The Symmetric Minimal Surface Equation

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1 Introduction

The Symmetric Minimal Surface Equation (SME) is the equation

\[ M(u) = \frac{m - 1}{u\sqrt{1 + |Du|^2}} \]

on an open set \( \Omega \subset \mathbb{R}^n \), where

\[ M(u) = \sum_{i=1}^n D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right) \]

is the mean curvature operator on \( \mathbb{R}^n \), \( m \geq 2 \) is an integer, and \( u > 0 \) is of class \( C^2 \). Notice that this equation is geometrically scale invariant: that is if \( G(u) \subset \Omega \times (0, \infty) \) is the graph

\[ G(u) = \{(x, u(x)) : x \in \Omega\} \]

of a solution \( u \) on \( \Omega \) then, for each \( \lambda > 0 \), \( \lambda G(u)(= \{\lambda(x, u(x)) : x \in \Omega\}) \) is also the graph of a solution; indeed it is the graph of the solution \( u_\lambda(x) = \lambda u(x/\lambda) \) on the domain \( \lambda \Omega \). Analytically this is just the statement

\[ u \text{ satisfies 1.1 on } \Omega \subset \mathbb{R}^n \iff \lambda u(x/\lambda) \text{ satisfies 1.1 on } \lambda \Omega \text{ for each } \lambda > 0. \]

In fact geometrically the equation 1.1 expresses the fact that the graph \( G(u) \) of \( u \) is a hypersurface with mean curvature \( H = \frac{m - 1}{u\sqrt{1 + |Du|^2}} \) at each of its points \((x, u(x))\).

The chief motivation here for the study of 1.1 is the fact that if \( u \) is a positive \( C^2 \) solution of 1.1 and if \( S(u) \) is the “symmetric graph” of \( u \), defined by

\[ S(u) = \{(x, \xi) \in \Omega \times \mathbb{R}^m : |\xi| = u(x)\}, \]

then \( S(u) \) is a minimal hypersurface (i.e. has zero mean curvature) in \( \mathbb{R}^{n+m} \).

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This is easily checked by observing that 1.1 is the Euler-Lagrange equation for the functional

$$A(u) = \sigma_{m-1} \int_{\Omega} \sqrt{1 + |Du|^2} u^{n-1} \, dx \quad (\sigma_{m-1} = \mathcal{H}^{m-1}(S^{m-1}))$$

and, geometrically, $A(u)$ represents the area functional for $S(u)$; that is, $A(u)$ is the $(n + m - 1)$-dimensional Hausdorff measure $\mathcal{H}^{n+m-1}(S(u))$. This is clear because the integrand $\sqrt{1 + |Du|^2} u^{m-1}$ for $A(u)$ is the Jacobian of the map $(x, \omega) \in \Omega \times S^{m-1} \mapsto (x, u(x)\omega) \in \Omega \times \mathbb{R}^m$, and this map is a local coordinate representation for the symmetric graph $S(u)$. Since 1.1 expresses the fact that $u$ is stationary with respect to $A$, we see that $S(u)$ is stationary with respect to smooth symmetric deformations, and hence stationary with respect to all deformations by a well-known principle (see e.g. [Law72]). (The latter principle here is just the natural generalization of the fact that if a smooth hypersurface $\Sigma$ is rotationally symmetric about an axis and if $\Sigma$ is stationary with respect to smooth rotationally symmetric compactly supported perturbations, then $\Sigma$ is minimal—i.e. stationary with respect to all smooth compactly supported perturbations whether symmetric or not.) Thus the smooth submanifold $S(u)$ is stationary as a multiplicity 1 varifold in $\Omega \times (\mathbb{R}^m \setminus \{0\})$ and hence is a smooth minimal submanifold of $\Omega \times (\mathbb{R}^m \setminus \{0\})$ as claimed.

We say that a non-negative $C^0(\Omega)$ function is a singular solution of the SME if it is locally the uniform limit of $C^2$ positive solutions of the SME. More precisely:

1.5 Definition. $u : \Omega \to [0, \infty)$ is a singular solution of the SME in $\Omega$ if $u$ is continuous on $\Omega$, if $\{x \in \Omega : u(x) = 0\} \neq \emptyset$, and if $u = \lim_{j \to \infty} u_j$, uniformly on each compact subset of $\Omega$, where each $u_j$ is a positive $C^2(\Omega)$ solution of 1.1.

If $u$ is a regular or singular solution of the SME in $\Omega$ then sing $u$, the singular set of $u$, is defined to be $u^{-1}\{0\}$; that is,

$$\text{sing } u = \{x \in \Omega : u(x) = 0\}.$$  

Since by definition singular solutions are continuous at all points of $\Omega$, we see that sing $u$ is closed as a subset of $\Omega$, and of course sing $u = \emptyset$ in case $u > 0$ everywhere on $\Omega$.

1.7 Remark: For any regular or singular solution $u$ of 1.1 the symmetric graph $S(u)$ (as in 1.4) is a stationary multiplicity 1 varifold in the cylinder $\Omega \times \mathbb{R}^m$ and the singular set of $S(u)$ indeed coincides with sing $u \times \{0\}$; that is if $G$ is the graph of a singular solution $u$ of 1.1 on $\Omega$, and if sing $S(u)$ is (as usual for stationary varifolds) defined to be the set of points $z \in S(u)$ such that there is no $\sigma > 0$ such that $S(u) \cap B_\sigma(z)$ is an $(n + m - 1)$-dimensional embedded $C^1$ submanifold of $\Omega \times \mathbb{R}^m$, then

$$\text{sing } S(u) = \text{sing } u \times \{0\} = \{(x, \xi) \in \Omega \times \mathbb{R}^m : \xi = 0 \text{ and } u(x) = 0\}.$$  

We will check this in Corollary 2.5 after we have established the necessary preliminary area bounds.

It is not quite clear a-priori (but nevertheless true) that singular solutions $u$ of 1.1 are in fact weak solutions of 1.1, i.e. if $u$ is a singular solution of 1.1 then $u \in W^{1,1}_{\text{loc}}(\Omega)$, $1/u \in L^1_{\text{loc}}(\Omega)$, and

$$1.1' \quad \int_{\Omega} \left( \sum_{i=1}^n \frac{D_i u D_i \zeta}{\sqrt{1 + |Du|^2}} + \frac{(m - 1)\zeta}{u \sqrt{1 + |Du|^2}} \right) = 0$$  

for each $\zeta \in C^1(\Omega)$.

We shall not explicitly use this fact here (although we will use 1.1' for regular solutions $u$), but the interested reader can check that singular solutions $u$ also satisfy 1.1' by using the main regularity theorem of §9. (See Remark 9.2 following Theorem 9.1.)
As far as the existence of singular solutions is concerned, we first note that there are no singular solutions in case \( n = 1 \): Indeed when \( n = 1 \) there is a single variable \( x \) and any positive solution \( u \) on an interval \((a, b)\) satisfies the ODE \( \frac{u''}{1+u^2} = \frac{m-1}{u} \), so \( u \) is strictly convex on \((a, b)\), and (after multiplying by \( u' \) and integrating) we see that \( u^{1-m}(1+(u')^2)^{1/2} = C \) (a positive constant). Thus \( u \) is bounded below (indeed \( u^{m-1} \geq C^{-1} \)). By the uniqueness and extension theorems for ODE’s and the convexity of \( u \) we then easily check that, modulo a translation of the independent variable \( x \), any positive solution extends to a maximal interval \((-d, d)\) with \( 0 < d \leq \infty \), where \( u(x) = u(-x) \), \( u \) takes its unique minimum at \( x = 0 \) and \( u(x) \to \infty \) as \(|x| \to d \). Finally since the ODE is geometrically scale invariant it follows (again using the uniqueness theorem) that, after a translation of the independent variable \( x \), all solutions \( u \) are just geometric rescalings \( u(x) = \lambda \varphi(x/\lambda) \) for some \( \lambda > 0 \), where \( \varphi \) is the unique (maximally extended) solution of the ODE with \( \varphi(0) = 1 \), \( \varphi'(0) = 0 \) with \( \varphi \) defined over some interval \((-d_0, d_0)\) where \( 0 < d_0 \leq \infty \). Now it is evident that there can be no singular solution \( u \), since otherwise there would be a sequence \( \lambda_j \varphi((x-x_j)/\lambda_j) \) with \( \lambda_j \downarrow 0 \) and \( x_j \to 0 \) which converges uniformly on some open interval of \( \mathbb{R} \). This is impossible because if \( d_0 < \infty \) then \( \varphi(x/\lambda_j) \) is defined only over the interval \((-d_0\lambda_j, d_0\lambda_j) \to \{0\} \), while if \( d_0 = \infty \) then \( \lambda_j \varphi(x/\lambda_j) \) has derivative \( \varphi'(x/\lambda_j) \) which tends to \(+\infty\) for \( x > 0 \) and \(-\infty\) for \( x < 0 \).

On the other hand in case \( n \geq 2 \) it is easy to give examples of singular solutions. For instance one sees by direct computation that \( u(x) \equiv \left(\frac{n-1}{n-1}\right)^{1/2}|x| \) is a solution of of the SME on \( \mathbb{R}^n \setminus \{0\} \), and, with a little more effort using ODE theory, it is straightforward to show that this \( u \) is locally the uniform limit of positive solutions in a neighborhood of 0, and so is a singular solution in the sense introduced above. Notice that for this example the symmetric graph \( S(u) \) is just the minimal cone \(|\xi| = (\frac{n-1}{n-1})^{1/2}|x| \) or in other words the cone \((n-1)|\xi|^2 = (m-1)|x|^2 \). If \( m = n = 4 \) this is the 7-dimensional “Simons cone” over \( S^3 \times S^3 \) which was the first example known of a singular area minimizing hypersurface ([BDG69], [Sis68]).

We show in §4, via a Leray-Schauder argument, that in fact there is a very rich class of singular solutions for \( n, m \geq 2 \). In addition, one can quite easily modify the method of [CHS84] to show that the special example \( u_0(x) = (\frac{n-1}{n-1})^{1/2}|x| \) generates a rich class of examples with isolated singular points at 0, each of which is asymptotic to \( u_0 \) on approach to the singular point 0.

Regular solutions of equation 1.1 have been studied in [DH90], [DH96], with a different motivation than the present one—the emphasis here is on the study of singular solutions of 1.1 and the corresponding singular minimal surfaces obtained from the symmetric graphs of solutions. Our aim is to develop the basic theory of such solutions, showing on the one hand that there is a convenient general theory governing their qualitative behavior, and that on the other hand, as mentioned above, there is a rich class of singular solutions, suggesting that this setting could be valuable in improving our understanding of singular behavior of minimal submanifolds.

The main regularity results here (see §8 and §9 below) are that singular solutions \( u \) of 1.1 are always locally Lipschitz in \( \Omega \) with \( \text{sing } u \) having Hausdorff dimension less than or equal \( n-2 \).

2 Volume Bounds

Here and subsequently, for \( R > 0 \) and \( y \in \mathbb{R}^N \),

\[
B_R^N(y) = \{x \in \mathbb{R}^N : |x - y| < R\} \quad \text{(possibly abbreviated } B_R(y) \text{ if } N \text{ is evident).}
\]

We first want to present a lemma describing the basic volume bounds available for solutions of the SME (i.e. solutions of 1.1).
As a preliminary to this, recall (see e.g. \cite{Sim83}) that if \( U \subset \mathbb{R}^{n+1} \) is open and \( \Sigma \) is a smooth embedded hypersurface (for the moment with no singular set in \( U \), i.e. \((\Sigma \setminus \Sigma) \cap U = \emptyset\) then
\[
e^{\Lambda \rho^-n} \mathcal{H}^n(\Sigma \cap B_\rho(y)) \text{ is increasing in } \rho, \rho \in (0, R],
\]
provided \( \overline{B}_R(y) \subset U \) and \( \sup_{B_R(y)} |H| \leq \Lambda \), where \( H \) is the mean curvature of \( \Sigma \). In particular
\[
\mu(\Sigma \cap B_\rho(y)) \geq e^{-\Lambda \rho} \omega_n \rho^n, \quad \rho \in (0, R], \ y \in \text{spt } \mu.
\]

The main volume bounds for solutions of 1.1 are as follows:

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2.1
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\]

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\[ 1.1 \]

2.3 **Lemma.** There is a constant \( C = C(m,n) \) such that if \( u \) is a regular or singular solution of 1.1 on a domain \( \Omega \subset \mathbb{R}^n \) and if \( y = (x_0, u(x_0)) \in G = \text{graph } u \) and \( \overline{B}_\rho(y) \cap G \) is compact (so the boundary of \( G \) does not intersect \( B_\rho(y) \)), then
\[
\begin{align*}
(1) \quad & C^{-1} \rho^n \leq \int_{S_{\rho/2}(y)} 1 + |Du|^2 \, dx \leq C \rho^n \\
(2) \quad & \int_{S_{\rho/2}(y)} 1 + |Du|^2 u^{m-1} \, dx \geq C^{-1} (\rho + u(x_0))^{m-1} \rho^n,
\end{align*}
\]
where we use the notation \( S_\sigma(y) = \{ x : (x, u(x)) \in B_\sigma(y) \cap G \} \); that is,
\[
S_\sigma(y) = \{ x \in \Omega : \sqrt{|x - x_0|^2 + (u(x) - u(x_0))^2} < \sigma \}.
\]

2.4 **Remark:** Observe that the above lemma implies that there is \( \varepsilon_0 = \varepsilon_0(m,n) > 0 \) such that if \( u \) is a regular or singular solution of 1.1 on a ball \( B_\rho(x_0) \subset \mathbb{R}^n \) then there is at least one point \( x \) in \( B_{\rho/2}(x_0) \) with \( u(x) > \varepsilon_0 \rho \), because otherwise by the right inequality of (i) we would have
\[
\int_{S_{\rho/2}(x_0,u(x_0))} \sqrt{1 + |Du|^2} u^{m-1} \, dx \leq C \varepsilon_0^{-1} \rho^{n+m-1},
\]
contradicting the bound (ii) of the lemma.

We can now check the claim made in Remark 1.7:

2.5 **Corollary.** If \( u \) is a regular or singular solution of 1.1 in an open \( \Omega \subset \mathbb{R}^n \), then the symmetric graph \( S(u) \) (as in 1.4) is a multiplicity 1 stationary \((n+m-1)\)-dimensional varifold in \( \Omega \times \mathbb{R}^m \) with
\[
sing S(u) = \text{sing } u \times \{ 0 \} = \overline{S(u)} \setminus S(u), \text{ where } \overline{S(u)} \text{ is the closure of } S(u) \text{ in } \Omega \times \mathbb{R}^m.
\]
Furthermore in both the regular and singular case \( u \) is \( C^\infty \) on the (open) set of points where it is positive, and \( G(u) \cap (\Omega \times (0, \infty)) \), \( S(u) \cap (\Omega \times (\mathbb{R}^m \setminus \{ 0 \})) \) are \( C^\infty \) hypersurfaces in \( \mathbb{R}^{n+1} \) and \( \mathbb{R}^{n+m} \) respectively.

**Proof:** In case \( u > 0 \) in \( \Omega \) (i.e. the case when \( \text{sing } u = \emptyset \)), \( S(u) \) is a smooth \((n+m-1)\)-dimensional minimal submanifold of \( \Omega \times (\mathbb{R}^m \setminus \{ 0 \}) \), as discussed in §1. Hence if \( u > 0 \) in \( \Omega \) then \( S(u) \) is stationary, as a multiplicity 1 varifold, in \( \Omega \times \mathbb{R}^m \) in accordance with the discussion of §1.

