HOMOLOGY SPHERES WITH $E_8$-FILLINGS AND ARBITRARILY LARGE CORRECTION TERMS

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Abstract. In this paper we construct families of homology spheres which bound 4-manifolds with intersection forms isomorphic to $-E_8$. We show that these families have arbitrarily large correction terms. This result says that among homology spheres, the difference between the maximal rank of any minimal sub-lattice of a definite filling and the maximal rank of any even definite filling is arbitrarily large.

1. Introduction

1.1. Definite fillings and homology cobordism invariants. If a 3-manifold $Y$ bounds $X$, then we call $X$ a filling of $Y$. If a filling $X$ of $Y$ has a definite or even intersection form, or spin structure, then $X$ is called a definite or even filling of $Y$, or a spin filling of $Y$ respectively. Under the assumption that the homology of a filling has no 2-torsion, an even filling is equivalent to a spin filling. If a definite filling has a positive (or negative) definite intersection form, then we call the filling a positive-definite filling (or a negative-definite filling respectively).

Let $Y$ be an integral homology sphere. The Rohlin invariant $\mu(Y)$ is defined to be $\sigma(W)/8 \in \mathbb{Z}/2\mathbb{Z}$ for a spin filling $W$ of $Y$. We can assume that the spin filling $W$ has $H_1(W, \mathbb{Z}) = \{0\}$ (we say $W$ is homologically 1-connected). In this article we mainly consider homologically 1-connected definite fillings.

Ozsváth and Szabó defined a homology cobordism invariant $d$ in [9]. If a 3-manifold $Y$ has a negative-definite filling of $Y$, then the $d$-invariant has the following restriction.

**Theorem 1.1** ([9]). Let $Y$ be an integral homology three-sphere, then for each negative-definite four-manifold $X$ with boundary $Y$, we have the inequality

$$\xi^2 + \text{rk}(H_2(X, \mathbb{Z})) \leq 4d(Y)$$

for each characteristic vector $\xi$. 

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Furthermore, if a homology sphere \( Y \) has an even negative-definite filling \( W \), then \( b_2(W) \leq 4d(Y) \) is satisfied. For example, \( \Sigma(2, 3, 5) \) is the boundary of the \( -E_8 \)-plumbing. Here \( -E_8 \) is the unique unimodular, even, negative-definite, rank 8 quadratic form. The computation \( d(\Sigma(2, 3, 5)) = 2 \) means that \( b_2 \) of any even negative-definite filling is at most 8. The plumbing realizes a negative-definite filling with \( b_2 = 8 \). If \( Y \) has a definite filling with intersection form an \( nE_8 \) for some integer \( n \), then the filling is called an \( nE_8 \)-filling.

On the other hand, the \( d \)-invariant of \( \Sigma(2, 3, 7) \) is 0. Thus, if there exists an even negative-definite filling, then \( b_2 \) of the filling has to be 0. Since \( \mu(\Sigma(2, 3, 7)) = 1 \), it has no homologically 1-connected even definite filling. The plumbing of \( \Sigma(2, 3, 7) \) with all weights \(-2\) can give an even filling with even intersection form \((-E_8) \oplus H\), where \( H \) is the hyperbolic intersection form. This filling is a homologically 1-connected even (equivalently spin) indefinite filling.

In [14] the author defined the following invariants. If \( Y \) has a homologically 1-connected \( nE_8 \)-filling, then we define \( g_8 \) (or \( g_8 \)) to be
\[
g_8(Y) = \max\{b_2(W)/8|W \text{ is an } nE_8\text{-filling of } Y, H_1(W) = \{0\}\},
\]
and
\[
g_8(Y) = \min\{b_2(W)/8|W \text{ is an } nE_8\text{-filling of } Y, H_1(W) = \{0\}\}.
\]
If \( Y \) has no homologically 1-connected \( nE_8 \)-fillings, then \( g_8(Y) = -\infty \).

We call the invariant \( g_8(Y) \) the \( E_8 \)-genus of \( Y \). If \( Y \) has a homologically 1-connected \( nE_8 \)-filling, then we can immediately see the following bound:
\[
2g_8(Y) \leq |d(Y)|.
\]
For example, for any integer \( n \), \( d(\Sigma(2, 3, 12n + 5)) = 2 \) holds. The author in [14] showed \( g_8(\Sigma(2, 3, 12n + 5)) = 1 \) when \( 0 \leq n \leq 13 \) or \( n = 15 \). In [14] we gave other examples with \( 2g_8(Y) = |d(Y)| \). A natural question is the following:

**Question 1.2.** Among integral homology spheres \( Y \) with positive \( E_8 \)-genus, is \( |d(Y)| - 2g_8(Y) \) bounded?

We construct families of Brieskorn homology spheres giving a negative answer to this question.

1.2. **Main results.** Here we give the main result:

**Theorem 1.3.** For any integer \( n \), let \((p, q, r)\) be one of twelve types of triples below. The triple consists of pairwise coprime integers. Then the Brieskorn homology sphere \( \Sigma([p],[q],[r]) \) has a homologically 1-connected \(-E_8\)-filling and has \( g_8 = -\bar{\mu} = 1 \).

- (i) \((2,8n-3,14n-5)\), (ii) \((2,14n+3,24n+5)\)
- (iii) \((2,16n+3,26n+5)\), (iv) \((2,10n-3,16n-5)\)
- (v) \((5,35n-2,50n-3)\), (vi) \((5,25n-2,40n-3)\)
- (vii) \((3,15n-2,36n-5)\), (viii) \((3,9n-2,24n-5)\)
- (ix) \((3,21n-4,36n-7)\), (x) \((3,27n-4,48n-7)\)
- (xi) \((4,28n-3,64n-7)\), (xii) \((4,32n-3,76n-7)\)
The invariant $\bar{\mu}$ is the Neumann-Siebenmann invariant (NS-invariant), which will be defined in Section 2.3. These examples can be useful for realizing desired fillings restricted by gauge theory. For example, see the recent work of Scaduto [12].

