PRODUCT FORMULAS ON POSETS, WICK PRODUCTS, AND A CORRECTION FOR THE q-POISSON PROCESS

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ABSTRACT. We give an example showing that the product and linearization formulas for the Wick product versions of the $q$-Charlier polynomials in [Ans04a] are incorrect. Next, we observe that the relation between monomials and several families of Wick polynomials is governed by “incomplete” versions of familiar posets. We compute Möbius functions for these posets, and prove a general poset product formula. These provide new proofs and new inversion and product formulas for Wick product versions of Hermite, Chebyshev, Charlier, free Charlier, and Laguerre polynomials. By different methods, we prove inversion formulas for the Wick product versions of the free Meixner polynomials.

1. INTRODUCTION

Let $\mathcal{M}$ be a complex star-algebra, and $\langle \cdot \rangle$ a star-linear functional on it. Let $\Gamma(\mathcal{M})$ be the complex unital star-algebra generated by non-commuting symbols \{\(X(a) : a \in \mathcal{M}\)\} and 1, subject to the linearity relations

\[ X(\alpha a + \beta b) = \alpha X(a) + \beta X(b). \]

Thus $\Gamma(\mathcal{M})$ is naturally isomorphic to the tensor algebra of $\mathcal{M}$, but we prefer to use the polynomial notation rather than this identification. In particular the algebra is filtered by the degree of polynomials. The star-operation on $\Gamma(\mathcal{M})$ is determined by the requirement that $X(a^*) = X(a)^*$.

In this article we will discuss six constructions of Wick products (four known and two new), that is, linear maps $W : \mathcal{M}^\otimes n \to \Gamma(\mathcal{M})$ whose ranges (together with the scalars) provide a grading compatible with the degree filtration of $\Gamma(\mathcal{M})$. These have also been called the Kailath-Segall polynomials. Note that in the literature, the term “Wick product” is also used for the multivariate Appell polynomials. See, for example, Sections 2.3 and 2.8 of [Ans04a] for differences and similarities between these two families.

As is well-known, there are three reasons to consider Wick products.

- Define a star-linear functional $\varphi$ of $\Gamma(\mathcal{M})$ by $\varphi[1] = 1$, $\varphi[W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)] = 0$ for $n \geq 1$. In many examples, $\varphi$ is positive, the $W$ operators have orthogonal ranges for different $n$, and we have a Fock representation of $\Gamma(\mathcal{M})$ on (a quotient of) $L^2(\Gamma(\mathcal{M}), \varphi)$. In this case the $W$ operators are indeed Wick products.
For $\mathcal{M} = (L^1 \cap L^\infty)(\mathbb{R})$, in many examples we have an Itô isometry which allows us to interpret $W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ as a stochastic integral

$$\int \ldots \int a_1(t_1) \ldots a_n(t_n) \, dX(t_1) \ldots dX(t_n).$$

This isometry may involve unfamiliar inner products on multivariate function spaces, see Remark 42.

Suppose all $a_1 = \ldots = a_n = a$. Setting $a = 1$, or more generally a multiple of a projection, $W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ becomes a polynomial in $X(a)$. In many examples, these polynomials are orthogonal for different $n$.

Our main interest is in mutual expansions between polynomials $W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ and monomials, product formulas

$$\prod_{i=1}^k W(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}) = \sum W,$$

and corresponding linearization coefficients. We show that the product formulas for $q$-Wick products claimed in [Ans04a] are incorrect. The rest of the article concerns related positive results.