If now \( u \) is a singular solution of the SME on \( \Omega \), then there is a sequence \( u_j \) of positive solutions of 1.1 with \( u_j \to u \) uniformly on compact subsets of \( \Omega \). Using the gradient estimates of \cite{Sim76} (applied in the case when the functions \( A_i, B \) of \cite{Sim76} satisfy \( A_i(x, u, Du) = D_i u / \sqrt{1 + |Du|^2}, |B(x, u, Du)| \leq C / \sqrt{1 + |Du|^2} \) together with quasilinear elliptic regularity theory \cite[Chapter 10]{GT}, we deduce that, for each \( k \), \( u \) is the \( C^k \) limit of \( u_j \) in a neighborhood of each point where \( u \) is non-zero. So indeed \( G(u) \cap (\Omega \times (0, \infty)) \), \( S(u) \cap (\Omega \times (\mathbb{R}^m \setminus \{ 0 \})) \) are \( C^\infty \) as claimed.

Since we have the local area bounds of Lemma 1 and since each \( S(u_j) \) is stationary as a multiplicity 1 varifold in \( \Omega \times \mathbb{R}^m \), by the Allard compactness theorem for integer multiplicity varifolds there is a subsequence and a limiting integer multiplicity varifold \( V \) which is stationary in \( \Omega \times \mathbb{R}^m \), and by the above discussion \( V \) is smooth in \( \Omega \times (\mathbb{R}^m \setminus \{ 0 \}) \). Furthermore if \( B_\rho^m(x_0) \subset \Omega \) then, for each \( \sigma \in (0, \rho/2) \), \( B_{\rho/2}^m(x_0, 0) \cap \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : |y| \leq \sigma/2 \} \) can be covered by
balls $B_\sigma(y_1,0), \ldots, B_\sigma(y_N,0)$ with $N \leq C(\rho/\sigma)^n$, and the upper bound of Lemma 2.3(i) gives $H^{n+m-1}(S(u_j) \cap B_\sigma(y_k)) \leq C\sigma^{n+m-1}$ for each $k$, so $H^{n+m-1}(S(u_j) \cap \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |y| \leq \sigma/2\}) \leq C\rho^n\sigma^{m-1} \to 0$ as $\sigma \to 0$. Since by the discussion above we also know the $S(u_j)$ converges to $S(u)$ in the $C^k$ sense on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |y| > \sigma/2\}$ for each $\sigma > 0$, we thus have that $S(u)$ is the varifold limit of $S(u_j)$ (assuming $S(u)$ is viewed as a multiplicity 1 varifold in $\mathbb{R}^{n+m}$), so indeed $S(u) = V$ and $S(u)$ is stationary as claimed.

To complete the proof we have to check sing $S(u) = u^{-1}(0) \times \{0\}$. Certainly sing $S(u) \subset u^{-1}\{0\} \times \{0\}$ because, as already mentioned, $S(u)$ is smooth in $\Omega \times (\mathbb{R}^n \setminus \{0\})$. So we have only to check that no point $(x,0) \in u^{-1}\{0\} \times \{0\}$ is a regular point of $S(u)$ and without loss of generality we just check the case $x = 0$. If $0$ is a regular point of $S(u)$ then there is $\sigma > 0$ with $S = S(u) \cap B_\sigma^{n+m}(0)$ a smooth $(n+m-1)$-dimensional embedded manifold of $\mathbb{R}^n \times \mathbb{R}^m$. Evidently, since $S$ is invariant under orthogonal transformations of the last $m$ variables, $T_xS + (\mathbb{R}^{n+1} \times \{0\}) = \mathbb{R}^{n+m}$ for each $x \in S \cap \{(\mathbb{R}^{n+1} \times \{0\}) \setminus (\mathbb{R}^n \times \{0\})$, and by 2.4 each $x \in S \cap (\mathbb{R}^n \times \{0\})$ is the limit of a sequence $w_j \in S \cap (\mathbb{R}^{n+1} \times \{0\})$, so in fact $T_xS + (\mathbb{R}^{n+1} \times \{0\}) = \mathbb{R}^{n+m}$ for each $x \in S \cap (\mathbb{R}^{n+1} \times \{0\})$.

Therefore by transversality theory $S$ intersects $\mathbb{R}^{n+1} \times \{0\}$ transversely with intersection a smooth embedded $n$-dimensional submanifold, but clearly $S \cap (\mathbb{R}^{n+1} \times \{0\}) = (G(u) \cup G(-u)) \cap B_\sigma^{n+1}(0)$, so $(G(u) \cup G(-u)) \cap B_\sigma^{n+1}(0)$ is a smooth embedded $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. By Remark 2.4 the tangent space of this submanifold at $0$ is distinct from $\mathbb{R}^n \times \{0\}$ and hence, again by transversality theory and taking a smaller $\sigma$ if necessary, we see that $B_\sigma^{n+1}(0) \cap (G(u) \cup G(-u)) \cap (\mathbb{R}^n \times \{0\})$ is a smooth connected embedded $(n-1)$-dimensional manifold $\Gamma$ and $B_\sigma^{n+1}(0) \cap G(u) \setminus \Gamma$ is connected. But $B_\sigma^{n+1}(0) \cap G(u) \setminus \Gamma$ can be written as the disjoint union $(B_\sigma^{n+1}(0) \cap \text{graph}(u|_{U_+})) \cup (B_\sigma^{n+1}(0) \cap \text{graph}(u|_{U_-}))$, where $U_\pm$ are the two connected components of $B_\sigma^{n+1}(0) \setminus \Gamma$, and, by continuity of $u$ at $0$, both sets in this union are non-empty, contradicting the connectedness of $B_\sigma^{n+1}(0) \cap G(u) \setminus \Gamma$.  

**Proof of Lemma 2.3:** First we prove the upper bound in (i). We can assume that $u$ is a regular solution, otherwise apply the argument to an approximating sequence of regular solutions $u_j$. Replace $\zeta$ in the weak form 1.1’ of 1.1 by $\gamma(u)\zeta$, where $\zeta \in C_0^1(B_\rho(y))$ is non-negative and $\gamma(t)$ is the piecewise linear function $\mathbb{R} \to \mathbb{R}$ which is zero for $t \leq (u(x_0) - \rho/2)_+$, slope 1 for $t \in (u(x_0) - \rho/2, u(x_0) + \rho/2)$ and $\gamma(t) = \rho$ for $t \geq u(x_0) + \rho/2$. Then since

$$\sum_{i=1}^n \frac{D_iu}{\sqrt{1+|Du|^2}} D_iu \geq \sqrt{1+|Du|^2} - 1,$$

we deduce that

$$\int_{\{|x| < \rho/2 < u(x) - u(x_0) < \rho/2\}} \zeta \sqrt{1+|Du|^2} \, dx \leq \int_{B_\rho(y)} (m\zeta + \gamma(u)\sum_{i=1}^n \frac{D_iu}{\sqrt{1+|Du|^2}} D_i\zeta) \, dx \leq C\rho^n + \rho \int_{B_\rho(y)} |D\zeta| \, dx,$$

and choosing $\zeta$ to be a standard cut-off function in $B_\rho(y)$ with $\zeta \equiv 1$ in $B_{\rho/2}(y)$ we then obtain the desired bound.

Next we observe that since the symmetric graph $S(u)$ of $u$ is a minimal hypersurface, then the lower bound 2.2 applies to give

$$H^{n+m-1}(S(u) \cap Q(B_\sigma^{n+m}(y,u(x_0),0))) \geq \omega_{m+n-1}\sigma^{m+n-1}$$

for any orthogonal transformation $Q$ of $\mathbb{R}^{m+n}$ which acts as the identity on the first $n$-coordinates (i.e. the $x$ coordinates) and any $\sigma \in (0,\rho)$, where $\omega_{m+n-1}$ is the measure of the unit ball in $\mathbb{R}^{m+n-1}$. Also, using the symmetry of $S(u)$, we see that

$$\{(x,\xi) : |x-y| < \rho/4, u(x_0) - \rho/4 < |\xi| < u(x_0) + \rho/4\} \subset S_{\rho/2}(x_0)$$
contains pairwise disjoint balls \( Q_j B^{n+m}_{\rho/4}(y, u(x_0), 0) \), where \( Q_j, j = 1, \ldots, N \), are orthogonal transformations of \( \mathbb{R}^{n+m} \) which act as the identity in the first \( n \) coordinates and
\[
N \geq \max\{1, C(u(x_0) + \rho)^{m-1}/\rho^{m-1}\}, \quad C = C(n, m) > 0.
\]

We then have by (1) and (2)
\[
\mathcal{H}^{n+m-1}(S_{\rho/2}(x_0)) \geq C(u(x_0) + \rho)^{m-1}\rho^n,
\]
which is the required inequality (ii).

Finally to prove the lower bound in (i), we consider cases \( u(x_0) \geq \rho/4, u(x_0) < \rho/4 \). If \( u(x_0) \geq \rho/4 \), the mean curvature \( H \) of graph \( G \) in \( S_{\rho/8}(y) \) satisfies \( \rho|H| \leq (m-1)\rho/u \leq 8(m-1) \), and a standard consequence of the monotonicity inequality for surfaces of such bounded mean curvature is exactly that
\[
e^{8(m-1)\sigma/\rho \sigma^{-n}}\mathcal{H}^n(G \cap S_\sigma(y)) \text{ is increasing for } \sigma \in (0, \rho/8],
\]
and so the inequality
\[
\int_{S_{\rho/2}(y)} \sqrt{1 + |Du|^2} \, dx \geq \int_{S_{\rho/8}(y)} \sqrt{1 + |Du|^2} \, dx \geq C^{-1}\rho^n
\]
follows. On the other hand if \( u(x_0) < \rho/4 \) then \( u < \rho/4 + \rho/2 = 3\rho/4 < \rho \) in \( S_{\rho/2} \) and we can use the bound (ii) to give
\[
C\rho^{m-1} \int_{S_{\rho/2}(y)} \sqrt{1 + |Du|^2} \, dx \geq \int_{S_{\rho/2}(y)} \sqrt{1 + |Du|^2} u^{m-1} \, dx \geq C^{-1}\rho^{m+n-1},
\]
and so again
\[
\int_{S_{\rho/2}(y)} \sqrt{1 + |Du|^2} \, dx \geq C^{-1}\rho^n.
\]
Thus the proof of Lemma 2.3 is complete. \( \square \)

3 H" older Continuity

Here we establish H"older estimates for regular and singular solutions of the SME (equation 1.1). These will be important in the proof of both the existence result in §4 and also in the proof of the gradient estimate in §8.

3.1 Theorem (H"older Continuity). Let \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfy 1.1 in \( \Omega \) and \( (u - \varphi)|\partial\Omega = 0 \), where \( \varphi \) is \( C^{1,1}(\mathbb{R}^n) \) and \( 0 \leq \varphi \leq u \) in \( \Omega \). Then \( u \) is H"older continuous with exponent \( \frac{1}{2} \) on \( \overline{\Omega} \), and in fact
\[
|u(x) - u(y)| \leq C|x - y|^\frac{1}{2}, \quad x, y \in \overline{\Omega},
\]
where \( C = C(M, n); C \) does not depend on \( \Omega \). Also
\[
\sup_{\Omega} |D(u - \varphi)^2| \leq C
\]
where again \( C \) depends only on \( M, n \).

Before giving the proof, we observe that the above theorem directly implies a local interior H"older estimate for solutions of 1.1 and a local bound on the gradient of \( u^2 \):

3.2 Corollary. If \( u \) is a regular or singular solution of 1.1 in \( B_{\rho}(z) \) then \( u \) is locally H"older continuous with exponent \( \frac{1}{2} \) in \( B_{\rho}(z) \), and
\[
|u(x) - u(y)| \leq C|x - y|^\frac{1}{2}, \quad x, y \in B_{\rho/2}(z)
\]
\[
\sup_{B_{\rho/2}(z)} |Du^2| \leq C,
\]
where \( C \) depends only on \( n, m \) and \( M/\rho \), where \( M \) is any upper bound for \( u \) on \( B_{\rho}(z) \).

Remark: We note also that the above corollary implies that the gradient of \( u \) is bounded on the subset of \( B_{\rho/2}(z) \) where \( u(x) \geq \varepsilon \) for each \( \varepsilon > 0 \).
Proof of the Corollary 3.2: We can suppose that $u$ is a regular solution on $B_\rho(z)$ (otherwise uniformly approximate $u$ by positive solutions $u_j$ and apply the estimates to $u_j$).

Choose $\psi$ to be any non-negative smooth function with $\psi \equiv 0$ on $B_{\rho/2}(z)$, $\psi|\partial B_\rho(z) = 2 \sup_{B_\rho(z)} u$ and

$$ |D^j\psi| \leq C(n)\rho^{-j} \sup_{B_\rho(z)} u \quad \text{for } j = 1, 2,$$

and then apply 3.1 with $\tilde{\Omega}$ in place of $\Omega$, where $\tilde{\Omega} = \{ x \in B_{\rho}(z) : u(x) > \psi(x) \}$. The lemma gives $\sup_{B_{\rho/2}(z)} |Du|^2 \leq C$ (where $C$ depends only on $\rho^{-1}M$ and $n$), which implies the Hölder estimate $|u(x) - u(y)| \leq C|x - y|^{1/2}$, $x, y \in B_{\rho/2}(z)$. □

Proof of the Theorem 3.1: We use a modification of the method of [KS89], which was used to establish such Hölder estimates for solutions of $M(u) = H(x, u)$ in the case when $H(x, z)$ is increasing in $z$, which is the correct sign for application of the maximum principle. The special form of the right side, including in particular the factor $1/\sqrt{1 + |Du|^2}$, allows the method of [KS89] to be successfully modified to the present setting, even though $1/u$ is a decreasing function of $u$, as we now show.

