NONCOMPACT $L_p$-MINKOWSKI PROBLEMS

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Abstract. In this paper we prove the existence of complete, noncompact convex hypersurfaces whose $p$-curvature function is prescribed on a domain in the unit sphere. This problem is related to the solvability of Monge-Ampère type equations subject to certain boundary conditions depending on the value of $p$. The special case of $p = 1$ was previously studied by Pogorelov [28] and Chou-Wang [10]. Here, we give some sufficient conditions for the solvability for general $p \neq 1$.

1. Introduction

Let $M$ be a compact, strictly convex $C^2$-hypersurface in $\mathbb{R}^{n+1}$. Since the Gauss map is a bijection between $M$ and the unit sphere $S^n$, $M$ can be parametrised by the inverse of the Gauss map, and consequently the Gauss curvature $K$ of $M$ can be regarded as a function on $S^n$. Let $H$ be the support function of $M$ (see definitions in §2). For $p \in \mathbb{R}$, $K_p := KH^{p-1}$ is called the $p$-curvature of $M$. The $L_p$-Minkowski problem introduced by Lutwak [20] asks that whether a given function $f$ on $S^n$ is the $p$-curvature of a unique compact convex hypersurface. This problem is related to the solvability of the following Monge-Ampère type equation

$$ \det (\nabla_{ij} H + H \delta_{ij}) = f H^{p-1} \quad \text{on} \ S^n, $$

where $\nabla$ is the covariant differentiation with respect to an orthonormal frame on $S^n$. When $p = 1$, one has the classical Minkowski problem [9, 27]. For general $p$, the $L_p$-Minkowski problem has been intensively studied in recent decades, for example, in [5, 11, 15, 18, 19, 20, 21, 23, 34, 35, 36] and many others. We refer the reader to the newly expanded book [29] by Schneider for a comprehensive introduction on related topics.

The same problem makes perfectly sense for complete, noncompact, convex hypersurfaces. In that case, by a suitable rotation, the spherical image of such a hypersurface is an open convex subset contained in the hemisphere $S^n_+ := \{ X \in S^n : X_{n+1} < 0 \}$. The corresponding
problem is then: Given an open convex subset $D$ of $\mathbb{S}^n_-$ and a positive function $K_p$ in $D$, does there exist a (unique) complete convex hypersurface with spherical image $D$ and $p$-curvature $K_p$?

When $p = 1$, Pogorelov [28] firstly proved the existence of such a hypersurface under certain decay conditions on $K$ near the boundary of $D$. Chou and Wang [10] considered it in more general cases. For $p \neq 1$, this problem becomes much more complicated, partly because the $p$-curvature $K_p$ involves the support function $H$, which depends on the position of hypersurface $M$ and thus is not translation-invariant. In this paper we give sufficient conditions for the solvability for general $p \neq 1$, and extend Chou-Wang’s results in [10] for $p = 1$.

Similarly as above, the problem in noncompact setting is related to the solvability of Equation (1.1) in $D$ associated with certain compatible boundary conditions, where $f = K_p^{-1}$ is prescribed. One can see clearly from Equation (1.1) that whether $p > 1$ or $p < 1$ makes a big difference, as the right hand side of equation goes to degenerate or singular when $H \to 0$, respectively. Correspondingly, in the subsequent context we shall consider these two cases separately.

When $p < 1$, from a geometric observation we show that if $f \geq 0$, there does not exist such a complete, noncompact, convex hypersurface (with $H \geq 0$) satisfying Equation (1.1) (see Lemma 3.1). Instead, we consider $H < 0$, namely the origin lies in the concave side of $M$, and the hypersurface $M$ satisfies

$$(1.2) \quad H = -\hat{f}K_p^{1-p} \quad (p < 1)$$

for a given positive function $\hat{f}$ in $D$. We remark that when $p < 1$, it is necessary to have $D$ strictly contained in $\mathbb{S}^n_-$, see §3. Our first result is

**Theorem 1.1.** Let $D$ be a uniformly convex $C^2$-domain strictly contained in $\mathbb{S}^n_-$, $\hat{f}$ a positive function in $C^\alpha(D) \cap L^{1-p}(D)$, where $\alpha \in (0, 1)$ and $p < 1$. Suppose there exists two positive functions $g$ and $h$ defined in $(0, r_0)$, $r_0 > 0$, satisfying

\begin{align*}
(a) \quad & \int_0^{r_0} \left( \int_0^{r_0} g^{1-p}(t) dt \right)^{1/n} ds < \infty, \\
(b) \quad & \int_0^{r_0} h^{1-p}(t) dt = \infty,
\end{align*}

so that

$$C^{-1}h(\text{dist}(X, \partial D)) \leq \hat{f}(X) \leq Cg(\text{dist}(X, \partial D))$$

near $\partial D$ for some constant $C > 0$. Then there exists a unique complete, noncompact, strictly convex hypersurface $M$ such that the support function $H \in C^{2,\alpha}(D) \cap C(\overline{D})$ satisfies (1.2) in $D$, and $H = 0$ on $\partial D$.

We remark that the assumption $\hat{f} \in L^{1-p}(D)$ ensures $H = 0$ on $\partial D$ and $M$ approaches to an asymptotic convex cone. Without this assumption, one can also obtain the existence
of $M$ with a general boundary condition $H = \Phi$ for a function $\Phi \in C^2(\partial D)$, but to show $M$ is complete, one needs a stronger assumption that $h(\text{dist}(X, \partial D))/f(X) \to 0$ as $X \to \partial D$. More details are contained in §3.

When $p > 1$, depending on the relative position of $D$ there are multiple cases for discussion. By suitably rotating axes, we may assume $D$ satisfies one and exactly one of the following conditions:

(I) $D$ is strictly contained in $\mathbb{S}^n$,

(II) $D = \mathbb{S}^n$,

(III) $D$ is a proper subset of $\mathbb{S}^n$ and it is not strictly contained in any hemisphere.

We shall say $M$ is of type I, II, or III when its spherical image $D$ satisfies (I), (II), or (III), respectively. Notice that by our choice of coordinates, $M$ is the graph of a convex function over a convex domain in the $(x_1, \ldots, x_n)$-space.

For type I hypersurfaces, it is clear that $M$ is complete if and only if $M$ is a graph over $\mathbb{C}^n$. Correspondingly, we impose a boundary condition to the support function $H = \Phi$ on $\partial D$, where $\Phi$ is a prescribed function.

**Theorem 1.2.** Let $D$ be a uniformly convex $C^2$-domain satisfying condition (I), $p \geq 1$ and $p \neq n + 1$. Assume $K_p$ is a positive function in $C^\alpha(D)$ and $\Phi$ is a function in $C^2(\partial D)$. Suppose there exists two positive functions $g$ and $h$ defined in $(0, r_0]$, $r_0 > 0$, satisfying

(a) $\int_0^{r_0} (\int_0^{r_0} g(t)dt)^{1/n} ds < \infty,$

(b) $\int_0^{r_0} h(t)dt = \infty,$

so that

$K_p(X)g(\text{dist}(X, \partial D)) \geq C^{-1}$ and $K_p(X)h(\text{dist}(X, \partial D)) \leq C$

near $\partial D$ for some constant $C > 0$. Then there exists a unique complete, noncompact, strictly convex hypersurface $M$ such that $K_p$ is the $p$-curvature of $M$ and $H = \Phi$ on $\partial D$.

**Remark 1.1.** We remark that the classification of types I–III follows from [10]. When $p = 1$, Theorem 1.2 corresponds to [10] Theorem A. However, when $p \neq 1$, equation (1.3) has an extra factor $H^{p-1}$, and we need some different approaches in constructing subsolutions, in particular for the case of $p > n + 1$ in §4.1.2.

In fact, for this reconstruction of complete convex hypersurface, Aleksandrov [1] firstly formulated the geometric problem with prescribed area of Gaussian mapping and its asymptotic cone. It amounts to the solvability of the boundary value problem for a Monge-Ampère equation, see [1, 2]. Bakelman [3, 4] established the existence and uniqueness of convex generalised solutions for the second boundary value problem of the Monge-Ampère equation

$\det D^2u = \frac{T(x)}{Q(Du)}$. (**1.3**)
It has been convinced that the necessary and sufficient condit on for the existence of those complete hypersurfaces project one-to-one on $\mathbb{R}^n$ with prescribed asymptotic cone $C$, is

$$
\int_{\mathbb{R}^n} T(x)dx = \int_{\mathcal{N}_C(\mathbb{R}^n)} Q(p)dp,
$$

where $\mathcal{N}_C(\mathbb{R}^n)$ is the normal image of $C$. See also Pogorelov [26, 28]. Moreover, Oliker [25] constructed a minimisation problem associated with the Monge-Kantorovich optimal mass transfer problem for this kind of reconstruction geometric problem.

For type II hypersurfaces, we prove the existence of such a complete hypersurface $M$ when $p > n + 1$, under an asymptotic growth assumption on the prescribed function $K_p$.