In combinatorics, linearization formulas are often proved using a weight-preserving sign-reversing involution, in other words a version of the inclusion-exclusion principle. We use a different generalization of this principle, namely Möbius inversion. For five out of our six examples, we define posets $\Pi$ such that

$$X(a_1) \ldots X(a_n) = \sum_{\pi \in \Pi} W(a_1 \otimes \ldots \otimes a_n)^\pi.$$

The posets which arise are “incomplete” versions of matchings, non-crossing matchings, set partitions, non-crossing partitions, and permutations. These posets, as posets, may be deserving of further study (some preliminary enumeration results for them are described in Appendix A). We compute their Möbius functions and thus obtain inversion formulas. We also prove a general product formula on posets, and apply it to obtain product formulas for Wick products. For the matchings and partitions the results are known but the proofs are new. For the permutations the results are new. For the non-crossing matchings and partitions, the results are known for the usual Wick products, but the poset method allows us to extend them to operator-valued Wick products with no difficulty. As an application, it was observed by ad hoc methods that in the case of incomplete partitions, inversion formulas involve general open blocks but only singleton closed blocks (see Proposition 8 for terminology). The Möbius function approach provides an explanation for this phenomenon. As expected, in the product formulas, in the partitions cases only inhomogeneous partitions appear, while in the permutation case we encounter “incomplete generalized derangements”.

It is well known that the number of non-crossing matchings on $2n$ points equals the number of non-crossing partitions of $n$ points (namely it is the Catalan number). As a combinatorial aside, we observe that the same numerical equality holds for their incomplete versions, but moreover, the collections of incomplete non-crossing matchings on $2n$ points and incomplete non-crossing partitions of $n$ points are isomorphic as posets. The reader interested only in combinatorial results may concentrate on Section 2 and the Appendices, while the reader not interested in combinatorics may skip most of these sections.
For our sixth and most interesting example, morally corresponding to “non-crossing permutations”, we do not know a natural poset structure governing Wick product expansions. So we perform the computations in a more direct way, using induction and generating functions. The combinatorial objects which govern these expansions are pairs $\sigma \ll \pi$ of non-crossing partitions in a relation first observed by Belinschi and Nica [BN08, Ans07, Nic10]. In Appendix B, we list other combinatorial structures corresponding to non-crossing permutations.

Finally, we discuss some analytic extensions of the algebraic results from earlier sections. It is natural to ask whether the map $(\mathcal{M}, \langle \cdot \rangle) \mapsto (\Gamma(\mathcal{M}), \varphi)$ can be interpreted as a functor, generalizing the well-known Gaussian/semicircular functors. For the case of the incomplete non-crossing partitions, we show that this is so, although a larger collection of morphisms would be desirable. We also observe that in all six of our examples, if the functionals $\langle \cdot \rangle$ are tracial states, so are the functionals $\varphi$. Finally, for the case of incomplete non-crossing partitions, we show that the product formulas hold when one of the factors is in $L^2$.

The paper is organized as follows. In Section 2, we prove a general linearization result on posets, and compute Möbius functions for five posets. In Section 3, we use these to obtain inversion and product formulas for five types of Wick products. In Section 4, we obtain the monomial expansions and inversion formulas for the free Meixner Wick products. In short Section 5 we give an explicit counterexample to the product formula for $q$-Wick products claimed in [Ans04a]. In Section 6, we collect the analytic results. In Appendix A, we list some enumerative properties of incomplete posts, and combinatorial consequences of the results above for ordinary polynomials. In short Appendix B we finish with variations on the approach in Section 4.

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## 2. Linearization on posets

Throughout the article, we will only consider finite posets.

**Remark 1.** Let $(\Pi_n, \leq)^{n=1,\infty}$ be a family of posets. As usual, we write $\sigma < \pi$ if $\sigma \leq \pi$ and $\sigma \neq \pi$. In all our examples, $\Pi_n$ will be a meet-semilattice, with the meet operation $\wedge$, and the smallest element, denoted by $\hat{0}_n$. Recall that for $\sigma \leq \pi$, the Möbius function $\mu(\sigma, \pi)$ on $\Pi$ is determined by the property that

\[
\sum_{\tau: \sigma \leq \tau \leq \pi} \mu(\sigma, \tau) = \begin{cases} 
1, & \sigma = \pi, \\
0, & \sigma \neq \pi.
\end{cases}
\]

