On Delaunay Ends in the DPW Method

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Abstract

We consider constant mean curvature 1 surfaces in $\mathbb{R}^3$ arising via the DPW method from a holomorphic perturbation of the standard Delaunay potential on the punctured disk. Kilian, Rossman and Schmitt have proven that such a surface is asymptotic to a Delaunay surface. We consider families of such potentials parametrised by the necksize of the model Delaunay surface and prove the existence of a uniform disk on which the surfaces are close to the model Delaunay surface and are embedded in the unduloid case.

Introduction

Beside the sphere, the simplest non-zero constant mean curvature (CMC) surface is the cylinder, which belongs to a one-parameter family of surfaces generated by the revolution of an elliptic function: the Delaunay surfaces, first described in [1]. Like the cylinder, Delaunay surfaces have two annular type ends, and Delaunay ends are the only possible embedded annular ends for a non-zero CMC surface. Indeed, as proven in [11] by Korevaar, Kusner and Solomon, if $\mathcal{M} \subset \mathbb{R}^3$ is a proper, embedded, non-zero CMC surface of finite topological type, then every annular end of $\mathcal{M}$ is asymptotic to a Delaunay surface and if $\mathcal{M}$ has exactly two ends which are of annular type, then $\mathcal{M}$ is a Delaunay surface. Thus, the status of Delaunay surfaces for non-zero CMC surfaces is very much alike the catenoid position in the study of minimal surfaces (see the result of Schoen in [17]), and one has to understand the behaviour of Delaunay ends in order to construct examples of non-compact CMC surfaces with annular ends, as Kapouleas did in 1990 [6].

For an immersion, having a constant mean curvature and having a harmonic Gauss map are equivalent. This is why the Weierstrass type representation of Dorfmeister, Pedit and Wu [2] has been used since the publication of their article to construct CMC surfaces. The DPW method can construct any conformal non-zero CMC immersion in $\mathbb{R}^3$, $\mathbb{H}^3$ or $\mathbb{S}^3$.
with three ingredients: a holomorphic potential which takes its values in a loop group, a loop group factorisation, and a Sym-Bobenko formula. Several examples of CMC surfaces with annular ends, like \( n \)-noids and bubbletons, have been made by Dorfmeister, Wu, Killian, Kobayashi, McIntosh, Rossman, Schmitt and Sterling [3, 16, 8, 9, 10, 15]. These constructions often rely on a holomorphic perturbation of the holomorphic potential giving rise to a Delaunay surface via the DPW method, and Kilian, Rossman and Schmitt [7] have proven that such perturbations always induce asymptotically a Delaunay end.

More precisely, any Delaunay embedding can be obtained with a holomorphic potential of the form \( \xi^D = Az^{-1}dz \) where

\[
A = \begin{pmatrix}
0 & r\lambda^{-1} + s \\
r\lambda + s & 0
\end{pmatrix}.
\]

The main result of [7] states that any immersion obtained from a perturbed potential of the form \( \xi = \xi^D + \mathcal{O}(z^0) \) is asymptotic to an embedded half-Delaunay surface in a neighbourhood of \( z = 0 \), provided that the monodromy problem is solved. In this paper, we allow the perturbed potential to move in the family of Delaunay potentials by introducing a real parameter \( t \), proportional to the weight of the model Delaunay surface, and consider \( \xi_t = \xi^D_t + \mathcal{O}_t(z^0) \) where \( \xi^D_t \) is a Delaunay potential of weight \( 8\pi t \). The main theorem of [7] states that for every \( t > 0 \), there exists a small neighbourhood of the origin on which the surface produced by the potential \( \xi_t \) is embedded and asymptotic to a half Delaunay surface. Unfortunately, without further hypotheses, this neighbourhood vanishes into a single point as \( t \) tends to zero. Adding a few assumptions, we prove here that there exists a uniform neighbourhood of the origin upon which the surfaces induced by the family \( \xi_t \) are all embedded and asymptotic to a half Delaunay surface for \( t > 0 \) small enough.

Hence, the point of our paper is not to show that the ends of the perturbed unduloid family are embedded (which is what [7] does), but that all the immersions of this family are embedded on a uniform punctured disk. Equipped with our result, Martin Traizet (in [22] and [21]) showed for the first time how the DPW method can be used to both construct CMC \( n \)-noids without symmetries and prove that they are Alexandrov embedded.

The theorem we prove is the following one (definitions and notations are clarified in Section 1):

**Theorem 1.** Let \( \Phi_t \) be a holomorphic frame arising from a perturbed Delaunay potential \( \xi_t \) defined on a punctured neighbourhood of \( z = 0 \). Suppose that \( \Phi_0(1, \lambda) = I_2 \) and that the monodromy of \( \Phi_t \) is unitary. Then, if \( f_t \) denotes the immersion obtained via the DPW method,
• There exists a family \( f_t^P \) of Delaunay immersions such that for all \( \alpha < 1 \) and \( |t| \)
small enough,
\[
\| f_t(z) - f_t^P(z) \|_{\mathbb{R}^3} \leq C_\alpha |t| \| z \|^\alpha
\]
on a uniform neighbourhood of \( z = 0 \).

• If \( t > 0 \) is small enough, then \( f_t \) is an embedding of a uniform neighbourhood of \( z = 0 \).

• The limit axis of \( f_t^P \) as \( t \) tends to 0 can be made explicit.

An outline of the proof is given in Section 1.9, together with an explanation of why
the convergence of \( t \) to 0 forbids us from using several key results of [7].

1 The DPW method

1.1 Loop groups

Our maps will often depend on a spectral parameter \( \lambda \) that can be in one of the following
subsets of \( \mathbb{C} \) \( (R > 1) \):
\[
\mathcal{D}_R = \{ \lambda \in \mathbb{C}, \ |\lambda| < R \}, \quad \mathcal{A}_R = \{ \lambda \in \mathbb{C}, \ \frac{1}{R} < |\lambda| < R \},
\mathcal{D}_1 = \{ \lambda \in \mathbb{C}, \ |\lambda| < 1 \}, \quad \mathcal{A}_1 = \{ \lambda \in \mathbb{C}, \ |\lambda| = 1 \} .
\]

For the coordinate \( z \), we will note \( (\epsilon > 0) \):
\[
\mathbb{D}_\epsilon = \{ z \in \mathbb{C}, \ |z| < \epsilon \}, \quad \mathbb{S}_\epsilon = \{ z \in \mathbb{C}, \ |z| = \epsilon \} .
\]

Let us define the following (untwisted) loop groups and algebras:

• \( \mathit{ASL}_2 \mathbb{C} \) is the set of smooth maps \( \Phi : \mathcal{A}_1 \rightarrow \mathit{SL}_2 \mathbb{C} \).

• \( \mathit{ASU}_2 \subset \mathit{ASL}_2 \mathbb{C} \) is the set of maps \( F \in \mathit{ASL}_2 \mathbb{C} \) such that \( F(\lambda) \in \mathit{SU}_2 \) for all \( \lambda \in \mathcal{A}_1 \).

• \( \Lambda_+ \mathit{SL}_2 \mathbb{C} \subset \mathit{ASL}_2 \mathbb{C} \) is the set of maps \( G \in \mathit{ASL}_2 \mathbb{C} \) that can be holomorphically
extended to \( \mathcal{D}_1 \) and such that \( G(0) \) is upper triangular.

• \( \Lambda_+^\mathbb{R} \mathit{SL}_2 \mathbb{C} \subset \Lambda_+ \mathit{SL}_2 \mathbb{C} \) is the set of maps \( B \in \Lambda_+ \mathit{SL}_2 \mathbb{C} \) such that \( B(0) \) has positive
elements on the diagonal.

• \( \mathit{ASl}_2 \mathbb{C} \) is the set of smooth maps \( A : \mathcal{A}_1 \rightarrow \mathit{sl}_2 \mathbb{C} \).
• $\Lambda \mathfrak{su}_2$ is the set of maps $m \in \Lambda \mathfrak{sl}_2 \mathbb{C}$ such that $m(\lambda) \in \mathfrak{su}_2$ for all $\lambda \in A_1$.

• $\Lambda_+ \mathfrak{sl}_2 \mathbb{C} \subset \Lambda \mathfrak{sl}_2 \mathbb{C}$ is the set of maps $g \in \Lambda \mathfrak{sl}_2 \mathbb{C}$ that can be holomorphically extended to $D_1$ and such that $g(0)$ is upper triangular.

• $\Lambda_+ \mathfrak{sl}_2 \mathbb{C} \subset \Lambda_+ \mathfrak{sl}_2 \mathbb{C}$ is the set of maps $b \in \Lambda_+ \mathfrak{sl}_2 \mathbb{C}$ such that $b(0)$ has real elements on the diagonal.

We also use the following notation:

$$\mathcal{O}(t^\alpha, z^\beta, \lambda^\gamma) = t^\alpha z^\beta \lambda^\gamma f(t, z, \lambda)$$

where $f$, on its domain of definition, is continuous with respect to $(t, z, \lambda)$ and holomorphic with respect to $(z, \lambda)$ for any $t$. If one variable is not specified, its exponent is assumed to be 0.

One step of the DPW method relies on the following Iwasawa decomposition (Theorem 8.1.1. in [14]):

**Theorem 2** (Iwasawa decomposition). Any element $\Phi \in \Lambda \text{SL}_2 \mathbb{C}$ can be uniquely factorised into a product

$$\Phi = F \times B$$

where $F \in \Lambda \text{SU}_2$ and $B \in \Lambda^R \text{SL}_2 \mathbb{C}$. Moreover, the map $\Lambda \text{SL}_2 \mathbb{C} \rightarrow \Lambda \text{SU}_2 \times \Lambda^R \text{SL}_2 \mathbb{C}$ is a $C^\infty$ diffeomorphism for the intersection of the $C^k$ topologies (see [7]).

The Iwasawa decomposition of a map $\Phi$ will often be written:

$$\Phi = \text{Uni}(\Phi) \times \text{Pos}(\Phi),$$

where $\text{Uni}(\Phi)$ is called “the unitary factor” of $\Phi$ and $\text{Pos}(\Phi)$ is “the positive factor” of $\Phi$. Using Corollary 3 of Appendix A, note that if $\Phi$ is holomorphic on $A_R$, then its unitary factor holomorphically extends to $A_R$ and its positive factor holomorphically extends to $D_R$.