We let $\Delta_G$ denote the Laplace-Beltrami operator on the graph $G = \{(x, u(x)) : x \in \Omega \}$ expressed in the natural coordinates $x \in \Omega$: thus using the notation $v = (1 + |Du|^2)^{1/2}$, we let $\nu = (\nu_1, \ldots, \nu_{n+1}) = v^{-1}(-Du, 1)$ be the upward pointing unit normal of $G(u)$ and $\Delta_G \psi = v^{-1} \sum_{i,j=1}^n D_i(v(\delta_{ij} - \nu_i \nu_j)D_j \psi)$. Keeping in mind the classical identity

$$ v^{-1} \sum_{i=1}^n D_i(v(\delta_{ij} - \nu_i \nu_j)) = H \nu_j $$

for the mean curvature $H = -\sum_{i=1}^n D_i \nu_i (= (m-1)\nu_{n+1}/u$ by 1.1), we see that this can in fact alternatively be written

$$ \Delta_G \psi = (\delta_{ij} - \nu_i \nu_j)D_i D_j \psi + (m-1)\nu_{n+1}/u \nu_j D_j \psi, $$

where, here and subsequently, repeated indices are summed from 1 to $n$. Using the abbreviations $g^{ij} = \delta_{ij} - \nu_i \nu_j$, $u_k = D_k u$, $u_{ik} = D_k D_i u$, we also directly compute

$$ \Delta_G \nu_{n+1} = v^{-1} D_j(v g^{ij} D_i \nu_{n+1}) = -v^{-1} D_j(v^{-1} g^{ij} u_k/v \nu_{ki}) $$

$$ = -u_k/v^2 D_j(v^{-1} g^{ij} u_{ki}) - v^{-3} g^{ij} g^{k\ell} u_{ki} u_{\ell j} \quad \text{(as } D_j(vk/v) = v^{-1} g^{k\ell} u_{\ell j}) $$

$$ = -u_k/v^2 D_j(v^{-1} g^{ij} u_{ki}) - |A|^2 \nu_{n+1} \quad \text{(where } |A|^2 = v^{-2} g^{ij} g^{k\ell} u_{ki} u_{\ell j}) $$

$$ = -u_k/v^2 D_j(D_k(u_k/v)) - |A|^2 \nu_{n+1} \quad \text{(since } D_k(u_k/v) = v^{-1} g^{ij} u_{ki}) $$

$$ = -u_k/v^2 D_k(D_j(u_j/v)) - |A|^2 \nu_{n+1} $$

$$ = -u_k/v^2 D_k((m-1)\nu_{n+1}/u) - |A|^2 \nu_{n+1} \quad \text{(by 1.1)} $$

$$ = \nu_{n+1} D_k((m-1)\nu_{n+1}/u) - |A|^2 \nu_{n+1}. \quad \text{(|A|^2 = v^{-2} g^{ij} g^{k\ell} u_{ik} u_{j\ell} is geometrically the squared length of the second fundamental form of G).} $$

Thus, in summary,

$$ \Delta_G \nu_{n+1} + |A|^2 \nu_{n+1} = (m-1)\nu_{n+1} \nu_j D_j(\nu_{n+1}/u). \quad \text{(1)} $$
Now we define
\[ \eta = e^{K(u - \varphi)} - 1 \quad (K > 0 \text{ to be chosen}) \]
and let \( M = \sup_{\varepsilon + \nu_n+1} \) (where \( \varepsilon > 0 \) will be allowed to approach 0 shortly) and we observe that then \( \eta - M(\varepsilon + \nu_n+1) \) has a maximum value of 0 which is attained at some point \( x_0 \in \Omega \). Thus
\[ D(\eta - M(\varepsilon + \nu_n+1))(x_0) = 0 \text{ and } \Delta_G(\eta - M(\varepsilon + \nu_n+1))(x_0) \leq 0. \]

On the other hand we can directly compute \( \Delta_G(\eta - M(\varepsilon + \nu_n+1))(x_0) \). In this computation we let \( h(u) = (m - 1)/u \) (so the mean curvature \( H = -Du \nu_1 \) of \( G \) is just \( h(u)\nu_{n+1} \)), \( g^{ij} = (\delta_{ij} - \nu_i \nu_j) \), subscripts (like \( i, j \) in \( \varphi_i, \varphi_{ij} \)) denote partial derivatives, and we use the summation convention that repeated indices are summed from 1 to \( n \):

\[ \Delta_G(\eta - M(\varepsilon + \nu_n+1)) = g^{ij} \eta_{ij} + \nu_{n+1} \nu_j \eta_j - M \Delta_G \nu_{n+1} \]
\[ \geq g^{ij} \eta_{ij} + \nu_{n+1} \nu_j \eta_j - M \nu_{n+1} \nu_j D_j (h \nu_{n+1} + M \nu_{n+1}) - \nu_{n+1} \nu_j \eta_j D_j h \]
\[ = K^2 e^{K(u - \varphi)} g^{ij} (u_i - \varphi_i) (u_j - \varphi_j) + K e^{K(u - \varphi)} g^{ij} (u_{ij} - \varphi_{ij}) \]
\[ - \nu_{n+1} \nu_j \eta_j D_j h + \nu_{n+1} \nu_j D_j (h - M \nu_{n+1}) \]
\[ = K^2 e^{K(u - \varphi)} g^{ij} (u_i - \varphi_i) (u_j - \varphi_j) - K e^{K(u - \varphi)} g^{ij} \varphi_{ij} \]
\[ + K e^{K(u - \varphi)} - \nu_{n+1} \nu_j D_j h + \nu_{n+1} \nu_j \eta_j D_j (h - M \nu_{n+1}) \]
where we used the fact that \( g^{ij} u_{ij} = h \) (by 1.1). Now at the point \( x_0 \) where \( \eta - M(\varepsilon + \nu_n+1) \) has its maximum value of zero, we have \( D_j (\eta - M \nu_{n+1}) = 0 \) and \( \eta - M \nu_{n+1} = M \varepsilon \), so
\[ \nu_{n+1} \nu_j D_j (h - M \nu_{n+1}) = M \varepsilon \nu_{n+1} \nu_j D_j h = M(m - 1)\varepsilon |Du|^2 u^{-2} v^{-2} \geq 0, \]
and thus the crucial remaining point is in the sign of the term \( K e^{K(u - \varphi)} h - \nu_{n+1} \nu_j D_j h \); since \( e^{Kt} - 1 \leq Kte^{Kt} \) for \( t \geq 0 \) (which in particular guarantees \( \eta \leq K(u - \varphi) e^{K(u - \varphi)} \leq Kue^{K(u - \varphi)} \)), and since \( \nu_{n+1} \nu_j D_j h = (m - 1)u^{-2} v^{-2} |Du|^2 \), we see that in fact
\[ K e^{K(u - \varphi)} - \nu_{n+1} \nu_j D_j h \geq Khe^{K(u - \varphi)} \left( 1 - \frac{|Du|^2}{v^2} \right) = Khe^{K(u - \varphi)} v^{-2} \geq 0. \]
Thus, at the point \( x_0 \) where \( \eta - M(\varepsilon + \nu_n+1) \) takes its zero maximum value, (2) gives
\[ 0 \geq \Delta_G(\eta - M(\varepsilon + \nu_n+1)) \geq K^2 e^{K(u - \varphi)} g^{ij} (u_i - \varphi_i) (u_j - \varphi_j) - K e^{K(u - \varphi)} g^{ij} \varphi_{ij} \]
\[ = K^2 e^{K(u - \varphi)} \left( \frac{|Du|^2}{1 + |Du|^2} - 2 \frac{u_{ij} \varphi_{ij}}{1 + |Du|^2} + g^{ij} \varphi_{ij} \right) - K e^{K(u - \varphi)} g^{ij} \varphi_{ij} \]
\[ \geq K e^{K(u - \varphi)} \left( K \left( \frac{|Du|^2}{1 + |Du|^2} - \gamma \frac{2|Du|}{1 + |Du|^2} \right) - \gamma \right), \]
where \( \gamma = \sup \{ |\varphi_{ij}| + \sum_i \varphi_i \} \), so we conclude
\[ \frac{|Du|^2}{1 + |Du|^2} - \frac{2\gamma |Du|}{1 + |Du|^2} \leq \gamma/K \]
at the point \( x_0 \) where \( \eta/(\varepsilon + \nu_n+1) \) has its maximum. Using Cauchy’s inequality on the left we see that then
\[ \frac{1}{2} \frac{|Du(x_0)|^2}{1 + |Du(x_0)|^2} - 2\gamma^2 \leq \gamma/K, \]
and selecting $K = 4\gamma$ we see that then

$$|Du(x_0)|^2 \leq 8\gamma^2 + 1,$$

hence $|Du(x_0)| \leq 4(\gamma + 1)$. Thus, with the above choice $K = 4\gamma$, we get $M = \sup_{\Omega} \eta/(\varepsilon + \nu_{n+1}) \leq \sup_{\Omega} e^{4\gamma(u-\varphi)/(\varepsilon + (4\gamma + 4)^{-1})}$, so that letting $\varepsilon \downarrow 0$ we have

$$\sup_{\Omega}(|Du|e^{4\gamma(u-\varphi)} - 1) \leq \sup_{\Omega} \left(\nu_{n+1}^{-1}(e^{4\gamma(u-\varphi)} - 1)\right) \leq 4(\gamma + 1) \sup_{\Omega} e^{4\gamma(u-\varphi)}.$$

The required inequality $|D(u-\varphi)^2| \leq C$ now follows because $e^{4\gamma(u-\varphi)} - 1 \geq 4\gamma(u-\varphi)$. □

4 Existence Results

In this section $n \geq 2$ and we show there exists quite a rich class of singular solutions of the SME (i.e. 1.1) on any uniformly convex $C^{2,\alpha}$ domain $\Omega \subset \mathbb{R}^n$.

We start with the following definitions:

4.1 Definitions: Boundary data $\varphi \in C^0(\partial\Omega)$ is said to be strongly positive if there exists $\eta > 0$ such that $\inf_{\Omega} u \geq \eta$, whenever $u \geq 0$ is a $C^0(\overline{\Omega}) \cap C^2(\Omega)$ solution of 1.1 with $\min_{\partial\Omega} u \geq \varphi$, and $\varphi$ is said to be non-solvable if there exists no $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution of 1.1 with $u|\partial\Omega = \varphi$.

Of course there exists such non-solvable data; indeed if $\varepsilon > 0$ and $\sup_{\Omega} \varphi < \varepsilon$ then by the maximum principle any solution $u$ of 1.1 with boundary data $\varphi$ would also satisfy $\sup_{\Omega} u \leq \varepsilon$, and for small enough $\varepsilon = \varepsilon(m,\Omega) > 0$ this is impossible by Remark 2.4.

There also exists strongly positive data for any given $C^2$ domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$—indeed there is $K > 0$ such that any $\varphi$ with $\min_{\partial\Omega} \varphi \geq K$ is strongly positive, as one easily checks as follows:

First consider solutions of the SME 1.1 on $\{x \in \mathbb{R}^n : |x| > 1\}$ and which are functions of $r = |x|$, so $u_0(x) = \psi(r)$ with $\psi \in C^\infty(0, \infty)$. One can check using ODE theory that such solutions exist and indeed there is a unique such solution which satisfies

$$\psi(1+) = \lim_{r \to 1^+} \psi(r) = 0, \quad \psi'(1+) = +\infty, \quad \text{and} \quad \lim_{r \to \infty} |\psi(r) - \sqrt{\frac{m-1}{n-1}} r| = 0,$$

With this solution $\psi$ and $\lambda > 0$, let

$$\psi_\lambda(r) = \lambda \psi(r/\lambda), \quad r \geq \lambda.$$

Let $u$ be a $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution of 1.1, and let

$$\beta = \sup_{r > 1} \psi(r)/r \quad \text{(which is the same as} \quad \beta = \sup_{r > \lambda} \psi_\lambda(r)/r \quad \text{for each} \quad \lambda > 0).$$

We remark that actually $\beta = (\frac{m-1}{n-1})^{1/2}$ if $m + n - 1 \geq 7$, because the solution $\psi(r)$ remains below $(\frac{m-1}{n-1})^{1/2} r$ for all $r$ in this case; but in any case for any $m, n \geq 2$ such a finite $\beta$ exists because $\frac{\psi(r)}{r} \to (\frac{m-1}{n-1})^{1/2}$ as $r \uparrow \infty$. Now assume $u = \varphi$ on $\partial\Omega$, where

$$\varphi(x) > \sup_{x_0 \in \Omega} \beta |x - x_0|$$

Then evidently $\varphi(x) \geq \psi_\lambda(|x-x_0|)$ for each $(x, \lambda) \in \partial\Omega \times (0, \infty)$ with $|x-x_0| > \lambda$, and we can select the largest $\lambda > 0$ such that $u(x) \geq \psi_\lambda(|x-x_0|)$ for every $x \in \Omega \cap \{y : |y-x_0| > \lambda\}$, and, for such a $\lambda$, we have $\xi \in \Omega \cap \{x : |x-x_0| > \lambda\}$ with $u(\xi) = \psi_\lambda(|\xi-x_0|)$ and $u(x) \geq \psi_\lambda(|x-x_0|)$ for all $x$ in some neighborhood of $\xi$. But then, by taking the difference of the SME for $u(x)$ and the SME for $\psi_\lambda(|x-x_0|)$, we see that then $v(x) = u(x) - \psi_\lambda(|x-x_0|)$ has a local minimum value of zero at $x = \xi$.
and in a neighborhood of $\xi$ it satisfies an equation of the form $\sum_{i,j=1}^{n} a_{ij} D_i D_j \psi + \sum_{i=1}^{n} b_i D_i \psi + c \psi = 0$, with $(a_{ij})$ positive definite and $a_{ij}, b_i, c$ continuous. This evidently contradicts the Hopf maximum principle. So we have proved that any $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution of 1.1 with $u(x) \geq \sup \{ \beta |x - x_0| : x_0 \in \Omega \}$ for each $x \in \partial \Omega$ automatically satisfies $u(x) \geq \sup \{ \psi_\lambda(|x - x_0|) : x_0 \in \Omega, \lambda \in (0, \infty) \}$ for each $x \in \Omega$. Since $\sup_{\lambda \in (0, \infty)} \psi_\lambda(|x - x_0|) \geq \left( \frac{m-1}{n-1} \right)^{1/2} |x - x_0|$ we see that

$$\inf_{x \in \partial \Omega} \sup_{x_0 \in \Omega, \lambda \in (0,1)} \psi_\lambda(|x - x_0|) \geq \inf_{x \in \partial \Omega} \sup_{x_0 \in \Omega} \left( \frac{m-1}{n-1} \right)^{1/2} |x - x_0| \geq \gamma$$

for suitable $\gamma = \gamma(m, \Omega) > 0$, so indeed any continuous data $\varphi$ satisfying 4.3 is strongly positive.

Now, we can state the main existence result of this section

**4.4 Theorem.** Let $\Omega$ be a uniformly convex $C^{2,\alpha}$ domain in $\mathbb{R}^n$, $n \geq 2$, and $\{ \varphi_\lambda \}_{\lambda \in [0,1]} \subset C^{2,\alpha}(\overline{\Omega})$ such that $(x, \lambda) \rightarrow \varphi_\lambda(x)$ is $C^0$ map $\overline{\Omega} \times [0,1] \rightarrow \mathbb{R}$, $\varphi_1 > \varphi$, where $\varphi$ is any strongly positive data (as in 4.1), and $\varphi_0$ non-solvable (also as in 4.1). Then there is $\lambda \in (0,1)$ such that there is a singular solution $u$ of 1.1 with $u = \varphi_\lambda$ on $\partial \Omega$ and dist($\text{sing} u, \partial \Omega$) > 0.

**Remark:** With some more effort it is possible to replace the uniform convexity hypothesis with the hypothesis that $\partial \Omega$ is mean convex, but we shall not discuss that here.

The proof of Theorem 4.4 involves an application of a standard Leray-Schauder degree argument, aided by the Hölder continuity results of the previous section. The Leray-Schauder component of the proof is presented in the following lemma:

**4.5 Lemma.** Let $V$ be an open (not necessarily bounded) subset of a Banach space $B$ and let $T_\lambda : \overline{V} \rightarrow B$, $0 \leq \lambda \leq 1$ be such that the map $(x, \lambda) \mapsto T_\lambda(x)$, $(x, \lambda) \in \overline{V} \times [0,1]$, is a continuous compact map (i.e. a continuous map taking bounded subsets of $\overline{V} \times [0,1]$ into compact subsets of $B$), and assume

(a) $T_1$ is a constant map $\overline{V} \rightarrow B$ with constant value $p_0 \in V$
(b) $T_0$ has no fixed points in $\overline{V}$
(c) $\sup \{ \|u\| : u \in \bigcup_{\lambda \in [0,1]} \{ v \in \overline{V} : T_\lambda(v) = v \} \} < \infty$.