As a consequence, we have the Möbius inversion formula: if $F, G$ are two functions on $\Pi$ such that

\[
F(\pi) = \sum_{\sigma \geq \pi} G(\sigma),
\]

then

\[
G(\pi) = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) F(\sigma).
\]
Theorem 2. Consider a family of finite posets \((\Pi_i)_{i=1}^n\). Fix \(s(1), s(2), \ldots, s(k) \geq 1\), and denote \(n = s(1) + \ldots + s(k)\). Suppose we have an order-preserving injection

\[ \alpha : \Pi_{s(1)} \times \ldots \times \Pi_{s(k)} \rightarrow \Pi_n \]

with the property that for any \(\tau \in \Pi_n\) there exists a \(\tau_{s(1),\ldots,s(k)} \in \Pi_n\) with

\[ \{ \sigma \in \alpha(\Pi_{s(1)} \times \ldots \times \Pi_{s(k)}), \sigma \leq \tau \} = \{ \sigma \in \Pi_n : \sigma \leq \tau_{s(1),\ldots,s(k)} \}, \]

where \(\tau_{s(1),\ldots,s(k)} = \tau\) if \(\tau \in \alpha(\Pi_{s(1)} \times \ldots \times \Pi_{s(k)})\). Let \(G\) be a function on the disjoint union \(\prod_{n=1}^\infty \Pi_n\). Denote, for \(\pi \in \Pi_i\),

\[ F(\pi) = \sum_{\sigma \geq \pi} G(\sigma). \]

Suppose that for \(\pi_i \in \Pi_{s(i)}\),

\[ F(\pi_1)F(\pi_2)\ldots F(\pi_k) = F(\alpha(\pi_1, \ldots, \pi_k)). \]

Then

\[ G(\hat{0}_{s(1)}) \ldots G(\hat{0}_{s(k)}) = \sum_{\tau \in \Pi_n} G(\tau). \]

Proof. Taking first \(\tau = \alpha(\sigma_1, \ldots, \sigma_k)\), we see that since \(\tau = \tau_{s(1),\ldots,s(k)}\),

\[ [\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)] = [\hat{0}_n, \tau] = \{ \sigma \in \Pi_n : \sigma \leq \tau_{s(1),\ldots,s(k)} \} \]

\[ = \{ \sigma \in \alpha(\Pi_{s(1)} \times \ldots \times \Pi_{s(k)}), \sigma \leq \alpha(\sigma_1, \ldots, \sigma_k) \} \]

\[ \simeq \{ \sigma \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}, \sigma \leq (\sigma_1, \ldots, \sigma_k) \} \]

\[ = [\hat{0}_{s(1)}, \sigma_1] \times \ldots \times [\hat{0}_{s(k)}, \sigma_k]. \]

Thus, since the Möbius function is multiplicative,

\[ \mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) = \prod_{i=1}^k \mu(\hat{0}_{s(i)}, \sigma_i). \]
The rest of the proof is similar to that of Theorem 4 in [RW97]. Using various assumptions,

\[ G(\hat{0}_{s(1)}) \ldots G(\hat{0}_{s(k)}) = \sum_{\sigma_1 \in \Pi_{s(1)}} \ldots \sum_{\sigma_k \in \Pi_{s(k)}} \prod_{i=1}^{k} \mu(\hat{0}_{s(i)}, \sigma_i) \prod_{i=1}^{k} F(\sigma_i) \]

\[ = \sum_{(\sigma_1, \ldots, \sigma_k) \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}} \mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) F(\alpha(\sigma_1, \ldots, \sigma_k)) \]

\[ = \sum_{(\sigma_1, \ldots, \sigma_k) \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}} \mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) \sum_{\tau \supseteq \alpha(\sigma_1, \ldots, \sigma_k)} G(\tau) \]

\[ = \sum_{\tau \in \Pi_n} G(\tau) \sum_{(\sigma_1, \ldots, \sigma_k) \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}} \mu(\hat{0}_n, \sigma) \]

\[ = \sum_{\tau \in \Pi_n} G(\tau) \sum_{\tau \supseteq \alpha(\sigma_1, \ldots, \sigma_k)} \mu(\hat{0}_n, \sigma) \]

\[ = \sum_{\tau \in \Pi_n} G(\tau) \sum_{\tau \supseteq \alpha(\sigma_1, \ldots, \sigma_k)} \mu(\hat{0}_n, \sigma) \]