**Theorem 1.3.** Assume that $K_p$ satisfies an asymptotic growth condition

(1.4) \[ K_p(X) \sim |X_{n+1}|^{2q}, \]

for some constant $q \in (0, 1)$. When $p > n + 1$, there exists a complete noncompact convex hypersurface $M$ such that $K_p$ is the $p$-curvature of $M$.

We further remark that when $M$ is a graph over a bounded domain $\Omega^*$, (1.4) is necessary for the solvability, see §4.2.

For type III hypersurfaces, we have a similar result as Theorem 1.2, however, the boundary condition is imposed on part of $\partial D$ that is away from $\partial \mathbb{S}^n$. The corresponding statement is postponed to Theorem 4.2 in §4.3.

Last, we point out that the $L_p$-Minkowski problem is related to the expanding Gauss curvature flow when $p > 1$, and the contracting Gauss curvature flow when $p < 1$, as Equation (1.1) describes homothetic solutions in each case, respectively. For complete, noncompact hypersurfaces, one may consult Urbas [31, 32] for works in this direction, also [10, 14] and references therein.

This paper is organised as follows: In §2, we derive the Monge-Ampère equation (1.1). Although this has been done in some literatures, we would like to provide a more general and unified treatment, which includes (1.1) as a special case when the metric on $\mathbb{S}^n$ is orthonormal. In §3, we consider the case of $p < 1$ and prove Theorem 1.1. Moreover, when the prescribed function $\hat{f}$ satisfies certain boundedness conditions, we can give a different proof without the uniform convexity assumption on $D$. In §4, we consider the case of $p > 1$, Theorems 1.2, 1.3 and 4.2 are proved in §4.1–§4.3, respectively.

2. Preliminaries

2.1. Support function. Let $M$ be a strictly convex $C^2$-hypersurface in $\mathbb{R}^{n+1}$ and let $D \subset \mathbb{S}^n$ be its spherical image. Assume that $M$ is parametrised by the inverse Gauss map
The support function of $M$ is defined by
\[
H(\xi) = \langle \xi, X(\xi) \rangle, \quad \xi \in D, \tag{2.1}
\]
where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{n+1}$.

The metric and the second fundamental form of $M$ can be represented in terms of its support function $H$. To see that, let $(u_1, \ldots, u_n)$ be a local coordinate chart and \{e_i = \partial_i \xi\} be the local frame field on $D$, where $\partial_i = \partial/\partial u_i$, $i = 1, \ldots, n$. By differentiating (2.1) we obtain that
\[
\partial_i H = \langle \partial_i \xi, X \rangle + \langle \xi, \partial_i X \rangle = \langle e_i, X \rangle, \tag{2.2}
\]
since $\partial_i X(\xi)$ is tangential to $M$ at $X(\xi)$, and $\xi$ is the normal to $M$ at $X(\xi)$. Differentiating once again, we have
\[
\partial_{ij} H = \langle \partial_j e_i, X \rangle + \langle e_i, \partial_j X \rangle = \langle \partial_j e_i, X \rangle + h_{ij}, \tag{2.3}
\]
where $h_{ij}$ is the second fundamental form of $M$.

To compute $\langle \partial_j e_i, X \rangle$, we use the Gauss derivation formulas to get
\[
\partial_j e_i = \Gamma_k^{ij} e_k - \sigma_{ij} \xi, \tag{2.4}
\]
where $\Gamma_k^{ij}$ are the Christoffel symbols, and $\sigma_{ij} = \langle e_i, e_j \rangle$ is the metric on $D \subset S^n$, respectively. Combining (2.2), (2.3) and (2.4), we can obtain
\[
\partial_{ij} H = \Gamma_k^{ij} \langle e_k, X \rangle - \sigma_{ij} \langle \xi, X \rangle + h_{ij} = \Gamma_k^{ij} \partial_k H - H \sigma_{ij} + h_{ij}, \tag{2.5}
\]
and thus
\[
h_{ij} = \partial_{ij} H - \Gamma_k^{ij} \partial_k H + H \sigma_{ij} = \nabla_{ij} H + H \sigma_{ij},
\]
where $\nabla_{ij} H$ is the covariant differentiation of $H$ with respect to the frame \{e_i\} on $S^n$.

Let’s now compute the metric $g_{ij}$ of $M$. Using the Gauss-Weingarten relations
\[
\partial_i \xi = h_{ik} g^{kl} \partial_l X, \tag{2.6}
\]
we have
\[
\sigma_{ij} = \langle \partial_i \xi, \partial_j \xi \rangle = h_{ik} g^{kl} h_{jm} g^{ms} (\partial_l X, \partial_s X) = h_{ik} h_{jl} g^{kl}. \tag{2.7}
\]
Hence, the metric satisfies
\[
g_{ij} = h_{ik} h_{jl} \sigma^{kl}. \tag{2.8}\]

The principal radii of curvature are the eigenvalues of the matrix $b_{ij} = h_{ik} g_{jk}$, which, by virtue of (2.5) and (2.7), is given by
\[
b_{ij} = h_{ik} \sigma^{kj} = (\nabla_{ik} H + H \sigma_{ik}) \sigma^{kj}. \tag{2.8}
\]
Therefore, the Gauss curvature $K$ of $M$ is given by
\begin{equation}
\frac{1}{K} = \det b_{ij} = \frac{\det (\nabla_{ij} H + H \sigma_{ij})}{\det (\sigma_{ij})}.
\end{equation}

For a general $p \in \mathbb{R}$, the $p$-curvature of $M$ is defined by $K_p := KH^{p-1}$. When $p = 1$, $K_1 = K$ is the Gauss curvature. When $p \neq 1$, $K_p$ involves the support function $H$, and thus is not intrinsic. The $L^p$-Minkowski problem asks for the existence of $M$ with a prescribed $p$-curvature $K_p$. Let $f := K_p^{-1} = H^{1-p}/K$, from (2.9) one can obtain the Monge-Ampère equation satisfied by $H$,
\begin{equation}
\det (\nabla_{ij} H + H \sigma_{ij}) = f H^{p-1} \det (\sigma_{ij}) \quad \text{on} \quad D.
\end{equation}

In particular, under a smooth local orthonormal frame field on $\mathbb{S}^n$, namely $\sigma_{ij} = \delta_{ij}$, the above equation becomes (1.1), namely
\begin{equation}
\det (\nabla_{ij} H + H \delta_{ij}) = f H^{p-1} \quad \text{on} \quad D.
\end{equation}

2.2. **Homogeneous extension.** In order to study the solvability of $L^p$-Minkowski problems, it is convenient to express the above equations for $H$ in a local coordinate chart. Extend $H$ to be a function of homogeneous degree one over the cone $\{ \lambda \xi : \xi \in D, \lambda > 0 \}$, let $\Omega := \{ \lambda \xi : \xi \in D, \lambda > 0 \} \cap \{ x_{n+1} = -1 \}$, and $u(x) = H(x, -1)$, where $x = (x_1, \cdots, x_n)$. Denote
\begin{equation}
\mu(x) = \left( 1 + \sum_{i=1}^n x_i^2 \right)^{1/2}.
\end{equation}

By the homogeneity,
\begin{equation}
u(x) = \mu(x) H(\xi(x)), \quad x \in \Omega,
\end{equation}
where $\xi(x) \in D$ is given by
\begin{equation}
\mu(x) \xi(x) = (x, -1).
\end{equation}

In order to rewrite equation (2.10) in terms of $u$, we adopt the following computations from [22]. Differentiating (2.14) we have
\begin{equation}
(\partial_i \mu) \xi + \mu \partial_i \xi = (0, \cdots, 0, 1, 0, \cdots, 0).
\end{equation}

Differentiating once again yields
\begin{equation}
(\partial_{ij} \mu) \xi + (\partial_i \mu) \partial_j \xi + (\partial_j \mu) \partial_i \xi + \mu \partial_{ij} \xi = 0.
\end{equation}