Then there is a $u \in \partial V$ and a $\lambda \in (0,1)$ with $T_\lambda(u) = u$.

**Proof of Lemma 4.5:** The proof is a standard application of the Leray-Schauder degree of completely continuous maps (i.e. maps of the form $t \rightarrow T$, where $T$ is continuous and compact and where $t$ is the identity map on $B$). Specifically we use the fact (see e.g. [Dei85]) that if $U$ is a bounded open subset of a Banach space $B$, then there is a well-defined topological degree $d$ for completely continuous transformations of $\overline{U}$ into $B$ as follows:

(i) If $T : \overline{U} \rightarrow B$ is continuous and compact then the topological degree $d(t, T, U, q)$ is a well-defined integer for $q \in B \setminus (t, T)(\partial U)$, $d(t, T, U, q)$ remains constant for $q$ in a given connected component of $B \setminus (t, T)(\partial U)$, and $d(t, T, U, q) \neq 0 \Rightarrow q \in (t, T)(U)$.

(ii) If $T_\lambda : \overline{U} \rightarrow B$ are given for $\lambda \in [0,1]$ such that the map $(p, \lambda) \mapsto T_\lambda(p)$, $(p, \lambda) \in \overline{U} \times [0,1]$, is a compact continuous map, and if $q \in B \setminus (\bigcup_{\lambda \in [0,1]} (t, T_\lambda)(\partial U))$, then $d(t, - T_\lambda, U, q) = d(t, - T_0, U, q)$ for each $\lambda \in [0,1]$.

(iii) If $T$ is a constant map with constant value $q_0 \in U$, then $d(t, - T, U, 0) = 1$.

To prove the lemma, we first use hypothesis (c) to choose $R > 0$ such that

$$R > \sup \left\{ \|u\| : u \in \overline{V} \text{ and } T_\lambda(u) = u \text{ for some } \lambda \in [0,1] \right\}$$
and then we apply the above properties (i), (ii), (iii) of the topological degree with \( U = V \cap \{ u \in B : \| u \| < R \} \) as follows:

Either \( \exists \lambda \in [0, 1] \) and \( u \in \partial (V \cap \{ u : \| u \| < R \}) \) with \( T_{\lambda}(u) = u \), or else there is no such \( \lambda \). But by property (ii) of the degree the latter alternative implies \( d(u - T_{\lambda}, V \cap \{ u : \| u \| < R \}, 0) \) is constant for \( \lambda \in [0, 1] \). By hypothesis (a) and property (iii) the constant value must be 1, and hence, by property (i), \( T_{0}(u) = u \) for some \( u \in V \cap \{ u : \| u \| < R \} \), contradicting hypothesis (b) of the lemma. Thus the former alternative holds, and, since \( \| u \| < R \) whenever \( T_{\lambda}(u) = u \) with \( u \in \nabla \) and \( \lambda \in [0, 1] \), we deduce that there is a \( \lambda \) with \( T_{\lambda}(u) = u \) for some \( u \in \partial V \).

Proof of Theorem 4.4: Let \( \delta \in (0, 1] \). We apply the Lemma 4.5 with \( B = C^{1, \alpha}(\Omega) \) and \( V = \{ u \in C^{1, \alpha}(\Omega) : u > \delta \} \). Let \( a_{ij}(p) = \delta_{ij} - p_{i}p_{j} / (1 + |p|^2) \), so that

\[
M(u) = \frac{1}{\sqrt{1 + |Du|^2}} a_{ij}(Du) D_{ij} u,
\]

and consider the following family of problems for given \( v \in S_{\delta} \):

\[
Q_{\lambda}^{v} = \begin{cases} 
  a_{ij}(Dv) D_{ij} u = \frac{(m - 1)}{v}, & u_{|\partial \Omega} = \varphi_{2\lambda}, & 0 \leq \lambda \leq \frac{1}{2} \\
  a_{ij}(Dv) D_{ij} u = \frac{(m - 1)}{v}, & u_{|\partial \Omega} = \varphi_{1} + 4(\lambda - \frac{1}{2})(K - \varphi_{1}), & \frac{1}{2} < \lambda \leq \frac{3}{4} \\
  a_{ij}(Dv) D_{ij} u = \frac{4(m - 1)(1 - \lambda)}{v}, & u_{|\partial \Omega} = K, & \frac{3}{4} < \lambda \leq 1
\end{cases}
\]

where \( K \) is to be chosen (large).

Each of these equations is a inhomogeneous linear second order elliptic equation with Hölder continuous coefficients and with no first or zero order terms, so by the theory of such equations (see [GT]) each problem has a unique solution in \( C^{2, \alpha}(\Omega) \). Define the map \( T_{\lambda} : \nabla \to C^{1, \alpha}(\Omega) \) by

\[ T_{\lambda}(v) = u, \]

where \( u \) is the solution to the problems above. As the coefficients of \( Q_{\lambda}^{v} \) are in \( C^{0, \alpha}(\Omega) \), by virtue of global Schauder estimates in [GT, Chapter 6] \( T_{\lambda} \) maps bounded sets in \( \nabla \) to bounded sets in \( C^{2, \alpha}(\Omega) \), which are precompact in \( C^{1, \alpha}(\Omega) \). Therefore, \( T_{\lambda} \) is a compact mapping. The continuity and compactness of \( T : \nabla \times [0, 1] \to C^{1, \alpha}(\Omega) \) (with \( T(u, \lambda) = T_{\lambda}(u) \)) follows from a similar standard argument. Also by the maximum principle for elliptic equations, \( T_{1} \equiv K \). By assumption \( T_{0} \) has no fixed points in \( \nabla \). Using exactly the same barrier argument discussed in [GT, Chapter 14] (capitalizing on the fact that the fixed points satisfy \( M u = \frac{m-1}{u \sqrt{1 + |Du|^2}} \) and

\[
0 < \frac{m-1}{u \sqrt{1 + |Du|^2}} \leq \delta^{-1} \frac{m-1}{\sqrt{1 + |Du|^2}} \text{ for } u \in \nabla
\]

the boundary gradient estimate

\[
(1) \quad \sup_{\partial \Omega} |Du| \leq C,
\]

holds for the fixed points of \( T_{\lambda} \), where \( C = C(m, n, \delta, \sup_{\mu} |\varphi_{\mu}|) \).

Then by applying the gradient estimates of [Sim76] (in the case when the functions \( A_{j}, B \) of [Sim76] satisfy \( A_{j}(x, u, Du) = D_{j}u / \sqrt{1 + |Du|^2} \) and \( |B(x, u, Du)| \leq C / \sqrt{1 + |Du|^2} \)), we have

\[
(2) \quad \sup_{\Omega} |Du| \leq C, \quad C = C(M, \Omega, \delta),
\]

where \( M \) is any upper bound for \( \sup_{\lambda \in [0, 1]} |\varphi_{\lambda}|_{C^{2}(\Omega)} \). (Alternatively one can use the argument of [DH90] for this.)

Combining (2) and the regularity theory for quasilinear elliptic equations in [GT] ensures that

\[
(3) \quad \sup_{\{ u \in \nabla : T_{\lambda}(u) = u \text{ for some } \lambda \in [0, 1] \}} \| u \| < \infty.
\]
Now we can apply Lemma 4.5 to conclude that there is \( u_\delta \in \partial V \) and \( \lambda_\delta \in (0, 1] \) with \( T_{\lambda_\delta}(u_\delta) = u_\delta \).

Next we claim that for suitably large \( K \) (depending on \( \delta \)) we can arrange that

\[
\text{(4)} \quad \text{if } \delta < \eta \text{ with } \eta = \eta(\varphi_1) > 0 \text{ as in 4.1 then there are no fixed points of } T_\lambda \text{ on } \partial V \text{ for } \lambda \in [\frac{1}{2}, 1]
\]

(so that all fixed points of \( T_\lambda \) occur for \( \lambda \in (0, \frac{1}{2}] \)).

To see (4), first consider the case \( \lambda \in [\frac{1}{2}, 1] \). Assuming \( K > \sup \varphi_1 \), we have

\[
\varphi_1 + 4(\lambda - \frac{1}{2})(K - \varphi_1) \geq \varphi_1,
\]

and so \( u \geq \eta > \delta \) by 4.1, contradicting the fact that \( u \in \partial V \).

Next we consider the case where \( \lambda \in (\frac{2}{3}, 1) \). Recall that \( a_{ij}(p) = \delta_{ij} - p_i p_j / (1 + |p|^2) \), so we have

\[
\sum_{i=1}^{n} a_{ii} > n - 1.
\]

Supposing that there is a fixed point \( u \in \partial V \), after dividing by \( K \) in 1.1 we get

\[
\sum_{i,j=1}^{n} a_{ij}(Du) D_{ij}(\frac{u}{K}) = \frac{4(1 - \lambda)(m - 1)}{uK} , \quad u \in \partial V.
\]

Assume without loss of generality that \( 0 \in \Omega \) and let \( \theta = \frac{1}{2} \text{diam}^{-2}(\Omega) \). Then \( w(x) = \frac{u}{K} - \theta |x|^2 \) satisfies

\[
w|_{\partial \Omega} \geq \frac{1}{2} \quad (\text{since } u \geq K \text{ on } \partial \Omega)
\]

and, by (5) and (6),

\[
\sum_{i,j=1}^{n} a_{ij}(Du) D_{ij}w < \frac{4(1 - \lambda)(m - 1)}{uK} - 2(n - 1)\theta.
\]

Thus if \( K > \max \left\{ \frac{2(n-1)\theta}{(m-1)\delta}, \sup \varphi_1 \right\} \) then \( a_{ij}(Du) D_{ij}w < 0 \) in \( \Omega \), so the maximum principle implies that

\[
\inf_{\Omega} w \geq \frac{1}{2}.
\]

In particular, \( \inf_{\Omega} u \geq K/2 > \delta \) contradicting the fact that \( \inf_{\Omega} u = \delta \). Thus (4) is established.

On the other hand Lemma 4.5 implies there is indeed a fixed point in \( \partial V \) of \( T_{\lambda} \) for some \( \lambda \in (0, 1) \), and (4) ensures that this \( \lambda \) is in the interval \((0, 1/2]\), which means that \( u \) satisfies 1.1 with \( u|_{\partial \Omega} = \varphi_\mu \) for some \( \mu \in (0, 1] \). Taking \( \delta = \delta_j < \inf_{\mu \in [0, 1]} \min_{\partial \Omega} \varphi_\mu \) with \( \delta_j \to 0 \), we thus see that there are \( u_j \) such that \( u_j \) satisfies 1.1 and \( u_j = \varphi_{\mu_j} \) for some \( \mu_j \in (0, 1) \) and \( \min_{\partial \Omega} u_j = \delta_j \).

We claim that there is \( \eta > 0 \) such that

\[
\text{(7)} \quad u_j(y) \geq \eta \text{ for all } j \text{ and all } y \in \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \eta \}.
\]

This is easily checked using the natural variant of the argument, involving the Hopf maximum principle and the family \( \psi_\lambda \) of ODE solutions of 1.1, mentioned earlier in this section, as follows:

Let \( x_0 \in \partial \Omega \), \( \nu_0 \) be the inward pointing unit normal of \( \partial \Omega \) at \( x_0 \), and take a point \( y_0 = x_0 + t_0 \nu_0 \) on the ray \( \{ x_0 + t \nu_0 : t > 0 \} \) with \( t_0 \) sufficiently large to ensure that the sphere \( S_{y_0} \) with center \( y_0 \) and radius \( t_0 \) satisfies \( S_{y_0} \cap \partial \Omega = \{ x_0 \} \). Then \( \psi_\lambda(x - y_0) \) is a family of solutions defined over the region \( |x - y_0| \geq \lambda \) and, if \( \varepsilon > 0 \) is small enough, each \( \lambda \in (t_0 - \varepsilon, t_0) \) gives a solution of 1.1 which satisfies \( \psi_\lambda \leq \inf_{\mu \in [0, 1]} \varphi_\mu \) on \( \{ x \in \partial \Omega : |x - y_0| \geq \lambda \} \) for each \( \lambda \) in the interval \( (t_0 - \varepsilon, t_0) \). We then claim that the solutions \( u_j \) constructed above must satisfy, for each \( \lambda \) in the interval \( (t_0 - \varepsilon, t_0) \),

\[
\text{(8)} \quad u_j(x) \geq \psi_\lambda(x - y_0), \quad \forall x \text{ with } |x - y_0| \geq t_0 - \lambda.
\]
Otherwise we pick $\lambda_j$ to be the smallest $\lambda$ in the interval $(t_0 - \varepsilon, t_0)$ where this is true. Then, analogous to the argument used above to prove the existence of strongly positive data, we contradict the Hopf maximum principle for the difference $u_j(x) - \psi_{\lambda_j}(|x - y_0|)$. So, since (8) applies for each point $x_0 \in \partial \Omega$, (7) is proved.

Now in the region \{\(x \in \Omega : \text{dist}(x, \partial \Omega) \geq \eta\}\) we have equicontinuity of the $u_j$ by Corollary 3.2, and on the other hand in the region \{\(x \in \Omega : \text{dist}(x, \partial \Omega) < \eta\}\) we obtain the same local gradient estimates (using [Sim76]) as mentioned above (since $u_j \geq \eta$ in this region), hence we also obtain equicontinuity of the $u_j$ \{\(x \in \Omega : \text{dist}(x, \partial \Omega) < \eta\}\}. So there is a subsequence $u_j'$ converging uniformly on $\overline{\Omega}$ to a singular solution $u$ of 1.1 with $\text{sing} \cap \{x \in \Omega : \text{dist}(x, \partial \Omega) \leq \eta\} = \emptyset$ and $u = \varphi_{\lambda}$ for some $\lambda \in (0, 1)$. This completes the proof of Theorem 4.4. \(\Box\)

### 5 Compactness and Regularity Results

In this section we discuss the regularity of limits of the graphs of the solutions of the SME (i.e. equation 1.1) in the open upper half space $x_{n+1} > 0$ by showing that the regularity theory of [SS81] can be applied.

We let

$$G_R = \{ G(u) : \text{with } u > 0 \text{ a C}^2 \text{ solution of 1.1 on } B^n_R \},$$

and

$$\overline{G}_R = \{ \text{varifolds } V \text{ on } B^n_R \times \mathbb{R} \text{ expressible as } V = \lim \left. G_j \right|_{B^n_R \times \mathbb{R}} \text{ for some sequence } u_j \in G_R, \text{ where } G_j = G(u_j) \in G_R \}.$$ 

Of course here $G_j$ is interpreted as the multiplicity 1 varifold corresponding to the graph $G(u_j)$ of $u_j$ and the varifold convergence guarantees the measure convergence of $\mathcal{H}^n \ll G_j$ to a Radon measure on $B^n_R \times \mathbb{R}$. In view of the volume bounds of 2.3(i), the compactness theorem for Radon measures on the space $G_n(B^n_R \times \mathbb{R}) = (B^n_R \times \mathbb{R}) \times G(n, n + 1)$ (where $G(n, n + 1)$ denotes the $n$-dimensional subspaces in $\mathbb{R}^{n+1}$ with the metric $\rho(S, T) = |ps - pr|$) guarantees that every sequence $G_j = G(u_j)$ automatically has a subsequence $G_{j'}$ with $\lim G_{j'} = V$ for some varifold $V$—i.e. for some Radon measure $V$ on $G_n(B^n_R \times \mathbb{R})$. For further discussion of the theory of varifolds we refer e.g. to [Sim83, Chapter 8].