### 1.2 The $\mathfrak{su}_2$ model of $\mathbb{R}^3$

In the DPW method, immersions are given in a matrix model. The euclidean space $\mathbb{R}^3$ is thus identified with the Lie algebra $\mathfrak{su}_2$ by

$$x = (x_1, x_2, x_3) \simeq X = \frac{-i}{2} \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix}.$$
The canonical basis of $\mathbb{R}^3$ identified as $\mathfrak{su}_2$ is denoted $(e_1, e_2, e_3)$. In this model, the euclidean norm is given by
\[ \|x\|^2 = 4 \det(X). \tag{1} \]
Linear isometries are represented by the conjugacy action of $SU_2$ on $\mathfrak{su}_2$:
\[ H \cdot X = HXH^{-1}. \]

### 1.3 The recipe

The DPW method takes for input data:

- A Riemann surface $\Sigma$;
- A $\Lambda \mathfrak{sl}_2 \mathbb{C}$-valued holomorphic 1-form $\xi = \xi(z, \lambda)$ on $\Sigma$ called “the DPW potential” which extends meromorphically to $D_1$ with a pole only at $\lambda = 0$, and which must be of the form
  \[ \xi(z, \lambda) = \sum_{j=-1}^{\infty} \xi_j(z)\lambda^j \]
  where each matrix $\xi_j(z)$ depends holomorphically on $z$ and all the entries of $\xi_{-1}(z)$ are zero except for the upper right entry which must never vanish;
- A base point $z_0 \in \Sigma$;
- An initial condition $\Phi_{z_0} \in \Lambda \mathfrak{sl}_2 \mathbb{C}$.

Given such data, here are the three steps of the DPW method for constructing CMC-1 surfaces in $\mathbb{R}^3$ (in the untwisted setting):

1. Solve for $\Phi$ the Cauchy problem with parameter $\lambda \in \mathcal{A}_1$:
   \[
   \begin{align*}
   d_{\lambda} \Phi(z, \lambda) &= \Phi(z, \lambda)\xi(z, \lambda), \\
   \Phi(z_0, \lambda) &= \Phi_{z_0}(\lambda).
   \end{align*}
   \]
   The solution $\Phi(z, \cdot) \in \Lambda \mathfrak{sl}_2 \mathbb{C}$ is called the “holomorphic frame” of the surface. In general, $\Phi(\cdot, \lambda)$ is only defined on the universal cover $\tilde{\Sigma}$ of $\Sigma$ (see Section 1.6). Note that if $\xi(z, \cdot)$ can be holomorphically extended to $\mathcal{A}_R$ ($R > 1$), then $\Phi(z, \cdot)$ can also be holomorphically extended to $\mathcal{A}_R$ provided that $\Phi_{z_0}$ is holomorphic on $\mathcal{A}_R$.

2. For all $z \in \tilde{\Sigma}$, Iwasawa decompose $\Phi(z, \lambda) = F(z, \lambda)B(z, \lambda)$. The decomposition is done pointwise in $z$, but $F(z, \lambda)$ and $B(z, \lambda)$ depend real-analytically on $z$. The map $F$ is called the “unitary frame” of the surface.
3. Define \( f : \tilde{\Sigma} \rightarrow \mathfrak{su}_2 \) by the Sym-Bobenko formula:

\[
f(z) = \text{Sym}(F) = i \frac{\partial F}{\partial \lambda}(z, 1) F(z, 1)^{-1}.
\]

The map \( f \) is then a conformal CMC-1 immersion whose normal map is given by

\[
\mathcal{N}(z) = \frac{-i}{2} F(z, 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F(z, 1)^{-1}.
\]

Its metric and Hopf differential are

\[
ds = 2\rho^2 |\xi_{12}^0| |dz|,
\]
\[
Q = -2\xi_{12}^0 \xi_j^{21} dz^2
\]

where \( \xi_{kl}^j \) is the \((k, l)\)-entry of the matrix \( \xi_j(z) \) and \( \rho \) is the upper-left entry of \( B(z, 0) \).

The theory states that every conformal CMC-1 immersion can be obtained this way.

### 1.4 Rigid motions of the surface

Let \( \xi \) be a DPW potential and \( \Phi \in \text{ASL}_2 \mathbb{C} \) a solution of \( d\Phi = \Phi \xi \). Take a loop \( H \in \text{ASU}_2 \) that does not depend on \( z \). Then \( \tilde{\Phi} = H \Phi \) also satisfies \( d\tilde{\Phi} = \tilde{\Phi} \xi \) and gives rise to a rigid motion of the original surface given by \( \tilde{\Phi} \). Let \( f = \text{Sym} \circ \text{Uni}(\Phi) \) and \( \tilde{f} = \text{Sym} \circ \text{Uni}(\tilde{\Phi}) \). Then,

\[
\tilde{f}(z) = H(1) \cdot f(z) + \text{Sym}(H).
\]

This enjoins us to extend the action of section 1.2 to affine isometries by

\[
H(\lambda) \cdot X = H(1) X H(1)^{-1} + i \frac{\partial H}{\partial \lambda}(1) H(1)^{-1}.
\]

Note that \( \text{ASU}_2 \) also acts on the tangent bundle of \( \mathbb{R}^3 \) via:

\[
H \cdot (p, \vec{v}) = (H \cdot p, H(1) \cdot \vec{v}).
\]

This action will be useful to follow the axis of our surfaces: oriented affine lines are generated by pairs \((p, \vec{v})\) and the action of \( \text{ASU}_2 \) on a given oriented affine line corresponds to the action (3) on its generators.
1.5 Gauging

Let \((\Sigma, \xi, z_0, \Phi_{z_0})\) be a set of DPW data with \(d\Phi = \Phi\xi\). Let \(G(z, \lambda)\) be a holomorphic map with respect to \(z \in \Sigma\) such that \(G(z, \cdot) \in \Lambda_{+}\SL_2\mathbb{C}\) (such a map is called an “admissible gauge”). If we define \(\Phi = \Phi G\), then \(\Phi\) and \(\Phi\) give rise to the same immersion \(f\). This operation is called “gauging” and one can retrieve \(\Phi\) by applying the DPW method to the data \((\Sigma, \xi \cdot G, z_0, \Phi_{z_0} G(z_0, \cdot))\) where

\[
\xi \cdot G = G^{-1} \xi G + G^{-1} dG
\]

is the action of gauges on potentials.

1.6 The monodromy problem

Since \(\Phi\) is defined as the solution of a Cauchy problem on \(\Sigma\), it is only defined on the universal cover \(\tilde{\Sigma}\) of \(\Sigma\). For any deck transformation \(\tau\) of \(\tilde{\Sigma}\), we define the monodromy matrix \(M_\tau(\Phi) \in \SL_2\mathbb{C}\) as follow:

\[
\Phi(\tau(z), \lambda) = M_\tau(\Phi)(\lambda)\Phi(z, \lambda).
\]

Note that \(M_\tau(\Phi)\) does not depend on \(z\). The standard sufficient condition for the immersion \(f\) to be be well-defined on \(\Sigma\) is the following set of equations, called the monodromy problem in \(\mathbb{R}^3\):

\[
\begin{cases}
M_\tau(\Phi) \in \SU_2, & (i) \\
M_\tau(\Phi)(1) = \pm I_2, & (ii) \\
\frac{\partial}{\partial \lambda} M_\tau(\Phi)(1) = 0. & (iii)
\end{cases}
\]

Remark 1. In this paper, the Riemann surface \(\Sigma\) will always be a punctured neighbourhood \(\mathbb{D}_z^\circ\) of \(z = 0\). Thus, all the deck transformations \(\tau\) will be associated to a closed loop around \(z = 0\) and we will write \(M(\Phi)\) instead of \(M_\tau(\Phi)\).

Remark 2. Let \(\Phi : \mathbb{C}^* \rightarrow \SL_2\mathbb{C}\) such that \(M(\Phi) \in \SU_2\). Let \(\Phi = H(h^*\Phi) \cdot G\) where \(H \in \SL_2\mathbb{C}\), \(G\) is holomorphic at \(z = 0\) and \(h\) is a Möbius transformation that leaves \(z = 0\) invariant. Then

\[
M(\Phi) = H M(\Phi) H^{-1}.
\]

Thus, if the monodromy problem for \(\Phi\) is solved, a sufficient condition for the monodromy problem for \(\Phi\) to be solved is that \(H \in \SU_2\).
1.7 The Delaunay family

Delaunay surfaces come in a one-parameter family: for all \( t \in (-\infty, \frac{1}{16}] \setminus \{0\} \), there exists a unique Delaunay surface, whose weight (as defined in [5]) is \( 8\pi t \). The DPW method can retrieve these surfaces using the following data:

\[
\Sigma = \mathbb{C}^*, \quad \xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz, \quad z_0 = 1, \quad \Phi_{z_0} = I_2,
\]

where

\[
A_t(\lambda) = \begin{pmatrix} 0 & r\lambda^{-1} + s \\ r\lambda + s & 0 \end{pmatrix}
\]

and \( r, s \) are functions of \( t \in (-\infty, \frac{1}{16}] \) satisfying

\[
\begin{cases} 
  r, s \in \mathbb{R}, \\
  r + s = \frac{1}{2}, \\
  rs = t.
\end{cases} \tag{4}
\]

Note that the system (4) admits two solutions, whether \( r \geq s \) or \( r \leq s \). For a fixed value of \( t \), these two solutions give two different parametrisations of the same surface (up to a translation). If \( r \geq s \), the unit circle of \( \mathbb{C}^* \) is mapped onto a parallel circle of maximal radius: a bulge of the Delaunay surface. If \( r \leq s \), the unit circle of \( \mathbb{C}^* \) is mapped onto a parallel circle of minimal radius: a neck of the Delaunay surface. As \( t \) tends to 0 and in the case \( r \geq s \), the immersions tend towards the parametrisation of a sphere on every compact subset of \( \mathbb{C}^* \), which is why we call this setting the “spherical case”. On the other hand, when \( r \leq s \) and \( t \) tends to 0, the immersions degenerate into a point on every compact subset of \( \mathbb{C}^* \). Nevertheless, we call this setting the “catenoidal case” because applying a blowup to the immersions makes them converge towards a catenoid on every compact subset of \( \mathbb{C}^* \) (see [21] for further details).

In any case, the corresponding holomorphic frame is explicit:

\[
\Phi_t(z, \lambda) = z^{A_t(\lambda)}
\]

as is its monodromy around \( z = 0 \):

\[
\mathcal{M}(\Phi_t)(\lambda) = \exp(2i\pi A_t(\lambda)) = \cos(2\pi \mu_t(\lambda)) I_2 + \frac{i \sin(2\pi \mu_t(\lambda))}{\mu_t(\lambda)} A_t(\lambda) \tag{5}
\]

where

\[
\mu_t(\lambda)^2 = -\det A_t(\lambda) = \frac{1}{4} + t\lambda^{-1}(\lambda - 1)^2. \tag{6}
\]

Note that the conditions (4) have been chosen in order for the monodromy problem of Section 1.6 to be solved. The axis of the surface is given by \( \{(x, 0, -2r), \ x \in \mathbb{R}\} \) and its weight is \( 8\pi t \). Thus, the induced surface is an unduloid if \( t > 0 \) and a nodoid if \( t < 0 \).
Remark 3. In order to deal with a single-valued square root of $\mu_t(\lambda)^2$ and to avoid some resonance cases in Section 3, we set $T > 0$ and $R > 1$ small enough for

$$|\mu_t(\lambda)^2 - \frac{1}{4}| < \frac{1}{4}$$

to hold for all $(t, \lambda) \in (-T, T) \times A_R$.

