2\pi i q_k \alpha}|}{|1 - e^{2\pi i q_k \beta}|} = \sum_k |\hat{f}_{q_k}| \frac{\sin(q_k \pi \alpha)}{\sin(q_k \pi \beta)} = \sum_k |\hat{f}_{q_k}| \frac{\sin(q_k \pi \alpha)}{\sin(q_k \pi \beta)} \leq \frac{\pi}{2} \sum_k |\hat{f}_{q_k}| \frac{q_k \alpha}{q_k \beta}. \]

Thus we have found $g \in C(\mathbb{T})$ with $(I - T_\beta)g = (I - T_\alpha)f$.

Now, we want to show that $(I - T_\alpha)f$ is not a double coboundary even in $L_1$: that is, there is no $h \in L_1$ such that $(I - T_\alpha)f = (I - T_\alpha)(I - T_\beta)h$. Suppose there is; it then implies the following restrictions on the corresponding Fourier coefficients of $h$:

\[ (1 - e^{2\pi i n \alpha}) \hat{f}_n = (1 - e^{2\pi i n \alpha})(1 - e^{2\pi i n \beta}) \hat{h}_n \quad \text{for every } n. \]
The same computation as above, with \( \hat{f}_n = 0 \) for \( n \notin \{q_k\} \), yields:

\[
|h_{q_k}| = \frac{|\hat{f}_{q_k}|}{1 - e^{2\pi i q_k \beta}} = \frac{2|\hat{f}_{q_k}|}{2\sin(\pi ||q_k\beta||)} > \frac{1}{2\pi}.
\]

Since \( h \in L_1 \), the Riemann-Lebesgue lemma yields \( |h_{q_k}| \to 0 \), which is a contradiction. \( \square \)

**Corollary 4.6.** Let \( \alpha \) and \( \beta \) be irrational numbers. Then there exist continuous functions \( f, g \in C(\mathbb{T}) \) such that \( (I - T_\alpha)f = (I - T_\beta)g \), but there is no measurable \( h \) satisfying \( (I - T_\alpha)(I - T_\beta)h = (I - T_\alpha)f \) (i.e. \( (I - T_\alpha)f \) is not a measurable double coboundary).

**Proof.** In the construction of \( f \) and \( g \) in the proof of Theorem 4.5, we can assume, by taking a subsequence, that \( \{q_k\} \) is lacunary: for some \( Q > 1 \), we have \( q_{k+1} / q_k \geq Q \) for every \( k \). Then \( f \) and \( g \) are continuous with lacunary Fourier series, and the same holds for \( (I - T_\alpha)f \). Without loss of generality, we may assume \( \int_T f = 0 \).

If \( h \) is measurable with \( (I - T_\alpha)(I - T_\beta)h = (I - T_\alpha)f \), then \( (I - T_\alpha)[f - (I - T_\beta)h] = 0 \), and ergodicity of \( T_\alpha \) implies that \( f = (I - T_\beta)h + \text{const} \). Hence \( (I - T_\beta)h \) is continuous, and by Anosov [4, Theorem 1] \( \int f (I - T_\beta)h = 0 \). Since \( \int f = 0 \), the constant is zero, and \( f = (I - T_\beta)h \). But since \( f \) is continuous with lacunary Fourier series, Herman’s theorem [29] says that \( h \in L_2(\mathbb{T}) \), contradicting Theorem 4.5. \( \square \)

J.P. Conze has noted that the work of Conze and Marco [15] yields an interesting result, in the spirit of Corollary 4.6, with very simple \( L_2 \) joint coboundaries.

**Proposition 4.7.** For any irrational \( \alpha \in (0, 1) \) with unbounded quotients there exist uncountably many pairs \( (\beta, \gamma) \) of irrational numbers in \( (0, 1) \), such that \( \{1, \gamma, \alpha\} \) are linearly independent over \( \mathbb{Q} \), and \( (I - T_\beta)1_{[0, \gamma]} \) is an \( L_2 \) joint coboundary of \( T_\alpha \) and \( T_\beta \) which is not a measurable double coboundary.

**Proof.** By [15, Theorem 2.2], if \( \alpha \) has unbounded partial quotients, then there is an uncountable set of pairs of irrational numbers \( \beta \) and \( \gamma \) in \( (0, 1) \) such that \( \varphi_{\beta, \gamma} := 1_{[0, \gamma]} - T_\beta 1_{[0, \gamma]} \in (I - T_\alpha)L_2 \), i.e. \( \varphi_{\beta, \gamma} = (I - T_\alpha)\psi \) with \( \psi = \psi_{\beta, \gamma} \) in \( L_2 \). Hence \( f = 1_{[0, \gamma]} - \gamma \) satisfies \( (I - T_\beta)f = (I - T_\alpha)\psi \). Since the number of pairs \( (\beta, \gamma) \) is uncountable, there is an uncountable subset of pairs with \( \{1, \alpha, \gamma\} \) linearly independent over the rationals. For such pairs, by Oren [47], the skew product \( T_f(x, y) = (x + \alpha, y + f(x)) \) on \( \mathbb{T} \times \mathbb{R} \) is ergodic. We prove that there is no measurable \( h \) such that \( (I - T_\alpha)(I - T_\beta)h = (I - T_\beta)f \); indeed, if such \( h \) existed, we would have \( f = (I - T_\alpha)h \) (shown similarly to the proof of Corollary 4.6). Then the set \( E(f) \) of essential values of \( f \) is \( \{0\} \) [54, Theorem 3.9(4)]. But ergodicity of \( T_f \) yields that \( \mathbb{R} \subset E(f) \) [54, Corollary 5.4] – a contradiction. \( \square \)

**Remarks.**

1. In the proof it is shown that if \( \alpha, \gamma \in (0, 1) \) with \( \{1, \alpha, \gamma\} \) linearly independent over \( \mathbb{Q} \), then \( f = 1_{[0, \gamma]} - \gamma \) is not a measurable coboundary of \( T_\alpha \). Petersen’s result [51] shows only \( f \notin (I - T_\alpha)L_2 \).