We claim that

$$\mathcal{H}^n(\text{spt } V \cap (B^n_R \times \{0\})) = 0 \text{ for each } V \in \overline{G}_R.$$

To prove this, suppose $V = \lim G(u_j)$ with $u_j > 0$ a solution of 1.1 on $B^n_R$, and on the contrary that $\mathcal{H}^n(K) > 0$, where

$$K = \text{spt } V \cap (B^n_R \times \{0\}).$$

We claim $u_j(x) \to 0$ for $\mathcal{H}^n$-a.e. $(x, 0) \in K$. Otherwise $\limsup u_j(x) > 0$ on a subset of $K$ with positive $\mathcal{H}^n$ measure, hence there would be $\varepsilon_0 > 0$ and a set $K_0 \subset K$ with $\mathcal{H}^n(K_0) > 0$ and $\limsup_{x \in K_0} u_j(x) \geq \varepsilon_0$, for every $x \in K_0$. Also for every $\sigma > 0$ we must have some $\xi_j \in G_j \cap (B^n_\sigma(x) \times [0, \varepsilon_0/2])$ for all sufficiently large $j$ (otherwise, since $G_j \to V$ in the varifold sense on $B^n_R \times \mathbb{R}$, we would have $(x, 0) \notin \text{spt } V$). So then $\{u_j(tx + (1 - t)\xi_j) : t \in [0, 1]\}$ includes every value between $\varepsilon_0/2$ and $3\varepsilon_0/4$ for infinitely many $j$. Thus for each $\sigma > 0$ there are infinitely many $j$ with $G_j \cap (B^n_\sigma(x) \times \{t\}) \neq \emptyset$ for every $t \in [\varepsilon_0/2, 3\varepsilon_0/4]$, so in particular, since $G_j \to V$ locally in the Hausdorff distance sense on $B^n_\sigma \times (0, \infty)$, the vertical segment $\{(x, t) : t \in [\varepsilon_0/2, 3\varepsilon_0/4]\}$ is
contained in \( \text{spt} V \). So \( K_0 \times [\varepsilon_0/2, 3\varepsilon_0/4] \subset \text{spt} V \) contradicting the fact that \( \text{spt} V \cap (B^n_\rho \times (0, \rho)) \) has finite \( n \)-dimensional measure for each \( \rho < R \) by the volume bounds of Lemma 2.3(i).

So indeed \( u_j(x) \to 0 \) for \( \mathcal{H}^n \)-a.e. \( (x, 0) \in K \) and by Egoroff’s theorem there is then a subset \( S \subset K \) of positive measure with \( u_j \to 0 \) uniformly on \( S \). Pick \( (y, 0) \in S \) and \( \rho > 0 \) with \( B^+_{\rho/4}(y, 0) \subset B^n_\rho \times \mathbb{R} \)
and \( \mathcal{H}^n(S \cap B^+_{\rho/4}(y, 0)) > 0 \). Then, with \( \tilde{S} = S \cap B_{\rho/4}(y, 0) \), by Cauchy-Schwarz and the fact that \( \{(x, u_j(x)) : x \in \tilde{S}\} \subset B^+_{\rho/2}(y, u_j(y)) \) for sufficiently large \( j \),

\[
\left( \mathcal{H}^n(\tilde{S}) \right)^2 = \left( \int_{\tilde{S}} (1 + |Du_j|^2)^{1/4} (1 + |Du_j|^2)^{-1/4} \right)^2 \\
\leq \int_{\tilde{S}} (1 + |Du_j|^2)^{1/2} \int_{\tilde{S}} (1 + |Du_j|^2)^{-1/2} \\
\leq C \int_{\tilde{S}} (1 + |Du_j|^2)^{1/2} \int_{\tilde{S}} (1 + |Du_j|^2)^{-1/2} \quad \text{(by 2.3(i))} \\
\leq C \sup_{\tilde{S}} u_j \int_{\tilde{S}} u_j^{-1} (1 + |Du_j|^2)^{-1/2} \\
\leq C \sup_{\tilde{S}} u_j \int_{B^+_{\rho/2}(y)} u_j^{-1} (1 + |Du_j|^2)^{-1/2}
\]

for all sufficiently large \( j \). On the other hand by using the weak version 1.1’ of 1.1 with \( u_j \) in place of \( u \) and with \( \zeta \) a cut-off function with \( \zeta = 1 \) on \( B_{\rho/2}(y) \) and \( \zeta = 0 \) outside \( B_{\rho}(y) \), we conclude that

\[
\int_{B^+_{\rho/2}(y)} u_j^{-1} (1 + |Du_j|^2)^{-1/2} \leq C,
\]
with \( C \) independent of \( j \). Combining the previous two inequalities we have

\[
\left( \mathcal{H}^n(\tilde{S}) \right)^2 \leq C \sup_{\tilde{S}} u_j \to 0,
\]
a contradiction. So 5.1 is proved.

We also define

\[
\overline{G}_\infty = \text{the set of varifolds } V \text{ on } \mathbb{R}^n \times \mathbb{R} \text{ with } V \prec (B^n_R \times \mathbb{R}) \in \overline{G}_R \forall R > 0.
\]

As already mentioned above, by the volume bounds of Lemma 2.3 we can use the compactness theorem for Radon measures to prove that an arbitrary sequence \( G(u_j) \in G_R \) always has a subsequence which converges to \( V \in \overline{G}_R \) in the sense of Radon measures in \((B^n_R \times \mathbb{R}) \times G(n, n + 1)\).

Additionally, since the mean curvature of \( G_j \) is bounded by \( C/\sigma \) in the region \( \{x_{n+1} > \sigma\} \) (\( \sigma > 0 \) arbitrary), by the Allard compactness theorem any \( V \in \overline{G}_R \) is an integer multiplicity varifold with (generalized) mean curvature bounded by \( C/\sigma \) in \( B^n_R \times (\sigma, \infty) \) for each \( \sigma > 0 \).

We proceed to show that the regularity theory of [SS81] can be applied locally in \( B^n_R \times (0, \infty) \).

First take an arbitrary \( C^{3}(\Omega) \) function \( u \) (not necessarily satisfying 1.1), and let \( G = G(u) \), \( H = \) mean curvature of \( G = \sum^n_{i=1} D_i(D_iu/\sqrt{1 + |Du|^2}) \), \( w = -\log \nu_{n+1}, \nu_{n+1} = 1/\sqrt{1 + |Du|^2} \) (so that \( w \geq 0 \) because \( 0 < \nu_{n+1} \leq 1 \)). (1) in the proof of Theorem 3.1 says

\[
\Delta_G \nu_{n+1} + |A_G|^2 \nu_{n+1} = -e_{n+1} \cdot \nabla_G H,
\]
and (using \( \nabla_G w = -\nu^{-1}_{n+1} \nabla_G \nu_{n+1} \)) the correspond identity for \( \Delta_G w \):

\[
\Delta_G w - (|A_G|^2 + |\nabla_G w|^2) = \nu^{-1}_{n+1} e_{n+1} \cdot \nabla_G H,
\]
where \( A_G \) is the second fundamental form of \( G \) and \( \nabla_G \) denotes the gradient on \( G \), so that

\[
|A_G|^2 = \nu^2_{n+1} \sum^n_{i,j=1} g^{ij} D_i D_j u D_k D_\ell u,
\]
where $g^{ij} = \delta_{ij} - \nu_i \nu_j$.

When $u$ is a regular or singular solution of 1.1 we have, in $G(u) \cap (B_R^n \times (0, \infty))$, $H = (m-1)\nu_{n+1}/u$ (which can be expressed $H = (m-1)\nu_{n+1}/x^{n+1}$, and so the identities 5.2 and 5.3 can be written

5.4\hspace{1cm} \Delta_G \nu_{n+1} + |A_G|^2 \nu_{n+1} = -(m-1)\epsilon_{n+1} \cdot \nabla_G ((x^{n+1})^{-1} \nu_{n+1})

5.5\hspace{1cm} \Delta_G w - (|A_G|^2 + |\nabla_G w|^2) = -(m-1)(x^{n+1})^{-2} \epsilon_{n+1} \cdot \nabla_G x^{n+1} - (m-1)(x^{n+1})^{-1} \epsilon_{n+1} \cdot \nabla_G w.

Rewriting 5.5 in the weak form, we have

5.6\hspace{1cm} \int_G (|\nabla_G w|^2 + |A|^2) \zeta^2 = -\int_G 2 \zeta \nabla_G w \cdot \nabla_G \zeta - (m-1) \int_G (x^{n+1})^{-2} \epsilon_{n+1} \cdot \nabla_G x^{n+1} + (x^{n+1})^{-1} \epsilon_{n+1} \cdot \nabla_G w \zeta^2

provided $\zeta \in C^1_{\epsilon}(\mathbb{R}^{n+1})$ with spt $\zeta \cap G$ is a compact subset of $G$. Using the Cauchy-Schwarz inequality we have $2 \zeta \nabla w \cdot \nabla G \zeta \leq (1-\epsilon)\zeta^2 |\nabla_G w|^2 + (1-\epsilon)^{-1} |\nabla G \zeta|^2$ and $(x^{n+1})^{-1} \epsilon_{n+1} \cdot \nabla_G w \zeta^2 \leq \epsilon \zeta^2 + (x^{n+1})^{-1} \epsilon^{-1} \delta^{-2}$, so 5.6 gives

5.7\hspace{1cm} \int_G |A|^2 \zeta^2 \leq \frac{1}{1-\epsilon} \int_G |\nabla_G \zeta|^2 + \epsilon^{-1} \delta^{-2} \int_G \zeta^2

provided spt $\zeta \cap G$ is compact and contained in the region $x^{n+1} \geq \delta$.

The inequality 5.7 enables us to use the regularity theory of SS81 for solutions $u$ in the region $x^{n+1} > 0$, which we now discuss:

Define $F : \mathbb{R}^{n+1}_+ \times \mathbb{R}^n \to \mathbb{R}$ by

$$F(x, x_{n+1}, p) = x_{n+1}^{m-1} |p|$$

With this notation the Symmetric Minimal Surface equation 1.1 can be viewed as the Euler-Lagrange equation for the functional

$$F(G) = \int_G F(x, x^{n+1}, \nu) d\mathcal{H}^n$$

where $\nu = (-Du, 1)/\sqrt{1+|Du|^2}$ is the upward pointing unit normal on $G = G(u)$. If we restrict the domain to where $x_{n+1} \geq \delta$, for some fixed $\delta > 0$, then $F$ satisfies all the properties (1.2) to (1.6) in SS81. Note that, in the region $x_{n+1} > 0$, the inequality 5.7 replaces the $F$-stability in SS81, because 5.7 implies the stability inequality [SS81, (1.17)] required in the proof of regularity results in SS81; in fact 5.7 has an additional factor $(1-\epsilon)^{-1}$ in front of the principal term on the right side, but this causes no complication in the discussion of SS81 so long as we take $\epsilon = \epsilon(n)$ sufficiently small.

Thus with $V = \lim G_j \in \overline{G}_R$ as above, we can apply the regularity and compactness theory of SS81 and in particular Theorem 4 in SS81 implies that $V$ has a singular set in $x_{n+1} > 0$ (i.e. sing $V \cap (B_R \times (0, \infty))$) of codimension at least 7 (empty for $n \leq 6$, discrete for $n = 7$).

We henceforth write

$$\Sigma = \text{reg } V \cap (B_R^n \times (0, \infty)),$$

and observe that, in view of the regularity statements above,

$$\dim \text{sing } \Sigma \leq \max\{n-7, 0\},$$

where $\text{sing } \Sigma \cap (B_R^n \times (0, \infty)) = (\Sigma \setminus \Sigma) \cap B_R^n \times (0, \infty)$.

An additional part of the theory established in SS81 shows that, in the region where $x_{n+1} > 0$, the approximating graphs $G_j$ actually converge in the $C^k$ sense to $\text{reg } V$ for each $k$ locally near points of $\text{reg } V$. More precisely:
5.8 Lemma ("Sheeting Lemma.") Suppose that \( G_j \) is the sequence of graphs of (possibly singular) solutions \( u_j \) of 1.1 converging to \( V \) in the sense of varifolds in the cylinder \( B^n_R \times (0, \infty) \), let \( \Sigma = \text{reg} \ V \cap (B^n_R \times (0, \infty)) \) and let \( (\xi, \tau) \in \Sigma \). Then there is \( \rho > 0 \) and an integer \( j_0 \) such that for all \( j \geq j_0 \) there is an integer \( L \geq 1 \) and \( C^\infty \) functions \( u^1_j < \cdots < u^L_j \) on \( \Sigma \cap B^{n+1}_\rho (\xi, \tau) \) such that

\[
G_j \cap B^{n+1}_\rho (\xi, \tau) = \bigcup_{i=1}^L G_{\Sigma}(u^i_j) \cap B^{n+1}_\rho (\xi, \tau),
\]

and \( \sup_i |u^1_j|_{C^k(\Sigma \cap B^{n+1}_\rho (\xi, \tau))} \to 0 \) as \( j \to \infty \) for each \( k = 0, 1, 2, \ldots \). Here \( G_{\Sigma}(u^i_j) \) denotes the graph of \( u^i_j \) defined by

\[
G_{\Sigma}(u^i_j) = \{ x + u^i_j(x) \nu(x) : x \in \Sigma \cap B^{n+1}_\rho (\xi, \tau) \},
\]

where \( \nu \) is a smooth unit normal for \( \Sigma \).

Of course in view of the \( C^k \) convergence of the above lemma, we can take limits in 5.7 in order to deduce the inequality

\[
\int_{\Sigma} |A_{\Sigma}|^2 \zeta^2 \leq \frac{1}{\varepsilon^2} \int_{\Sigma} |\nabla \zeta|^2 + \varepsilon^{-1} \delta^{-2} \int_{\Sigma} \zeta^2
\]

for each \( \varepsilon, \delta > 0 \) and each \( \zeta \in C^1_c(\Sigma \cap B^\infty_\rho \times (\delta, \infty)) \), where \( A_{\Sigma} \) denotes the second fundamental form of \( \Sigma \) and \( \nabla \zeta \) is the gradient operator on \( \Sigma \).

As a further consequence of Lemma 5.8 we can prove that \( V \) has multiplicity \( N \leq 2 \) at all points of \( \Sigma \):

5.10 Lemma. The varifold \( V \) has multiplicity \( \leq 2 \) at each point of \( \Sigma = \text{reg} \ V \cap (B^n_R \times (0, \infty)) \).

Remark: Notice that, since \( \text{reg} \ V \) is an embedded \( C^2 \) submanifold, by the constancy theorem for integer multiplicity varifolds with bounded mean curvature the multiplicity of \( V \) is constant on each connected component \( \Sigma_* \) of \( \Sigma = \text{reg} \ V \cap (B^n_R \times (0, \infty)) \).