1.8 Perturbed Delaunay DPW data

We take a Delaunay potentials family as in section 1.7 and we perturb it for $z$ in a small uniform neighbourhood of 0:

Definition 1 (Perturbed Delaunay potential). Let $\epsilon > 0$. A perturbed Delaunay potential is a one-parameter family $\{\xi_t\}_{t \in (-T, T)}$ of DPW potentials, holomorphic on $\mathbb{D}_\epsilon^* \times A_R$ and of the form

$$\xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz + R_t(z, \lambda)dz$$

where $A_t$ is a Delaunay residue as in Section 1.7 and $R_t(z, \lambda) \in \mathbb{C}^2$ with respect to $(t, z, \lambda)$, is holomorphic on $\mathbb{D}_\epsilon^* \times A_R$ for all $t$ and satisfies $R_0(z, \lambda) = 0$.

The following set of hypotheses will be used to make sure that our holomorphic frames have a $\mathcal{C}^0$ regularity, are holomorphic with respect to $(z, \lambda)$, and solve the monodromy problem:

Hypotheses 1. Let $\xi_t$ be a perturbed Delaunay potential. Let $\Phi_t$ be a holomorphic frame associated to it. We suppose that

- For some $t \in (-T, T)$ and $z \in \mathbb{D}_\epsilon^*$, $\Phi_t(z, \cdot)$ is holomorphic on $A_R$,
- $\Phi_t(z, \lambda)$ is continuous with respect to $(t, z, \lambda)$,
- The monodromy is unitary: $\mathcal{M}(\Phi_t) \in \Lambda SU_2$.

Remark 4. When needed, one can replace $R > 1$ by a smaller value in order for $\Phi_t$ to be holomorphic on $A_R$ and continuous on $\overline{A_R}$.

The theorem we prove in this paper is the following:

Theorem 3. Let $\xi_t$ be a perturbed Delaunay potential and $\Phi_t$ a holomorphic frame associated to $\xi_t$ satisfying Hypotheses 1 and such that $\Phi_0(1, \lambda) = I_2$. Let $f_t = \text{Sym}(\text{Uni}(\Phi_t))$. Then,
1. For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\|f_t(z) - f_t^D(z)\|_{\mathbb{R}^3} \leq C|t||z|^\alpha$$

where $f_t^D$ is a Delaunay immersion of weight $8\pi t$.

2. There exist $T' > 0$ and $\epsilon' > 0$ such that for all $0 < t < T'$, $f_t$ is an embedding of $\{0 < |z| < \epsilon'\}$.

3. If $r \geq s$, the limit axis as $t$ tends to 0 of $f_t^D$ is the oriented line generated by $(-e_3, -\vec{e}_1)$.

If $r \leq s$, the limit axis as $t$ tends to 0 of $f_t^D$ is the oriented line generated by $(0, -\vec{e}_1)$.

**Remark 5.** We do not have to assume that $1 \in \mathbb{D}_t$ for $\Phi_0$ to be defined at $z = 1$. This only comes from the fact that $\xi_0$ is defined on $\mathbb{C}^*$, which implies that $\Phi_0$ is defined on the universal cover $\mathbb{C}^*$.

### 1.9 Outline of the proof and comparison with [7]

In Section 3 we start the proof of Theorem 3 by gauging the potential and changing coordinates. Starting from

$$\xi_t = A_t z^{-1} dz + O(t, z^0) dz$$

we gain an order on $z$ and obtain the following new potential:

$$\tilde{\xi}_t = A_t z^{-1} dz + O(t, z) dz.$$  

We then use the Fröbenius method and the new holomorphic frame is

$$\tilde{\Phi}_t = M_t z^{A_t} \left( I_2 + O(t, z^2) \right).$$

In Section 4, we use this estimate on $\tilde{\Phi}_t$ to prove the convergence of the immersions:

$$\left\| \tilde{f}_t(z) - \tilde{f}_t^D(z) \right\|_{\mathbb{R}^3} \leq C|t||z|^\alpha, \quad \alpha < 1$$

where $\tilde{f}_t^D$ is a Delaunay immersion whose axis can be explicitly computed. To do so, we need to know the asymptotic behaviour of the positive part $\text{Pos}(\tilde{\Phi}_t)$, which we compute using the fact that $\tilde{f}_t^D(\mathbb{C}^*)$ is a surface of revolution.
Finally, Section 5 proves that perturbations of unduloids are embedded on a uniform neighbourhood of the origin.

Although the method of this paper is inspired by what Kilian, Rossman and Schmitt did in [7], their results cannot be used to prove our theorem. This is mainly because the asymptotics given in [7] for a fixed value of our parameter $t$ do not hold as $t$ tends to 0. As an example, consider the proof of Lemma 2.5 in [7]: with our hypotheses, the constant they call $\kappa$ becomes a function of $t$ such that (with our notation of Section 3.2)

$$\kappa \big|_{t=0} = \frac{c_{12}(0,0)}{4} \neq 0.$$  

Later in the proof, computing the determinant of the linear map $\mathcal{L}_1$ gives

$$\det \mathcal{L}_1 = \mathcal{O}(t)$$

and their gauged potential is then of the form

$$\hat{\xi}_t = A_t z^{-1} dz + \mathcal{O}(t^{-1}, z) dz,$$

the corresponding holomorphic frame being

$$\hat{\Phi}_t = \hat{M}_t z^{A_t} \left( I_2 + \mathcal{O}(t^{-1}, z^2) \right).$$

Applying the Sym-Bobenko formula would give at best

$$\left\| \hat{f}_t(z) - \hat{f}_t^D(z) \right\|_{\mathbb{R}^3} \leq C \frac{1}{|t|} |z|^\alpha, \quad \alpha < 1$$

which is not enough to show the convergence of the immersions on the compact sets of $\mathbb{C}^\alpha$ as $t$ tends to 0. Note that gaining one order on $|t|$ in the estimate (7) is still not enough to show the embeddedness of $\hat{f}_t$, since the first catenoidal neck of $\hat{f}_t^D$, which has a size of the order of $t$, is attained for $|z| \sim |t|$ as $t$ tends to 0.

Finally, some bounds used in [7] such as (see Lemma 1.11 in [7])

$$c_1(\lambda) = \max_{x \in (0,\rho)} \|B(x, \lambda)\|$$

depend on $t$ in our framework and may explode as $t$ tends to 0.
2 An application

Before proving Theorem 3, we must take account of the fact that one of its hypotheses is too restrictive. Indeed, \( \Phi_0(1, \lambda) = I_2 \) has no reason to hold when one wants to construct examples, as Martin Traizet did in [22] and [21]. We thus show here on a specific example how to ensure this hypothesis by gauging the potential and changing coordinates.

In all the section, \( \xi_t \) is a perturbed Delaunay potential with \( r \geq s \) and \( \Phi_t \) is a holomorphic frame associated to \( \xi_t \), satisfying Hypotheses 1 and such that \( \Phi_0(1, \lambda) = M(\lambda) \) where
\[
M(\lambda) = \begin{pmatrix} a & b \lambda^{-1} \\ c \lambda & d \end{pmatrix} \in \text{ASL}_2(\mathbb{C}) \quad (a, b, c, d \in \mathbb{C}). \tag{8}
\]

After some simplification, we will be able to apply Theorem 3 even though \( \Phi_0(1, \lambda) \neq I_2 \). The only difference in the conclusion will be in the third point: the limit axis as \( t \) tends to 0 of the model Delaunay surface \( f_t^D \) will be the oriented line generated by \( Q \cdot (0, e_3^2) \) where
\[
Q = \text{Uni} [MH] \tag{9}
\]
with
\[
H(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}. \tag{10}
\]
The method involves gauging, changing coordinates and applying an isometry, and relies on the fact that one can explicitly compute the Iwasawa decomposition of \( MH \). Indeed, for all \( a, b, c, d \in \mathbb{C} \) such that \( ad - bc = 1 \),
\[
\begin{pmatrix} a & b \lambda^{-1} \\ c \lambda & d \end{pmatrix} = \frac{1}{\sqrt{|b|^2 + |d|^2}} \begin{pmatrix} 1 & b \lambda^{-1} \\ 0 & \frac{d}{-b \lambda} \end{pmatrix} \times \frac{1}{\sqrt{|b|^2 + |d|^2}} \begin{pmatrix} 1 & 0 \\ -\frac{1}{ab + cd} \lambda & 0 \end{pmatrix} \tag{11}
\]
is the Iwasawa decomposition of the left-hand side term. Note that if the matrix \( M \) is explicit, then this formula makes both the matrix \( Q \) in Equation (9) and the limit axis of \( f_t^D \) explicit because \( MH \) and \( M \) have the same form.

**Lemma 1.** Let \( \xi_t \) be a perturbed Delaunay potential as in Definition 1 with \( r \geq s \). Let \( \Phi_t \) be a holomorphic frame associated to it, satisfying Hypotheses 1 and such that \( \Phi_0(1, \lambda) = M(\lambda) \) as in (8). Then there exists a Möbius transformation that leaves \( z = 0 \) invariant and a gauge \( G \) such that:

1. the new potential \( \tilde{\xi}_t = (h^* \xi_t) \cdot G \) is also a perturbed Delaunay potential with the same residue than \( \xi_t \),

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2. the holomorphic frame \( \Phi_t \) associated to \( \xi_t \) satisfies Hypotheses 1 with \( \Phi_0(1, \lambda) \in \Lambda \).