**Remark 3.** We will show that for the five posets in the next series of propositions, the conditions above are satisfied, and compute the corresponding \( \tau_{s(1), \ldots, s(k)} \). In all the examples, \( \alpha \) combines objects defined on each of the subintervals

(4) \( J_1 = [1, \ldots, s(1)] \), \( J_2 = [s(1) + 1, \ldots, s(1) + s(2)] \), \ldots , \( J_k = [n - s(k) + 1, \ldots, n] \)

into a single object on the interval \([1, \ldots, n]\), in a natural way. We will denote by \((\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})\) the partition on \([n]\) whose blocks are these intervals. Note that \( \Pi_i \) is *not* assumed to have a maximal element, so \( \hat{1}_i \) only denotes the maximal element of \( \mathcal{P}(i) \).

**Notation 4** (Background on partitions). Denote \( \text{Sing}(\pi) \) the single-element blocks of a partition \( \pi \), and \( \text{Pair}(\pi) \) the two-element blocks.

A partition \( \pi \) of an ordered set \( \Lambda \) is *non-crossing* if there are no two blocks \( U \neq V \) of \( \pi \) with \( i, k \in U \), \( j, l \in V \), and \( i < j < k < l \). The set of non-crossing partitions is denoted by \( \mathcal{NC}(\Lambda) \), or \( \mathcal{NC}(n) \) in case \( \Lambda = [n] = \{1, 2, \ldots, n\} \). A block \( U \in \pi \in \mathcal{NC}(n) \) containing \( i_0 \) has depth \( k \) if \( k \) is the largest integer (starting with 0) for which there is exist \( i_k < i_{k-1} < \ldots < i_1 < i_0 < j_1 < \ldots < j_k \) such that all \( i_0, i_1, \ldots, i_k \) belong to different blocks of \( \pi \) while \( i_u \sim j_\sim u \), \( u = 1, \ldots, k \). A block is *outer* if it has depth zero, and *inner* otherwise. Denote \( \text{Out}(\pi) \) the outer blocks of \( \pi \). Denote \( \mathcal{NC}_{\geq 2}(\Lambda) \) the non-crossing partitions with no singletons. The *interval partitions* \( \text{Int}(\Lambda) \) are partitions whose blocks are intervals.

**Proposition 5.** Denote by \( \mathcal{P}_{1,2}(n) \), the incomplete matchings, the partitions of \([n]\) into pairs and singletons. Equip it with the poset structure it inherits from the usual refinement order on partitions. Then the Möbius function on this poset is

\[ \mu(\pi, \sigma) = (-1)^{|\text{Pair}(\sigma)| - |\text{Pair}(\pi)|} \]

Also for this poset, \( \tau_{s(1), \ldots, s(k)} = \tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) \).
Proof. It suffices to note that, denoting by $U$ the pairs and $V$ the singleton blocks,

$$[(U_1, \ldots, U_u, V'_1, \ldots, V'_v), (U_1, \ldots, U_k, V_1, \ldots, V_l)] \simeq [(\hat{0}, (U_{u+1}, \ldots, U_k)) \simeq \mathcal{P}(2)^{k-u}. \quad \Box$$

Proposition 6. Denote by $\mathcal{INC}_{1,2}(n)$, the incomplete non-crossing matchings, the non-crossing partitions of $[n]$ into pairs and singletons, such that all singletons are outer. Equip it with the poset structure inherited from $\mathcal{P}_{1,2}(n)$. Then the Möbius function on this poset is

$$\mu((U_1, \ldots, U_u, V'_1, \ldots, V'_v), (U_1, \ldots, U_k, V_1, \ldots, V_l)) = \begin{cases} (-1)^{k-u}, & \forall (u+1) \leq i \leq k : U_i \in \text{Out}(\pi), \\ 0, & \text{otherwise.} \end{cases}$$

For this poset, $\tau_{s(1), \ldots, s(k)}$ is given by the same expression as for $\mathcal{P}_{1,2}$.