By the Gauss derivation formulas,
\begin{equation}
\partial_{ij} \xi = \Gamma_{ij}^k \partial_k \xi - \sigma_{ij} \xi,
\end{equation}
where \( \sigma_{ij} = \langle \partial_i \xi, \partial_j \xi \rangle \) is the metric of \( \mathbb{S}^n \). Taking the inner product of (2.16) with \( \partial_s \xi \), and noting that \( \langle \partial_s \xi, \xi \rangle = 0 \), we get
\[
\partial_i \mu \sigma_{sj} + \partial_j \mu \sigma_{si} + \mu \Gamma^k_{ij} \sigma_{ks} = 0.
\]
While taking the inner product of (2.16) with \( \xi \), we also get
\[
\partial_{ij} \mu = \mu \sigma_{ij}.
\]
Now, differentiating (2.13) we obtain
\[
\partial_i u = \partial_i \mu H + \mu \partial_i H,
\]
then by (2.18) and (2.19)
\[
\begin{align*}
\partial_{ij} u &= \partial_{ij} \mu H + \partial_i \mu \partial_j H + \partial_j \mu \partial_i H + \mu \partial_{ij} H \\
&= \partial_{ij} \mu H + (\partial_i \mu \sigma_{js} + \partial_j \mu \sigma_{is}) \sigma^{st} \partial_t H + \mu \partial_{ij} H \\
&= \partial_{ij} \mu H - \mu \Gamma^t_{ij} \partial_t H + \mu \partial_{ij} H \\
&= \partial_{ij} \mu H + \mu \nabla_{ij} H \\
&= \mu (\nabla_{ij} H + H \sigma_{ij}).
\end{align*}
\]
On the other hand, by straightforward computations we have
\[
\sigma_{ij} = (1 + |x|^2)^{-1} \left( \delta_{ij} - \frac{x_i x_j}{1 + |x|^2} \right),
\]
and
\[
\det (\sigma_{ij}) = (1 + |x|^2)^{-n+1} = \mu^{-2(n+1)}.
\]
Substituting (2.13), (2.20) and (2.22) into (2.10), we obtain the standard Monge-Ampère equation satisfied by \( u \),
\[
\det D^2 u = (1 + |x|^2)^{-\frac{n+1}{2}} f \left( \frac{x, -1}{\sqrt{1 + |x|^2}} \right) u^{p-1}, \quad x \in \Omega,
\]
where \( D^2 u = (\partial_{ij} u) \) is the Hessian matrix of \( u \). Therefore, the solvability of \( L^p \)-Minkowski problems is equivalent to the solvability of the Monge-Ampère equation (2.23).

In the complete, noncompact case with certain boundary conditions, whenever a convex solution of (2.23) is given, as a rescaled support function it determines the hypersurface \( M \) in the following way (see [9, 27]): Let \( \Omega^* = Du(\Omega) \) and
\[
\begin{align*}
u^* (y) &= \sup \{ \langle x, y \rangle - u(x) : x \in \Omega \}, \quad y \in \Omega^*.
\end{align*}
\]
Then \( M \) is the graph \( \{(y, u^*(y)) : y \in \Omega^*\} \), and its \( p \)-curvature is equal to \( f^{-1} = K_p \) as prescribed.

By straightforward computations, the dual function \( u^* \) satisfies
\[
\begin{align*}
\det D^2 u^* &= \frac{(1 + |Du^*|^2)^{\frac{n+1}{2}}}{(y \cdot Du^* - u^*)^{p-1}} f^{-1}(\gamma), \quad y \in \Omega^*.
\end{align*}
\]
where $\gamma = \frac{(Du^*, -1)}{\sqrt{1 + |Du^*|^2}}$ is the unit normal of $M$ at the point $(y, u^*(y))$.

### 3. The Case of $p < 1$

In this section we first show a nonexistence result for hypersurface $M$ with support function $H \geq 0$. Then alternatively, we consider the hypersurface satisfying (1.2) with $H < 0$ and prove Theorem 1.1. When the prescribed function $\hat{f}$ satisfies further boundedness conditions, we also give some independent and interesting results for the existence and completeness. Throughout this section we assume $p < 1$.

Originally, one asks for a strictly convex $C^2$-hypersurface $M$ in $\mathbb{R}^{n+1}$ such that

$$
(3.1) \quad H = (fK)^{\frac{1}{1-p}},
$$

for some prescribed function $f \geq 0$ on the spherical image $D \subset \mathbb{S}^n$, where $H$ is the support function of $M$.

**Lemma 3.1 (Nonexistence).** If $f \in L^1(D)$ is a nonnegative function, there does not exist any complete, noncompact, strictly convex hypersurface $M$ satisfying (3.1).

**Proof.** Since $M$ is convex, $K \geq 0$. From assumption $f \geq 0$, the support function $H \geq 0$ by (3.1). If $H = 0$ on $\partial D$, then either $M$ is a cone or $M$ is degenerate with zero $n$-dimensional Hausdorff measure, $\mathcal{H}^n(M) = 0$. Therefore, we assume that $\sup_{\partial D} H \geq \delta$ for some positive constant $\delta$. By continuity of $H$, there exists a subset $E \subset D$ such that $H \geq \delta/4$ in $E$. Let $G := X(E) \subset M$, where $X$ is the inverse Gauss map, see (2.1). Since $M \in C^2$ is strictly convex, the map $X : D \to M$ is a bijection. Since $M$ is complete noncompact and $E \cap \partial D \neq \emptyset$, we have $\mathcal{H}^n(G) = \infty$, [33]. Then by integration we obtain

$$
\infty = \int_G (\delta/4)^{1-p} \, d\mathcal{H}^n \leq \int_E \frac{H^{1-p}}{K} \, dx
$$

$$
\leq \int_D \frac{H^{1-p}}{K} \, dx = \int_D f \, dx < \infty,
$$

where $dx$ is the spherical measure of $\mathbb{S}^n$. The last equality is due to (3.1). This is a contradiction to the assumption $f \in L^1(D)$, and thus Lemma 3.1 is proved. \hfill \square

We remark that in proving the above lemma, one can in fact show that the set $\{ \xi \in D : H(\xi) = 0 \}$ has zero $\mathcal{H}^n$ measure, where $H \geq 0$ is the support function of a complete, noncompact, strictly convex hypersurface $M$. Hence, the contradiction (3.2) will occur under the assumption of Lemma 3.1.

Therefore, in the case of $p < 1$, it is reasonable to consider the existence of hypersurface $M$ satisfying (1.2) and $H \leq 0$, that is

$$
(3.3) \quad H = -\hat{f} K^{\frac{1}{1-p}}.
$$
where \( \hat{f} \geq 0 \) is a prescribed function on the spherical image \( D \). Then one’s aim is to look for a complete, noncompact, strictly convex hypersurface \( M \) satisfying (3.3). And in such cases, \( K_p = \hat{f}^{p-1} = K(-H)^{p-1} \) is the \( p \)-curvature of \( M \).

Note that \( H \leq 0 \) implies that any tangent hyperplane \( T \) to \( M \) either contains the origin, or else, the origin lies on the opposite side of \( T \) from \( M \). If \( 0 \in M \), it follows that every tangent hyperplane to \( M \) must contains 0 and \( H \equiv 0 \), which implies that \( M \) is a hyperplane containing the origin or a cone with vertex at the origin, but this contradicts with the strict convexity of \( M \). Now assume that \( 0 \not\in M \), the origin lies on the opposite sides of all tangent hyperplanes from \( M \), or equivalently, \( H < 0 \) in \( D \).

Let \( C \) be the intersection of all closed halfspaces \( P \) of \( \mathbb{R}^{n+1} \) with \( 0 \in \partial P \) and \( M \subset P \). Then \( C \) is a closed convex cone with vertex at the origin with nonempty interior. Moreover, \( \partial C \) can be represented as the graph of a convex degree one homogeneous function \( \psi \) with \( \psi \geq 0 \) in \( \mathbb{R}^n - \{0\} \) and \( |D\psi| \) bounded. Recall that \( M \) is a graph of \( u^* \) over \( \Omega^* \subset \mathbb{R}^n \), from the construction of \( C \), \( Du^*(\Omega^*) \subset D\psi(\mathbb{R}^n) \). Because \( M \) is complete and \( Du^* \) is bounded, we must have \( \Omega^* = \mathbb{R}^n \), namely \( M \) is an entire graph over \( \mathbb{R}^n \). By parallel translating supporting hyperplanes between \( M \) and \( C \), one also has

\[
Du^*(\mathbb{R}^n) = D\psi(\mathbb{R}^n),
\]

see [31] for more geometric details. Therefore, the spherical image of \( M \),

\[
D = \frac{(Du^*,-1)}{\sqrt{1 + |Du^*|^2}}(\mathbb{R}^n)
\]

must be strictly contained in \( S^n_+ \), and its projection image \( \Omega = \{\lambda \xi : \xi \in D, \lambda > 0\} \cap \{x_{n+1} = -1\} \) must be a bounded, convex domain in \( \mathbb{R}^n \).

Next lemma shows that under hypotheses of Theorem [11], \( M \) is asymptotically approaching to \( C \) in the sense that \( H(X) \to 0 \) as \( X \to \partial D \).

**Lemma 3.2 (Asymptotic).** Under the hypotheses of Theorem [11], let \( M \) be a solution satisfying (3.3). If \( \hat{f} \in L^{1-p}(D) \), then \( H(X) \to 0 \) as \( X \to \partial D \).