**Proof.** Let \( A_t \) and \( R_t \) be as in Definition 1. Then

\[
\xi_t = G^{-1} \left( A_t h^{-1} dh + (h^* R_t) dh \right) G + G^{-1} dG.
\]

The Möbius transformation we are looking for satisfies \( h(0) = 0 \) and thus

\[
h^{-1} dh = z^{-1} dz + O(z) dz.
\]

Wanting \( \xi_t \) to have a simple pole at \( z = 0 \), we look for a gauge \( G \) that is holomorphic at \( z = 0 \). Wanting the residue of \( \tilde{\xi}_t \) to be \( A_t \), we suppose that \( G(0, \lambda) = I_2 \). These two conditions together with \( \xi_0 = A_0 z^{-1} dz \) enjoin us to solve the following Cauchy problem:

\[
\begin{cases}
dG = G A_0 z^{-1} dz - A_0 G h^{-1} dh \\
G(0) = I_2.
\end{cases}
\]

(12)

If we write

\[
h(z) = \frac{z}{p z + q}, \quad p \in \mathbb{C}, \quad q \in \mathbb{C}^*,
\]

then the only solution of (12) is given (by Maple) by:

\[
G(z, \lambda) = \begin{pmatrix}
\sqrt{\frac{q}{p z + q}} & 0 \\
\frac{\lambda p z}{\sqrt{q (p z + q)}} & \sqrt{\frac{p z + q}{q}}
\end{pmatrix}
\]

and a straightforward computation allows us to check that \( G \) satisfies (12). Setting \( 0 < \epsilon' < \epsilon \) with \( \epsilon' < \frac{|q|}{|p|} \) if necessary, this proves the first point of the lemma.

In order to prove the second point, diagonalise \( A_0 = HDH^{-1} \) with \( H \) as in (10) and compute

\[
\tilde{\Phi}_0(1, \lambda) = M(\lambda) H(\lambda) \left( h(1)^D H(\lambda)^{-1} G(1, \lambda) H(\lambda) \right) H(\lambda)^{-1}
\]

(13)

where

\[
D = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{1}{2}
\end{pmatrix}.
\]

Hence \( \tilde{\Phi}_0(1, \cdot) \) is holomorphic on \( \mathcal{A}_R \). Moreover, the fact that \( \tilde{\xi}_t \) is \( C^2 \) in \( (t, z, \lambda) \) together with remark 2 imply that \( \tilde{\Phi}_t \) satisfies Hypotheses 1. Finally, compute

\[
h(1)^D H(\lambda)^{-1} G(1, \lambda) H(\lambda) = \begin{pmatrix}
\frac{1}{\sqrt{q}} & 0 \\
\lambda \frac{p}{\sqrt{q}} & \sqrt{q}
\end{pmatrix}
\]

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and, using Equation (11),
\[ \text{Pos}(MH) = \begin{pmatrix} \rho & 0 \\ \lambda \mu & \rho^{-1} \end{pmatrix} \]
where
\[ \rho = \frac{\sqrt{2}}{\sqrt{|b-a|^2 + |d-c|^2}}, \quad \mu = \frac{1}{\sqrt{2}} \times \frac{(a+b)(\bar{b} - \bar{a}) + (c+d)(\bar{d} - \bar{c})}{\sqrt{|b-a|^2 + |d-c|^2}}. \]
Then, setting
\[ p = -\rho \mu, \quad q = \rho^2, \]
Equation (13) becomes (\( Q \) is defined in (9))
\[ \tilde{\Phi}_0(1, \lambda) = QH^{-1} \in \Lambda SU_2 \]
because \( H \in \Lambda SU_2 \).

If one wants to apply Theorem 3, it then suffices to set
\[ \tilde{\Phi}_t = HQ^{-1} \tilde{\Phi}_t \]
where \( \tilde{\Phi}_t \) is constructed by Lemma 1. Let \( \tilde{f}_t^P \) be the model Delaunay immersion towards which the immersion \( \text{Sym}(\text{Uni}(\tilde{\Phi}_t)) \) converges. Theorem 3 then states that the limit axis at \( t \) tends to 0 of \( \tilde{f}_t^P \) is the oriented line generated by \((-e_3, -\bar{e}_1)\). Compute
\[ H^{-1} \cdot (-e_3, -\bar{e}_1) = (-e_3, e_3) \simeq (0, e_3) \]
to prove that \( \text{Sym}(\text{Uni}(\Phi_t)) \) converges to a model Delaunay surface whose limit axis as \( t \) tends to 0 is \( Q \cdot (0, e_3) \). The following corollary summarises this section:

**Corollary 1.** Let \( \xi_t \) be a perturbed Delaunay potential with \( r \geq s \) and \( \Phi_t \) a holomorphic frame associated to \( \xi_t \) satisfying Hypotheses 1 and such that \( \Phi_0(1, \lambda) \) is of the form given by (8). Let \( f_t = \text{Sym}(\text{Uni}(\Phi_t)) \). Then,

1. For all \( \alpha < 1 \) there exist constants \( \epsilon > 0, T > 0 \) and \( C > 0 \) such that for all \( 0 < |z| < \epsilon \) and \( |t| < T \),
\[ \|f_t(z) - f_t^P(z)\|_{\mathbb{R}^3} \leq C|t||z|^\alpha \]
where \( f_t^P \) is a Delaunay immersion of weight \( 8\pi t \).

2. There exist \( T' > 0 \) and \( \epsilon' > 0 \) such that for all \( 0 < t < T' \), \( f_t \) is an embedding of \( \{0 < |z| < \epsilon'\} \).

3. The limit axis at \( t \) tends to 0 of \( f_t^P \) is the oriented line generated by \( Q \cdot (0, e_3) \) where \( Q \) is given by Equation (9).
3 The $z^AP$ form of $\Phi_t$

Let us start the proof of Theorem 3: let $\xi_t$ be a perturbed Delaunay potential and $\Phi_t$ a holomorphic frame associated to $\xi_t$ satisfying Hypotheses 1 and such that $\Phi_0(1, \lambda) = I_2$.

In this section, we want to apply the Fröbenius method and write $\Phi_t$ in a $z^AP$ form. Unfortunately, the underlying Fuchsian system seems to admit resonance points. Our goal is to avoid them and to gain an order of convergence in the matrix $P$ of the $z^AP$ form. We will obtain the following result:

**Proposition 1.** There exist a change of coordinate $h_t$ and a gauge $G_t$ such that, denoting

$$\tilde{\Phi}_t = h_t^*(\Phi_t G_t)$$

and

$$\tilde{\xi}_t = h_t^*(\xi_t G_t),$$

$\tilde{\xi}_t$ is a perturbed Delaunay potential and $\tilde{\Phi}_t$ is a holomorphic frame associated to $\tilde{\xi}_t$ satisfying Hypotheses 1 and such that $\tilde{\Phi}_0(1, \lambda) = I_2$. Moreover,

$$\tilde{\Phi}_t(z, \lambda) = \tilde{M}_t(\lambda) z^{A_t(\lambda)} \tilde{P}_t(z, \lambda)$$

(14)

where $\tilde{M}_t \in \text{ASL}_2\mathbb{C}$ is continuous and holomorphic on $\mathcal{A}_R$ for all $t$ and $\tilde{P}_t : \mathbb{D}_k \to \text{ASL}_2\mathbb{C}$ is $C^2$, holomorphic on $\mathbb{D}_k \times \mathcal{A}_R$ for all $t$ and satisfies $\tilde{P}_t(z, \lambda) = I_2 + O(t, z^2)$.

3.1 Extending to the resonance points

In this section, we use the Fröbenius method to write $\Phi_t$ in a $z^AP$ form, and extend this form to the resonance points. We will thus prove:

**Proposition 2.** There exist $M_t \in \text{ASL}_2\mathbb{C}$ continuous and holomorphic on $\mathcal{A}_R$ for all $t$ and $P_t : \mathbb{D}_k \to \text{ASL}_2\mathbb{C}$ continuous and holomorphic on $\mathbb{D}_k \times \mathcal{A}_R$ for all $t$ satisfying $P_t(0, \lambda) = I_2$ and

$$\Phi_t(z, \lambda) = M_t(\lambda) z^{A_t(\lambda)} P_t(z, \lambda).$$

Let us first recall the Fröbenius method in the non-resonant case (see [19] and [18]). Let $\epsilon > 0$ and $\xi$ be a holomorphic 1-form from $\mathbb{D}_k^\epsilon$ to $\mathcal{M}_2(\mathbb{C})$ defined by

$$\xi(z) = Az^{-1}dz + \sum_{k \in \mathbb{N}} C_k z^k dz.$$
For all \( k \in \mathbb{N} \), let \( P_k \) solve
\[
\begin{align*}
\mathcal{L}_k(P_{k+1}) &= \sum_{i+j=k} P_i C_j \\
P_0 &= I_2,
\end{align*}
\]
where for all \( n \in \mathbb{N} \),
\[
\mathcal{L}_n: \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_2(\mathbb{C})
\]
\[
X \mapsto [A, X] + nX.
\]
Then \( P(z) = \sum_{k \in \mathbb{N}} P_k z^k \) is holomorphic on \( \mathbb{D}_r \) and \( \Phi(z) = z^A P(z) \) is holomorphic on the universal cover \( \hat{\mathbb{D}}_r \) of \( \mathbb{D}_r \) and solves \( d\Phi = \Phi \xi \).

Let us now recall Lemma 2.2 of [7] in our framework:

**Lemma 2.** Let \( A \in \mathfrak{sl}_2 \mathbb{C} \) such that \( A^2 = \mu^2 I_2 \). Then for all \( n \in \mathbb{N} \),
\[
\det \mathcal{L}_n = n^2 (n^2 - 4\mu^2)
\]
and
\[
\mathcal{L}_n^{-1}(X) = \frac{1}{n} \left( X - \frac{1}{n^2 - 4\mu^2} (nI_2 - 2A) [A, X] \right)
\]

Corollary 2 follows from Remark 3 and Equation (16).

**Corollary 2.** Let \( \mathcal{L}_{t,n}(X) = [A_t(\lambda), X] + nX \).

- For all \( n \geq 2 \), \( \mathcal{L}_{t,n} \) is invertible on \( (t, \lambda) \in (-T, T) \times \mathcal{D}_R^* \).
- For \( n = 1 \), \( \mathcal{L}_{t,1} \) is invertible on \( (t, \lambda) \in (-T, T) \setminus \{0\} \times \mathcal{D}_R^* \setminus \{1\} \).