**Theorem 1.4.** For a positive integer $n$ the correction terms of Brieskorn homology spheres (i), (ii) (iii) and (iv) in Theorem 1.3 have the following inequalities:

\[
2 \left\lceil \frac{n}{2} \right\rceil \leq d(\Sigma(2, 8n - 3, 14n - 5)), \quad 2 \left\lceil \frac{n+1}{2} \right\rceil \leq d(\Sigma(2, 14n+3, 24n+5)) ,
\]

\[
2 \left\lceil \frac{n+1}{2} \right\rceil \leq d(\Sigma(2, 16n+3, 26n+5)), \quad 2 \left\lceil \frac{n}{2} \right\rceil \leq \Sigma(2, 10n - 3, 16n - 5).
\]

These theorems say that for any positive integer $n$, the Brieskorn homology spheres (i), (ii), (iii), and (iv) have $\mathbb{E}_8$-fillings and $d(Y) - 2g_8(Y) = d(Y) + 2\bar{\mu}(Y)$ are arbitrarily large.

**Remark 1.5.** Let $(Y, c)$ be a pair consisting of a Seifert rational homology sphere $Y$ and a spin structure $c$. According to [18], the NS-invariant $\bar{\mu}(Y, c)$ is equivalent to the Fukumoto-Furuta invariant $w(Y, c)$.

Manolescu in [6] defined homology cobordism invariants $\alpha$, $\beta$, and $\gamma$ in the framework of $\text{Pin}(2)$-equivariant Seiberg-Witten Floer homology. A result in [13] says that for any Brieskorn homology sphere $Y$ (with the usual orientation) $\beta(Y) = \gamma(Y) = -\bar{\mu}(Y)$ and $\alpha(Y) = d(Y)/2$ or $(Y)/2 + 1$. Hence, our result implies the existence of integral homology spheres that have $\beta(Y) = 1$ but for which $\alpha(Y)$ is arbitrarily large.

**Remark 1.6.** We conjecture that the inequalities in Theorem 1.4 are actually equalities for any positive integer $n$. The evidence is due to Karakurt’s program [5]. Similarly, for any positive integer $n$ we predict that the following equalities for other Brieskorn homology spheres in Theorem 1.3 hold.

- For $(p, q, r) = (5, 35n - 2, 50n - 3), (5, 25n - 2, 40n - 3)$, we have
  \[
  d(\Sigma(p, q, r)) = 6n.
  \]

- For $(p, q, r) = (3, 15n - 2, 36n - 5), (3, 9n - 2, 24n - 5), (3, 21n - 4, 36n - 7), (3, 27n - 4, 48n - 7)$, we have
  \[
  d(\Sigma(p, q, r)) = 2n.
  \]

- For $(p, q, r) = (4, 28n - 3, 64n - 7), (4, 32n - 3, 76n - 7)$, we have
  \[
  d(\Sigma(p, q, r)) = 4 \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil.
  \]

Further, we predict that $d(\Sigma([p],[q],[r])) = 2$ for any triple $(p, q, r)$ in the eight families above with any non-positive integer $n$.

Here we compare the following result [17] by Ue and the result above.

**Theorem 1.7** ([17]). Let $(S, c)$ be a pair of a spherical 3-manifold and a spin structure on it. Then $d(S, c) = -2\bar{\mu}(S, c)$. 

In fact, in [17] it is shown that the general correction term \( d(S,t) \) of a spin\(^c\) spherical 3-manifold coincides with the Fukumoto-Furuta invariant, which is defined by using the index of the Dirac operator of a line bundle over a 4-orbifold.

We remark that the NS-invariant \( \bar{\mu} \) in this paper is defined to be the value obtained by dividing \( \bar{\mu} \) in [17] by 8. The examples in Theorem 1.3 are integral homology spheres that have \( d(\Sigma) + 2\bar{\mu}(\Sigma) \) arbitrarily large. The relationship between the correction term \( d \) and the NS-invariant \( \bar{\mu} \) for non-spherical 3-manifolds is not well-understood even in the case of Seifert 3-manifolds.

1.3. Other invariants related to definite fillings.

1.3.1. Homologically 1-connected fillings. In [14] the homology cobordism invariant \( \mathfrak{d}_s(Y) \) is defined to be

\[
\text{the maximal } b_2(W)/8 \text{ among homologically 1-connected, even definite fillings } W \text{ of } Y.
\]

A homologically 1-connected even filling gives a spin filling. Any invariant related to a kind of filling is defined to be \(-\infty\), if there exists no such fillings. We define \( o(Y) \) to be

\[
\text{the maximal rank of the minimal definite lattice } L \text{ such that } L \oplus \langle \pm 1 \rangle^n \text{ is the intersection form of a homologically 1-connected definite filling } W \text{ of } Y.
\]

Here ‘minimal’ means that any element with square ±1 is not included in the lattice and \( n \) is some non-negative integer. We have

\[
8g_8(Y) \leq 8\mathfrak{d}_s(Y) \leq o(Y),
\]

due to the definitions of the invariants. For example, in the case of \( Y = \Sigma(2,5,9) \), according to Corollary 1.2 in [12], \( g_8(Y) = \mathfrak{d}_s(Y) = 1 \) and \( o(Y) = 12 \) holds. We have the following question.

**Question 1.8.** Are the differences of these invariants bounded or unbounded?

We can show that the homology spheres in Theorem 1.4 satisfy the following unbounded property.

**Corollary 1.9.** Among homology spheres \( Y \), \( o(Y) - 8\mathfrak{d}_s(Y) \) is unbounded.

These integral homology spheres satisfy \( \mathfrak{d}_s(Y) = g_8(Y) \). It is not known whether for some integral homology spheres \( Y \) the differences \( \mathfrak{d}_s(Y) - g_8(Y) \) are positive or unbounded.