**Proof.** Suppose if not true, by continuity of \( H \), there exists some \( X_0 \in \partial D \) and a positive constant \( \delta \) such that \(-H \geq \delta \) in a neighborhood of \( X_0 \), \( E \subset D \). Let \( G := X(E) \subset M \), where \( X \) is the inverse Gauss map, which is a bijection from \( D \) to \( M \). As \( X_0 \in E \cap \partial D \), \( E \cap \partial D \neq \emptyset \). Since \( M \) is complete noncompact, one has \( \mathcal{H}^n(G) = \infty \). [33]. Then similarly to Lemma [5.1] by integration we have

\[
\infty = \int_G (\delta/4)^{1-p} d\mathcal{H}^n \leq \int_E \frac{(-H)^{1-p}}{K} dx \leq \int_D \frac{(-H)^{1-p}}{K} dx = \int_D \hat{f}^{1-p} dx < \infty,
\]

(3.4)
where $d\mathbf{x}$ is the spherical measure of $S^n$. The last equality is due to (3.3). By assumption $\hat{f} \in L^{1-p}(D)$, this contradiction thus implies that $H(X) \to 0$ as $X \to \partial D$. \hfill $\square$

In a local coordinate chart, by (2.13),

$$u = \mu H < 0 \quad \text{in } \Omega.$$ Since $M$ is enclosed by the asymptotic cone $C$, $u = 0$ on $\partial \Omega$. It is also clear that $M$ is complete if and only if $Du(\Omega) = \mathbb{R}^n$.

From the computations in §2, the above question is related to a variant of Monge-Ampère equation

$$\mu^{-1}u = -\hat{f}\mu^{-\frac{n+2}{1-p}}(\det D^2u)^{-\frac{1}{1-p}} \quad \text{in } \Omega.$$ Hence, by Lemma 3.2 we pose the following boundary value problem

\begin{align*}
(3.5) \quad & \det D^2u = \mu^{-(n+p+1)}\hat{f}^{1-p}\left(\frac{1}{u}\right)^{1-p} \quad \text{in } \Omega, \\
(3.6) \quad & u = 0 \quad \text{on } \partial \Omega, \\
(3.7) \quad & |Du(x)| \to \infty \quad \text{as } x \to \partial \Omega.
\end{align*}

Here, $\hat{f}(x)$ is interpreted as $\hat{f}\left(\frac{x}{\sqrt{1+|x|^2}}\right)$ for $x \in \Omega$.

Therefore, in order to prove Theorem 1.1, it suffices to prove the following result.

**Theorem 3.1.** Let $\Omega$ be a bounded, uniformly convex smooth domain in $\mathbb{R}^n$, and $\hat{f} \geq 0$ be a smooth function in $\Omega$. Suppose there exists two positive functions $g$ and $h$ satisfying conditions (a) and (b) in Theorem 1.1 such that

$$C^{-1}h(\text{dist}(x,\partial \Omega)) \leq \hat{f}(x) \leq Cg(\text{dist}(x,\partial \Omega))$$

near $\partial \Omega$ for some constant $C > 0$. Then (3.5)–(3.7) has a unique smooth solution $u$.

The Dirichlet problem (3.5)–(3.6) was previously studied by Cheng-Yau [9]. They proved that if $\Omega$ satisfies a uniform enclosing sphere condition and $\hat{f}(x) \leq C\text{dist}(x,\partial \Omega)^{-\beta-1}$, $\beta > 0$, then there admits a unique solution. If $\hat{f} \equiv 1$, while $\Omega$ is merely a bounded convex domain, Urbas [31] also obtained the unique existence of convex solution. One can easily check that the assumption on $\hat{f}$ in [9] is contained in condition (a) of Theorem 3.1. We divide the proof of Theorem 3.1 into two parts: Prove the solvability of Dirichlet problem (3.5)–(3.6), and verify the solution satisfies boundary condition (3.7).

### 3.1. Existence

For the existence part, we consider a general Dirichlet boundary condition

\begin{align*}
(3.8) \quad & u = \phi \quad \text{on } \partial \Omega,
\end{align*}

where $\phi \leq 0$ is a convex function in $\Omega$. Write

\begin{align*}
(3.9) \quad & R(x) = \mu^{-(n+p+1)}\hat{f}^{1-p}(x), \quad x \in \Omega.
\end{align*}
Equation (3.5) can be simplified to

\[(3.10) \quad \det D^2u = R(x) \left(\frac{-1}{u}\right)^{1-p}.\]

Let \(\Omega_r = \{ x \in \Omega : \text{dist}(x, \partial\Omega) > r \} \). When \(\Omega\) is uniformly convex, for \(r_0 > 0\) small depending on the geometry of \(\Omega\), \(\Omega_r\) is still uniformly convex. For \(x \in \Omega \setminus \Omega_{r_0}\), \(x\) can be represented uniquely by \(x_b + dn(x_b)\), where \(x_b \in \partial\Omega\), \(d = \text{dist}(x, \partial\Omega)\), and \(n(x_b)\) is the unit inner normal at \(x_b\). For a function \(f\) defined near \(\partial\Omega\) we write \(f(x) = f(x_b, d)\).

**Lemma 3.3.** Let \(\Omega\) be a bounded, uniformly convex \(C^2\)-domain. Suppose there exists a positive function \(g\) in \((0, r_0)\) satisfying

\[(3.11) \quad \int_0^{r_0} \left( \int_s^{r_0} g(t)dt \right)^{1/n} ds < \infty\]

such that

\[R(x_b, d) \leq g(d)\]

Then \(3.10\) admits a unique generalised solution \(u\) satisfying \(3.8\).

**Proof.** Our proof is inspired by [10]. For \(x = x_b + dn(x_b)\) in \(\Omega \setminus \Omega_{r_0}\), we define

\[(3.12) \quad v(x) = \tilde{\rho}(d) := -(-\rho(d))^{\varepsilon},\]

where \(\varepsilon \in (0, 1)\) is a constant to be determined and

\[\rho(d) := -\int_0^d \left( \int_s^{r_0} g(t)dt \right)^{1/n} ds.\]

By computations, see [10] Lemma 1,

\[\det D^2v(x) = \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b)d} (-\tilde{\rho}'(d))^{n-1} \tilde{\rho}''(d)\]

in \(\Omega \setminus \Omega_{r_0}\), where \(k_i(x_b), i = 1, \cdots, n - 1\), are the principal curvatures of \(\partial\Omega\) at \(x_b\).

By differentiation

\[\tilde{\rho}' = \varepsilon(-\rho)^{\varepsilon-1}\rho', \]

\[\tilde{\rho}'' = \varepsilon(-\rho)^{\varepsilon-1}\rho'' + \varepsilon(1 - \varepsilon)(-\rho)^{\varepsilon-2}(\rho')^2.\]

Hence,

\[(3.13) \quad \det D^2v(x) \geq \varepsilon^n \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b)d} \left((-\rho)^{\varepsilon-1}\rho' \right)^{n-1} \left[(-\rho)^{\varepsilon-1}\rho'' + (1 - \varepsilon)(-\rho)^{\varepsilon-2}(\rho')^2 \right] \]

\[\geq \varepsilon^n \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b)d} (-\rho')^{n-1} \rho''(-\rho)^{n(\varepsilon-1)} \]

\[\geq \varepsilon^n C(n, \Omega) g(d)(-\rho)^{n(\varepsilon-1)}.\]
By setting $\varepsilon = \frac{p}{n+1-p}$ and rescaling $v$ to $bv$ for a constant $b$ satisfying $b^{n+1-p} \varepsilon^n C(n, \Omega) \geq 1$, one can see that $\det D^2v \geq R(x)(-1/v)^{1-p}$ in $\Omega \setminus \Omega_{r_0}$.

Observing that $v$ is a negative constant on $\partial \Omega$ for $r \in (0, r_0)$, we can extend $v$ to $\Omega_{r_0}$ so that $\det D^2v \geq \varepsilon (-1/v)^{1-p}$ in $\Omega_{r_0}$. For $\varepsilon$ small, $v$ is uniformly convex in $\Omega$. Similarly, by a rescaling we have $v$ is a subsolution of (3.10) in $\Omega$ and $v = 0$ on $\partial \Omega$.

Let $w = \phi + v$, where $\phi$ is a nonpositive, convex function in $\Omega$. Since $w \leq v \leq 0$, $(-1/v)^{1-p} \geq (-1/w)^{1-p}$, one can see that $w$ is a subsolution of (3.10) in $\Omega$ and satisfies $w = \phi$ on $\partial \Omega$.

Last step is to use the Perron method as in [11]. Denote $\Phi$ by the set of all subsolutions of (3.10) and (3.8), and let $u(x) = \sup \{ \tilde{u}(x) : \tilde{u} \in \Phi \}$. One can easily verify that $u$ is a generalised solution of (3.10). Since $w \in \Phi$ we conclude that $u = \phi$ on $\partial \Omega$. The uniqueness is due to the comparison principle [9, 12].

One example for $g$ satisfying (3.11) is that $g(d) \leq Cd^{\beta - n - 1}$ for some $\beta > 0$, which is also the case considered in [9]. Notice that when $\beta \geq n + 1$, $g$ is bounded in $(0, r_0]$ and thus $R$ in (3.10) is bounded in $\Omega$. In such a case, we can reduce the uniform convexity assumption on $\Omega$ in Lemma 3.3 following the work in [31].