**Remark 6.** If we use the Ansatz given by the Fröbenius method and write
\[
\Phi_t(z, \lambda) = M_t(\lambda) z^{A_t(\lambda)} P_t(z, \lambda)
\]
where
\[
P_t(z, \lambda) = \sum_{k=0}^{\infty} P_{t,k}(\lambda) z^k,
\]
note that the resonance points only occur in the computation of \( P_{t,1}(\lambda) \) because \( \mathcal{L}_{t,n} \) is invertible on \( (t, \lambda) \in (-T, T) \times \mathcal{A}_R \) for all \( n \geq 2 \). Thus, we only need to extend \( P_{t,1}(\lambda) \) at \( t = 0 \) and \( \lambda = 1 \) to extend the \( z^AP \) form of \( \Phi_t \). According to (15),
\[
P_{t,1}(\lambda) = \mathcal{L}_t^{-1}(tC_t(\lambda))
\]
and the form of \( \det \mathcal{L}_{t,1} \) shows that \( P_{t,1} \) has at most a pole of order 2 at \( \lambda = 1 \). Moreover, \( \det \mathcal{L}_{t,1} = \mathcal{O}(t) \) and \( tC_t = \mathcal{O}(t) \), so we already know that \( P_t \) (and as a consequence, \( M_t \)) extends to \( t = 0 \).
It remains to extend the $z^A P$ form (18) to $\lambda = 1$. To do this, we adapt the techniques used in Lemma 2.5 of [15] to prove the following unitary $\times$ commutator lemma:

**Lemma 3.** Let $M : A_R\{1\} \rightarrow \text{SL}_2\mathbb{C}$ holomorphic on $A_R\{1\}$ with at most a pole at $\lambda = 1$. Let $t \neq 0$, $Q = \exp(2i\pi A_t) \in \Lambda\text{SU}_2$ and suppose that for all $\lambda \in A_1\{1\}$, $MQM^{-1} \in \text{SU}_2$. Then there exist $U \in \Lambda\text{SU}_2$ and $K : A_R\{1\} \rightarrow \text{SL}_2\mathbb{C}$ holomorphic such that

$$\begin{align*}
M &= UK \\
[A_t, K] &= 0.
\end{align*}$$

*Proof.* We first apply Lemma 2.5 of [15] to construct $U$ and $K$ satisfying $M = UK$ and $[Q, K] = 0$ on $A_1\{1\}$. The map $U$ is holomorphic on a small neighbourhood of $A_1$. Without loss of generality, let this neighbourhood be $A_R$. Then, $K$ is meromorphic on $A_R\{1\}$ with at most a pole at $\lambda = 1$. Hence the map $\lambda \mapsto [Q(\lambda), K(\lambda)]$ is holomorphic on $A_R\{1\}$ and vanishes on $A_1\{1\}$. Thus, for all $\lambda \in A_R\{1\}$,

$$[Q(\lambda), K(\lambda)] = 0. \tag{20}$$

Recalling Equation (5),

$$Q = \cos(2\pi \mu_t)I_2 + \frac{i \sin(2\pi \mu_t)}{\mu_t} A_t.$$ 

Hence Equation (20) implies that $[A_t, K] = 0$ whenever $\mu_t(\lambda)^2 \neq \frac{1}{4}$. Using (6), $[A_t(\lambda), K(\lambda)] = 0$ for all $(t, \lambda) \in (-T, T) \setminus \{0\} \times A_R\{1\}$. \hfill $\square$

We can now extend the $z^AP$ form of $\Phi_t$ to $\lambda = 1$. For $t \neq 0$ and $\lambda \in A_1\{1\}$, use Lemma 3 to write

$$\Phi_t(z, \lambda) = U_t(\lambda)z^{A_t(\lambda)}K_t(\lambda)P_t(z, \lambda).$$

Let $\epsilon > 0$ small enough for $P_t(\cdot; \lambda)$ to be defined on $\overline{D}_\epsilon$. On $S_\epsilon \times A_1\{1\}$, $\Phi_t$ and $z^{A_t}$ are bounded. Then the map $(z, \lambda) \mapsto K_tP_t$ is bounded on $S_\epsilon \times A_1\{1\}$ and holomorphic on $D_\epsilon \times A_1\{1\}$, so it is bounded on $D_\epsilon \times A_1\{1\}$. But $P_t(0, \lambda) = I_2$, so $K_t$ is bounded on $A_1\{1\}$. Thus, $P_t$ is bounded on $D_\epsilon \times A_1\{1\}$. But $P_t$ is holomorphic on $D_\epsilon \times A_R\{1\}$ with at most a pole at $\lambda = 1$, so $P_t$ is holomorphic on $D_\epsilon \times A_R$ and $M_t$ is holomorphic on $A_R$. This ends the proof of Proposition 2.

### 3.2 A property of $\xi_t$

The fact that there exists a holomorphic frame $\Phi_t$ associated to $\xi_t$ such that $M(\Phi_t) \in \Lambda\text{SU}_2$ and $\Phi_0(1, \lambda) = I_2$ gives us a piece of information on the potential $\xi_t$. Let $C_t(\lambda) \in \mathfrak{sl}_2\mathbb{C}$ so that

$$\xi_t(z, \lambda) = A_t(\lambda)z^{-1}dz + tC_t(\lambda)dz + \mathcal{O}(t, z)dz$$

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and write
\[ C_t(\lambda) = \begin{pmatrix} c_{11}(t, \lambda) & \lambda^{-1} c_{12}(t, \lambda) \\ c_{21}(t, \lambda) & -c_{11}(t, \lambda) \end{pmatrix}. \] (21)

Define
\[ p_t = \frac{sc_{12}(t, 0) + rc_{21}(t, 0)}{2}. \] (22)

Lemma 4. The quantity \( p_t \) vanishes at \( t = 0 \).

Proof. First, note that \( \Phi_0(1, \lambda) = I_2 \) implies that \( \Phi_0(z, \lambda) = z^{A_0(\lambda)} \), and thus \( \mathcal{M}(\Phi_0) = -I_2 \). Let \( \gamma \subset \mathbb{D}_\epsilon^* \) be a closed loop around 0. Apply Proposition 5 of Appendix B to get (\( X' \) denotes the derivative of \( X \) at \( t = 0 \) and \( R_\epsilon \) is the holomorphic part of \( \xi_\epsilon)\)

\[ \mathcal{M}(\Phi_t)' = \int_\gamma z^{A_0} \xi' z^{-A_0} \times \mathcal{M}(\Phi_0) \]
\[ = -\int_\gamma z^{A_0} (A' z^{-1}) z^{-A_0} dz - \int_\gamma z^{A_0} R' z^{-A_0} dz \]
\[ = \mathcal{M}(z^{A_1})' - \int_\gamma z^{A_0} R' z^{-A_0} dz. \]

But \( \mathcal{M}(\Phi_t), \mathcal{M}(z^{A_1}) \in \Lambda SU_2 \) and \( \mathcal{M}(\Phi_0) = \mathcal{M}(z^{A_0}) = -I_2 \). Thus, \( \mathcal{M}(\Phi_t)', \mathcal{M}(z^{A_1})' \in \Lambda su_2 \) and
\[ \int_\gamma z^{A_0} R' z^{-A_0} dz \in \Lambda su_2. \] (23)

Diagonalise \( A_0 = HDH^{-1} \) with
\[ D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \]
and \( H \in \Lambda SU_2 \) to be expressed later. Then
\[ z^D = \frac{1}{\sqrt{z}} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \]
and
\[ \int_\gamma z^{A_0} R' z^{-A_0} dz = \int_\gamma H z^D H^{-1} (C_0 + \mathcal{O}(z)) H z^{-D} H^{-1} \]
\[ = H \left( \text{Res}_{z=0} z^D H^{-1} C_0 H z^{-D} \right) H^{-1}. \]

Equation (23) and \( H \in \Lambda SU_2 \) imply that
\[ \text{Res}_{z=0} \left( z^D H^{-1} C_0 H z^{-D} \right) \in \Lambda su_2. \] (24)
Denoting by $c(\lambda)$ the bottom-left entry of $H^{-1}C_0H$ and looking at the product $z^D(H^{-1}C_0H)z^{-D}$, Equation (24) gives
\[
\begin{pmatrix}
0 & 0 \\
c(\lambda) & 0
\end{pmatrix} \in \Lambda_{\mathfrak{su}_2}
\]
and thus,
\[c(\lambda) = (H^{-1}C_0H)_{21} \equiv 0 \quad \text{(25)}\]
Two cases can occur:

- If $r \geq s$,
  \[
  H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix} \in \Lambda_{\mathfrak{su}_2}
  \]
  and computation gives
  \[c(\lambda) = -\lambda \left( c_{11}(0, \lambda) + \frac{c_{12}(0, \lambda)}{2} \right) + \frac{c_{21}(0, \lambda)}{2}.\]

  Using Equation (25), $c_{21}(0, 0) = 0$ and $p_0 = 0$.

- If $r \leq s$, the same reasoning applies with
  \[
  H(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad c(\lambda) = -\lambda^{-1} \frac{c_{12}(0, \lambda)}{2} + \frac{c_{21}(0, \lambda)}{2} - c_{11}(0, \lambda).
  \]

  Thus, $c_{12}(0, 0) = 0$ and $p_0 = 0$.

\[\square\]

3.3 Gaining an order of convergence

We can now prove Proposition 1 by following the method used in Section 2.2 of [7]: gauging the potential. The gauge we will use is of the following form:
\[G_t(z, \lambda) = \exp \left( g_t(\lambda)z \right) \quad \text{(26)}\]
which is an admissible gauge provided that $g_t \in \Lambda_{\mathfrak{sl}_2 \mathbb{C}}$. This is why we need the following lemma:

**Lemma 5.** Let
\[g_t(\lambda) = p_t A_t(\lambda) - P_{t,1}(\lambda)\]
where $P_{t,1}$ is defined in Equation (19). Then
1. The map $g_t$ is in $\Lambda_+ \mathfrak{sl}_2 \mathbb{C}$. 