Proof of Lemma 5.10: If \( V \) has multiplicity \( N \geq 3 \) at a point \( (\xi, \tau) \in \Sigma \), then for suitable \( \rho \in (0, \delta), \delta < \tau/2 \), we have the conclusions of Lemma 5.8 with \( L = N \geq 3 \). Since \( |u^i_j|_{C^1} \to 0 \) for each \( i = 1, \ldots, N \) and \( \bigcup_{i=1}^N \text{graph} \ u^j_j \subset G_j \) for each \( j \), for \( \sigma \in (0, \rho/2) \) small enough to ensure that \( \Sigma \) is a graph of a function with \( C^1 \) norm less than \( \frac{1}{2} \) over the affine tangent hyperplane of \( \Sigma \) at the point \( (\xi, \tau) \), and for \( j \) sufficiently large, we then have that \( U_j = \{ x + t \nu(x) : x \in \Sigma \cap B_\sigma (\xi, \tau) \} \) and \( u^1_j < t < u^2_j \), \( \bar{U}_j = \{ x + t \nu(x) : x \in \Sigma \cap B_\rho (\xi, \tau) \} \) and \( u^2_j < t < u^3_j \) are piecewise \( C^1 \) open sets. Also, since \( N \geq 3 \) and \( u^1_j < u^2_j < u^3_j < \cdots < u^N_j \), we have \( G_j \cap (U_j \cup \bar{U}_j) = \emptyset \), so \( U_j, \bar{U}_j \) are on opposite sides of \( G_j \). Since it is then just a matter of notation, we can, and we shall, assume that \( U_j \) is contained in the region below the graph \( G_j \) for each \( j \). For each \( j \) let \( \nu^j \) denote the upward pointing unit normal function of \( G_j \), thought of as a function defined in the whole cylinder \( B^n_R \times (0, \infty) \) which is independent of the \( x_{n+1} \). Thus \( \nu^j(x, x_{n+1}) = (1 + |Du_j(x)|^2)^{-1/2}(-Du_j(x), 1) \), and the equation 1.1 can be written

\[
-\text{div}_{R^{n+1}} \nu^j = -\sum_{i=1}^{n+1} D_i \nu^j_i(x, x_{n+1}) = H_j(x, x_{n+1}),
\]

valid in the cylinder \( B^n_R \times (0, \infty) \), where \( H_j \) denotes the mean curvature function defined on \( B^n_R \times (0, \infty) \) by

\[
H_j(x, x_{n+1}) = (m - 1)(1 + |Du_j(x)|^2)^{-1/2}/u_j(x),
\]

so that \( H_j(x, x_{n+1}) \) is also independent of the \( x_{n+1} \). Observe also that, since \( U_j \) is contained in the region below the graph, the outward pointing unit normal of \( \partial U_j \) on graph \( u^1_j \), graph \( u^2_j \)
agrees with \( \nu^j \) and the remaining part of \( \partial U_j \) has measure \( \leq C \sup(u_j^2 - u_j^1) \to 0 \). So applying the divergence theorem over \( U_j \) gives
\[
\int_{U_j} H_j = \int_{U_j} \operatorname{div}\nabla u^j = \mathcal{H}^n(\bigcup_{\ell=1}^m \text{graph}(u_j^\ell \cap B_\sigma(\xi, \tau))) + E_j,
\]
where \( E_j \to 0 \). On the other hand \( H_j \) is bounded independent of \( j \) on \( U_j \) (because \( U_j \) is contained in the region below the graph \( G_j \) so at each point \((x, x_{n+1}) \in U_j \) we have \( u_j(x) \geq x_{n+1} \) and hence, since \( x_{n+1} > \tau - \rho > \delta \) for any \((x, x_{n+1}) \in U_j \) and all sufficiently large \( j \), we have \( u_j(x) > \delta \) whenever \((x, x_{n+1}) \in U_j \), so in particular \( H_j(x, x_{n+1}) \leq (m-1)/\delta \) for all sufficiently large \( j \) if \((x, x_{n+1}) \in U_j \). Thus the above identity gives \( \mathcal{H}^n(\text{graph}(u_j^1 \cap B_\sigma(\xi, \tau)) \cup \text{graph}(u_j^2 \cap B_\sigma(\xi, \tau))) \to 0 \), which of course is impossible because \( u_j^1 \cap B_\sigma(\xi, \tau) \to 0 \) in the \( C^1 \) sense and hence \( \liminf_{j \to \infty} \mathcal{H}^n(\text{graph}(u_j^1 \cap B_\sigma(\xi, \tau)) \cup \text{graph}(u_j^2 \cap B_\sigma(\xi, \tau))) \geq 2 \mathcal{H}^n(\Sigma \cap B_\sigma(\xi, \tau)) > 0 \).
\( \Box \)

5.11 Lemma. Let \( \Sigma_* \) be a connected component of \( \Sigma = \text{reg} V \cap (B^m_R \times (0, \infty)) \), equipped with the same multiplicity as \( V \) at each of its points.

(i) If \( \Sigma_* \) is a vertical cylinder \( \Sigma_0 \times (0, \infty) \), then it has zero mean curvature (i.e. minimal) and is stable (i.e. it satisfies the stability inequality \( \int_{\Sigma_*} |A_{\Sigma_*}|^2 \geq \int_{\Sigma_*} |\nabla_{\Sigma_*} \varsigma|^2 \) for each \( \varsigma \in C^1_c(B^m_R \times (0, \infty)) \)), and furthermore \( \Sigma_* \cap \Sigma = \emptyset \) (where \( \Sigma \) means closure of \( S \) in \( B^m_R \times \mathbb{R} \)), and \( \Sigma_* \) has multiplicity 2.

(ii) If \( \Sigma_* \) has multiplicity 2 then it is a vertical cylinder \( \Sigma_0 \times (0, \infty) \)

Proof of (i): The mean curvature of \( \Sigma \) is \( \leq C/\sigma \) in the region \( x^{n+1} > \sigma \); but of course since \( \Sigma \) is a vertical cylinder this shows that \( \Sigma \) is minimal (i.e. mean curvature zero).

To prove that \( \Sigma \) is stable in \( \mathbb{R}^n \times (0, \infty) \) (or equivalently \( \Sigma_0 \) is stable in \( \mathbb{R}^n \)) we let \( \varsigma \in C^1_c(\mathbb{R}^n \times (0, \infty)) \) be arbitrary and for any given \( K \) let \( \varsigma_K(x, x^{n+1}) = \varsigma(x, x^{n+1} - K) \), so that \( \text{support} \varsigma_K \subset (K, \infty) \).

Then by the inequality 5.9 we have
\[
\int_{\Sigma} |\nabla_{\Sigma} \varsigma_K|^2 \leq (1 - \varepsilon)^{-1} \int_{\Sigma} |A_{\Sigma}|^2 \varsigma_K^2 + \varepsilon^{-2} K^{-2} \int_{\Sigma} \varsigma_K^2.
\]

Since \( \Sigma \) is cylindrical this can be written
\[
\int_{\Sigma} |\nabla_{\Sigma} \varsigma|^2 \leq (1 - \varepsilon)^{-1} \int_{\Sigma} |A_{\Sigma}|^2 \varsigma^2 + \varepsilon^{-2} K^{-2} \int_{\Sigma} \varsigma^2
\]
so by first letting \( K \to \infty \) and then letting \( \varepsilon \downarrow 0 \) we obtain the stability inequality as claimed.

Let \( S(\Sigma) = \{(x, \xi) \in B^m_R \times (\mathbb{R}^n \setminus \{0\}) : (x, |\xi|) \in \Sigma \} \), interpreted as an integer multiplicity varifold equipped at each point \((x, \xi) \in S(\Sigma) \) with the multiplicity of \( \Sigma \) at the point \((x, |\xi|) \). Then \( S(\Sigma) \) is the varifold limit of \( S(u_j) \) for some sequence \( u_j \) of singular solutions \( u_j \) of 1.1 on \( B^m_R \) and hence \( S(\Sigma) \) is a stationary integer multiplicity varifold in \( B^m_R \times \mathbb{R}^n \). Notice that, since \( m \geq 2 \), \( \mathbb{R}^n \times 0 \) has \( \mathcal{H}^{m+1} \)-dimensional measure zero, so in terms of varifold convergence \( S(\Sigma) = \lim S(u_j) \), and each \( S(u_j) \) is stationary in \( B^m_R \times \mathbb{R}^n \), and hence \( S(\Sigma) \) is indeed stationary in \( B^m_R \times \mathbb{R}^n \) rather than merely (locally) stationary in \( B^m_R \times (\mathbb{R}^n \setminus \{0\}) \), although a-priori we need to allow the possibility that \( S(\Sigma) \) might include a subset of \( B^m_R \times \{0\} \), even perhaps a subset of positive \( n \)-dimensional Hausdorff measure; ultimately we shall prove in §9 that at most a set of Hausdorff dimension \( n-2 \) occurs here.

Next observe that we can apply the maximum principle of Ilmanen [I96] to assert that \( S(\Sigma_* \cap S(\Sigma) \setminus S(\Sigma_*)) = \emptyset \) as claimed; notice that, since we are working with \( N = n + m - 1 \) dimensional hypersurfaces, to literally apply [I96] we need to check that \( S(\Sigma) \setminus S(\Sigma_* \cap S(\Sigma_*)) \) has \( N-2 (= n+m-3) \)-dimensional Hausdorff measure zero, but that hypothesis was only used in [I96]
to justify application of the regularity theory [SS81], and that really only uses that the “capacity” of any compact $K \subset S(\Sigma)$ is zero, meaning that there is a sequence $\zeta_j$ of $C^1$ functions with $\zeta_j = 0$ in a neighborhood of $K$, $\zeta_j \equiv 1$ in $\{x : \text{dist}(x, K) \geq 1/j\}$ and $\int_{S(\Sigma)} |\nabla \zeta_j|^2 \to 0$. Thus it suffices to check that $\overline{S(\Sigma)} \setminus S(\Sigma_1) \cap S(\Sigma_2)$ has locally finite (rather than zero) $(N-2)$-dimensional Hausdorff measure, because a compact set $K$ with finite $(N-2)$-dimensional Hausdorff measure automatically has capacity zero. (See e.g. [EG, Theorem 4.16].) In the present case we have $\overline{S(\Sigma_1)} \cap (B_R^0 \times \{0\})$ is a set of locally finite $n-1$ dimensional Hausdorff measure, and $n-1 = (n+m-1) - m = N-m \leq N-2$, so indeed [I96] can be applied, and $S(\Sigma) \setminus S(\Sigma_1) \cap S(\Sigma_2) = \emptyset$ as claimed. We should mention here that in any case recent work of Wickramasekera ([Wic14]) shows that the application of the maximum principle requires only that the $(N-1)$-dimensional Hausdorff measure of the intersection of the supports is zero and of course we have that there. So technically we did not need to include the discussion concerning capacity in relation to Hausdorff $(N-2)$ dimensional measure, but it was included for the reader’s convenience since the work of Wickramasekera is lengthy and deep.

Finally we have to check that $\Sigma_s = \Sigma_0 \times (0, \infty)$ has multiplicity 2. By Lemma 5.10 the multiplicity is either 1 or 2. Of course since the multiplicity is constant on the regular set it suffices to assume $0 \in \Sigma_0 \times \{0\}$ and show that it is not possible for $\Sigma_s \cap (B_0^m \times (0,1))$ to have multiplicity 1. In view of what has already been established above, we can choose $\sigma > 0$ small enough to ensure that $B_0^m \times (0,1) \cap \overline{(S(\Sigma) \setminus S(\Sigma_1)) \cap S(\Sigma_2)} = \emptyset$. Let $u_j$ be as above, so we have variifold convergence $\Sigma_s = \lim G_j$ in $B_0^m \times (-1,1)$, where $G_j = G(u_j)$ considered as a multiplicity 1 varifold. We can of course use the orientation of $G_j$ given by the upward pointing unit normal and view $G_j$ as a multiplicity 1 current in $B_0^m \times (-1,1)$ with $\partial G_j = 0$. The variifold convergence $G_j \rightarrow \Sigma_s$ is multiplicity 1, so using the local $C^k$ convergence guaranteed by Lemma 5.8, we can appropriately orient $\Sigma_s$ so that the convergence of $G_j$ to $\Sigma_s$ in $B_0^m \times (-1,1)$ is also in the weak sense of currents (thus $\int_{G_j} \omega \rightarrow \int_{\Sigma_s} \omega$ for each fixed smooth $n$-form $\omega$ with compact support in $B_0^m \times (-1,1)$), and hence $\partial \Sigma_s = \lim \partial G_j = 0$. But of course $\partial \Sigma_s = \pm \Sigma_0 \times \{0\}$, a contradiction. □

**Proof of (ii):** Take $(\xi, \tau) \in \Sigma_s$ and let $u_j^1 < u_j^2$ be as in the sheeting lemma 5.8, so that $|u_j|^2 \to 0$ for $j = 1, 2$. Now if $\Sigma_s$ has normal $\nu$ with $\nu_{n+1}(\xi, \tau) > 0$ (i.e. $\Sigma_s$ has a non-vertical tangent hyperplane at $(\xi, \tau)$) then the $C^1$ convergence of $u_j^1, u_j^2$ ensures that the vertical line $\{\xi \times (0, \infty)\}$ intersects both graph $u_j^1$ and graph $u_j^2$ for sufficiently large $j$, which contradicts the fact that $G_j$ is a graph over $B_R^m$. Thus $\nu_{n+1} \equiv 0$ and in particular the mean curvature of $\Sigma_s$ (which is $(m - 1)\nu_{n+1}/\nu_{n+1}$) is identically zero. Thus $\Sigma_s$ is vertical (i.e. $\nu_{n+1} \equiv 0$ on $\Sigma_s$) and has zero mean curvature.

To show that $\Sigma_s$ is actually a cylinder $\Sigma_0 \times (0, \infty)$ we have simply to prove that $(\xi, \tau) \in \Sigma_s \Rightarrow$ the whole ray $\{\xi \times (0, \infty)\} \subset \Sigma_s$. Since $\mathcal{H}^{n-1}(\text{sing} \ V \cap (B_R^m \times (0, \infty))) = 0$ we must have $\mathcal{H}^{n-1}(P(\Sigma_s \setminus \Sigma_s) \cap (B_R^m \times \{0\})) = 0$, where $P$ is the orthogonal projection of $\mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n \times \{0\}$. With $S = \overline{\Sigma_s \setminus \Sigma_s}$ we then have $\mathcal{H}^n(P(S) \times (0, \infty)) = 0$, and since $\Sigma_s$ is smooth embedded with $\nu_{n+1} \equiv 0$ it is clear that $\Sigma_s \setminus (P(S) \times (0, \infty))$ is a union of vertical rays $\{\xi \times (0, \infty)\}$ and hence is a cylinder which agrees up to a set of $\mathcal{H}^m$-measure zero with $\Sigma_s$, and in particular is dense in $\Sigma_s$. Thus $\Sigma_s$ is the closure (in $B_R^m \times (0, \infty)$) of a cylinder and hence is a cylinder. □

### 6 Tangent Cones of Singular Solutions

Let $u$ be an arbitrary singular solution of 1.1 in $B_R^m$ and suppose $0 \in \text{sing} \ u$. Since $S(u)$ is a stationary integer multiplicity varifold we can take tangent cones of $S(u)$ at $0$. Thus for each sequence $\lambda_j \downarrow 0$ we can take a subsequence (still denoted $\lambda_j$) such that $\lambda_j^{-1}S(u)$ converges in the varifold sense in $\mathbb{R}^n \times \mathbb{R}^m$ to a cone $\mathbb{S}$ (i.e. $h\mathbb{S} = \mathbb{S}$ for each $h > 0$), and by construction $\mathbb{S}$ is invariant
under all orthogonal transformations of $\mathbb{R}^n \times \mathbb{R}^m$ which leave the first $n$ coordinates fixed.