**Lemma 3.4.** If $R$ in (3.10) is bounded from above and $\Omega$ is a bounded convex domain, the Dirichlet problem (3.10) and (3.8) admits a unique generalised solution.

**Proof.** First, we consider the zero boundary condition $\phi \equiv 0$. Let $\{ \Omega_k \}$ be an increasing sequence of smooth uniformly convex subdomains of $\Omega$ with $\Omega = \bigcup \Omega_k$. Let $\{ v_k \}$ be the sequence of convex solution of (3.10) in $\Omega_k$ and $v_k = 0$ on $\partial \Omega_k$. Since $\{ \Omega_k \}$ is an increasing sequence, $\{ v_k \}$ is a decreasing sequence, by the comparison principle [9, 12]. We will show that $v^* = \lim_{k \to \infty} v_k$ exists, and $v^* = 0$ on $\partial \Omega$.

Under a suitable coordinate we may assume that $0 \in \partial \Omega$ and $\Omega \subset \{ x_n > 0 \}$. Since $\Omega$ is bounded, there exists a large $K > 0$ such that $\Omega \subset B_K^+(0) = B_K(0) \cap \{ x_n > 0 \}$. Define the barrier function

$$w(x) = (|x'|^2 - A)x_n^\delta,$$

where $x' = (x_1, \cdots, x_{n-1})$ and $A > R^2$, $\delta \in (0, 1)$ are to be fixed. One can see that $w$ is convex, and by computation [31]

$$\det D^2w = \left\{ 2^{n-1}\delta(1-\delta)(A - |x'|^2) - 2n\delta^2|x'|^2 \right\} \times (A - |x'|^2)^\frac{\delta-n}{w} \left( -\frac{1}{w} \right)^{\frac{\delta-n}{w}}.$$
If \( n \geq 2 \) we choose \( \delta = \left( \frac{2}{n+1} \right)^p \in (0,1) \), and then fix \( A > K^2 \) sufficiently large, so that
\[
\det D^2 w \geq R(x) \left( \frac{1}{w} \right)^{1-p} \quad \text{in } B_{K}(0).
\]
When \( n = 1 \), if \( p < 0 \) we obtain a similar inequality with \( \delta = \frac{2}{2-p} < 1 \), and if \( 0 \leq p < 1 \) we can choose any \( \delta \in (0,1) \). Since \( \Omega_k \subset B_{K}^+ \) and \( v_k = 0 \) on \( \partial \Omega_k \), by the comparison principle we have \( w \leq v_k \) in \( B_k + K(0) \).

When \( n = 1 \), if \( p < 0 \) we obtain a similar inequality with \( \delta = \frac{2}{2-p} < 1 \), and if \( 0 \leq p < 1 \) we can choose any \( \delta \in (0,1) \). Since \( \Omega_k \subset B_{K}^+ \) and \( v_k = 0 \) on \( \partial \Omega_k \), by the comparison principle we have \( w \leq v_k \) in \( B_k + K(0) \).

For general boundary value \((3.8)\), let \( w^* = v^* + \phi \), where \( \phi \leq 0 \) is convex in \( \Omega \). Then \( w^* \) is a subsolution of \((3.10)\) and \( w = \phi \) on \( \partial \Omega \). Using the Perron method as in Lemma 3.3 we then obtain the generalised solution of \((3.10)\) and \((3.8)\). The uniqueness is due to the comparison principle \([9, 12]\). \( \square \)

### 3.2. Completeness

Since the spherical image of \( M \) is strictly contained in \( S^n_0 \), in order to be complete, \( M \) must be an entire graph, namely the scaled support function \( u \) must satisfy \( |Du(x)| \to \infty \) as \( x \to \partial \Omega \).

**Lemma 3.5.** Let \( \Omega \) be a bounded, uniformly convex \( C^2 \)-domain. Suppose that there exists a positive function \( h \) in \((0,r_0]\) satisfying
\[
\int_0^{r_0} h(t) dt = \infty,
\]
such that
\[
\frac{h(d)}{R(x_b,d)} = o(1) \quad \text{as } d \to 0.
\]
Then the solution \( u \) of \((3.10)\) and \((3.8)\), produced by Lemma 3.3 satisfies \( |Du(x)| \to \infty \) as \( x \to \partial \Omega \). In particular, if \( \phi = 0 \) on \( \partial \Omega \), condition \((3.15)\) can be reduced to
\[
h(d) \leq C R(x_b,d)
\]
for a constant \( C > 0 \).

**Proof.** Adopting the notations from §3.1. In \( \Omega \setminus \Omega_{r_0} \) we define
\[
w(x) = -a \rho(d) + \phi = -a \int_0^d \left( \int_0^{r_0} h(t) dt \right)^{1/n} ds + \phi,
\]
which is uniformly convex for \( a > 0 \), \( w = \phi \) on \( \partial \Omega \) and \( |Dw(x)| \to \infty \) as \( x \to \partial \Omega \). Since \( \partial \Omega \in C^2 \), for sufficiently large \( a > 0 \) we have an estimate
\[
\det D^2 w \leq (2a)^n \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1-k_i(x_b)} (\rho'(d))^{n-1} \rho''(d)
\]
\[
\leq (2a)^n C h(d),
\]
where \( C \) is a constant depending on \( n \) and \( \partial \Omega \).
Recall that $\phi \leq 0$. Let $\Omega' := \{ x \in \Omega : w(x) < \inf_{\partial \Omega} \phi - \varepsilon \}$ be a sub-level set of $w$, which is uniformly convex. Choose $\varepsilon > 0$ sufficiently small such that $\Omega_{r_0} \Subset \Omega'$ and $\Omega' \Subset \Omega$. Note that $w$ is constant on $\partial \Omega'$, we can extend $w$ inside $\Omega'$ similarly as before and then modify $w$ to get a uniformly convex function $\tilde{w} \in C^2(\Omega)$ such that $\tilde{w} = w$ in $\Omega \setminus \Omega'$. Near $\partial \Omega$, observe that in $\Omega \setminus \Omega'$

$$R(x_b, d) \left( -\frac{1}{w} \right)^{1-p} \geq R(x_b, d)C_1,$$

where $C_1 = (-\inf_{\partial \Omega} \phi + \varepsilon)^{p-1} > 0$ is a finite constant. Therefore, $\tilde{w}$ is a supersolution of (3.10)–(3.8), and by the comparison principle, $\tilde{w} \geq u$ in $\Omega$. Hence, $|Du(x)| \to \infty$ as $x \to \partial \Omega$.

In the special case $\phi \equiv 0$, we set $a = 1$ in (3.17), and at the end, replace $\tilde{w}$ by $b\tilde{w}$ with $b > 0$ sufficiently small, so that we can obtain a supersolution.

A example for $h$ satisfying (3.14) is that $h(t) = t^{-\alpha}$ for some $\alpha > 1$. Alternatively, when studying the homothetic solutions to Gauss curvature flow, Urbas [31] proved that if $\hat{f} \equiv 1$ in $\Omega$, $\phi = 0$ on $\partial \Omega$, then for a range of $p$, $|Du(x)| \to \infty$ as $x \to \partial \Omega$. Inspired by that, we have the following results.

**Lemma 3.6.** Let $\Omega$ be a bounded convex domain and $\partial \Omega \in C^{1,1}$. Assume that $\phi = 0$ on $\partial \Omega$, $\hat{f} > 0$ in $\Omega$. Then, when $p \leq 0$, the solution $u$ of (3.10)–(3.8) satisfies $|Du(x)| \to \infty$ as $x \to \partial \Omega$.

**Proof.** The proof follows from [31]. Let $x_0 \in \partial \Omega$ and let $B$ be an interior ball at $x_0$, i.e., $B \subset \Omega$ and $\partial B \cap \partial \Omega = \{x_0\}$. From assumptions, $R = \inf_{x \in B} R(x)$ is a positive constant, where $R$ is defined in (3.9). Let $w$ be the unique convex solution of the Dirichlet problem

$$(3.19) \quad \begin{cases} \det D^2 w = R \left( \frac{-1}{w} \right)^{1-p} & \text{in } B, \\ w = 0 & \text{on } \partial B, \end{cases}$$

The solvability is due to Lemma 3.3 and the solution $w$ is radially symmetric since the above problem has at most one convex solution. Hence, $w$ is a supersolution of (3.10) and (3.8), and $w \geq u$ in $B$ by the comparison principle. So, it suffices to show that $|Dw(x)| \to \infty$ as $x \to x_0$. Suppose on the contrary that $N = \sup_{B} |Dw| < \infty$, then $|w| \leq N d$ where
\[ d = \text{dist}(x, \partial B), \]
\[
\omega_n N^n = |Dw(B)| = \int_B \det D^2w = R \int_B \left( \frac{-1}{w} \right)^{1-p} \geq RN^{p-1} \int_B d^{p-1}.
\]
(3.20)

When \( p \leq 0 \), the last integral is infinite, which gives a contradiction. Therefore, \( N = \infty \), and Lemma 3.6 is proved.