2. An irrational \( \alpha \) has unbounded partial quotients if and only if \( \liminf_n n||n\alpha|| = 0 \) (e.g. [24, Section 11.10]). The set of such \( \alpha \) in \( 0, 1 \) has Lebesgue measure 1, as its complement in \( 0, 1 \) has measure zero [24, Theorem 196]. Thus Proposition 4.7 applies to almost every (irrational) \( \alpha \in (0, 1) \).

The following result concerning measurable joint coboundaries is a special case of a result of Conze and Marco [15, Proposition 1.5]. An example is given in [15, Theorem 2.1].
Proposition 4.8. Let \( u \) be a measurable function on \( \mathbb{T} \) and \( \alpha \in (0, 1) \) irrational. If the set of \( \beta \in (0, 1) \), for which \((I - T_\beta)u = (I - T_\alpha)v_\beta \) for some measurable \( v_\beta \), has positive measure, then there exist a measurable \( h \) and a constant \( C \) such that \( u = (I - T_\alpha)h + C \). If \( u \) is integrable, then \( C = \int_\mathbb{T} u \).

Let \( u \) and \( \alpha \) be as in Proposition 4.8. Then \( u - C \) satisfies the same assumptions, so we may assume \( C = 0 \) (even without integrability of \( u \)). For each irrational \( \beta \) as in the proposition we then have that the joint coboundary \((I - T_\beta)u = (I - T_\alpha)v_\beta \) is a (measurable) double coboundary: \((I - T_\beta)u = (I - T_\beta)(I - T_\alpha)h; \) if \( h \) is non-integrable, then any \( w \) satisfying \((I - T_\alpha)(I - T_\beta)w = (I - T_\beta)u \) is non-integrable, by Lemma 4.2.

Proposition 4.9. Let \( \phi \in C(\mathbb{T}) \) with \( \int_\mathbb{T} \phi = 0 \) have Fourier coefficients satisfying \( \sum_k |\hat{\phi}_k| < \infty \) and \( \sum_{|k| > 0} |\hat{\phi}_k| \log(1/|\hat{\phi}_k|) < \infty \). Then \( \phi \in (I - T_\alpha)C(\mathbb{T}) \) for almost every \( \alpha \). Hence for almost every pair \((\alpha, \beta)\), there exist \( f, g \in C(\mathbb{T}) \) such that \((I - T_\alpha)f = \phi = (I - T_\beta)g \) (\( \phi \) is a joint coboundary in \( C(\mathbb{T}) \)).

Proof. By Kac and Salem [31], the series
\[
\sum_{|k| = 1}^\infty \frac{|\hat{\phi}_k|}{|\sin(\pi k x)|}
\]
converges a.e. For \( x = \alpha \) for which the series converges, define \( c_0 = 0 \) and \( c_k := \hat{\phi}_k/2\sin(\pi k x) \) for \( |k| > 0 \). Then \( \sum_k |c_k| < \infty \), and the function \( f(z) := \sum_k c_k z^k \) is in \( C(\mathbb{T}) \), and satisfies \((I - T_\alpha)f = \phi \). \( \square \)

Remarks. 1. Proposition 4.9 applies when \( |\hat{\phi}_k| = O(1/|k|(|\log |k|)|^{2+\epsilon}) \). This improves a result of Herman [28, p. 230, Proposition 8.2.1].

2. If \( |\hat{\phi}_k| \) and \( |\hat{\phi}_k - \hat{\phi}|- (k > 0) \) are non-increasing, then by Muromskii [45] (with \( \alpha = 1 \)), the condition \( \sum_{|k| > 0} |\hat{\phi}_k| \log |k| < \infty \) implies that \( \phi \in (I - T_\alpha)C(\mathbb{T}) \) for almost every \( \alpha \).

Definition. A (necessarily irrational) real number \( \alpha \) is said to be badly approximable (bad for short) if there exists \( \epsilon > 0 \) such that
\[
\|q\alpha\| := \min\{|q\alpha - p| : p \in \mathbb{Z}\} > \frac{\epsilon}{q} \quad \forall q \in \mathbb{N}.
\]
The set of badly approximable numbers is known to have Lebesgue measure 0 and Hausdorff dimension 1. Rozhdestvenskii [53] constructed mean zero \( L_2 \) functions such that whenever \( \alpha \) is bad, there is no measurable \( h \) satisfying \((I - T_\alpha)h = f \).

Lemma 4.10. Let the Fourier coefficients of \( f \) satisfy \( \hat{f}_0 = 0 \).

(i) If \( \sum_{k \neq 0} |k| \cdot |\hat{f}_k| < \infty \), then \( f \in (I - T_\alpha)C(\mathbb{T}) \) for every badly approximable \( \alpha \).

(ii) If \( \sum_{k \neq 0} |k|^2 |\hat{f}_k|^2 < \infty \), then \( f \in C(\mathbb{T}) \cap (I - T_\alpha)L_2(\mathbb{T}) \) for every badly approximable \( \alpha \).

Proof. We prove (i):
\[
2 \sum_{|k| = 1}^\infty \frac{|\hat{f}_k|}{|1 - e^{2\pi i k \alpha}|} = \sum_{|k| = 1}^\infty |\hat{f}_k| \leq \sum_{|k| = 1}^\infty \frac{|\hat{f}_k|}{2\|k\alpha\|} \leq \sum_{|k| = 1}^\infty \frac{|k| |\hat{f}_k|}{2c} < \infty.
\]
Hence the function \( g(z) = \sum_{|k| = 1}^\infty \frac{\hat{f}_k}{1 - e^{2\pi i k \alpha}} z^k \) is in \( C(\mathbb{T}) \) and satisfies \((I - T_\alpha)g = f \).
(ii) By Cauchy-Schwarz, $\sum_k |f_k| < \infty$. A computation similar to (i) yields that the above $g$ is in $L_2(\mathbb{T})$.