In terms of $u$ this means that, with $u_j(x) = \lambda_j^{-1} u(\lambda_j x)$ for $x \in B^n_{R/\lambda_j}$ and $G_j = G(u_j)$ (the graph of $u_j$), $G_j$ converges in the varifold sense to an integer multiplicity cone $\mathbb{C}$ in $\mathbb{R}^n \times \mathbb{R}$. Such $\mathbb{C}$ is called a tangent cone to $G(u)$ at 0 and of course $\mathbb{C}$ is indeed a cone in the sense that $h\mathbb{C} = \mathbb{C}$ for each $h > 0$.

6.1 Lemma. If $u$ is a singular solution of 1.1 on the ball $B^n_R$, and if $0 \in \text{sing} u$, then any tangent cone $\mathbb{C}$ of $G(u)$ at 0 (obtained as described above) is of multiplicity 1.

Proof: Let $u$ be any singular solution of 1.1 on $B^n_R$ with $0 \in \text{sing} u$, and, with the terminology introduced above, let $\mathbb{C}$ be any tangent cone to $G(u)$ at 0.

By the regularity discussion of §5, in the region $x^{n+1} > 0$ sing $\mathbb{C}$ has Hausdorff dimension $\leq n - 7$ for $n \geq 7$, empty for $n \leq 6$, discrete for $n = 7$. Since $h \text{reg} \mathbb{C} = \text{reg} \mathbb{C}$ for each $h > 0$, each connected component of $\text{reg} \mathbb{C}$ has zero in its closure taken in $\mathbb{R}^n \times \mathbb{R}$. Thus if $\mathbb{C} \cap (\mathbb{R}^n \times (0, \infty))$ has a vertical component $\Sigma_0 \times (0, \infty)$ then by Lemma 5.11 it has multiplicity 2 and is all of $\mathbb{C} \cap (\mathbb{R}^n \times (0, \infty))$.

We claim that in this case every tangent cone would also have to be vertical. Indeed the set of all tangent cones of $G(u)$ at 0 is evidently connected, so if there is a non-vertical tangent cone of $G(u)$ at 0 then there would have to be a sequence of $\mathbb{C}_j$ of non-vertical tangent cones converging to a vertical tangent cone $\mathbb{C}$ in the varifold sense. Also, again using Lemma 5.11, each $\mathbb{C}_j$ must have multiplicity 1, so using standard Harnack theory locally in $\Sigma = \text{reg} \mathbb{C} \cap (\mathbb{R}^n \times (0, \infty))$ and the sheeting lemma 5.8 in the manner of [Il96] we would conclude that there is a positive smooth solution $v$ of the Jacobi field equation $\Delta_v v + |\Sigma|^2 v = 0$ on $\Sigma$. Since $\Sigma$ is obtained by a rescaling of the difference in heights of the “sheets” (i.e. the difference $u_1' - u_2'$ in the terminology of Lemma 5.8), and each $\mathbb{C}_j$ is a cone, we know that $v$ is homogeneous of degree 1 on $\Sigma$. Thus $v$ is homogeneous degree 1 superharmonic (i.e. $\Delta_v v \leq 0$) on $\Sigma$. Then of course $\text{min} \{v, K\}$ is a bounded weakly superharmonic on $\Sigma$ for each constant $K > 0$. According the argument of [Il96], since $\mathbb{C}$ is a cone we can then use the mean value inequality for $\text{min} \{v, K\}$:

$$\int_{\Sigma \cap B^{n+1}_R} \text{min} \{v, K\} \leq \rho^{-n} \int_{\Sigma \cap B^{n+1}_R} \text{min} \{v, K\}, \quad \forall \rho \in (0, 1), \ K > 0.$$ 

Since $\Sigma$ is a cone and since $v$ is homogeneous of degree 1 we can change variable in the integral on the right side to give

$$\rho^{-n} \int_{\Sigma \cap B^{n+1}_R} \text{min} \{v, K\} = \int_{\Sigma \cap B^{n+1}_R} \text{min} \{\rho v, K\}$$

and so the above inequality gives

$$\int_{\Sigma \cap B^{n+1}_R} (\text{min} \{v, K\} - \text{min} \{\rho v, K\}) \leq 0.$$ 

But of course $\text{min} \{\rho v, K\} \leq \text{min} \{v, K\}$ so the integrand is non-negative here and hence

$$\text{min} \{v, K\} = \text{min} \{\rho v, K\} \ H^n\text{-a.e. on } \Sigma,$$

which of course is impossible for sufficiently large $K$.

So we conclude that either all the tangent cones of $G(u)$ at 0 are vertical (hence by Lemma 5.8 also have connected multiplicity 2 regular set), or else there are no tangent cones which are vertical. We claim that the former case, when all tangent cones of $G(u)$ at 0 are vertical, multiplicity 2, and with connected regular set cannot occur. Indeed in this case we have that each tangent cone $\mathbb{S}$ of $S(u)$ at 0 corresponds to such a cylindrical tangent cone $\mathbb{C}$ of $G(u)$ at 0 according to $\mathbb{S} = \{ (x, \xi) : (x, [\xi]) \in \mathbb{C} \}$. Thus, with $N = n + m - 1$, in particular every tangent cone $\mathbb{S}$ of $S(u)$ at 0 has connected regular set and a singular set (in $\mathbb{R}^{N+1}$) of locally finite $\mathcal{H}^{N-2}$-measure. But, according to [Il96, Theorem B] in these circumstances tangent cones of multiplicity 1 minimal
submanifolds must in fact also be multiplicity 1. To be strictly correct we should mention that Imanen actually proved his Theorem B only for stable multiplicity 1 minimal hypersurfaces with singular sets of $\mathcal{H}^{N-2}$ measure zero, but, as explained in the discussion in the proof of Lemma 5.11, Imanen’s argument applies in the present setting where we have the sheeting lemma 5.8 and the bound $\mathcal{H}^{N-2}(K) < \infty$ for each tangent cone $\mathbb{S}$ of $S(u)$ at 0 and each compact subset $K \subset \text{sing}\mathbb{S}$.

\[\Box\]

7 Multiplicity 1 Cones in $\mathcal{G}_\infty$

7.1 Lemma. Let $C$ be the family of all multiplicity 1 cones $C \in \mathcal{G}_\infty$ (thus each $C \in C$ is a multiplicity 1 cone—$C$ has multiplicity 1 at each point of $\text{reg}\ C$ and $\lambda C = C$ for each $\lambda > 0$—and there exists a sequence $u_j$ of solutions of 1.1 on $B_R^n$ with $R_j \to \infty$ and $G(u_j)$ converging $C$ in the varifold sense in $\mathbb{R}^{n+1}$).

Then $C$ is a compact subset of $\mathcal{G}_\infty$; thus for each sequence $C_j \subset C$ there is $C \in C$ and a subsequence $C_{j'}$ converging to $C$ in the varifold sense.

Proof: In view of the volume bounds 2.3 the Allard compactness theorem guarantees that for any sequence $C_j \subset C$ there is a subsequence $C_{j'}$ converging to $C \in \mathcal{G}_\infty$ in the varifold sense.

So to complete the proof of compactness we just have to prove that $C$ is closed in $\mathcal{G}_\infty$; i.e. if $C_j \subset C$ with $C_j \to C$ in the varifold sense, then $C$ is a multiplicity 1 cone in $\mathcal{G}_\infty$. Certainly $C_j \to C$ implies that $C$ is in $\mathcal{G}_\infty$ (because $\mathcal{G}_\infty$ is closed by definition), and trivially $C$ is a cone, so we just have to check that $C$ has multiplicity 1.

Otherwise, using the sheeting lemma 5.8 exactly as in the proof of Lemma 6.1 we would have a component $\Sigma$ of $\text{reg}\ C$ and homogeneous degree one smooth positive $v$ on $\Sigma$ with $\Delta_{\Sigma} v \leq 0$, and by the same argument as in the proof of Lemma 6.1, using the mean value inequality on $\Sigma$, this is impossible. \[\Box\]

The following lemma establishes a Harnack theory for certain supersolutions on domains in $C, C \in C$.

7.2 Lemma (Harnack for Supersolutions.) There is $\lambda = \lambda(m, n) \in (0, \frac{1}{4}]$ such that if $\beta > 0$, $C \in C$ and if $v$ is a bounded positive $C^1$ function on $\Sigma \cap B_{1/2}^{n+1}(e_{n+1})$ ($\Sigma = \text{reg}\ C$) which satisfies an inequality of the form

$$\Delta_{\Sigma} v + b \cdot \nabla_{\Sigma} v + cv \leq 0 \text{ on } \Sigma \cap B_{1/2}^{n+1}(e_{n+1}),$$

with $|b|, |c| \leq \beta$, then

$$\int_{\Sigma \cap B_{\lambda}^{n+1}(e_{n+1})} v \leq C \inf_{\Sigma \cap B_{\lambda}^{n+1}(e_{n+1})} v, \quad C = C(m, n, \beta).$$

Remark: Notice that this is uniformly applicable over all $C \in C$, because the constant $C$ depends only on $m, n, \beta$ and not on the particular $C \in C$.

Proof of Lemma 7.2: Since we have a suitable Sobolev inequality (see [MicS73]) it is well known (see e.g. the discussion in [BG72]) that one can apply a Harnack theory in $\Sigma \cap B_{1/2}^{n+1}(e_{n+1})$, $\Sigma = \text{reg}\ C$, for positive bounded $v$ satisfying $\Delta_{\Sigma} v + b \cdot \nabla_{\Sigma} v + cv \leq 0$ in $B_{1/2}^{n+1}(e_{n+1})$, provided $|b|, |c| \leq \beta$ and provided we have a suitable Poincaré inequality

$$\min_{\mu \in \mathbb{R}} \int_{\Sigma \cap B_{\lambda}^{n+1}(e_{n+1})} |h - \mu| \leq \gamma \int_{\Sigma \cap B_{1/4}^{n+1}(e_{n+1})} |\nabla_{\Sigma} h|$$

for $h \in C^1(\Sigma)$. As is well known, such an inequality is implied by the geometric inequality

$$\min\{\mathcal{H}^n(E \cap B_{\lambda}^{n+1}(e_{n+1})), \mathcal{H}^n(\Sigma \cap B_{\lambda}^{n+1}(e_{n+1}) \setminus E)\} \leq \gamma \mathcal{H}^{n-1}(\Gamma \cap B_{1/4}^{n+1}(e_{n+1})),$$
for sets $E \subset \Sigma \cap B^{n+1}_{1/4}(e_{n+1})$ where $\Gamma = (E \setminus E) \cap B^{n+1}_{1/4}(e_{n+1})$. (The inequality (1) follows from (2) by taking $E = E_t = \{x : h(x) < t\}$ and then integrating with respect to $t$ and applying the coarea formula.)

Using the Sobolev inequality [MicS73] and the volume bounds of 2.3, it is a standard fact that there is $\eta = \eta(m, n) > 0$ such that the inequality (2) holds with $\gamma = \gamma(m, n)$ if

$$\min\{\mathcal{H}^n(E \cap B^{n+1}_{2\lambda}(e_{n+1})), \mathcal{H}^n(\Sigma \cap B^{n+1}_{2\lambda}(e_{n+1}) \setminus E)\} \leq \eta \lambda^n.$$

So we only have to prove (2) subject to the extra assumption that

$$\min\{\mathcal{H}^n(E \cap B^{n+1}_{2\lambda}(e_{n+1})), \mathcal{H}^n(\Sigma \cap B^{n+1}_{2\lambda}(e_{n+1}) \setminus E)\} \geq \eta \lambda^n.$$

As a preliminary to this we first claim that there is $\lambda_0 = \lambda_0(m, n) \in (0, 1/3)$ such that if $C \in C$ and if $\Sigma$ is a connected component of reg $C$ then

$$\Sigma^1, \Sigma^2 \text{ distinct connected components of } \Sigma \cap B^{n+1}_{1/4}(e_{n+1}) \implies$$

$$\text{either } \Sigma^1 \cap B^{n+1}_{\lambda_0}(e_{n+1}) = \emptyset \text{ or } \Sigma^2 \cap B^{n+1}_{\lambda_0}(e_{n+1}) = \emptyset.$$

Indeed otherwise there would be a sequence $C_j \in C$ and components $\Sigma_j^1, \Sigma_j^2$ of $\Sigma_j \cap B^{n+1}_{1/4}(e_{n+1})$ ($\Sigma_j = \text{reg } C_j$) such that $\Sigma_j^1 \cap B^{n+1}_{1/2}(e_{n+1}) \neq \emptyset$ for $i = 1$ and $i = 2$. But then by the volume bounds 2.3 and the Allard compactness theorem, $\Sigma_j^i$ converges in the varifold sense to a bounded mean curvature integer multiplicity varifold $V^1$ in $B^{n+1}_{1/4}(e_{n+1})$ and $e_{n+1} \in \text{spt } V^1 \cap \text{spt } V^2$ and of course $\text{spt } V^1 \subset \text{spt } C$ for $i = 1, 2$. Then by the maximum principle [Il96] we must have $\mathcal{H}^{n-2}(\text{spt } V^1 \cap \text{spt } V^2) \neq 0$. But, since $\mathcal{H}^{n-2}(\text{sing } C \cap \{x^{n+1} > 0\}) = 0$, we must then have at least one point $\xi \in \text{reg } C \cap \text{spt } V^1 \cap \text{spt } V^2$. Since reg $C$ is smooth the constancy theorem implies that both $V^1$ and $V^2$ are positive integer multiples of $\mathcal{H}$ in a neighborhood of $\xi$ and by construction $V^1 + V^2 \leq C \subset B^{n+1}_{1/4}(e_{n+1})$, so $C$ would have multiplicity $\geq 2$ in a neighborhood of $\xi$, which contradicts Lemma 7.1.

Thus there is indeed a $\lambda_0 = \lambda_0(m, n)$ as in (5). With this $\lambda_0$ we claim that (2) holds with $\lambda = \lambda_0/3$ and for some $\gamma = \gamma(m, n)$.