The following lemma shows that in order for \( M \) to be complete, it is necessary to have \( \hat{f}(\xi) \to \infty \) as \( \xi \to \partial D \), where \( D \subset S^n_\pm \) is the spherical image of \( M \).

**Lemma 3.7.** If \( 0 < p < 1, \phi = 0 \) on \( \partial \Omega \), \( \hat{f} \) is bounded above in \( \Omega \), and \( \Omega \) satisfies a uniform enclosing sphere condition (namely, there exists a \( K > 0 \) such that for each \( x_0 \in \partial \Omega \) there is a ball \( B \) of radius \( K \) with \( \Omega \subset B \) and \( \partial B \cap \partial \Omega = \{x_0\} \)), then the solution \( u \) of (3.10)–(3.8) satisfies \( \sup \Omega |Du| \leq C \).

**Proof.** Let \( B \) be an enclosing ball at any point \( x_0 \in \partial \Omega \). From assumptions, \( \overline{R} = \sup_{x \in \Omega} R(x) \) is a positive constant. Replacing \( R \) in (3.19) by \( \overline{R} \), the convex solution \( w \) will be a subsolution of (3.10) and (3.8), and a gradient bound for \( u \) follows if we can prove \( N = \sup \Omega |Dw| < \infty \). Since \( w \) is convex, \( w \neq 0 \) and \( w = 0 \) on \( \partial B \), we have \( |w| \geq \theta d \) for some positive constant \( \theta \), where \( d = \text{dist}(x, \partial B) \). Proceeding as above, we now obtain

\[
\omega_n N^n \leq \overline{R} \theta^{p-1} \int_B d^{p-1}.
\]

The last integral is finite if \( p > 0 \). Therefore, we have a gradient bound for \( w \), and hence also for the solution \( u \). This completes the proof of Lemma 3.7.

4. The case of \( p > 1 \)

Recall that in §1 we know that for a complete, noncompact, convex hypersurface \( M \) in \( \mathbb{R}^{n+1} \), by a suitable rotation of coordinates its spherical image \( D \subset S^n_\pm \) satisfies one and exactly one of three cases (I), (II) and (III). Given a function \( f \) on \( D \), we investigate the existence of \( M \) such that \( f = H^{1-p}/K \) is the \( p \)-curvature function of \( M \), where \( H \) is the support function and \( K \) is the Gauss curvature of \( M \). When \( M \) is \( C^2 \) smooth, a function \( f \) is the \( p \)-curvature function of \( M \) if it satisfies equation (2.10), or (2.11) under an orthonormal frame field on \( S^n_\pm \). By the homogeneous extension (2.13), one has \( u \) satisfies equation (2.23).
in the domain $\Omega$, and $u^*$ satisfies (2.25) in $\Omega^*$. The hypersurface $M$ is then the graph of $u^*$ over $\Omega^*$.

Notice that by the convexity of $M$, $K$ is always nonnegative. However, the sign of the support function $H$ depending on the relative position of $M$ and the origin. As seen in §3, the above problem is equivalent to (3.1) that

\[ \frac{1}{H} = \tilde{f} K^{\frac{1}{p-1}}, \]

where $\tilde{f}$ is the given function on $D$. Similar to the nonexistence result in the case of $p < 1$, i.e. Lemma 3.1 when $p > 1$ we have the following analogous result.

**Lemma 4.1** (Nonexistence). If $\tilde{f} \leq 0$ is a nonpositive function on $D$, satisfying $\tilde{f} \in L^{p-1}(D)$, there does not exist any complete, noncompact, strictly convex hypersurface $M \in C^2$ satisfying (4.1).

**Proof.** Since $M$ is in the class $C^2$, by (4.1) we have $H < 0$ and $0 \notin M$. Hence, every tangent hyperplane $T$ of $M$ must pass between 0 and $M$, so $\text{dist}(0, T) \leq \text{dist}(0, M) = d$. Thus, $-H^{-1} \geq d^{-1}$. Since $M$ has infinite $n$-dimensional Hausdorff measure $\mathcal{H}^n$, by integrating we have

\[
\int_D |\tilde{f}|^{p-1}dx = \int_M (-\tilde{f})^{p-1}K d\mathcal{H}^n \\
= \int_M (-\frac{1}{H})^{p-1}d\mathcal{H}^n \\
\geq \int_M (\frac{1}{d})^{p-1}d\mathcal{H}^n = \infty,
\]

where $dx$ is the spherical measure of $\mathbb{S}^n$. The above inequality contradicts the assumption and thus completes the proof of Lemma 4.1. \qed 

In the subsequent context, we assume $\tilde{f} \geq 0$. Let $f = \tilde{f}^{p-1}$, we consider the equation

\[ \frac{1}{K} = f H^{p-1}, \]

where $p > 1$. This is a counterpart of Equation (3.3) in the case of $p < 1$. By the rescaling (2.13), Equation (4.2) is equivalent to (2.23), namely

\[ \det D^2 u = (1 + |x|^2)^{-\frac{n+1}{2}} f \left( \frac{x-1}{\sqrt{1 + |x|^2}} \right) u^{p-1}, \quad x \in \Omega, \]

where $\Omega = \{ \lambda \xi : \xi \in D, \lambda > 0 \} \cap \{ x_{n+1} = -1 \}$, $u(x) = H(x, -1)$, and $x = (x_1, \cdots, x_n)$. Depending on the spherical image $D \subset \mathbb{S}_-^n$ of $M$, let’s consider the cases (I), (II) and (III) separately in the following.
4.1. **Type I.** For type I hypersurfaces, $D$ is strictly contained in $S^n$, $\Omega$ is a bounded convex domain in $\mathbb{R}^n$. It is clear that $M$ is complete if and only if $\Omega^* = \mathbb{R}^n$, where $\Omega^* = Du(\Omega)$. Thus we pose the following boundary conditions associated with equation (4.3)

\begin{align}
|Du(x)| &\to \infty \quad \text{as } x \to \partial \Omega, \\
u(x) &= \phi(x) \quad x \in \partial \Omega,
\end{align}

where $\phi$ is assumed to be a positive, convex function in $\Omega$. We remark that in [32], when studying homothetic solutions of negative powered Gauss curvature flows, Urbas considered the above boundary value problem with $f \equiv 1$ and $\phi = \infty$ on $\partial \Omega$.

The solvability of Dirichlet problem (4.3) and (4.5) has been previously obtained in [7, §7], [13] and [17] under appropriate assumptions, especially $f$ is required to be positive and bounded. However, if the solution $u$ satisfies (4.4), by integrating equation (4.3),

$$
\int_\Omega \det D^2 u \leq C \int_\Omega f u^{p-1}.
$$

Notice that $0 < u \leq \sup_{\partial \Omega} \phi$. If $\sup_{\partial \Omega} \phi < \infty$, then it is necessary to have $f(x) \to \infty$ as $x \to \partial \Omega$. In the following, due to some technical differences, we consider two cases $p \in (1, n+1)$ and $p \in (n+1, \infty)$ separately. In each case, we first show the solvability of Dirichlet problem (4.3) and (4.5), and then prove that such an obtained solution $u$ satisfies (4.4). Hence, Theorem 1.2 is proved. Our approach is inspired by the work of Chou-Wang [10], in which they considered the special case $p = 1$, see also [28] and references therein.

4.1.1. **The case of $1 < p < n + 1$.** Write

$$
R(x) = (1 + |x|^2)^{-\frac{n+1}{2}} f \left( \frac{x, -1}{\sqrt{1 + |x|^2}} \right), \quad x \in \Omega.
$$

Equation (4.3) can be reduced to

\begin{equation}
\det D^2 u = u^{p-1} R(x), \quad x \in \Omega.
\end{equation}

As in §3, let $\Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \}$. For $x \in \Omega \setminus \Omega_{r_0}$, $r_0 > 0$ small, $x$ can be represented uniquely by $x_b + dn(x_b)$, where $x_b \in \partial \Omega$, $d = \text{dist}(x, \partial \Omega)$, and $n(x_b)$ is the unit inner normal at $x_b$. For a function $f$ defined near $\partial \Omega$ we write $f(x) = f(x_b, d)$.

**Lemma 4.2.** Assume that $1 \leq p < n+1$ and $\Omega \in C^2$ is uniformly convex. Suppose there exists a positive function $g$ in $(0, r_0]$, satisfying

\begin{equation}
\int_0^{r_0} \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds < \infty,
\end{equation}

such that

$$
R(x_b, d) \leq g(d).
$$

Then (4.6) admits a unique generalised solution $u$ in $C(\Omega)$ and $u = \phi$ on $\partial \Omega$. 


Proof. Similarly as in [10], for $x = x_b + dn(x_b)$ in $\Omega \setminus \Omega_{r_0}$ we define
\begin{equation}
(4.8) \quad v(x) = \rho(d) = -\int_0^d \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds,
\end{equation}
and have
\begin{equation}
(4.9) \quad \det D^2 v(x) = n - 1 \prod_{i=1}^{n-1} k_i(x_b) \left( -\rho'(d) \right)^{n-1} \rho''(d)
\end{equation}
in $\Omega \setminus \Omega_{r_0}$, where $k_i(x_b)$, $i = 1, \cdots, n - 1$, are the principal curvatures of $\partial \Omega$ at $x_b$.