**Proposition 4.11.** Let $a_k \downarrow 0$ satisfy $\sum_{k=1}^{\infty} ka_k^2 < \infty$. If the Fourier coefficients of $f \in L_2(\mathbb{T})$ satisfy $\hat{f}_0 = 0$ and $|\hat{f}_k| = O(a_k)$ for $k \neq 0$, then $f \in (I - T_\alpha)L_2$ for every badly approximable $\alpha$.

**Proof.** When $a_k \downarrow 0$, then by a result of Muromskii [45], (with $\alpha = 2$ and $c_k = a_k^2$), we have

$$\sum_{|k|=1}^{\infty} \frac{|\hat{f}_k|^2}{|\sin(\pi k x)|^2} \leq C \sum_{|k|=1}^{\infty} \frac{|a_k|^2}{|\sin(\pi k x)|^2} < \infty,$$

for every $x$ having a continued fraction expansion with bounded elements (quotients). It is known (e.g. [24, Section 11.10]) that badly approximable numbers have bounded quotients.

**Remarks.**

1. Proposition 4.11 applies to $f \in L_2$ with $|\hat{f}_k| = O(1/|k|(\log |k|)^{1/2+\delta})$.

2. Let $\hat{f}_k = 1/k(\log k)^{3/2+\delta}$ for $k > 0$ and $\hat{f}_k = 0$ for $k \leq 0$. Then $f \in (I - T_\alpha)L_2$ for any bad $\alpha$, $\sum_k k|f_k|^2 < \infty$, but $\sum_k k^2|f_k|^2 = \infty$.

**Corollary 4.12.** Let $f \in L_2(\mathbb{T})$ satisfy $\hat{f}_0 = 0$ and $|\hat{f}_k| = O(1/k^2(\log |k|)^\gamma)$ with $\gamma > 1$. Then for any badly approximable numbers $\alpha$ and $\beta$, $f \in (I - T_\alpha)C(\mathbb{T}) \cap (I - T_\beta)C(\mathbb{T})$ and $f \in (I - T_\alpha)(I - T_\beta)L_2(\mathbb{T})$.

**Proof.** By Lemma 4.10, $f \in (I - T_\alpha)C(\mathbb{T}) \cap (I - T_\beta)C(\mathbb{T})$. Let $g(z) = \sum_{|k|=1}^{\infty} \frac{\hat{f}_k}{1 - e^{2\pi ik\alpha}} z^k$, which satisfies $(I - T_\alpha)g = f$. For $k > 0$ put $a_k = 1/k(\log k)^\gamma$, so $|\hat{g}_k| = O(a_k)$. Then $\{a_k\}$ satisfies the assumptions of Proposition 4.11, so $g \in (I - T_\alpha)(I - T_\beta)L_2(\mathbb{T})$, which shows that $f \in (I - T_\alpha)(I - T_\beta)L_2(\mathbb{T})$.

**Remark.** When the Fourier coefficients of $f$ satisfy $\hat{f}_0 = 0$ and the stronger condition $\sum_{k \neq 0} k^2|\hat{f}_k| < \infty$, we define, for $\alpha$ and $\beta$ bad, $c_k = \hat{f}_k/(1 - e^{2\pi ik\alpha})(1 - e^{2\pi ik\beta})$ for $k \neq 0$ and $c_0 = 0$. Similarly to the proof of Lemma 4.10, we obtain that $\sum_k |c_k| < \infty$, and then $h(z) = \sum_k c_k z^k$ satisfies $f = (I - T_\alpha)(I - T_\beta)h$.

**Proposition 4.13.** Let the Fourier coefficients of $f \in L_2(\mathbb{T})$ satisfy $\liminf_{|n| \to \infty} |n\hat{f}_n| > 0$. Then:

(i) For any $\beta$ irrational, $f \not\in (I - T_\beta)L_1$ (so $f$ is not a joint coboundary of $T_\alpha$ and $T_\beta$).

(ii) For $\beta$ irrational, if $(I - T_\beta)f$ is a joint coboundary in $L_2$ with $T_\alpha$, i.e. $(I - T_\beta)f \in (I - T_\alpha)L_2$, then $\beta = k\alpha + n$ with $k, n \in \mathbb{Z}$ (i.e. $T_\alpha^k = T_\beta$).

(iii) If $\alpha$ is irrational and $T_\beta = T_\alpha^k$, then $(I - T_\beta)f$ is a joint coboundary of $T_\alpha$ and $T_\beta$ which is not a double coboundary in $L_1$.

**Proof.** (i) Assume $f = (I - T_\beta)h$ with $h \in L_1$. Then, as in (**), $|\hat{h}_n| \geq \frac{|\hat{f}_n|}{2\pi ||n\beta||}$. For $n > N$ and some $C > 0$ we have

$$\frac{C}{2\pi ||n\beta||} < C \frac{|\hat{f}_n|}{2\pi ||n\beta||} = C|\hat{h}_n| \to 0,$$

using the Riemann-Lebesgue lemma. Hence $\liminf_{n \to \infty} n||n\beta|| = \infty$, a contradiction to Dirichlet’s theorem [24, Theorem 185], which yields $\liminf_{n \to \infty} n||n\beta|| \leq 1$. 
(ii) Let \( g \in L_2 \) satisfy \((I - T_\alpha)g = (I - T_\beta)f\). Computing Fourier coefficients we obtain
\[
\sum_{n \neq 0} |\hat{f}_n|^2 \left| \frac{\sin(n\pi\beta)}{\sin(n\pi\alpha)} \right|^2 = \sum_n |\hat{g}_n|^2 < \infty,
\]
so the assumption \( \liminf_{|n| \to \infty} |n\hat{f}_n| > 0 \) yields \( \sum_{n \neq 0} \frac{1}{n^2} \left| \frac{\sin(n\pi\beta)}{\sin(n\pi\alpha)} \right|^2 < \infty \). By Petersen [51], \( \beta \in \mathbb{Z}\alpha \mod 1 \).