1.3.2. General definite fillings. Define \( E(Y) \) to be

\[
\text{the maximal } b_2(W)/8 \text{ among even definite fillings } W \text{ of } Y.
\]

Note the filling is possibly non-spin. In the same way, we define \( O(Y) \) to be

\[
\text{the maximal rank of the minimal sub-lattice of definite fillings } W \text{ of } Y \text{ and } G_8(Y) \text{ to be}
\]

\[
\text{the maximal } |n| \text{ among } nE_8\text{-fillings}
\]
of $Y$. Since the fillings used to define invariants $G_8(Y)$, $E(Y)$, and $O(Y)$ do not assume homologically 1-connected, we have $g_8(Y) \leq G_8(Y)$, $\partial \delta(Y) \leq E(Y)$, and $\delta(Y) \leq O(Y)$ naturally. We have the similar inequalities to (2):

$$(3) \quad 8G_8(Y) \leq 8E(Y) \leq O(Y).$$

For example, consider the case of $Y = \Sigma(2, 3, 7)$. As we mentioned above, $Y$ has no even definite fillings which are homologically 1-connected, i.e., $g_8(Y) = \partial \delta(Y) = -\infty$. On the other hand, Fintushel and Stern constructed a rational homology ball with boundary $Y$ in [2], i.e., $E(Y), G_8(Y) \geq 0$. Let $W$ be a negative-definite filling of $Y$ and $\Xi$ the set of characteristic vectors in $H_2(W)$. The computation $d(Y) = 0$ (for example, see [9]) means $\max_{c \in \Xi}(c^2 + b_2(W)) \leq 0$. The Elkies theorem in [1] concludes the inequality is an equality and the negative-definite lattice must be diagonalized. In particular, the even definite fillings of $Y$ must have the trivial intersection form, i.e., $E(Y) = G_8(Y) = 0$ holds. Considering the positive-definite fillings of $Y$, one has only to consider negative-definite fillings of $-Y$.

The definite plumbing lattice of $\Sigma(2, 3, 7)$ is diagonalized, therefore, $O(Y)$, $\delta(Y) \geq 0$. In the same way as above, the result $d(Y) = 0$ implies that $O(Y) = \delta(Y) = 0$.

Acknowledgements

This study was started by Christopher Scaduto’s question in the Gauge Theory in Fukuoka in 2018 February: Does $\Sigma(2, 5, 9)$ have a $-E_8$-filling? The author is grateful for the motivation. His question is also answered by Scaduto and Golla in [3]. The author is grateful to Macro Golla for providing many useful comments and advice for writing this article. The last corollary was suggested by Christopher Scaduto.

2. Notations and preliminaries

2.1. Plumbing diagram. For $i = 1, 2$ let $V_i \to S^2$ be two $D^2$-bundles over $S^2$ or let $V \to S^2$ be a $D^2$-bundle. For the $D^2$-bundles $V_1$ and $V_2$ we take sub-$D^2$-bundles over each disk in two base spaces $S^2$. For $V \to S^2$ we take sub-$D^2$-bundles over two disjoint disks in $S^2$. A plumbing process is a surgery obtained by identifying two $D^2$-bundles in such a way that one exchanges the roles of their sections and fibers. We call the plumbing of $V \to S^2$ with itself a self-plumbing. Actually, to define the plumbing process we need to choose one of the two possibilities of the orientation of the identification as in p.201 in [4]. Since we only deal with tree-type graphs later, we do not explain the choices.

We define a plumbing diagram (or graph) as explained in [11]. Let $V$ be the set of vertices with a weight function $m : V \to \mathbb{Z}$. We assign for $v \in V$ the $D^2$-bundle over $S^2$ with the Euler number $m(v)$. Let $E$ be the set of edges. Each edge $\{v, w\} \in E$ of a plumbing diagram signifies a plumbing process between the $D^2$-bundles over $S^2$. If $v = w$, i.e., it is a loop edge,
then the edge signifies a self-plumbing. A \( \mathbb{Z} \)-weighted graph \( (V, E, m) \) is called a plumbing graph or plumbing diagram.

Let \( (V, E, m) \) be a plumbing diagram. The plumbing process along a plumbing graph \( \Gamma = (V, E, m) \) gives a 4-manifold \( P(\Gamma) \) and we call \( P(\Gamma) \) a plumbed 4-manifold. The boundary \( \partial P(\Gamma) \) of \( P(\Gamma) \) is called a plumbed 3-manifold. Here \( [v] \) is the class represented by the core sphere of the \( D^2 \)-bundle corresponding to the vertex \( v \). The intersection form \( (\cdot, \cdot) : H_2(P(\Gamma)) \times H_2(P(\Gamma)) \to \mathbb{Z} \) on \( P(\Gamma) \) is computed from the linear extension of the following definition:

\[
(v, w) = \begin{cases} 
m(v) & v = w \\
1 & v \neq w \text{ and } \{v, w\} \in E, \\
0 & \text{otherwise.}
\end{cases}
\]

A tree-type graph with at most one vertex of degree larger than two is called a star-shaped graph. For a star-shaped graph we call a vertex with degree larger than two a central vertex. A plumbed 3-manifold for a star-shaped graph is called a Seifert manifold. Here we consider Seifert manifolds with one central vertex, i.e., not lens spaces. For \( i = 1, 2, \ldots, n \), let \( (\alpha^{(i)}_1, \alpha^{(i)}_2, \ldots, \alpha^{(i)}_{r_i}) \) be a sequence of weights of vertices of the \( i \)-th branch of a star-shaped graph with \( \alpha^{(i)}_1 \) adjacent to the central vertex. We compute the continued fraction for the sequence as follows:

\[
a_i/b_i = [\alpha^{(i)}_1, \alpha^{(i)}_2, \ldots, \alpha^{(i)}_{r_i}],
\]

where \( \alpha^{(i)}_j \) is some integer and \((a_i, b_i)\) is a pair of coprime integers. Here the continued fraction is defined to be

\[
[c_1, c_2, \ldots, c_m] = c_1 - \frac{1}{c_2 - \cdots - \frac{1}{c_m}}.
\]