Next, we extend $v$ inside $\Omega_{r_0}$. Note that $v = -G_0$ is a constant on $\partial \Omega_{r_0/2}$. We extend $v$ to $\Omega_{r_0/2}$ so that $\det D^2 v = \varepsilon > 0$ in $\Omega_{r_0/2}$. For $\varepsilon$ small, $v$ is uniformly convex in $\Omega$. By the uniform estimate, we have
\begin{equation}
(4.10) \quad \sup |v| \leq G_0 + C |\Omega|^{2/n},
\end{equation}
for some constant $C$ depending only on $n, \varepsilon$.

Let
\begin{equation}
(4.11) \quad w(x) = \phi(x) + Av(x), \quad x \in \Omega.
\end{equation}
Then $w$ is convex in $\Omega$, $w = \phi$ on $\partial \Omega$, and
\begin{equation}
\sup |w| \leq |\phi|_0 + A|v|
\end{equation}
\begin{equation}
\leq |\phi|_0 + A(G_0 + C|\Omega|^{2/n}).
\end{equation}

By computation we have the left hand side of equation (4.6)
\begin{equation}
(4.12) \quad \det D^2 w \geq A^n \det D^2 v \geq \begin{cases} A^n \varepsilon & \text{in } \Omega_{r_0} \\ A^n C_1 g & \text{in } \Omega \setminus \Omega_{r_0}, \end{cases}
\end{equation}
where $C_1$ is a constant depending on $n, r_0$ and $\partial \Omega$. Meanwhile, the right hand side
\begin{equation}
(4.13) \quad w^{p-1} R(x) \leq \sup |w|^{p-1} R(x) \leq \begin{cases} \sup |w|^{p-1} \tilde{R} & \text{for } x \in \Omega_{r_0} \\ \sup |w|^{p-1} R(x) & \text{for } x \in \Omega \setminus \Omega_{r_0}, \end{cases}
\end{equation}
where $\tilde{R} = \sup_{x \in \Omega_{r_0}} R(x)$ is finite.

Since $p < n + 1$, we can choose $A$ sufficiently large such that $\det D^2 w \geq w^{p-1} R(x)$ in $\Omega$. This means that $w$ is a subsolution of (4.6) and $w = \phi$ on $\partial \Omega$. Last step is to use the Perron method, which requires $p \geq 1$. Denote $\Phi$ by the set of all subsolutions of (4.6) and (4.5), and let $u(x) = \sup \{ \hat{u}(x) : \hat{u} \in \Phi \}$. One can easily verify that $u$ is a generalised solution of (4.6). Since $w \in \Phi$ we conclude $u = \phi$ on $\partial \Omega$. \hfill \Box

Lemma 4.3. Suppose that there exists a positive function $h$ in $(0, r_0]$ satisfying
\begin{equation}
(4.14) \quad \int_0^{r_0} h(t) dt = \infty,
\end{equation}
such that
\begin{equation}
R(x_b, d) \geq h(d).
\end{equation}
Then the solution $u$ produced by the above lemma satisfies (4.4).
**Proof.** For this proof we need an upper barrier function. Recall that \( \phi > 0 \) on \( \partial \Omega \). Introduce the function \( v \) as before, where \( g \in (L_8) \) is now replaced by \( h \), namely

\[
v(x) = \rho(d) = -\int_0^d \left( \int_s^{r_0} h(t) dt \right)^{1/n} ds, \quad x \in \Omega \setminus \Omega_{r_0}.
\]

Then \( |Dv(x)| \to \infty \) as \( x \to \partial \Omega \). Extend \( v \) to \( \Omega_{r_0} \) as in the previous proof and then modify \( v \) to get a uniformly convex function \( \tilde{v} \in C^2(\Omega) \) so that \( \tilde{v} = v \) in \( \Omega \setminus \Omega_{r_0/2} \).

For any point \( x_0 \in \partial \Omega \) we shall assume \( x_0 = 0 \) and the positive \( x_n \)-axis is in the inner normal direction. Define

\[
\hat{v}(x) := \eta \tilde{v}(x) + \phi(0) + x \cdot D\phi(0) + Kx_n, \quad x \in \Omega,
\]

where \( K > 0 \) is a constant. As \( \phi \in C^2(\Omega) \) and \( \partial \Omega \in C^2 \) is uniformly convex, we can choose \( K \) large enough such that \( \hat{v} \geq \phi \) on \( \partial \Omega \) and \( \hat{v}(x_0) = \phi(x_0) \). Then by choosing \( \eta > 0 \) small enough, we also have \( \hat{v} \geq v_0 > 0 \) in \( \Omega \).

Using similar computations as before, we have

\[
\det D^2 \hat{v} \leq \eta^n \det D^2 \tilde{v} \leq \left\{ \begin{array}{ll} \eta^n \varepsilon & \text{in } \Omega_{r_0} \\ \eta^n C h & \text{in } \Omega \setminus \Omega_{r_0}, \end{array} \right.
\]

where \( C_1 \) is a constant depending on \( n, r_0 \) and \( \partial \Omega \). For the right hand side we have

\[
\hat{v}^{p-1} R(x) \geq v_0^{p-1} R(x) \geq \left\{ \begin{array}{ll} v_0^{p-1} R & \text{for } x \in \Omega_{r_0} \\ v_0^{p-1} R(x) & \text{for } x \in \Omega \setminus \Omega_{r_0}, \end{array} \right.
\]

where \( R = \inf_{x \in \Omega_{r_0}} R(x) \) is positive and finite.

Therefore, by choosing \( K \) sufficiently large and \( \eta \) sufficiently small, using the comparison principle [12] we obtain \( \hat{v} \geq u \) in \( \Omega \). Hence \( |Du(x)| \to \infty \) as \( x \to x_0 \).

4.1.2. The case of \( p > n + 1 \). To obtain existence, we adopt a different approach of constructing subsolutions. Let’s define

\[
\rho(d) = -\int_0^d \left( \int_s^{r_0} g(t) dt \right)^{1/n} ds.
\]

Assume \( \phi = \phi_0 > 0 \) is a constant on \( \partial \Omega \). Define

\[
v(x) = (-\rho(d) + \phi_0^{-\frac{1}{p}})^{-\delta}, \quad \text{in } \Omega \setminus \Omega_{r_0},
\]

where \( \delta > 0, A > 0 \) are constants to be determined.

Setting \( \delta = \frac{n}{p-n-1} \), by computation we have

\[
\det D^2 v \geq A^n \delta^n v^{\frac{n(p+1)}{d}} \prod_{i=1}^{n-1} \frac{\kappa_i}{1-\kappa_i d} (-\rho')^{n-1} \rho^n \geq A^n \delta^n C(n, \Omega) g(d) v^{p-1},
\]

where
where $C$ is a constant depending only on $n$ and $\Omega$. Choosing $A$ sufficiently large and by extending $v$ inside $\Omega_{r_0}$ as before, we then obtain a subsolution. Note that $0 < v \leq \phi_0$ in $\Omega$ and $v = \phi_0$ on $\partial \Omega$. The existence of solution $u$ thus follows by the Perron process.

For a general $\phi > 0$ on $\partial \Omega$, we need modify $v$ in (4.17). For a point $x_0 \in \partial \Omega$, we may assume $x_0 = 0$ and the positive $x_n$-axis is in the inner normal direction. Let $\phi_0 = \phi(0)$.

Define
\[
(4.19) \quad v(x) = (-A\rho(d) + \phi_0^{-\frac{1}{2}} + Kx_n - \frac{1}{2}x \cdot D\phi(0))^{-\frac{1}{2}}, \quad \text{in } \Omega \setminus \Omega_{r_0},
\]

where $K > 0$ is chosen sufficiently large such that $v \leq \phi$ on $\partial \Omega$ and $v = \phi$ at $x_0$. By choosing $\delta = \frac{n}{p-n-1}$ and $A$ sufficiently large as above, we have $v$ is a solution. Therefore, the existence of solution $u$ follows.

For completeness, Lemma 5.2 applies in this case, so we have $|Du(x)| \to \infty$ as $x \to \partial \Omega$, and obtain the completeness.

4.2. Type II. Next, we consider type II hypersurfaces. In this case, we investigate the entire solution of (4.3), i.e.
\[
(4.20) \quad \det D^2 u = (1 + |x|^2)^{-\frac{n+1}{2}} f \left( \frac{x}{\sqrt{1 + |x|^2}} \right) u^{p-1} \quad \text{in } \Omega = \mathbb{R}^n.
\]

When $p > n + 1$, we prove the existence of a solution by constructing suitable upper and lower barriers.