(iii) We may assume \( k > 0 \). Since \((I - T_\alpha^k) = (I - T_\alpha)(I - T_\alpha^{k-1})\), \((I - T_\beta)f\) is a joint coboundary, and we may assume \( \hat{f}_\beta = 0 \). If there is \( h \in L_1 \) with \((I - T_\alpha)(I - T_\beta)h = (I - T_\beta)f\), then \( f - (I - T_\alpha)h \) is a constant, which is \( \hat{f}_\beta = 0 \). This contradicts part (i); hence \((I - T_\beta)f\) is not a double coboundary in \( L_1 \).

\[\square\]

Remarks. 1. If \( f \in L_2 \) satisfies \( \liminf_{|n| \to \infty} |n\hat{f}_n| > 0 \), then by [16] (see also [33, p. 283]) there exists a function \( \phi \in C(\mathbb{T}) \) with \( \liminf_{|n| \to \infty} |n\hat{\phi}_n| \geq \liminf_{|n| \to \infty} |n\hat{f}_n| > 0 \).

2. Compared with Theorem 2.9, Proposition 4.13(iii) yields an explicit construction of joint coboundaries in \( L_2 \) which are not double coboundaries (even in \( L_1 \)). It also yields, via the above mentioned result of [16], joint coboundaries in \( C(\mathbb{T}) \) which are not double coboundaries in \( L_1 \) (see Corollary 4.4).

3. Part (ii) of Proposition 4.13 proves the following special case of Kornfeld’s result [36]: If \( \alpha \) and \( \beta \) are irrational and \((I - T_\beta)L_2(\mathbb{T}) \subset (I - T_\alpha)L_2(\mathbb{T})\), then \( T_\beta = T_\alpha^k \) for some \( k \in \mathbb{Z} \).

Proposition 4.14. Let the Fourier coefficients of \( f \in L_2(\mathbb{T}) \) satisfy
\[
C := \liminf_{n \to \infty} n^\delta |\hat{f}_{n^2}| > 0 \quad \text{for some fixed} \quad \delta \in \left(\frac{1}{2}, \frac{2}{3}\right).
\]
Then for any \( \beta \) irrational, \( f \not\in (I - T_\beta)L_1 \).

Proof. Assume \( f = (I - T_\beta)h \) with \( h \in L_1 \). Then, as in (**) \( |\hat{h}_n| \geq \frac{\hat{f}_n}{2\pi n^{\delta/2}} \). By Zaharescu [60, Theorem 1], there exists an increasing subsequence \( (n_k) \) with \( \|n_k^2\beta\| < n_k^{-\delta} \). Then for \( n_k > N \) we have
\[
|\hat{h}_{n_k^2}| \geq \frac{|\hat{f}_{n_k^2}|}{2\pi \|n_k^2\beta\|} \geq \frac{|\hat{f}_{n_k^2}| n_k^\delta}{2\pi} \geq C \frac{n_k^\delta}{3\pi},
\]
which contradicts the Riemann-Lebesgue lemma. Hence \( f \not\in (I - T_\beta)L_1 \). \[\square\]

Remarks. 1. The requirement \( \delta > 0.5 \) follows from \( f \in L_2 \).

2. Propositions 4.13 and 4.14 are not comparable. In Proposition 4.14 we may have \( \hat{f}_n = 0 \) for infinitely many \( n > 0 \), while in Proposition 4.13(i), which holds also if \( \liminf_{n \to \infty} n|\hat{f}_n| > 0 \), this assumption implies \( \hat{f}_n \neq 0 \) from some place on. The price we pay in Proposition 4.14 is that the coefficients at \( n^2 \) have to be larger, of order \( 1/n^\delta \) (instead of \( 1/n^2 \)).

Definition. A pair \((\alpha, \beta)\) of irrational numbers is said to be \textit{badly approximable} if
\[
C(\alpha, \beta) := \liminf \sqrt{q} \max\{\|q\alpha\|, \|q\beta\|\} > 0.
\]
The set \( \text{Bad}_2 \) of bad pairs is not empty, by Perron [50]. A consequence of Khintchine’s theorem is that it has Lebesgue measure zero. \( \text{Bad}_2 \) is uncountable, since it has maximal
Proof. By the two-dimensional Dirichlet theorem, $C(\alpha, \beta) \leq 1$. Let
\[ \lim_k \sqrt{q_k} \max\{\|q_k \alpha\|, \|q_k \beta\|\} = C(\alpha, \beta) = C > 0. \]
Without loss of generality, we may assume $\|q_k \beta\| \geq \|q_k \alpha\|$ for infinitely many $q_k$. We take an increasing subsequence of these $q_k$, still denoted by $(q_k)$, such that
\[ \frac{C}{2} \leq \sqrt{q_k} \|q_k \beta\| \leq 2C \quad \forall k > 0, \quad \text{and} \quad \sum_k \frac{1}{\sqrt{q_k}} < \infty. \]
We take a further subsequence, still denoted by $(q_k)$, such that $\inf_k \frac{q_{k+1}}{q_k} \geq Q > 1$ ($(q_k)$ is lacunary). By the choice of $q_k$, we have $\frac{\|q_k \alpha\|}{\|q_k \beta\|} \leq 1$, so for any $f$ with $\hat{f}_{q_k} = a_k$ and $\hat{f}_n = 0$ otherwise, we get, as in (*), that $g$ with $\hat{g}_n$ defined by (12) is in $C(\mathbb{T})$, and $(I - T_\alpha)f = (I - T_\beta)g$. The condition $\limsup \sqrt{q_k} |a_k|$ is used to obtain that $(I - T_\alpha)f$ is not a double coboundary in $L_1$, by a contradiction to the Riemann-Lebesgue lemma, similar to (**). If $(I - T_\alpha)f = (I - T_\alpha)(I - T_\beta)h$, then $(I - T_\beta)h = f$ (since $\hat{f}_n = 0$). Since $(q_k)$ is lacunary, by Herman [29] $h \in L_2$, contradicting the fact that $(I - T_\alpha)f$ is not an $L_1$ double coboundary.