Let \( e \) be the weight of the central vertex. We present the Seifert manifold with the data \( e, (a_1, b_1), \ldots, (a_n, b_n) \) as

\[
S(e; (a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)).
\]

We call the data of these integers the Seifert invariant. Instead of \((a_i, b_i)\), we also present it as \( (\alpha^{(i)}_1 \cdot \alpha^{(i)}_2 \cdots \alpha^{(i)}_{r_i}) \). Thus, we denote the plumbing process by a dot ‘\.' We present several consecutive integers by a power as follows:

\[
\overbrace{m}^{\begin{array}{c}\cdots \\ \vdots \end{array}} = \cdots 2^m \cdots.
\]

The Brieskorn homology sphere \( \Sigma(p, q, r) \) is defined to be

\[
\{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^p + z_2^q + z_3^r = 0 \} \cap S^5,
\]

where \( p, q, \) and \( r \) are pairwise coprime positive integers. This manifold is a plumbed 3-manifold with a star-shaped graph having three branches. The Seifert invariant is

\[
S(e; (p, p'), (q, q'), (r, r')),
\]
where \( e - (p'/p + q'/q + r'/r) = -1/pqr \). For example, the plumbing diagram of \( \Sigma(2,3,5) \) is described as follows:

\[
S(-2; (-2)^4, (-2)^2, -2).
\]

In [14], the author showed that a Brieskorn homology sphere \( \Sigma(p,q,r) \) whose intersection matrix of the minimal plumbed 4-manifold is isomorphic to \(-E_8\) is either \( \Sigma(2,3,5) \) or \( \Sigma(3,4,7) \).

2.2. Notations. We explain two new notations below. The first notation is the following:

\[
(L(a_1 \cdot a_2 \cdots a_n)^{(k)} b_m \cdot b_{m-1} \cdots b_1).
\]

Here \( k \) is an integer with \( |k| > 1 \) and \( a_i \) and \( b_j \) are integers. This notation presents a ‘plumbing’ with a higher linking as in Figure 1. The dot with \((k)\) means the two components with framing \( a_n \) and \( b_m \) are linked by \( k \) times as in Figure 1. Here, the box with the integer \( k \) means the full \( k \)-twist. We call the notation a linear diagram and the center special linking a central linking.

As an example, we consider a linear diagram of \( \Sigma(2,3,5) \). Sliding the first branch of \( S(-2; (-2)^4, (-2)^2, -2) \) to the third branch, we have the following linear diagram:

\[
S(-2; (-2)^4, (-2)^2, -2) = L((-2)^2 \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2)^3).
\]

The 4-manifold having the framed link as in Figure 1 is called the 4-manifold having linear diagram (5).

The second notation is a linear diagram with a torus knot component. Consider a linear diagram where one of the framings nearest to the central linking \((k)\) is zero. Then, the surgery diagram can be deformed as in Figure 2. We present the deformation of the linear diagram as follows:

\[
Y = L(\cdots q \cdot n \cdot 0)^{(k)} p \cdots) = L(\cdots q^{(k)} p + nk^2)^{(k,nk+1)} \cdots).
\]

The component with the underline having index \((k,nk + 1)\) stands for the \((k,nk+1)\)-torus knot with framing \( p + nk^2 \). The last surgery diagram gives a 4-manifold \( X \) bounded by \( Y \). Clearly, the intersection form of \( X \) is isomorphic to the intersection form of the 4-manifold having a linear diagram as follows:

\[
L(\cdots q^{(k)} (p + nk^2) \cdots).
\]
2.3. An estimate of the $\bar{\mu}$-invariant. The NS-invariant $\bar{\mu}$ in [7] is defined for any plumbed 3-manifold $M = \partial P(\Gamma)$. We fix a weighted graph $\Gamma = (V, E, m)$. We assume that the plumbing graph is tree-type and $\partial P(\Gamma)$ is a homology sphere for simplicity. We define the Wu class $w(\Gamma) \in H_2(P(\Gamma), \mathbb{Z})$ as follows:

1. The class $w(\Gamma)$ is written by $w(\Gamma) = \sum_{v \in V} \epsilon_v [v]$ for $\epsilon_v = 0$ or 1.
2. For any $v \in V$ we have $(w(\Gamma), [v]) = ([v], [v]) \mod 2$.

Let $\sigma(\Gamma)$ be the signature of the intersection form $(\cdot, \cdot)$ associated with $\Gamma$. Then we define $\bar{\mu}$ of $M$ to be

$$\bar{\mu}(M) = \frac{\sigma(\Gamma) - w(\Gamma)^2}{8}.$$ 

Due to [7], in the case of rational plumbed spin 3-manifold $(M, c)$, we can generalize the Wu class to an element $w(\Gamma, c) \in H_2(P(\Gamma), \mathbb{Z})$ as the obstruction to extending spin structure $c$ to the plumbed 4-manifold. Then, in the same way, we can define the NS-invariant to be $\bar{\mu}(M, c) = (\sigma(\Gamma) - w(\Gamma, c)^2)/8$. Here we state Theorem 2.9 in [17] in the form restricted to the Seifert homology spheres.

**Theorem 2.1 ([17]).** Suppose that a Seifert homology 3-sphere $M$ bounds a negative-definite spin 4-manifold $Y$ with a spin structure. Then

$$b_2(Y) \equiv -8\bar{\mu}(M) \mod 16.$$
In particular, if a Seifert homology sphere $M$ has a spin negative-definite filling $Y$, then $b_2(Y) \leq -8\bar{\mu}(M)$ holds.

3. The families of Brieskorn homology spheres in Theorem 1.3.

3.1. Proof of Theorem 1.3. We prove Theorem 1.3.

Proof. Below, we write the Brieskorn homology spheres in Theorem 1.3 again.