2. The map $g_t$ extends to $t = 0$ with $g_0 = 0$.

Proof. To prove the first point, let $t \neq 0$ and use Equations (19), (21), (17) and (22) to compute (this is a tedious calculation)

$$P_{t,1}(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & r p_t \\ 0 & 0 \end{pmatrix} + \lambda^0 \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix} + \mathcal{O}(\lambda).$$

Thus,

$$g_t(\lambda) = p_t A_t(\lambda) - P_{t,1}(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \lambda^0 \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix} + \mathcal{O}(\lambda).$$

For the second point, use Equations (19) and (17) to write for $t \neq 0$:

$$P_{t,1} = t \mathcal{L}_{t,1}^{-1} (C_t) = t \left( C_t - \frac{1}{1 - 4\mu_t^2} (I_2 - 2A_t) [A_t, C_t] \right).$$

Note that $C_t$ is continuous at $t = 0$ because $\xi_t \in \mathcal{C}^2$ and that $1 - 4\mu_t^2 = \mathcal{O}(t)$ to extend $P_{t,1}$ to $t = 0$. Moreover, recall Lemma 4, Equation (6) and diagonalise $A_0 = HDH^{-1}$ to get:

$$g_0 = \frac{-\lambda}{4(\lambda - 1)^2} H (I_2 - 2D) \left[ D, H^{-1} C_0 H \right] H^{-1}.$$

A straightforward computation gives

$$(I_2 - 2D) \left[ D, H^{-1} C_0 H \right] = \begin{pmatrix} 0 & 0 \\ -2c(\lambda) & 0 \end{pmatrix}$$

with $c(\lambda)$ as in Equation (25). Hence $g_0 = 0$. $\square$

Let $G_t$ be the gauge defined by (26). Then the gauged potential has the form

$$\xi_t \cdot G_t(z, \lambda) = A_t(\lambda) z^{-1}dz + ([A_t(\lambda), g_t(\lambda)] + g_t(\lambda) + tC_t(\lambda)) dz + \mathcal{O}(t, z)dz + \mathcal{O}(g_t^2 z)dz$$

$$= A_t(\lambda) z^{-1}dz + (\mathcal{L}_{t,1}(g_t(\lambda)) + tC_t(\lambda)) dz + \mathcal{O}(t, z)dz$$

$$= A_t(\lambda) z^{-1}dz + p_t A_t(\lambda) dz + \mathcal{O}(t, z)dz,$$

because of Equation (19). This gauge has been chosen to fit with the following change of coordinate:

$$h_t(z) = \frac{z}{1 + p_t z}.$$
The resulting potential (defined in Proposition 1) is then
\[ \zeta_t = A_t \frac{dz}{1 + p_t z} + p_t A_t \frac{dz}{(1 + p_t z)^2} + \mathcal{O}(t, z) dz = A_t z^{-1} dz + \mathcal{O}(t, z) dz \]
because \( p_0 = 0 \). Apply the Fröbenius method to \( \zeta_t \) to obtain (14) and choose \( \epsilon' \leq \epsilon \) such that for all \( t \neq 0 \), \( \epsilon' < |p_t|^{-1} \) to end the proof of Proposition 1.

4 Convergence of immersions

In this section, we prove the first and third points of Theorem 3. In the end, we want to compare \( \Phi_t(z, \lambda) = M_t(\lambda) z^{A_t(\lambda)} (I_2 + \mathcal{O}(t, z^2)) \) to
\[ \Phi_t^D(z, \lambda) = M_t(\lambda) z^{A_t(\lambda)}. \]
We will denote
\[ F_t^D = \text{Uni}(\Phi_t^D) \]
and
\[ f_t^D = \text{Sym}(F_t^D). \]
We first want to make sure that \( \Phi_t^D \) induces a Delaunay surface for all \( t \). For this purpose, recall Lemma 1.12 in [7], which implies that \( f_t^D \) is a Delaunay surface of weight \( 8\pi t \). Hence, there exists a rigid motion \( \phi \) of \( \mathbb{R}^3 \) such that \( \phi \circ f_t^D \) has the following parametrisation:
\[ \phi \circ f_t^D : \quad z = e^{x+i y} \quad \rightarrow \quad (\tau_t(x), \sigma_t(x) \cos y, \sigma_t(x) \sin y) \]
where \( (\tau_t(x), \sigma_t(x)) \) is the profile curve of the surface. Recalling that the coordinates are isothermal gives the following metric:
\[ ds_t^2 = \sigma_t^2 \frac{|dz|^2}{|z|^2}. \]

Let us compare the asymptotic behaviours of the unitary parts of \( \Phi_t \) and \( \Phi_t^D \) for \( \lambda \in A_1 \) using, as in [7], a Cauchy formula. We will use the following norms:

- For \( v = (v_1, v_2) \in \mathbb{C}^2 \), \( |v| = (|v_1|^2 + |v_2|^2)^{1/2} \).
- For \( M \in \mathcal{M}_2(\mathbb{C}) \), \( ||M|| = \sup_{|v|=1} |Mv| \).
• For $\Psi : \mathcal{E} \to \mathcal{M}_2(\mathbb{C})$, $\|\Phi\|_{\mathcal{E}} = \sup_{\lambda \in \mathcal{E}} \|\Psi(\lambda)\|$.

**Lemma 6.** For all $\alpha < 1$ there exist constants $\epsilon > 0$, $T > 0$ and $C > 0$ such that for all $0 < |z| < \epsilon$ and $|t| < T$,

$$\left\| (F_t^P)^{-1} F_t - I_2 \right\|_{A_1} \leq C |t| |z|^\alpha$$  \hspace{1cm} (28)

and

$$\left\| \frac{\partial}{\partial \lambda} \left[ (F_t^P)^{-1} F_t \right] \right\|_{A_1} \leq C |t| |z|^\alpha.$$  \hspace{1cm} (29)

**Proof.** The first step is to estimate the norm of the positive part $B_t^P$ of $\Phi_t^P$. We first estimate $\Phi_t^P$ for $|z| < 1$: noting that $A_t$ is diagonalisable, that its eigenvalues tend to $\pm 1/2$ as $t \to 0$, and recalling that $M_t$ is continuous at $t = 0$ ensure that for all $\alpha < 1$ there exists $(T, R)$ and $C_1 > 1$ such that for all $|t| < T$,

$$\left\| \Phi_t^P(z, \lambda) \right\|_{A_R} \leq C_1 |z|^{-\frac{1}{2} - \frac{1-\alpha}{4}}.$$  

We then estimate $F_t^P$: let $\gamma \subset \mathbb{C}^*$ be a path from $z$ to 1, use Equation (39) of Appendix C and Equation (27) to get

$$\left\| F_t^P(z, \lambda) \right\|_{A_R} \leq C_2 \left\| F_t^P(1, \lambda) \right\|_{A_R} \times \exp \left( \frac{(R - 1)}{2} \int_{\gamma} \frac{\sigma_t(\log |z|)}{|z|} \right).$$

But $\sigma_t$ is uniformly bounded because so is the distance between the profile curve and the axis of a Delaunay surface. Moreover, the unitary frame at $z = 1$ is also bounded. Hence the existence, for $R > 1$ small enough, of a constant $C_3 \geq 1$ such that

$$\left\| F_t^P(z, \lambda) \right\|_{A_R} \leq C_3 |z|^{-\frac{1-\alpha}{4}}.$$  

We can now estimate the positive factor: for all $\alpha < 1$ there exist $T > 0$, $R > 1$ and $C_4 \geq 1$ such that for all $|t| < T$ and $|z| < 1$

$$\left\| B_t^P(z, \lambda) \right\|_{A_R} \leq \left\| F_t^P(z, \lambda)^{-1} \right\|_{A_R} \times \left\| \Phi_t^P(z, \lambda) \right\|_{A_R} \leq C_4 |z|^{\frac{2}{2} - 1}.$$  

We then define

$$\tilde{\Phi}_t := \left( (F_t^P)^{-1} F_t \right) \times \left( B_t (B_t^P)^{-1} \right) = B_t^P \left( \Phi_t^P \right)^{-1} \Phi_t \left( B_t^P \right)^{-1}$$  \hspace{1cm} (28)

$$= \tilde{F}_t \times \tilde{B}_t$$  \hspace{1cm} (29)
with \( \tilde{F}_t \in \Lambda SU_2 \) and \( \tilde{B}_t \in \Lambda^{P}_{\mathbb{R}} SL_2 \mathbb{C} \) and thus have

\[
\left\| \tilde{\Phi}_t(z, \lambda) - I_2 \right\|_{A_R} = \left\| B_t^D(z, \lambda) \left( P_t(z, \lambda) - I_2 \right) \left( B_t^D(z, \lambda) \right)^{-1} \right\|_{A_R} \\
\leq \left\| B_t^D(z, \lambda) \right\|_{A_R}^2 \mathcal{O}(t, |z|^2) \\
\leq C |t||z|^\alpha.
\]

Let \( n_k \) denote the seminorms

\[
n_k(X) = \sum_{j=0}^{k} \left\| \frac{\partial^k X}{\partial \lambda^k} \right\|_{A_1}.
\]

Apply Cauchy formula with \( \lambda \in \partial A_R \) to get

\[
n_k \left( \tilde{\Phi}_t - I_2 \right) \leq c_k |t||z|^\alpha, \quad \forall k \in \mathbb{N}
\]

where \( c_k > 0 \) are uniform constants. But \( \text{Uni}(\tilde{\Phi}_t) = \tilde{F}_t = (F_t^D)^{-1} F_t \) and Iwasawa decomposition is a \( C^1 \)-diffeomorphism, so \( n_0 \left( \tilde{F}_t - I_2 \right) \leq C |t||z|^\alpha \) and \( n_1 \left( \tilde{F}_t - I_2 \right) \leq C |t||z|^\alpha. \) We then have (28) and (29).

The asymptotic behaviour of \( \frac{\partial \tilde{F}_t}{\partial \lambda} \) allows us to prove the convergence of immersions as stated in the first point of Theorem 3. The Sym-Bobenko formula for \( \mathbb{R}^3 \) implies that (we omit the index \( t \))

\[
iF(z, 1) \frac{\partial (F^{-1}F^D)}{\partial \lambda}(z, 1)F^D(z, 1)^{-1} = i \frac{\partial F^D}{\partial \lambda}(z, 1)F^D(z, 1)^{-1} - i \frac{\partial F}{\partial \lambda}(z, 1)F(z, 1)^{-1} \\
= f^D(z) - f(z).
\]

We can then compute

\[
\left\| f_t(z) - f_t^D(z) \right\|_{\mathbb{R}^3}^2 = 4 \det (f_t(z) - f_t^D(z)) \\
= -4 \det \frac{\partial (F_t^{-1}F_t^D)}{\partial \lambda}(z, 1) \\
\leq C_2 t^2 |z|^{2\alpha}.
\]

And then for all \( \alpha < 1 \) there exist constants \( \epsilon > 0, T > 0 \) and \( C > 0 \) such that for all \( 0 < |z| < \epsilon \) and \( |t| < T \),

\[
\left\| f_t(z) - f_t^D(z) \right\|_{\mathbb{R}^3} \leq C |t||z|^\alpha.
\]

(30)
To prove the third point of Theorem 3, use (4) and note that \( M_0 = I_2 \). So the axis of \( f_t^D \) as \( t \to 0 \) is the same that the axis of the unperturbed Delaunay surface induced by \( z^{A_t} \).

In order to prove that the surface is embedded, we will need the convergence of the normal maps:

**Proposition 3.** For all \( \alpha < 1 \) there exist constants \( \epsilon > 0 \), \( T > 0 \) and \( C > 0 \) such that for all \( 0 < |z| < \epsilon \) and \( |t| < T \),

\[
||N_t(z) - N_t^D(z)||_{\mathbb{R}^3} \leq C|t||z|^{\alpha}
\]

*Proof.* Use the definition of the normal maps in Equation (2) to write

\[
N_t(z) - N_t^D(z) = -\frac{i}{2} F_t^D(z,1) \left[ AM\tilde{A} + AM + M\tilde{A} \right] F_t^D(z,1)^{-1}
\]

where

\[
A = F_t^D(z,1)^{-1} F_t(z,1) - I_2 = O(t, |z|^\alpha),
\]

\[
\tilde{A} = F_t(z,1)^{-1} F_t^D(z,1) - I_2 = O(t, |z|^\alpha)
\]

and

\[
M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Use equation (1) to get the conclusion. \( \square \)

It remains to show that the surface is embedded if \( t > 0 \).