5. COBOUNDARY SUMS AND UNIFORM ERGODICITY OF COMMUTING CONTRACTIONS

Let $\theta$ and $\tau$ be commuting ergodic measure preserving transformations of the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and let $f \in L_2(\mathbb{P})$ with $\int_{\Omega} f \, d\mathbb{P} = 0$. The central limit theorem (CLT) problem is to find conditions for the convergence in distribution of $\frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(\theta^k \tau^j \omega)$. The Koopman operators $Tg = g \circ \theta$ and $Sg = g \circ \tau$ commute, so the CLT problem is the convergence in distribution of $\frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j f$. When this latter expression converges in $L_2$-norm to zero, we have a degenerate CLT (a zero asymptotic variance). This motivates the results of this section.

Proposition 5.1. Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$ with $F(T) = F(S)$, and let $z := (I - T)x + (I - S)y$. Then $\|\frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z\|$ converges to zero.

Proof. It is enough to prove when $z = (I - T)x$. Let $E_S := \lim_n \frac{1}{n} \sum_{j=0}^{n-1} S^j$ (in the strong operator topology). Since $E_S x$ is $S$-invariant it is also $T$-invariant, and we may replace
Let $x$ by $x - E_Sx$, so we assume $E_Sx = 0$. Then
\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \right\| = \left\| \frac{1}{n} \sum_{j=0}^{n-1} S^j (I - T^n) x \right\| \leq \left\| \frac{1}{n} \sum_{j=0}^{n-1} S^j x \right\| + \left\| T^n \left( \frac{1}{n} \sum_{j=0}^{n-1} S^j x \right) \right\| \leq 2 \left\| \frac{1}{n} \sum_{j=0}^{n-1} S^j x \right\| \to \|E_Sx\| = 0.
\]

**Remarks.** 1. If $F(S) \neq F(T)$, then for $x = Sx \neq Tx$, the proposition fails when $T^n x \not\rightarrow x$. For example, on $X$ reflexive take $S = I$ and $T$ such $T^n \to 0$ in the weak operator topology.

2. $z = (I - T)x + (I - S)y$ is a joint coboundary if and only if both $(I - T)x$ and $(I - S)y$ are.

3. If $T$ and $S$ are induced by commuting invertible ergodic probability preserving transformations and $S \neq T^k$ for any $k \in \mathbb{Z}$, then [36] there exists $y$ such that $(I - S)y \notin (I - T)x$, so for any $x$, $z = (I - T)x + (I - S)y$ is not a coboundary of $T$, so $z$ is not a joint coboundary.

4. A special case of the result of Lind [39] is that if $T$ and $S$ are induced by commuting invertible probability preserving transformations such that $T^m S^n \neq I$ when $m \neq n$, then for every measurable $h$ there exist measurable $f$ and $g$ such that $h = (I - T)f + (I - S)g$.

5. The proof of Proposition 5.1 can be easily modified to show that for any sector $S := \{(m, n) \in \mathbb{N}^2 : 0 < \alpha \leq \frac{m}{n} \leq \beta < \infty\}$ we have
\[
\lim_{n \land m \to \infty, (m, n) \in S} \left\| \frac{1}{\sqrt{mn}} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} T^k S^j z \right\| = 0.
\]

The following proposition is well-known, and its proof is similar to the proof for a single operator; the Hahn-Banach theorem is used to show the ”only if” in (i).

**Proposition 5.2.** let $T$ and $S$ be commuting power-bounded operators on a Banach space $X$. Then:

(i) $\left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \right\| \to 0$ if and only if $z \in (I - T)x + (I - S)x$.

(ii) $\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j$ converges strongly if and only if
\[
X = [F(T) \cap F(S)] \oplus (I - T)x + (I - S)x.
\]

**Theorem 5.3.** Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$. Then the following are equivalent:

(i) $(I - T)x + (I - S)x$ is closed in $X$.

(ii) $\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j$ converges in operator-norm, as $n \to \infty$. 
\[ (iii) \frac{1}{nm} \sum_{k=0}^{n-1} \sum_{j=1}^{m-1} T^k S^j \text{ converges in operator-norm as } \min(n, m) \to \infty. \]

**Proof.** Assume (i), and put \( Y := (I - T)X + (I - S)X \). By (i) \( Y = (I - T)X + (I - S)X \). Fix \( 1 < \alpha < 2 \). For \( z = (I - T)x \) we have

\[
\left\| \frac{1}{n^\alpha} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \right\| = \left\| \frac{1}{n^\alpha} \sum_{j=0}^{n-1} S^j (I - T^n)x \right\| \leq \left\| \frac{1}{n} \sum_{j=0}^{n-1} S^j \right\| \cdot \| (I - T^n)x \| \to 0.
\]

A similar computation for \( z = (I - S)y \) shows that for \( z \in (I - T)X + (I - S)X = Y \) we have \( \frac{1}{n^\alpha} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \to 0 \). Hence \( \sup_n \left\| \frac{1}{n^\alpha} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j \right\| \to 0 \) by the Banach-Steinhaus theorem, which yields \( \| \frac{1}{n^\alpha} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j \|_Y \to 0 \). Since \( T \) and \( S \) are mean ergodic, by Proposition 5.2(ii) (see also [13, Lemma 2.2]) we have \( X = [F(T) \cap F(S)] \oplus Y \). Let \( E \) be the corresponding projection on \( F(T) \cap F(S) \); then \( \| \frac{1}{n^\alpha} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j - E \| \to 0 \).