- (i) $\Sigma(2, 14n - 5, 8n - 3)$, (ii) $\Sigma(2, 24n + 5, 14n + 3)$
- (iii) $\Sigma(2, 26n + 5, 16n + 3)$, (iv) $\Sigma(2, 10n - 3, 16n - 5)$
- (v) $\Sigma(5, 35n - 2, 50n - 3)$, (vi) $\Sigma(5, 25n - 2, 40n - 3)$
- (vii) $\Sigma(3, 15n - 2, 36n - 5)$, (viii) $\Sigma(3, 9n - 2, 24n - 5)$
- (ix) $\Sigma(3, 21n - 4, 36n - 7)$, (x) $\Sigma(3, 27n - 4, 48n - 7)$
- (xi) $\Sigma(4, 28n - 3, 64n - 7)$, (xii) $\Sigma(4, 32n - 3, 76n - 7)$

The Seifert invariants $S(1; (p_1, 1), (p_2, q_2), (p_3, q_3))$ of these are the data in the following table:

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$(p_2, q_2)$</th>
<th>$(p_3, q_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>2</td>
<td>$(14n - 5, 7n - 6)$</td>
</tr>
<tr>
<td>(ii)</td>
<td>2</td>
<td>$(14n + 3, 7n - 2)$</td>
</tr>
<tr>
<td>(iii)</td>
<td>2</td>
<td>$(26n + 5, 13n - 4)$</td>
</tr>
<tr>
<td>(iv)</td>
<td>2</td>
<td>$(10n - 3, 5n - 4)$</td>
</tr>
<tr>
<td>(v)</td>
<td>5</td>
<td>$(35n - 2, 28n - 3)$</td>
</tr>
<tr>
<td>(vi)</td>
<td>5</td>
<td>$(40n - 3, 32n - 4)$</td>
</tr>
<tr>
<td>(vii)</td>
<td>3</td>
<td>$(15n - 2, 10n - 3)$</td>
</tr>
<tr>
<td>(viii)</td>
<td>3</td>
<td>$(24n - 5, 16n - 6)$</td>
</tr>
<tr>
<td>(ix)</td>
<td>3</td>
<td>$(21n - 4, 14n - 5)$</td>
</tr>
<tr>
<td>(x)</td>
<td>3</td>
<td>$(48n - 7, 32n - 10)$</td>
</tr>
<tr>
<td>(xi)</td>
<td>4</td>
<td>$(28n - 3, 21n - 4)$</td>
</tr>
<tr>
<td>(xii)</td>
<td>4</td>
<td>$(76n - 7, 57n - 10)$</td>
</tr>
</tbody>
</table>

In each case, we deform the presentation of the Seifert invariants of them with reverse orientation. The notation $\sim$ below stands for deformations of presentations preserving intersection forms. In the sequences below of the deformation we use the following type of the change:

$S(e; (a_1, a_2), (b_1, b_2), (c_1, c_2)) = S(e + 1; (a_1, a_2 + a_1), (b_1, b_2), (c_1, c_2))$.

This deformation is allowed to replace the changing branch $(a_1, a_2)$ with $(b_1, b_2)$ or $(c_1, c_2)$.

Recall the following deformation of the diagram as in Figure 3. This deformation is used in the sequence below.

The case of (i).

$S(-1; (2, -1), (14n - 5, -7n + 6), (8n - 3, -2))$

$= S(1; 2, 2 \cdot (n + 1) \cdot 2^{[6]}(2 - 4n) \cdot 2) = S(0; 2, (-2) \cdot n \cdot 2^{[6]}(2 - 4n) \cdot 2)$

$= L(2^{[6]} \cdot n \cdot 0 \cdot (2 - 4n) \cdot 2) = L(2^{[6]} \cdot \frac{1}{4}(2, 2n + 1) \cdot 2) \sim L(2^{[6]} \cdot (2 \cdot 4 : 2)$
The case of (ii).

\[ S(-1; (2, -1), (14n + 3, -7n + 2), (24n + 5, -6)) \]
\[ = S(1; 2, 2 \cdot (n + 1) \cdot 4 \cdot 2, (-4n) \cdot 2^{[5]} = S(0; 2, (-2) \cdot n \cdot 4 \cdot 2, (-4n) \cdot 2^{[5]} \]
\[ = L(2^{[5]} \cdot (2 - 4n)) \cdot (2) \cdot 0 \cdot n \cdot 4 \cdot 2 = L(2^{[5]} \cdot 2_{(2,2n+1)} \cdot (2) \cdot 2) \]
\[ \sim L(2^{[6]} \cdot (2) \cdot 4 \cdot 2) \]

The case of (iii).

\[ S(-1; (2, -1), (26n + 5, -13n + 4), (16n + 3, -4)) \]
\[ = S(1; 2, 2 \cdot (n + 1) \cdot 4 \cdot 2^{[3]}, (-4n) \cdot 2^{[3]} = S(0; 2, (-2) \cdot n \cdot 4 \cdot 2^{[3]}, (-4n) \cdot 2^{[3]} \]
\[ = L(2^{[3]} \cdot 4 \cdot n \cdot 0 \cdot (2) \cdot (2 - 4n) \cdot 2^{[3]} = L(2^{[3]} \cdot 4 \cdot (2) \cdot 2_{(2,2n+1)} \cdot 2^{[3]} \]
\[ \sim L(2^{[3]} \cdot 4 \cdot (2) \cdot 2^{[4]} \]

The case of (iv).