## 5 Embeddedness

We suppose in this section that \( 0 < t < T \). The asymptotic behaviour of \( f_t \) and the fact that \( f_t^D \) is an embedding for all \( t \) allow us to show that \( f_t \) is an embedding of a sufficiently small uniform neighbourhood of \( z = 0 \) for \( t \) small enough. We first give a general result of embeddedness and then apply this result to show that our surfaces are embedded.

**Proposition 4.** Let \( f_n^R : \mathbb{C}^* \rightarrow \mathcal{M}_n^R = f_n^R(\mathbb{C}^*) \subset \mathbb{R}^3 \) be a sequence of complete immersions with normal maps \( N_n^R \) and an end at \( z = 0 \). Suppose that for all \( n \) there exists \( r_n > 0 \) such that the tubular neighbourhood \( \text{Tub}_{r_n} \mathcal{M}_n^R \) of \( \mathcal{M}_n^R \) is embedded. Suppose that for all \( \epsilon > 0 \) there exists \( 0 < \epsilon' < \epsilon \) such that for all \( n \in \mathbb{N} \), \( x \in \mathbb{S}_\epsilon \) and \( y \in \mathbb{D}_{\epsilon'} \),

\[
||f_n^R(x) - f_n^R(y)||_{\mathbb{R}^3} > 2r_n.
\]

(31)
Let $U^* \subset \mathbb{C}^*$ be a punctured neighbourhood of $z = 0$ and $f_n : U^* \rightarrow \mathbb{R}^3$ a sequence of immersions with normal maps $N_n$ satisfying

$$
\sup_{n \in \mathbb{N}} \frac{\|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3}}{r_n} \xrightarrow{z \to 0} 0 \tag{32}
$$

and

$$
\sup_{z \in U^*} \|N_n(z) - N_n^R(z)\|_{\mathbb{R}^3} \xrightarrow{n \to \infty} 0. \tag{33}
$$

Then there exist $\epsilon' > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $f_n$ is an embedding of $\mathbb{D}_*^\epsilon$. 

**Proof.** Let us split the proof in several steps.

- **Claim 1:** there exists $\epsilon > 0$ such that the map

  $$
  \varphi_n : \mathbb{D}_*^\epsilon \rightarrow \mathbb{M}_n^R
  \quad
  z \mapsto \pi_n \circ f_n(z)
  $$

  (where $\pi_n$ is the projection from $\text{Tub}_{r_n} \mathbb{M}_n^R$ onto $\mathbb{M}_n^R$) is well-defined and satisfies

  $$
  \|\varphi_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < r_n \tag{34}
  $$

  for all $z \in \mathbb{D}_*^\epsilon$.

  To prove this first claim, use Hypothesis (32): there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$ and $z \in \mathbb{D}_*^\epsilon$

  $$
  \|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < \frac{r_n}{2}. \tag{35}
  $$

  So $f_n(\mathbb{D}_*^\epsilon) \subset \text{Tub}_{r_n} \mathbb{M}_n^R$ and $\varphi_n$ is well-defined. Moreover, using (35) and the triangle inequality, for all $z \in \mathbb{D}_*^\epsilon$

  $$
  \|\varphi_n(z) - f_n^R(z)\|_{\mathbb{R}^3} \leq \|\varphi_n(z) - f_n(z)\|_{\mathbb{R}^3} + \|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} < r_n
  $$

  and Equation (34) holds. We fix $\epsilon$ and $\epsilon'$ so that Equation (31) is satisfied.

- **Claim 2:** there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\varphi_n$ is a local diffeomorphism on $\mathbb{D}_*^\epsilon$.

Let $z \in \mathbb{D}_*^\epsilon$. In order to show that $\varphi_n$ is a local diffeomorphism, we show that

$$
\langle N_{\varphi_n}(z), N_n(z) \rangle > 0 \tag{36}
$$
where $\mathcal{N}_{\varphi_n}$ is defined by

\[
\mathcal{N}_{\varphi_n} : \mathbb{D}_\varepsilon^* \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3 \\
\quad z \longmapsto \eta_n^R(\varphi_n(z))
\]

and $\eta_n^R$ is the Gauss map of $\mathcal{M}_n^R$. First, let $\gamma \subset \mathcal{M}_n^R$ be a path joining $\varphi_n(z)$ to $f_n^R(z)$. Using the fact that $\text{Tub}_{r_n}\mathcal{M}_n^R$ is embedded, one has

\[
\|d\eta_n^R\| \leq \frac{1}{r_n}
\]

and

\[
\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\|_{\mathbb{R}^3} \leq \frac{1}{r_n} \times |\gamma|.
\]

Let $\sigma(t) = (1 - t)f_n(z) + tf_n^R(z)$, $t \in [0, 1]$. Then,

\[
\|\sigma(t) - f_n^R(z)\|_{\mathbb{R}^3} \leq (1 - t)\|f_n(z) - f_n^R(z)\|_{\mathbb{R}^3} \leq \frac{r_n}{2} \quad (37)
\]

because of Equation (35). Let $\gamma = \pi_n \circ \sigma$. Note that Equation (37) implies that $\sigma \subset \text{Tub}_{\varphi_n}\mathcal{M}_n^R$ and restricting $\pi_n$ to $\text{Tub}_{\varphi_n}\mathcal{M}_n^R$ gives

\[
\|d\pi_n\| \leq \frac{r_n}{r_n - \frac{r_n}{2}} = 2
\]

and thus $|\gamma| < r_n$. Hence,

\[
\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\| < 1.
\]

Use Hypothesis (33) to choose a uniform $N \in \mathbb{N}$ such that for all $n \geq N$,

\[
\|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n(z)\| \leq \|\mathcal{N}_{\varphi_n}(z) - \mathcal{N}_n^R(z)\| + \|\mathcal{N}_n^R(z) - \mathcal{N}_n(z)\| < \sqrt{2},
\]

which proves Equation (36) and this second claim. We fix such $N$ and $n$.

- **Claim 3:** the restriction

\[
\tilde{\varphi}_n : \varphi_n^{-1}(\varphi_n(\mathbb{D}_r^*)) \cap \mathbb{D}_\varepsilon^* \longrightarrow \varphi_n(\mathbb{D}_r^*) \\
\quad z \longmapsto \varphi_n(z)
\]

is a covering map.
It suffices to show that $\tilde{\varphi}_n$ is a proper map. Let $(x_i)_{i \in \mathbb{N}} \subset \varphi_n^{-1}(\varphi_n(D^c_n)) \cap D^c_n$ such that $(\tilde{\varphi}_n(x_i))_{i \in \mathbb{N}}$ converges to $p \in \varphi_n(D^c_n)$. Then $(x_i)_i$ converges to $x \in \overline{D}$. Using Equation (34) and the fact that $f^{R}_n$ has an end at 0, $x \neq 0$. If $x \in \partial D$, denoting $\tilde{x} \in D^c_n$ such that $\tilde{\varphi}_n(\tilde{x}) = p$, one has
\[
\|f_n^R(x) - f_n^R(\tilde{x})\|_{\mathbb{R}^3} < \|f_n^R(x) - p\|_{\mathbb{R}^3} + \|f_n^R(\tilde{x}) - \tilde{\varphi}_n(\tilde{x})\|_{\mathbb{R}^3} < 2r_n
\]
which contradicts the definition of $\epsilon'$. Thus, $\tilde{\varphi}_n$ is a proper local diffeomorphism between locally compact spaces, i.e. a covering map.

- **Claim 4**: this covering map is one-sheeted.

To compute the number of sheets, let $\gamma : [0, 1] \rightarrow D^c_n$ be a loop of winding number 1 around 0, $\Gamma = f_n^R(\gamma)$ and $\tilde{\Gamma} = \tilde{\varphi}_n(\gamma) \subset \mathcal{M}^R_n$ and let us construct a homotopy between $\Gamma$ and $\tilde{\Gamma}$. Let
\[
\sigma_t : [0, 1] \rightarrow \mathbb{R}^3,
\quad s \mapsto (1 - s)\tilde{\Gamma}(t) + s\Gamma(t).
\]
For all $t, s \in [0, 1]$,
\[
\|\sigma_t(s) - \Gamma(t)\|_{\mathbb{R}^3} < r_n
\]
which implies that $\sigma_t(s) \in \text{Tub}_{r_n}\mathcal{M}^R_n$ because $\mathcal{M}^R_n$ is complete. One can thus define the following homotopy between $\Gamma$ and $\tilde{\Gamma}$
\[
H : [0, 1]^2 \rightarrow \mathcal{M}^R_n,
(s, t) \mapsto \pi_n \circ \sigma_t(s)
\]
where $\pi_n$ is the projection from $\text{Tub}_{r_n}\mathcal{M}^R_n$ to $\mathcal{M}^R_n$. Using the fact that $f_n^R$ is an embedding, the degree of $\Gamma$ is one, and the degree of $\tilde{\Gamma}$ is also one. Hence, $\tilde{\varphi}_n$ is one-sheeted.

- **Conclusion**: the map $\tilde{\varphi}$ is a diffeomorphism, so $f_n(D^c_n)$ is a graph over $\mathcal{M}^R_n$ contained in its embedded tubular neighbourhood and $f_n(D^c_n)$ is thus embedded.

We can now apply Proposition 4 to each case. Let $(t_n)$ be any sequence in $(\mathbb{T}, T)$ such that $t_n \rightarrow 0$.

- If $r \geq s$, we set $\widehat{f}_n^R = f_n^P$ and $\widehat{f}_n = f_n$. We aim to apply Proposition 4 on $\widehat{f}_n^R$ and $\widehat{f}_n$. The tubular radius $r_n$ is of the order of $4t_n$ and Hypothesis (31) is satisfied because $\widehat{f}_n^R$ tends to an immersion of a sphere. Equation (30) and Proposition 3 ensure that Hypotheses (32) and (33) hold.
• If \( r \leq s \), we set \( \hat{f}_n^R = \frac{1}{r_n} f_n^P \) and \( \hat{f}_n = \frac{1}{r_n} f_n \). We aim to apply Proposition 4 on \( \hat{f}_n^R \) and \( \hat{f}_n \). The tubular radius \( r_n \) is of the order of 4 and Hypothesis (31) is satisfied because \( \hat{f}_n^R \) tends to an immersion of a catenoid (see [21]). Equation (30) and Proposition 3 ensure that Hypotheses (32) and (33) hold.