By Theorem 6.2, (iii) and (ii) are equivalent, and each implies (i). \( \square \)

**Remark.** Unlike Theorem 6.2, we do not need to assume in (i) that \( (I - T^*)X^* + (I - S^*)X^* \) is closed in order to obtain (ii), because we have assumed that \( T \) and \( S \) are mean ergodic.

**Theorem 5.4.** Let \( T \) and \( S \) be commuting mean ergodic contractions on a Banach space \( X \) with \( F(T) = F(S) \). Then the following are equivalent:

(i) \( (I - T)X + (I - S)X \) is closed.

(ii) For every \( z \in (I - T)(I - S)X \) we have

\[
(15) \quad \left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \right\| \to 0.
\]

(iii) \( \frac{1}{n^2} \sum_{k=1}^{n} \sum_{j=1}^{n} T^k S^j \) converges in operator-norm.

(iv) \( \frac{1}{nm} \sum_{k=0}^{n-1} \sum_{j=1}^{m-1} T^k S^j \) converges in operator-norm as \( \min(n, m) \to \infty \).

When \( (I - T)X + (I - S)X \) is closed, it equals \( (I - T)(I - S)X \), and the limit in (iii) is the projection \( E \) on \( F(T) \) with \( \ker(E) = (I - T)X \).

**Proof.** by [13, Remark 2.5], the assumption \( F(T) = F(S) \) implies

\[
(16) \quad (I - T)X = (I - S)X = (I - T)(I - S)X.
\]

Assume (i). By (16) \( (I - T)X + (I - S)X = (I - T)X = (I - T)(I - S)X \), so when \( (I - T)X + (I - S)X \) is closed Proposition 5.1 yields (15) for every \( z \in (I - T)(I - S)X \).

Assume (ii). Put \( Y = (I - T)X \), and assume that (15) holds for every \( z \in Y \). By (16), \( Y \) is \( T \) and \( S \) invariant, so we restrict ourselves to \( Y \). Since \( \sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \right\| < \infty \) for every \( z \in Y \), by the Banach-Steinhaus theorem \( \sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j \right\|_Y < \infty \).
Hence $\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j \|Y \rightarrow 0$. Let $E$ be the ergodic projection of $T$ on $Y$; then (iii) holds with $E$ the limit.

By Theorem 5.3, (i), (iii) and (iv) are equivalent. \hfill \Box

**Remark.** If in the theorem $T$ is uniformly ergodic, then $(I - T)X + (I - S)X$ is closed, since $(I - T)X$ is closed, and

$$(I - T)X \subset (I - T)X + (I - S)X \subset \overline{(I - T)X} = (I - T)X.$$ 

**Examples.** 1. Let $\nu$ be an absolutely continuous probability on $\mathbb{T}$. By [10, Theorem 3] (see also [5, Corollary 4.2]), $\|\nu^n - \lambda\| \rightarrow 0$ (in total variation norm), where $\lambda$ is the normalized Lebesgue measure on $\mathbb{T}$. Let $X = L_2(\mathbb{T})$ and define $Tf = \nu * f$. Then $\|T^n - E\|_2 \rightarrow 0$ (where $Ef = \int f d\lambda$), so $T$ is uniformly ergodic. Let $S$ be induced on $L_2(\mathbb{T})$ by a rotation by $\theta$ irrational. Then $F(T) = F(S)$ and $(I - T)X + (I - S)X$ is closed. Since $\sigma(S) = \mathbb{T}$, $S$ is not uniformly ergodic, so by Corollary 2.6 there is $z \in (I - T)X + (I - S)X = (I - T)X$ which is not a joint coboundary. Note that by Lemma 2.7(i), every joint coboundary is a double coboundary.

2. On $[0, 1]$ define $\theta t = 2t \mod 1$ and $\tau t = 3t \mod 1$, and let $T$ and $S$ be the corresponding isometries induced on $L_2[0, 1)$. Put $f(t) = e^{2\pi i t}$. Then orthogonality of the exponents yields

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j f \right\|_2^2 = \frac{1}{n^2} \left\| \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} e^{2\pi i 2^k 3^j t} \right\|_2^2 = 1.$$ 

Since $\int f(t) dt = 0$, by Theorem 5.4 $(I - T)L_2 + (I - S)L_2$ is not closed in $L_2$.

**Theorem 5.5.** Let $\theta$ and $\tau$ be commuting ergodic measure preserving transformations of a non-atomic probability space which generate a free $\mathbb{Z}_2^2$ action. For $1 \leq p < \infty$, let $U$ and $V$ be their corresponding isometries induced on $L_p$. Then:

(i) $(I - U)L_p + (I - V)L_p$ is not closed (both for the real and for the complex $L_p$).