\[ S(-1; (2, -1), (10n - 3, -5n + 4), (16n - 5, -4)) \]
\[ = S(1; 2, 2 \cdot (n + 1) \cdot 2^{[4]}, (2 - 4n) \cdot 2^{[3]} = S(0; 2, (-2) \cdot n \cdot 2^{[4]}, (2 - 4n) \cdot 2^{[3]} \]
\[ = L(2^{[4]} \cdot n \cdot 0 \cdot (2) \cdot (4 - 4n) \cdot 2^{[3]} = L(2^{[4]} \cdot (2) \cdot 2_{(2,2n+1)} \cdot 2^{[3]} \sim L(2^{[4]} \cdot (2) \cdot 2^{[3]} \]

The case of (v).

\[ S(-1; (5, -1), (35n - 2, -28n + 3), (50n - 3, -2)) \]
\[ = S(0; -5, 5 \cdot n \cdot (-7), (2 - 25n) \cdot 2 = L((-7) \cdot n \cdot 0 \cdot (-5) (-3 - 25n) \cdot 2) \]
\[ = L((-7) \cdot (-5) \cdot (-3) \cdot (-5, -5n+1) \cdot 2 = L(2 \cdot 1 \cdot (-5) \cdot (-5) \cdot (-3) \cdot (-5, -5n+1) \cdot 2) \]

Here we slide the \((-3)\)-framed \((-5, -5n + 1)\)-torus knot component to the component with a \((-5)\)-framed component. Then we obtain a 4-manifold with the intersection form of the plumbed 4-manifold for a Seifert manifold \(S(1; 2, 2^{[2]}, -5)\). By doing four blow-ups and one blow-down, we have the intersection form \(E_8\).

In the same way, the 4-manifolds that the last diagrams in the following equalities present can be deformed into 4-manifolds with intersection form \(E_8\). The results of the latter four equalities are a plumbed 4-manifold that reduces to the Seifert invariant \(S(2; 2^{[2]}, 2^{[4]}, 2 \cdot 4) = -\Sigma(3, 4, 7)\).
According to the definition of homology spheres, we can see

\[ S(-1; (5, -1), (40n - 3, -32n + 4), (25n - 2, -1)) = S(0; -5, 5 \cdot n \cdot (-8), 2 - 25n) = L((-3 - 25n) \cdot (-5) \cdot n \cdot (-5) \cdot 1 \cdot 2^2) = L((-5, -5n+1) \cdot (-5) \cdot 1 \cdot 2^2) \]

The case of (vi).

\[ S(-1; (3, -1), (15n - 2, -10n + 3), (36n - 5, -4)) = S(0; -3, 3 \cdot n \cdot (-8), 2 - 9n) = L((-5) \cdot n \cdot 0 \cdot (-1 - 9n) \cdot 2^3) = L(2 \cdot 1 \cdot (-3) \cdot (-3 - 3n+1) \cdot 2^3) \]

The case of (vii).

\[ S(-1; (3, -1), (24n - 5, -16n + 6), (9n - 2, -1)) = S(0; -3, 3 \cdot n \cdot (-8), 2 - 9n) = L(2^4 \cdot 1 \cdot (-3) \cdot (-3 - 3n+1)) \]

The case of (viii).

\[ S(-1; (3, -1), (21n - 4, -14n + 5), (36n - 7, -4)) = S(0; -3, 3 \cdot n \cdot (-7), 2 - 9n) = L(2^3 \cdot 1 \cdot (-3) \cdot (-3 - 3n+1) \cdot 4) \]

The case of (ix).

\[ S(-1; (3, -1), (48n - 7, -32n + 10), (27n - 4, -3)) = S(0; -3, 3 \cdot n \cdot (-5) \cdot 3, (2 - 9n) \cdot 2^2) = L(2^2 \cdot (-1 - 9n) \cdot (-3 - 3n+1) \cdot (-3) \cdot 1 \cdot 2^4) \]

The case of (x).

\[ S(-1; (4, -1), (28n - 3, -21n + 4), (64n - 7, -4)) = S(0; -4, 4 \cdot n \cdot (-7), (2 - 16n) \cdot 4) = L(4 \cdot (-2 - 16n) \cdot (-4) \cdot n \cdot (-4) \cdot 1 \cdot 2^3) = L(4 \cdot (-2 - 16n) \cdot (-4) \cdot (-4) \cdot (-4) \cdot 1 \cdot 2^2) \]

The case of (xi).

\[ S(-1; (4, -1), (76n - 7, -57n + 10), (32n - 3, -2)) = S(0; -4, 4 \cdot n \cdot (-6) \cdot 3, (2 - 16n) \cdot 2) = L(2 \cdot (-2 - 16n) \cdot (-4) \cdot n \cdot (-4) \cdot 1 \cdot 2^4) = L(2 \cdot (-2 - 16n) \cdot (-4) \cdot (-4) \cdot (-4) \cdot 1 \cdot 2^2) \]

According to the definition of \( \bar{\mu} \) as above, computing \( \bar{\mu} \) for these Brieskorn homology spheres, we can see \( \bar{\mu} = -1 \) easily. From the description under
Theorem 2.1 we obtain $g_8 \leq 1$. Namely, the homology spheres all have $g_8 = 1$. 

4. Brieskorn homology spheres with $E_8$-fillings and arbitrarily large correction terms.

4.1. Heegaard Floer homology and one preparation. In [9] for any spin$^c$ rational homology sphere $(Y, s)$ the Heegaard Floer homology $HF^+(Y, s)$ has the following exact sequence:

$$0 \rightarrow T^+_d(Y, s) \rightarrow HF^+(Y, s) \rightarrow HF_{\text{red}}(Y, s) \rightarrow 0.$$ 

$T^+_s$ is isomorphic to $T^+ := \mathbb{F}[U, U^{-1}] / U \cdot \mathbb{F}[U]$ with the minimal degree $s$. $HF_{\text{red}}(Y, s)$ is a finite dimensional torsion $\mathbb{F}[U]$-module. $d(Y, s)$ is called the correction term of $(Y, s)$. We call the submodule $T^+_d(Y, s)$ in $HF^+(Y, s)$ the $T^+$-part of $HF^+(Y, s)$.