The second point of our theorem is then proved.

## A Iwasawa extended

In this section, we note \( \mathcal{A}_{\hat{R}^{-1}} = \{ \lambda \in \mathbb{C} : \frac{1}{|\tau|} < |\lambda| < 1 \} \).

**Lemma 7.** Let \( F : \mathcal{A}_{\hat{R}^{-1}} \rightarrow \mathrm{SL}_2 \mathbb{C} \) be a holomorphic map that can be continuously extended to the circle \( \mathcal{A}_1 \) and such that \( F(\lambda) \in \mathrm{SU}_2 \) for all \( \lambda \in \mathcal{A}_1 \). Then \( F \) holomorphically extends to \( \mathcal{A}_R \) into a map that satisfies

\[
\begin{align*}
\bar{F}(1/\lambda) = F(\lambda)^{-1} \quad & \forall \lambda \in \mathcal{A}_R.
\end{align*}
\]

**Proof.** Apply Schwarz reflexion principle on each coefficient of the matrix

\[
\bar{F}(\lambda) = \begin{pmatrix}
F_{11}(\lambda) + F_{22}(\lambda) & F_{12}(\lambda) - F_{21}(\lambda) \\
\bar{i}(F_{12}(\lambda) + F_{21}(\lambda)) & \bar{i}(F_{11}(\lambda) - F_{22}(\lambda))
\end{pmatrix}
\]

where \( F_{ij} \) denote the entries of \( F \). The fact that \( F(\lambda) \in \mathrm{SU}_2 \) for all \( \lambda \in \mathcal{A}_1 \) ensures that \( \Im \bar{F} = 0 \) on \( \mathcal{A}_1 \). Thus, \( \bar{F} \) holomorphically extends to \( \mathcal{A}_R \) and satisfies for all \( \lambda \in \mathcal{A}_R \)

\[
\bar{F}(1/\lambda) = \overline{F(\lambda)}.
\]

Hence, \( F \) holomorphically extends to \( \mathcal{A}_R \) and satisfies

\[
F_{11}\left(\frac{1}{\lambda}\right) = F_{22}(\lambda), \quad F_{12}\left(\frac{1}{\lambda}\right) = -F_{21}(\lambda)
\]

which implies Equation (38) because \( F(\lambda) \in \mathrm{SL}_2 \mathbb{C} \). \( \Box \)

**Corollary 3.** Let \( \Phi : \mathcal{A}_R \rightarrow \mathrm{SL}_2 \mathbb{C} \) be a holomorphic map and let \( FB \) be the Iwasawa decomposition of its restriction to \( \mathcal{A}_1 \). Then \( F \) holomorphically extends to \( \mathcal{A}_R \), satisfies Equation (38), and \( B \) holomorphically extends to \( \mathcal{D}_R \).

**Proof.** Write \( F = \Phi B^{-1} \) to holomorphically extend \( F \) to \( \mathcal{A}_{\hat{R}^{-1}} \). Apply Lemma 7 to holomorphically extend \( F \) to \( \mathcal{A}_R \), and write \( B = F^{-1} \Phi \) to holomorphically extend \( B \) to \( \mathcal{D}_R \). \( \Box \)
B Derivative of the monodromy

The following proposition, used in Section 3, is derived from Proposition 8 in [22].

**Proposition 5.** Let $\xi_t$ be a $C^1$ family of matrix-valued 1-forms on a Riemann surface $\Sigma$, defined for $t$ in a neighbourhood of $t_0 \in \mathbb{R}$. Let $\tilde{\Sigma}$ be the universal cover of $\Sigma$. Fix a point $z_0$ in $\Sigma$ and let $\tilde{z}_0$ be a lift of $z_0$ to $\tilde{\Sigma}$. Let $\Phi_t$ be a continuous family of solutions of $d\Phi_t = \Phi_t \xi_t$ on $\tilde{\Sigma}$ such that for all $t$,

$$[\mathcal{M}(t_0), \Phi_{t_0}(z_0)\Phi_t(z_0)^{-1}] = 0,$$

where $\mathcal{M}(t)$ is the monodromy of $\Phi_t$ with respect to some $\gamma \in \pi_1(\Sigma, z_0)$. Let $\tilde{\gamma}$ be the lift of $\gamma$ to $\tilde{\Sigma}$ such that $\tilde{\gamma}(0) = \tilde{z}_0$. Then $\mathcal{M}$ is differentiable at $t_0$ and

$$\mathcal{M}'(t_0) = \left( \int_{\gamma} \Phi_{t_0} \frac{\partial \xi_t}{\partial t} \big|_{t=t_0} \Phi_{t_0}^{-1} \right) \times \mathcal{M}(t_0).$$

In particular, if $\mathcal{M}(t_0) = \pm I_2$ or if $\Phi_t(z_0)$ is constant, then (5) is satisfied.

**Proof.** Proposition 8 in [22] is proved in the case where $\Phi_t(z_0)$ is constant. Let $\tilde{\Phi}_t(z) = \Phi_t(z_0)^{-1}\Phi_t(z)$, so that $d\tilde{\Phi}_t = \tilde{\Phi}_t \xi_t$ and $\tilde{\Phi}_t(z_0) = I_n$. Let $\tilde{\mathcal{M}}(t)$ be the monodromy of $\tilde{\Phi}_t$ along $\gamma$. Then Proposition 5 of [22] applies and

$$\tilde{\mathcal{M}}'(t_0) = \left( \int_{\gamma} \tilde{\Phi}_{t_0}(z) \frac{\partial \xi_t(z)}{\partial t} \big|_{t=t_0} \tilde{\Phi}_{t_0}(z)^{-1} \right) \times \tilde{\mathcal{M}}(t_0).$$

On the other hand,

$$\mathcal{M}(t) = \Phi_t(z_0)\tilde{\mathcal{M}}(t)\Phi_t(z_0)^{-1}$$

and because of Equation (5),

$$\mathcal{M}(t_0) = \Phi_t(z_0)\tilde{\mathcal{M}}(t_0)\Phi_t(z_0)^{-1}.$$

Thus, $\mathcal{M}$ is differentiable at $t_0$ and

$$\mathcal{M}'(t_0) = \Phi_{t_0}(z_0)\tilde{\mathcal{M}}'(t_0)\Phi_{t_0}(z_0)^{-1}$$

which proves the proposition.  

\qed
C  A control formula on the unitary frame

The following proposition is used in Section 4.

**Proposition 6.** Let $(\Sigma, \xi, z_0, \Phi_{z_0})$ be a set of untwisted DPW data, holomorphic for $\lambda \in \mathcal{A}_R$ with $R \geq 1$. Then for all $z_1, z_2 \in \Sigma$ and $\gamma \subset \Sigma$ joining $z_1$ to $z_2$,

$$\|F(z_1, \lambda)\|_{\mathcal{A}_R} \leq C \|F(z_2, \lambda)\|_{\mathcal{A}_R} \times \exp \left( (R - 1) \int_\gamma \rho^2(w) |a_{-1}(w)||dw| \right)$$

where $C$ is a uniform positive constant, $a_{-1}(z)dz$ is the $\lambda^{-1}$ factor of $\xi$ and $\rho(z)$ is the upper-left entry of $\text{Pos} (\Phi)(z, 0)$.

**Proof.** Write

$$\xi(z, \lambda) = \lambda^{-1} \begin{pmatrix} 0 & a_{-1}(z) \\ 0 & 0 \end{pmatrix} dz + \lambda^0 \begin{pmatrix} c_0(z) & a_0(z) \\ b_0(z) & -c_0(z) \end{pmatrix} dz + \mathcal{O}(\lambda).$$

Let $\Phi = FB$ be the Iwasawa decomposition of $\Phi$. Untwisting formula (4.3.5) of [5] with the help of Remark 4.2.6 of [5] gives $dF = FL$ where

$$L(z, \lambda) = \begin{pmatrix} \rho^{-1} \rho_z & \lambda^{-1} \rho^2 a_{-1} \\ b_0 \rho^{-2} & -\rho^{-1} \rho_z \end{pmatrix} dz + \begin{pmatrix} -\rho^{-1} \rho_z & -b_0 \rho^{-2} \\ -\lambda \rho^2 a_{-1} & \rho^{-1} \rho_z \end{pmatrix} dz.$$

Let

$$\tilde{F}(z, \lambda) = F \left( z, \frac{\lambda}{|\lambda|} \right)$$

so that $\tilde{F}(z, \lambda) \in \text{SU}_2$ for all $\lambda \in \mathcal{A}_R$. Then $d\tilde{F} = \tilde{F}L$ where

$$\tilde{L}(z, \lambda) = L \left( z, \frac{\lambda}{|\lambda|} \right).$$

Using the variation of constants method, for all $z_1, z_2 \in \Sigma$ (we omit the variable $\lambda$),

$$F(z_1) = F(z_2) \tilde{F}(z_2)^{-1} \tilde{F}(z_1) + \left( \int_{z_2}^{z_1} F(w) \left( L(w) - \tilde{L}(w) \right) \tilde{F}(w)^{-1} \right) \tilde{F}(z_1).$$

But

$$L(w, \lambda) - \tilde{L}(w, \lambda) = \rho^2(w) \begin{pmatrix} 0 & a_{-1}(w)\lambda^{-1} (1 - |\lambda|) \, dw \\ -\bar{a}_{-1}(w) \lambda (1 - |\lambda|)^{-1} \, d\bar{w} & 0 \end{pmatrix}$$

30
so there exists a uniform constant $\tilde{C}$ such that

$$\|L(w, \lambda) - \tilde{L}(w, \lambda)\|_{A_R} \leq \tilde{C}(R - 1)\rho^2(w)|a_{-1}(w)||dw|$$

and the result follows from Gronwall’s inequality (Lemma 2.7 in [19]) using the fact that $\tilde{F} \in SU_2$ for all $\lambda \in A_R$.

As an application, recall that in the untwisted $\mathbb{R}^3$ setting, if $f = \text{Sym}(F)$, then $f$ is a CMC 1 conformal immersion whose metric is given by

$$ds = 2\rho^2|a_{-1}||dz|.$$ 

So let $z_1, z_2 \in \Sigma$ and $\gamma \subset \Sigma$ be a path joining $f(z_1)$ to $f(z_2)$. Then,

$$\|F(z_1, \lambda)\|_{A_R} \leq C \|F(z_2, \lambda)\|_{A_R} \exp\left(\frac{(R - 1)}{2}|\gamma|\right).$$ (39)
References


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