If (2) is false, then there would be a sequence $C_j \subset C$ and measurable subsets $E_j \subset \Sigma_j \cap B^{n+1}_{1/4}(e_{n+1})$, $\Sigma_j = \text{reg } C_j$, such that (4) holds with $E = E_j$ and $\mathcal{H}^n(\Gamma_j) \to 0$, where $\Gamma_j = (\Sigma_j \setminus E_j) \cap B^{n+1}_{1/4}(e_{n+1})$. Let $F_j = \Sigma_j \cap B^{n+1}_{1/4}(e_{n+1}) \setminus E_j$ and view $E_j, F_j$ as varifolds in $B^{n+1}_{1/4}(e_{n+1})$. Since the mean curvature of $C_j \leq 2(m-1)$ in $B^{n+1}_{1/4}(e_{n+1})$ the first variations $|\delta E_j(\zeta)|, |\delta F_j(\zeta)|$ are evidently $\leq 2(m-1)||\zeta||_{L^1} + \mathcal{H}^{n-1}(\delta \Gamma_j) \sup |\zeta| \to 2(m-1)||\zeta||_{L^1}$ for $\zeta \in C^1_c(B^{n+1}_{1/4}(e_{n+1}) ; \mathbb{R}^{n+m})$ (viewing $E_j, F_j$ as multiplicity 1 varifolds), so by the Allard compactness theorem $E_j, F_j$ converge to bounded mean curvature integer multiplicity varifolds $V^1, V^2$ in $B^{n+1}_{1/4}(e_{n+1})$ with both $\text{spt } V^1 \cap B^{n+1}_{2\lambda}(e_{n+1}) \neq \emptyset$ and $\text{spt } V^2 \cap B^{n+1}_{2\lambda}(e_{n+1}) \neq \emptyset$ and $\text{spt } V^1 \cap B^{n+1}_{1/4}(e_{n+1}) \subset \text{spt } C \cap B^{n+1}_{1/4}(e_{n+1})$ for $i = 1, 2$. By the constancy theorem $V^i$ is a sum of positive integer multiplicities of some components of $\text{reg } C \cap B^{n+1}_{1/4}(e_{n+1})$ for $i = 1, 2$ and for each $i = 1, 2$ one of these components must intersect $B^{n+1}_{2\lambda}(e_{n+1})$.

But since $2\lambda = 2\lambda_0/3 < \lambda_0$, we then conclude that these two components must coincide so again $C$ has a connected component of multiplicity 2 which again contradicts Lemma 7.1.

Thus we do have (2), hence (1), and so by the discussion of [BG72] we do indeed have the relevant Harnack theory for supersolutions, so the inequality for $v$ claimed in the lemma is proved. □

7.3 Theorem. If $C$ is as above then there is a bound, depending on $m, n$ only, on the slope of rays for the cones $\mathcal{C} \in C$; in fact there is a $K_0 = K_0(m, n) > 0$ such that if $C \in C$ then there is a $C^{0,1/2}(\mathbb{R}^n)$ homogeneous degree 1 singular solution $\varphi$ of 1.1 with $G(\varphi) = \mathcal{C}$ and $\varphi(x) \leq K_0|x|$ for all $x \in \mathbb{R}^n$. 


\textbf{Proof:} If the first claim is false then there is a sequence \( \mathbb{C}_j \in \mathcal{C} \) such that, for each \( \sigma > 0 \), \( \mathbb{C}_j \cap (B^n_\sigma(0) \times (1, \infty)) \neq \emptyset \) for all \( j \) sufficiently large (depending on \( \sigma \)), and by Lemma 7.1 there is a subsequence of \( \mathbb{C}_j \) (still denoted \( \mathbb{C}_j \)) and \( \mathbb{C} \in \mathcal{C} \) with \( \mathbb{C}_j \to \mathbb{C} \). By construction \( \text{spt} \mathbb{C} \cap (B^n_\sigma \times (1, \infty)) \neq \emptyset \) for each \( \sigma > 0 \), so \( \{(0, t) : t > 0\} \subset \text{spt} \mathbb{C} \). Also on reg \( \mathbb{C} \) we have by 5.5 that

\[ \Delta_C \nu_{n+1} + |A\nu|^2 \nu_{n+1} = -(m-1)e_{n+1} \cdot \nabla_C ((x^{n+1})^{-1} \nu_{n+1}), \]

and hence

\[ \Delta_C \nu_{n+1} + (m-1)e_{n+1} \cdot \nabla_C ((x^{n+1})^{-1} \nu_{n+1}) \leq 0. \]

But we can then apply Lemma 7.2, hence, with \( \Sigma \) any connected component of reg \( \mathbb{C} \cap (\mathbb{R}^n \times (0, \infty)) \),

\[ \int_{\Sigma \cap B^{n+1}_\lambda(e_{n+1})} \nu_{n+1} \leq C \inf_{\Sigma \cap B^{n+1}_\lambda(e_{n+1})} \nu_{n+1}. \]

In view of the inclusion \( \{(0, t) : t > 0\} \subset \text{spt} \mathbb{C} \) the right side here is zero, so \( \nu_{n+1} \) is identically zero on \( \Sigma \cap B^{n+1}_\lambda(e_{n+1}) \), hence on all of \( \Sigma \). Thus reg \( \mathbb{C} \cap (\mathbb{R}^n \times (0, \infty)) \) contains a vertical cylinder \( \Sigma_0 \times (0, \infty) \). But by Lemma 5.11 such a vertical cylinder has multiplicity 2, contradicting Lemma 7.1.

Now let \( \mathbb{C} \) be any tangent cone of \( G(u) \). Then there is a sequence \( \lambda_j \downarrow 0 \) and \( G(u_j) \to \mathbb{C} \) on \( B_1(0) \times (0, \infty) \), where \( u_j(x) = \lambda_j^{-1} u_j(y, x) \). In view of the uniform bound (by \( K_0 \) say) on the slope of the rays of \( \mathbb{C} \) established above, we have \( \text{spt} \mathbb{C} \cap (B^n_1 \times \mathbb{R}) \subset B^n_1 \times [0, K_0] \), so by Lemma 3.2 the \( u_j \) are uniformly bounded in \( C^{0, \frac{1}{2}}(B^n_\frac{1}{2}) \) and hence there is a subsequence (still denoted \( u_j \)) converging uniformly to \( \varphi \) on \( B^n_\frac{1}{2} \) (and of course the convergence is locally in the \( C^k \) sense for each \( k \) on the open set where \( \varphi > 0 \)), so \( \text{spt} \mathbb{C} \cap (B^n_\frac{1}{2} \times (0, \infty)) = \text{reg} \mathbb{C} \cap (B^n_\frac{1}{2} \times (0, \infty)) = \text{graph} \varphi|\{x : \varphi > 0\} \), and hence \( |D_r \varphi| \leq K_0 \) on \( \{x : \varphi > 0\} \) \( (D_r \varphi = |x|^{-1} x \cdot D \varphi) \). Evidently then \( \varphi(x) \leq K_0 |x| \) for all \( x \in B^n_\frac{1}{2} \). But since \( \mathbb{C} \) is a cone \( \varphi \) is the restriction to \( B^n_\frac{1}{2} \) of a homogeneous degree 1 function on \( \mathbb{R}^n \). This completes the proof of Theorem 7.3. \( \square \)

\section{8 Gradient Estimates}

We can now prove the gradient bounds mentioned in §1:

\textbf{8.1 Theorem.} Let \( u \) be a regular or singular solution of the SME (i.e. 1.1) on a ball \( B_\rho(y) \) with \( \sup_{B_\rho(y)} u < M \). Then, for any \( \theta \in (0, 1) \),

\[ \sup_{B_{\rho \theta}(y)} |Du| < C, \]

where \( C \) depends only on \( m, n, M/\rho, \) and \( \theta \).

\textbf{Proof:} By scaling it suffices to prove the theorem with \( \rho = 1 \). If this is false, there would exist constants \( M > 0, \theta \in (0, 1) \), a sequence \( \{x_j\} \in B_1(y) \), and a corresponding sequence \( \{u_j\} \) of (possibly singular) solutions of 1.1 defined on \( B_1(y) \) with \( \sup u_j < M \) such that \( u_j(x_j) > 0 \) and \( |Du_j(x_j)| \uparrow \infty \). Because the solutions of 1.1 are invariant under translations of \( \mathbb{R}^n \), translating the graph \( u_j \) by \( (x_j, 0) \), we can assume that \( x_j = 0 \) for all \( j \). Thus we have \( u_j \) defined at least over \( B_R(0) \), where \( R = 1 - \theta \), and \( \sup_{B_{\theta R}(0)} u_j \leq M \). \( |Du_j(0)| \to \infty \). But by virtue of the fixed bounds on the gradient of \( u_j^2 \) (Corollary 3.2), we have

\[ \sup_j u_j(0) |Du_j(0)| < \infty, \]

and hence

\[ u_j(0) \to 0. \]
Let $K > M/(1 - \theta)$ be a fixed constant (to be chosen later), and $W$ be the open circular cone
$$W = \{(\xi, \tau) \in \mathbb{R}^n \times (0, \infty) : \tau > K|\xi|\}$$
so that the boundary of $W$ is a union of rays $\tau = K|\xi|$ of slope $K$, and let
$$W_t = W \cap (\mathbb{R}^n \times \{t\}).$$
Note that $(0, u_j(0)) \in W$, but $\sup_{B_R} u_j \leq M$ and $K > M/(1 - \theta)$, so $G_j \cap W_t = \emptyset$ for $t = M$; thus $G_j$ intersects $W$ non-trivially at some heights $t \geq u_j(0)$, but eventually leaves $W$ completely. We let
$$h_j = \inf\{t : t > u_j(0) \text{ and } G_j \cap W_t = \emptyset\},$$
so then
$$G_j \cap W_t \neq \emptyset \text{ for each } t \in [u_j(0), h_j) \text{ and } \sup_{B_{h_j/K}(0)} u = h_j.$$ 
We claim that
$$\frac{h_j}{u_j(0)} \to \infty \text{ as } j \to \infty.$$ 
To check this, rescale to give $\tilde{G}_j$ according to
$$\tilde{G}_j = (u_j(0))^{-1}G_j$$
Then $\tilde{G}_j = \text{graph } \tilde{u}_j$, where $\tilde{u}_j = \lambda_j^{-1}u_j(\lambda_j x)$, with $\lambda_j = u_j(0)$, is a (possibly singular) solution of 1.1, $|D\tilde{u}_j(0)| = |Du_j(0)| \to \infty$, and $\sup_{B_{h_j/(K\lambda_j)}} \tilde{u}_j = h_j/u_j(0)$ and $\tilde{u}_j(0) = 1$. Then if $h_j/u_j(0)$ is bounded above by some constant $\beta < \infty$ we could deduce from Corollary 3.2 that $|Du_j(0)| = \tilde{u}_j(0)|D\tilde{u}_j(0)|$ is bounded above by a constant depending only on $m, n$, and $\beta$, a contradiction since $|Du_j(0)| \to \infty$. Thus (3) is proved.

Now with $h_j$ as above, consider the new rescaling
$$\tilde{G}_j = h_j^{-1}G_j.$$ 
Observe that then $\tilde{G}_j = \text{graph } \tilde{u}_j$, where $\tilde{u}_j(x) = h_j^{-1}u(h_j x)$ on $B_{1/K}(0)$, and
$$\tilde{u}_j(0) = u_j(0)/h_j \to 0, \quad |D\tilde{u}_j(0)| = |Du_j(0)| \to \infty, \quad \sup_{B_{1/K}(0)} \tilde{u}_j = 1$$
by (2) and (3).

By Corollary 3.2 the $\tilde{u}_j$ are equicontinuous on $B_{\rho}(0)$ for each $\rho < 1/K$ and hence a subsequence converges locally uniformly in $B_{1/K}(0)$ to a singular solution $\tilde{u}$ of 1.1 and the graph $\tilde{G}$ of $\tilde{u}$ intersects the cone $W$ at each height $0 < t < 1$, so every tangent cone of $\tilde{G}$ at $(0, 0)$ has rays of slope $\geq K$, thus contradicting Theorem 6.1, provided we choose $K = \max\{M/(1 - \theta), K_0 + 1\}$ with $K_0$ as in Lemma 7.3. □

9 dim sing $u \leq n - 2$

9.1 Theorem. Suppose $u$ is a singular solution of the SME (i.e. a singular solution of 1.1) in the domain $\Omega \subset \mathbb{R}^n$. Then sing $u(= \{x \in B_1 : u(x) = 0\})$ has Hausdorff dimension $\leq n - 2$; in fact, for each closed ball $\overline{B}_\rho(y) \subset \Omega$, sing $u \cap \overline{B}_\rho(y)$ can be written as a finite union of locally compact (i.e. intersection of a compact with an open set) subsets, each of which has finite $(n - 2)$-dimensional Hausdorff measure in a neighborhood of each of its points.

9.2 Remark. Using the bound on the singular set in the above theorem, together with the gradient estimate of Theorem 8.1, we can now check that singular solutions $u$ of 1.1 on a domain $\Omega$ automatically have $1/u \in L^1_{loc}(\Omega)$ and are weak solutions of 1.1; i.e. 1.1’ holds. To check this we
replace $\zeta$ in 1.1 by $\zeta\chi_j$ where $\chi_j \equiv 0$ in the $(1/j)$ neighborhood of support $\zeta \cap \{x \in \Omega : u(x) = 0\}$, $\chi_j \equiv 1$ outside the $(2/j)$ neighborhood of support $\zeta \cap \{x \in \Omega : u(x) = 0\}$, and $\int_{\mathbb{R}^n} |D\chi_j| \to 0$ as $j \to \infty$; such $\chi_j$ exist because $H^{n-1}(\text{sing } u) = 0$ by the above theorem.

**Proof of Theorem 9.1:** For $K > 0$ and $\Omega \subset \mathbb{R}^n$ open, let

$$\mathcal{M}_{K,\Omega} = \{u : u \text{ is a singular solution of 1.1 on } \Omega \text{ with } \sup \{Du\} \leq K\}$$

and

$$\mathcal{M}_K = \cup \{\Omega \subset \mathbb{R}^n \mathcal{M}_{K,\Omega}, \quad S(\mathcal{M}_K) = \{S(u) : u \in \mathcal{M}_K\},$$

where as usual $S(u)$ denotes the symmetric graph of $u$.

Then the gradient bound of Theorem 8.1 guarantees that each singular solution $u$ of the SME 1.1 on an open $\Omega \subset \mathbb{R}^n$ must have $u|_{B_{2r}(y)} \in \mathcal{M}_{K,B_{2r}(y)}$ for some $K$, provided $\overline{B_r}(y) \subset \Omega$. Also, with the aid of the Arzela-Ascoli lemma, one can readily check that, for each $K > 0$, $S(\mathcal{M}_K)$ is a “multiplicity one class” of stationary minimal hypersurfaces in $\mathbb{R}^n + n$ in the sense of [Sim93]. Then, as discussed in [Sim93], there is an integer $q \in \{0, \ldots, n + m - 2\}$ such that $\dim \text{sing } M \leq n + m - q$ for each $M \in S(\mathcal{M}_K)$, where $q$ is the integer maximum such that there is $C = C_0 \times \mathbb{R}^{n+m-1-q} \subset S(\mathcal{M}_K)$, where $C_0$ is a minimal hypercone in $\mathbb{R}^m$ with sing $C_0 = \{0\}$ which is invariant under rotations of the last $m$ coordinates. Then of course $q \neq 0$, and in terms of the functions $u \in \mathcal{M}_K$ this says that $\dim \text{sing } u \leq n - q$ where $q$ is the minimum integer such that there is a homogeneous degree 1 singular Lipschitz solution $u$ of the SME 1.1 on $\mathbb{R}^n$. $q$ is not equal to 1 because, by the discussion in §1, there are no singular solutions of 1.1 in case $n = 1$. On the other hand as discussed in §1 there is the homogeneous degree 1 singular solution $\sqrt{m-1}|x|$ of 1.1 in $\mathbb{R}^2$. So $q = 2$ and hence each $u \in \mathcal{M}_K$ has sing $u$ of Hausdorff dimension $\leq n - 2$ as claimed.

The remaining rectifiability claims are true by [Sim95]. □

**References**


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