Here we prepare a lemma to prove Theorem 1.4. We abbreviate $d(L(p, q), i)$ by $d(p, q, i)$. Here we define the lens space $L(p, q)$ to be the $p/q$-surgery of the unknot. The identification of the spin$^c$ structures of $L(p, q)$ with $\mathbb{Z} / p\mathbb{Z}$ is due to Fig.2 in [9]. Here the $p$-Dehn surgery of a knot $K$ in a homology sphere $Y$ is the surgery $(Y \setminus S^1 \times D^2) \cup V$, where $V \cong S^1 \times D^2$. The attaching meridian of the new solid torus $V$ is mapped to $p \cdot [m] + [l] \in H_1(\partial(S^1 \times D^2))$ where $m$ is the meridian of $K$ and $l$ is the homologically trivial longitude of $K$. We denote the $p$-Dehn surgery of a knot in $\Sigma$ by $\Sigma(p)$. Here we prove the following lemma.

**Lemma 4.1.** Let $\Sigma$ be a homology sphere and $K \subset \Sigma$ a knot. For some positive integer $p$, if $\Sigma(p - 1)$ is an $L$-space and $\Sigma(p)$ is a lens space $L(p, q)$, then the correction term $d(\Sigma)$ is computed as follows.

$$(6) \quad d(\Sigma) = \max \{ d(p, q, ki + c) - d(p, 1, i) \mid 0 \leq i < p \},$$

where $k$ is the dual class of $[K] \in H_1(L(p, q), \mathbb{Z})$ and $c = (k+1+p)(k-1)/2$, where $K$ is the surgery dual of the lens space surgery.

Let $C$ be a core circle of the genus one Heegaard decomposition of $L(p, q)$. Then the dual class $k$ (mod $p$) is defined by the equality $k[C] = [\bar{K}] \in H_1(L(p, q), \mathbb{Z})$. Some similar situations that dual classes are used, for example, appear in [16]. For understanding it, readers might as well read the paper.

**Proof.** We use the following surgery exact sequence (Corollary 9.13 in [9]):

$$\cdots \rightarrow HF^+(\Sigma) \rightarrow HF^+(\Sigma(p-1)) \rightarrow HF^+(\Sigma(p)) \rightarrow HF^+(\Sigma) \rightarrow \cdots .$$

First, we easily show that the corresponding map on $HF^\infty$, $F^\infty : HF^\infty(\Sigma(p)) \rightarrow HF^\infty(\Sigma)$ is surjective. Since $\Sigma(p - 1)$ and $\Sigma(p)$ are L-spaces and $F^\infty$ is surjective, $F^+$ is also surjective onto the $T^+$-part in $HF^+(\Sigma)$. The map $F^+$ is induced from the cobordism $\Sigma(p)$ to $\Sigma$ obtained by attaching a 0-framed
2-handle along the meridian of \( \tilde{K} \). The spin\(^c\) structures on \( \Sigma(p) \) are identified with \( \mathbb{Z}/p\mathbb{Z} \) due to the description in p.213 in [9]. For any integer \( j \) with \( 0 \leq j < p \) consider the surgery exact sequence in Theorem 9.19 in [8]:

\[
\cdots \to HF^+(\Sigma) \to HF^+(\Sigma(0), [j]) \to HF^+(\Sigma(p), j) \xrightarrow{F_j^+} HF^+(\Sigma) \to \cdots.
\]

\( F_j^+ \) is a component of \( F^+ \) restricted to the spin\(^c\) structure \( j \). It is also a sum of homogeneous maps \( f_i^+ \) with respect to the spin\(^c\) cobordism from \( (\Sigma(p), j) \) to the unique spin\(^c\) manifold on \( \Sigma \). Namely, \( F_j^+ \) is described by the sum \( F_j^+ = \sum_{j \equiv i \mod p} f_i^+ \). The degree shift of \( f_i^+ \) is \( (4p-(2i-p)^2)/(4p) \) due to [9]. The maximal degree shift among \( \{ f_i^+ | j \equiv i \mod p \} \) is \( (4p-(2j-p)^2)/(4p) = -d(p, 1, j) \). Since \( F_j^+ \) is a surjective \( U \)-equivariant map, for \( 0 \leq j < p \) we have

\[
d(\Sigma) \geq d(p, q, kj + c) - d(p, 1, j).
\]

The 1-1 correspondence \( \mathbb{Z}/p\mathbb{Z} \to \text{Spin}^c(L(p, q)) \) in Corollary 7.5 in [9] is described by \( ki + c \). See [15].

Suppose that \( d(\Sigma) > d(p, q, kj + c) - d(p, 1, j) \) for any integer \( j \) with \( 0 \leq j < p \). Then any element with the minimal degree in \( HF^+(\Sigma(p)) \) is included in the kernel of \( F^+ = \sum_{0 \leq j < p} F_j^+ \). Thus the kernel of \( F^+ \) includes at least \( p \) components. On the other hand, for a sufficiently large integer \( N, \ker(F^+)/U^N = 0 \) is \( (p-1) \)-fold direct sum of \( T^+ \) from the exact sequence of the version of \( HF^\infty \). Hence, this implies that in the image of \( G^+ \) there is a torsion \( \mathbb{F}[U] \)-module by at least one component. However, since \( \Sigma(p-1) \) is an L-space, the image of \( G^+ \) has no torsion \( \mathbb{F}[U] \)-module. This is a contradiction. Therefore for some integer \( j, \ d(\Sigma) = d(p, q, kj + c) - d(p, 1, j) \) holds. \( \square \)