(ii) There exists a real function $f \in L_p$ with integral zero such that

$$\limsup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} U^k V^j f \right\|_p > 0.$$ 

**Proof.** Assume that $(I - U)L_p + (I - V)L_p$ is closed, for the real $L_p$. Then it is also closed in the complex $L_p$, and we apply in that space Theorem 5.4, which yields that

$$\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} U^k V^j$$

converges in operator norm, with the limit $E$ a projection onto $F := F(U) = F(V) = \{\text{complex constants}\}$ with null space $Y := (I - U)L_p + (I - V)L_p = \{f \in L_p : \int f = 0\}$. By Corollary 6.3, $(1, 1)$ is not in $\sigma(U_Y, V_Y)$, where $U_Y$ and $V_Y$ are the restrictions to $Y$. The restrictions of $U$ and $V$ to $F(U)$ are both the identity, and for any complex Banach space $X$ we have $\sigma(I_X, I_X) = \{(1, 1)\}$. Since $L_p = F(U) \oplus Y$, by (10) we have

$$\sigma(U, V) \subset \sigma(I_U, I_V) \cup \sigma(U_Y, V_Y) = \{(1, 1)\} \cup \sigma(U_Y, V_Y).$$

Since the joint spectrum is closed, there is a neighborhood of $(1, 1)$ which is not in $\sigma(U_Y, V_Y)$, so $(1, 1)$ is isolated in $\sigma(U, V)$, a contradiction to Corollary 3.5. Hence (i) holds in the complex $L_p$ and therefore also in the real $L_p$. Hence (ii) of Theorem 5.4 fails, which yields (ii) of our theorem. \hfill \Box
Remarks. 1. The research leading to Theorem 5.5 was motivated by the result of Depauw [17, p. 168], who proved that for $U$ and $V$ induced on $L_2(\mathbb{T})$ by two irrational rotations, there exists $f \in L_2$ with integral zero which cannot be represented as $f = (I - U)g + (I - V)h$ with $g, h \in L_2$; this is (i) of Theorem 5.5 for $p = 2$.

2. Theorem 5.4 yields directly the result for $p = 2$, without the theory of joint spectra, when there exists $0 \neq f \in L_2$ with integral zero, such that all the orbit is orthogonal, i.e. the functions $\{U^kV^j f : k \geq 0, j \geq 0\}$ are orthogonal. See the second example following Theorem 5.4.

Derriennic and Lin [18] introduced the notion of fractional coboundaries of contractions. Let $0 < a < 1$, and let $(1 - t)^a = 1 - \sum_{j=1}^{\infty} a_j t^j$. It is known that $a_j > 0$ with $\sum a_j = 1$, so for any contraction $T$ on a Banach space we can define the operator $(I - T)^a = I - \sum_{j=1}^{\infty} a_j T^j$. The elements of $(I - T)^a X$ were called in [18] fractional coboundaries. If $T$ is not uniformly ergodic (i.e. $(I - T)X$ not closed), then the spaces $(I - T)^a X \subset (I - T)^b X$ for $0 < b < a \leq 1$ are all different, with closure $(I - T)X$.

Theorem 5.6. Let $T$ and $S$ be commuting mean ergodic contractions on a Banach space $X$ with $F(T) = F(S)$, and let $0 \leq a \leq 1$. If $z := (I - T)^a(I - S)^{1-a}x$, then $\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k (I - T)^a \right\| \rightarrow 0$.

Proof. Proposition 5.1, which requires $F(T) = F(S)$, yields the extreme cases $a = 0$ and $a = 1$, so we assume $0 < a < 1$. It was proved in [18, Corollary 2.15] that if $T$ is a mean ergodic contraction and $y \in (I - T)^a X$, then $\left\| (1/n^{1-a}) \sum_{k=1}^{n} T^k y \right\| \rightarrow 0$. It follows that

$$(17) \quad \sup_{n \geq 1} \left\| \frac{1}{n^{1-a}} \sum_{k=0}^{n-1} T^k (I - T)^a \right\| = K < \infty.$$ 

If $x \in F(T)$, then $(I - T)^a x = 0$, so we may assume $E_T x = 0$, i.e. $x \in (I - T)X$. By (17) and an application of [18, Corollary 2.15] to $S$ we obtain

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k S^j z \right\| = \left\| \frac{1}{n^a} \frac{1}{n^{1-a}} \sum_{k=0}^{n-1} T^k S^j (I - T)^a (I - S)^{1-a} x \right\| \leq K \left\| \frac{1}{n^a} \sum_{j=0}^{n-1} S^j (I - S)^{1-a} x \right\| \rightarrow 0.$$ 

Note that for $0 < a < 1$, the assumption $F(T) = F(S)$ is not used. 

Remark. By the theorem, if $z = \sum_{\ell=1}^{L} (I - T)^a \ell (I - S)^{1-a} x_\ell$, with $0 \leq a_1 < a_2 < \ldots a_L \leq 1$, then $\left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j z \right\| \rightarrow 0$.

Corollary 5.7. Let $\theta$ and $\tau$ be commuting ergodic measure preserving transformations of the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and let $f \in L_2(\mathbb{P})$ with $\int_{\Omega} f d\mathbb{P} = 0$. Let $T$ and $S$ be the corresponding Koopman operators. If

$$f = \sum_{\ell=1}^{L} (I - T)^a \ell (I - S)^{1-a} g_\ell,$$

with $0 \leq a_1 < a_2 < \ldots a_L \leq 1$ and $g_1, \ldots, g_L \in L_2$, then $\left\| \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(\theta^k \tau^j \omega) \right\|_2 \rightarrow 0$. 


6. Appendix: The Uniform Ergodic Theorem for Commuting Contractions

Mbekhta and Vasilescu [42] extended the uniform ergodic theorem of [37] to $d$ commuting operators on a complex Banach space. A special case of their result is the following.

**Theorem 6.1.** Let $T$ and $S$ be commuting power-bounded operators on a complex Banach space $X$. Then the following are equivalent:

(i) $(I - T)X + (I - S)X$ and $(I - T^*)X^* + (I - S^*)X^*$ are closed in $X$ and $X^*$, respectively.

(ii) $\frac{1}{nm} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j$ converges in operator-norm, as $\min(n, m) \to \infty$.

(iii) $\frac{1}{n^m} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j$ converges in operator-norm, as $\min(n, m) \to \infty$.