Assume that $f$ satisfies the asymptotic growth condition
\[
(4.21) \quad f(x) \sim (1 + |x|^2)^q \quad \text{as } x \to \infty,
\]

where $q \in (0, 1)$ is a constant. Note that this is equivalent to $f(X) \sim |X_{n+1}|^{-2q}$. We remark that it is necessary to have a growth condition on $f$. Otherwise, by integration one can see that when $\Omega^*$ is bounded, $f$ is bounded, there doesn’t exist a complete noncompact hypersurface $M$ satisfying (4.20), (see [32] for the case of $f \equiv 1$). To prove this claim, it is convenient to use Equation (2.25) for the dual function $u^*$. In that case, $M$ is a graph of $u^*$ over $\Omega^*$. If $\Omega^*$ is bounded, let $y_0 \in \partial \Omega^*$. Since $u^*$ is convex, there exists a constant $C_0 > 0$ such that
\[
u^* \geq -C_0 \quad \text{on } \Omega^* \cap B_1(y_0).
\]

Let $P \in \tilde{M} := M \cap (B_1(y_0) \times \mathbb{R})$. Assuming $M$ is a complete, noncompact hypersurface, we compute its support function $H$ at $P$ and have
\[
H|_P = \frac{y \cdot Du^* - u^*}{\sqrt{1 + |Du^*|^2}} \leq |y| + C_0 \leq 1 + |y_0| + C_0.
\]
Consequently, $H^{-1} \geq c_0 > 0$ in a neighbourhood $G \subset \tilde{M}$. Similarly to the nonexistence Lemmas 3.1 and 4.1 by integrating (4.1) we obtain

$$\epsilon_0^{p-1} \mathcal{H}^n(G) \leq \int_D f K d\mu = \int_D f dx,$$

where $D \subset \mathbb{S}^n$ is the spherical image of $G$, $d\mu = K^{-1} dx$ is the area measure, and $dx$ is the spherical measure. As $\mathcal{H}^n(G) = \infty$, so the function $f$ cannot be bounded.

From (4.20) and (4.21), an upper (or lower) barrier is a function satisfying

$$\det D^2 u \leq (1 + |x|^2)^{-\gamma} w^{p-1} \quad (\text{or} \geq)$$

as $x \to \infty$, where $\gamma := \frac{n+p+1}{2} - q$. In a bounded domain, one can always construct such a barrier by rescaling $u$ to $\lambda u$ for a suitable constant $\lambda$, provided $p \neq n + 1$.

Now, let’s consider the function

$$w(x) = (1 + |x|^2)^\delta$$

where $\delta > 1/2$ is to be chosen. Clearly $w$ is a convex function.

By computations

$$D_{ij} w = 2\delta(1 + |x|^2)^{\delta-1} \delta_{ij} + 4\delta(\delta - 1)(1 + |x|^2)^{\delta-2} x_i x_j,$$

where $\delta_{ij}$ is the Kronecker delta. Hence,

$$\det D^2 w = (2\delta)^n (1 + |x|^2)^{n(\delta-1)} \left( \frac{1 + (2\delta - 1)|x|^2}{1 + |x|^2} \right)$$

$$= C(n, \delta) w^{n(\delta-1)+\gamma}(1 + |x|^2)^{-\gamma},$$

where $C(n, \delta)$ is a positive constant bounded by $C_1 \leq C(n, \delta) \leq C_2$, and

$$C_1 := (2\delta)^n \inf_{x \in \mathbb{R}^n} \left\{ \frac{1 + (2\delta - 1)|x|^2}{1 + |x|^2} \right\},$$

$$C_2 := (2\delta)^n \sup_{x \in \mathbb{R}^n} \left\{ \frac{1 + (2\delta - 1)|x|^2}{1 + |x|^2} \right\}.$$

Choose $\delta = (\gamma - n)/(p - n - 1) > 0$ such that $\frac{n(\delta-1)+\gamma}{\delta} = p - 1$. One can see that as far as $q < 1$,

$$\delta > \frac{n+q+1}{2} - n - 1 = \frac{1}{2}.$$

By a rescaling we obtain that $\overline{w} = \mu w$ is a convex supersolution of (4.20) for $\mu^{p-n-1} \geq C_1$, while $\underline{w} = \mu w$ is a convex subsolution of (4.20) for $0 < \mu^{p-n-1} \leq C_2$. Since $\mu \leq \overline{w}$, we have $\underline{w} \leq \overline{w}$. Let $\phi$ be any smooth function such that $\underline{w} \leq \phi \leq \overline{w}$ in $\mathbb{R}^n$. By [7] the Dirichlet problem

$$\det D^2 w_k = (1 + |x|^2)^{-\frac{n+q+1}{2}} f w_k^{p-1} \quad \text{in } B_{2k}(0),$$

$$w_k = \phi \quad \text{on } \partial B_{2k}(0),$$
has a unique convex solution $w_k \in C^\infty(B_2)$, $\underline{w} \leq w_k \leq \overline{w}$ in $B_2$. From this there exists a subsequence of $\{w_k\}$ converges locally in any $C^l$ form to a convex solution $u \in C^\infty(\mathbb{R}^n)$ of (4.20), and $\underline{w} \leq u \leq \overline{w}$ in $\mathbb{R}^n$. Thus $u(x)/\sqrt{1+|x|^2} \to \infty$ as $|x| \to \infty$, $\Omega' = Du(\Omega) = \mathbb{R}^n$, and hence the corresponding hypersurface $M$ is complete.

In fact, any admissible solution $u$ of (4.20) with $Du(\Omega) = \mathbb{R}^n$ must satisfies
\begin{equation}
\frac{u(x)}{\sqrt{1+|x|^2}} \to \infty \quad \text{as} \quad |x| \to \infty.
\end{equation}
Otherwise, if this is not true, there exists a sequence $\{z_k\} \subset \Omega$ such that $|z_k| \to \infty$ and for each $k$, $u(z_k) \leq C|z_k|$ for some constant $C$. By choosing a subsequence and making a rotation of coordinates if necessary we may assume that $z_k/|z_k| \to e_n = (0, \ldots, 0, 1)$. Let $x_{n+1} = a_0 + \langle a, x \rangle = a_0 + \sum_{i=1}^n a_i x_i$
be the graph of any tangent hyperplane to graph $u$. Then
\begin{equation}
a_0 + \langle a, z_k \rangle \leq C|z_k|
\end{equation}
for each $k$, so dividing by $|z_k|$ and letting $k \to \infty$ we obtain $a_n \leq C$. This implies that $Du(\Omega) \cap \{x_n > C\} = \emptyset$, which contradicts with $Du(\Omega) = \mathbb{R}^n$. This proves (4.22).

Therefore, we have the following existence result, which is equivalent to Theorem 1.3.

**Theorem 4.1.** When $p > n + 1$, $f$ satisfies the asymptotic growth condition (4.21), there exists a complete noncompact hypersurface $M$ whose support function is a solution of (4.20).

**4.3. Type III.** This case can be handled similarly as in [10]. We observe that for a type III hypersurface, $\Omega$ is of the form $\omega \times \mathbb{R}^m$ for some $m < n$, where $\omega$ is a bounded convex domain in $\mathbb{R}^{n-m}$. Near $\partial \omega$ we may write $\tilde{x} = (x_1, \ldots, x_{n-m}) = \tilde{x}_b + dn(\tilde{x}_b)$ analogously as before. Correspondingly the boundary conditions (4.14) and (4.15) are imposed on $\partial \omega$
\begin{align}
|Du(x)| \to \infty & \quad \text{as} \quad x \to \partial \omega, \quad (4.23) \\
u(x) = \phi(x) & \quad x \in \partial \omega, \quad (4.24)
\end{align}
where $\phi$ is prescribed on $\partial \omega$. Then by following the lines in §4.1, we have

**Theorem 4.2.** Let $p \geq 1$, $n+1$, $\Omega = \omega \times \mathbb{R}^m$, where $\omega$ is a uniformly convex $C^2$-domain in $\mathbb{R}^{n-m}$. Suppose that $\phi$ can be extended to $\Omega$ so that $D^2\phi(x) \geq \delta_0 I$ for some positive constant $\delta_0$, where $I$ is the identity matrix. Suppose moreover there exist two positive functions $g$ and $h$ defined in $(0, r_0]$, $r_0 > 0$, satisfying
\begin{align}
\int_0^{r_0} \left( \int_s^{r_0} g(t)dt \right)^{1/(n-m)} ds < \infty, \quad (4.25) \\
\int_0^{r_0} h(t)dt = \infty, \quad (4.26)
\end{align}
such that
\begin{equation}
(4.27) \quad h(d) \leq R(\tilde{x}_b, d) \leq g(d), \quad \text{where } \tilde{x} = \tilde{x}_b + dn(\tilde{x}_b),
\end{equation}

near $\partial \Omega$. Then there exists a unique solution $u$ of (4.4), (4.23) and (4.24) in $C(\Omega) \cap C^{2,\alpha}(\Omega)$.

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