4.2. The \( d \)-invariants for the four families of Brieskorn homology spheres. We prove Theorem 1.4.

**Proof.** The Seifert presentations of Brieskorn homology spheres from (i) to (iv) in Theorem 1.3 are below:

| (i) | \( S(1; 2, 2 \cdot (-n + 1) \cdot 7, (4n - 1) \cdot 2) \) |
| (ii) | \( S(1; 2, 2 \cdot (-n) \cdot (-3) \cdot 2, (4n + 1) \cdot 6) \) |
| (iii) | \( S(1; 2, 2 \cdot (-n) \cdot (-3) \cdot 4, (4n + 1) \cdot 4) \) |
| (iv) | \( S(1; 2, 2 \cdot (-n + 1) \cdot 5, (4n - 1) \cdot 4) \) |

Let \( \Sigma_n \) be one of the Brieskorn homology spheres parametrized by \( n \) in the list above. We do the 0-surgery and +1-surgery of the homology sphere along the meridian of the singular fiber of multiplicity 2. We call the meridian \( K_n \). Note that the coefficients 0 and 1 are framings of the unknot \( K_n \) in the diagram. The 0-surgery and 1-surgery give lens spaces \( L(r_n, s_n) \) and
L(p_n, q_n). The results are the lens spaces in the list below.

<table>
<thead>
<tr>
<th>0-surgery ((r_n, s_n))</th>
<th>1-surgery ((p_n, q_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ((56n^2 - 41n + 7, 8n^2 - 7n + 2))</td>
<td>((56n^2 - 41n + 8, 8n^2 - 7n + 1))</td>
</tr>
<tr>
<td>(ii) ((168n^2 + 71n + 7, 72n^2 + 27n + 4))</td>
<td>((168n^2 + 71n + 8, 72n^2 + 27n + 1))</td>
</tr>
<tr>
<td>(iii) ((208n^2 + 79n + 7, 48n^2 + 17n + 2))</td>
<td>((208n^2 + 79n + 8, 48n^2 + 17n + 1))</td>
</tr>
<tr>
<td>(iv) ((80n^2 - 49n + 7, 16n^2 - 13n + 4))</td>
<td>((80n^2 - 49n + 8, 16n^2 - 13n + 1))</td>
</tr>
</tbody>
</table>

These examples satisfy \(p_n = r_n + 1\). As a result, the 0-surgery means a positive \(r_n\)-Dehn surgery along \(K_n\).

We set \(d_n := d(\Sigma_n)\). Here using Lemma 4.1, we compute the lower bound of \(d_n\). We argue the case of (i) only. Other cases are proven by similar arguments. Let \(\Sigma_n\) be a Brieskorn homology sphere of type (i). We set \(p_n = 56n^2 - 41n + 8, q_n = 8n^2 - 7n + 1, k_n = 14n - 5\) and \(c_n \equiv 42n^2 - 29n + 4 \mod p_n\).

The \(k_n\) is the dual class in the lens space \(L(p_n, q_n)\) which presents \(K_n\).

Here we assume \(i = \lfloor \frac{q_n+1}{2} \rfloor - n\). Then modulo \(p_n\) we have

\[
k_ni + c_n \equiv \begin{cases} 
\frac{7n-5}{2} & n: \text{odd} \\
\frac{-7n}{2} & n: \text{even}
\end{cases}
\]

If \(n\) is an odd number, then by using the reciprocity formula in [9], we have

\[
d\left(L(p_n, q_n), \frac{7n - 5}{2}\right) = \frac{224n^3 + 8n^2 - 95n + 25}{4p_n}
\]

and

\[
d(L(p_n, 1), i) = -\frac{52n^2 - 37n + 7}{4p_n}.
\]

Thus we have

\[
d(L(p_n, q_n), k_ni + c_n) - d(L(p_n, 1), i) = n + 1.
\]

If \(n\) is an even number, then we have

\[
d\left(L(p_n, q_n), -\frac{7n}{2}\right) = \frac{224n^3 - 216n^2 + 73n - 8}{4p_n}
\]

\[
d(L(p_n, 1), i) = -\frac{52n^2 - 41n + 8}{4p_n}.
\]

Thus we have

\[
d(L(p_n, q_n), k_ni + c_n) - d(L(p_n, 1), i) = n.
\]

Therefore we have \(d_n \geq 2\left\lceil \frac{n}{2} \right\rceil\). □

4.3. **Proof of Corollary 1.9.** Here we prove Corollary 1.9.

**Proof.** Let \(\{\Sigma_n\}_{n \in \mathbb{N}}\) be a family of homology spheres in Theorem 1.4. Since the correction term \(d(\Sigma_n)\) is positive, there is no even positive-definite filling of \(\Sigma_n\). A Seifert homology sphere \(\Sigma_n\) has a negative-definite plumbing \(P(\Gamma_n)\). Due to Corollary 1.5 in [10], \(4d(\Sigma_n) = \max_{c \in \Xi_n} (c^2 + \text{rank}(\Gamma_n))\) holds, where \(\Xi_n\) is the set of characteristic vectors in \(\Gamma_n\). If for a non-negative integer \(N, \Gamma_n = \Gamma_n' \oplus (-1)^N\) and \(\Gamma_n'\) is a minimal sub-lattice, then \(\max_{c \in \Xi_n} (c^2 + \text{rank}(\Gamma_n'))\) holds.
HOMOLOGY SPHERES WITH $E_8$-FILLINGS AND ARBITRARILY LARGE CORRECTION TERMS

rank($\Gamma_n$) is decomposed into the sum of the two maximal values according to the direct sum. Thus, we have

$$\max_{c \in \Xi_n}(c^2 + \text{rank}(\Gamma_n)) = \max_{c \in \Xi_n}(c^2 + \text{rank}(\Gamma'_n)) \leq \text{rank}(\Gamma'_n),$$

where $\Xi_n$ is the set of characteristic vectors of $L'$. Since $d(\Sigma_n)$ has no upper bound, the maximal of the rank of the minimal lattice $\Gamma'_n$ is also unbounded. On the other hand, the filling of $\Sigma_n$ we constructed in the proof of Theorem 1.3 implies $\partial n = 1$. Therefore $\partial(\Sigma_n) - \partial n$ is unbounded.

REFERENCES

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