The proof in [42] uses spectral theory, so does not apply directly in real Banach spaces.

**Theorem 6.2.** Let $T$ and $S$ be commuting power-bounded operators on a real or complex Banach space $X$. Then the following are equivalent:

(i) $(I - T)X + (I - S)X$ and $(I - T^*)X^* + (I - S^*)X^*$ are closed in $X$ and $X^*$, respectively.

(ii) $\frac{1}{nm} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j$ converges in operator-norm, as $\min(n, m) \to \infty$.

(iii) $\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j$ converges in operator-norm, as $n \to \infty$.

**Proof.** Obviously (ii) implies (iii). Assume (iii), and put $Y := (I - T)X + (I - S)X$. Then it is easy to compute that $\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j z \to 0$ for $z \in Y$ (using power-boundedness), and by (iii), the restrictions to $Y$ satisfy $\|\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j\|_Y \to 0$. Hence, for $n$ large enough

$$A_n := I_Y - \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} T^k S^j = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} (I - T^k S^j)$$

is invertible on $Y$. Since $(I - T^k)(I - S^j) = I - T^k + I - S^j - (I - T^k S^j)$, on $Y$ we have

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} (I - T^k) + \frac{1}{n} \sum_{j=0}^{m-1} (I - S^j) - \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} (I - T^k)(I - S^j).$$

Fix $n$ large. Denote by $A$ the restriction to $Y$ of $A_n$ and $B = A^{-1}$ (defined on $Y$). By the Neumann series expansion, $B$ is in the closed subalgebra of $B(Y)$ generated by the restrictions of $T$ and $S$ to $Y$. These restrictions satisfy

$$I_Y = BA = B \cdot \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=0}^{k-1} T^\ell [I_Y - \frac{1}{n} \sum_{j=0}^{m-1} (I_Y - S^j)] (I_Y - T) + B \cdot \frac{1}{n} \sum_{j=0}^{m-1} \sum_{\ell=0}^{j-1} S^\ell (I_Y - S).$$

Thus we have operators $C$ and $D$ in $B(Y)$, commuting with the restrictions $T_Y$ and $S_Y$, such that $I_Y = (I - T_Y)C + (I - S_Y)D$. Hence

$$Y = (I - T_Y)Y + (I - S_Y)Y \subset (I - T)X + (I - S)X \subset Y,$$
so (i) holds. Note that this proof is valid for $X$ real or complex, and unlike [42], spectral theory is not used.

When $X$ is complex, (i) implies (ii) by [42]. To prove that (i) implies (ii) when $X$ is real we will use the complexification of $X$, described below, and deduce the result from the complex case. We define $X_C = X \oplus X$, with the identification $(x, y) = x + iy$ which allows the definition of the multiplication by complex scalars. On $X_C$ we define the Taylor norm (see [44, Proposition 3])

$$
\|(x, y)\|_T := \sup_{0 \leq t \leq 2\pi} \|x \cos t - y \sin t\| = \sup_{\phi \in X^*, \|\phi\| \leq 1} \sqrt{\phi(x)^2 + \phi(y)^2}.
$$

In the sequel we write $\|(x, y)\|$ for $\|(x, y)\|_T$. Note that $\|(x, 0)\| = \|x\|$ and $\|(x, -y)\| = \|(x, y)\|$. Clearly

$$
\max\{\|x\|, \|y\|\} \leq \|(x, y)\| \leq \sqrt{\|x\|^2 + \|y\|^2} \leq \|x\| + \|y\|,
$$

which shows that $\{(x_k, y_k)\}$ converges if and only if both $\{x_k\}$ and $\{y_k\}$ converge. Given an operator $T$ on $X$, we extend it to $X_C$ by $T_C(x, y) = (Tx, Ty)$. By [44, Proposition 4] $\|(T_C)^n\| = \|(T^n)_C\| = \|T^n\|$, so $T_C$ is power-bounded when $T$ is.

Assume now that $T$ and $S$ on $X$ satisfy (i). If $(I - T_C)(x_k, y_k) + (I - S_C)(u_k, v_k)$ converges in $X_C$ to $(z, w)$, computations by the definitions yield that $(z, w) \in (I - T_C)X_C + (I - S_C)X_C$, so $(I - T_C)X_C + (I - S_C)X_C$ is closed.

By [44, Proposition 7], $(X_C)^*$ yields a reasonable complexification of $X^*$, which by [44, Proposition 3] is equivalent to the Taylor complexification of $X^*$. It is therefore easy to check that $(T_C)^* = (T^*)_C$, and the condition $(I - T^*)X^* + (I - S^*)X^*$ closed implies that $(I - T_C^*)X^* + (I - S_C^*)X^*$ is closed. Hence $T_C$ and $S_C$ on $X_C$ satisfy (i), so by Theorem 6.1 $\frac{1}{mn} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T_C^k S_C^j$ converges in operator norm on $X_C$, which implies (ii) of our theorem.

The above proof of (iii) implies (i) yields the following corollary.

**Corollary 6.3.** Let $T$ and $S$ be commuting power-bounded operators on a complex Banach space $X$. If $\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} T^k S^j$ converges in operator-norm, as $n \to \infty$, then $Y := (I - T)X + (I - S)X$ is closed, and for any subalgebra $A \subset B(Y)$ containing the restrictions $T_Y$ and $S_Y$, the point $(1, 1)$ is not in the Harte joint spectrum $\sigma_A(T_Y, S_Y)$.

**Remark.** The result of Corollary 6.3 for the Taylor spectrum was deduced in [42, Lemma 4] from the spectral mapping theorem [56]; the same proof could apply also for the Harte spectrum (in $Y$), by [26, Theorem 4.3]. Our proof for the Harte spectrum is simpler, since it uses only its definition [26].

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References


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