Proof. Clearly

$$[(U_1, \ldots, U_u, V'_1, \ldots, V'_v), (U_1, \ldots, U_k, V_1, \ldots, V_l)] \simeq [(\hat{0}, (U_{u+1}, \ldots, U_k)].$$

Moreover this interval is the product of intervals from $\hat{0}$ to a partition with a single outer block. If that partition is simply $\{U\}$, the Möbius function $\mu(\hat{0}, \{U\}) = (-1)$. On the other hand, if it is a larger partition with the single outer block $\{i, j\}$, recalling that all singletons in $\mathcal{INC}_{1,2}$ are outer, we see that

$$\sigma < (U'_1, U'_2, \ldots, \{i, j\}) \iff \sigma \leq (U'_1, \ldots, U'_{s-1}, \{i\}, \{j\}).$$

and so from property (1),

$$\mu(\hat{0}, (U'_1, U'_2, \ldots, \{i, j\})) = \sum_{\sigma \leq (U'_1, U'_2, \ldots, \{i, j\})} \mu(\hat{0}, \sigma) - \sum_{\sigma < (U'_1, U'_2, \ldots, \{i, j\})} \mu(\hat{0}, \sigma) = 0. \quad \Box$$

Definition 7. Denote by $\mathcal{IP}(n)$, the incomplete partitions, the collection

$$\{(\pi, S) : \pi \in \mathcal{P}(n), S \subset \pi\}.$$

Here, and in the subsequent examples, the elements of $S$ will be called open blocks, those of $\pi \setminus S$ closed blocks. Denote $\bigcup S = \bigcup_{V \in S} V$ the union of all the open blocks. Equip $\mathcal{IP}(n)$ with the poset structure

$$(\pi, S) \leq (\sigma, T) \text{ if } U \in \pi \setminus S \Rightarrow U \in \sigma \setminus T \text{ and } \pi \mid_{\bigcup S} \leq \sigma \mid_{\bigcup S}.$$

Proposition 8. The Möbius function on $\mathcal{IP}(n)$ is

$$\mu((\hat{0}, \hat{0}), (\pi, S)) = \begin{cases} (-1)^{n-[S\setminus V]} \prod_{V \in S} ([V] - 1)!, & \forall U \in \pi \setminus S : |U| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also for this poset, $(\tau, S)_{s(1), \ldots, s(k)} = (\tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}), T)$, where $U \in (\tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})) \setminus T$ if and only if $U \in \tau \setminus S$. In particular, $(\tau, S)_{s(1), \ldots, s(k)} = (\hat{0}_s, \hat{0}_n)$ if and only if $\tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n$ and all singletons of $\tau$ are open.

Proof. Clearly

$$[(\hat{0}, \hat{0}), (\pi, S)] \simeq \prod_{U \in \pi \setminus S} [(\hat{0}, \hat{0}), (\{U\}, \emptyset)] \times \prod_{V \in S} [(\hat{0}, \hat{0}), (\{V\}, \{V\})]$$
and \([\hat{0}, \hat{0}) \cup \{V\}, \{V\}] \simeq [\hat{0}, \{V\}] \simeq \mathcal{P}([V])\), so \(\mu((\hat{0}, \hat{0}), \{\{V\}, \{V\}\}) = (-1)^{|V|-1}(|V| - 1)!\). Also
\[
\{([\sigma, \theta]) < (\{U\}, \emptyset)\} = \{([\sigma, \theta]) \preceq (\{U\}, \{U\})\}
\]
so using property (1), \(\mu((\hat{0}, \hat{0}), \{\{U\}, \emptyset\}) = 0\) unless \(|U| = 1\). The formula for the Möbius function follows. The final formula follows from the definition of the order. 

\begin{proposition}
Denote by \(\mathcal{I}NC(n)\), the incomplete non-crossing partitions (called the linear non-crossing half-permutations in [KMS07]), the collection
\[
\{(\pi, S) : \pi \in \mathcal{N}C(n), S \subset Out(\pi)\}.
\]
Equip it with the poset structure inherited from \(\mathcal{I}P(n)\). Then the posets \(\mathcal{I}NC(n)\) and \(\mathcal{I}NC_{1,2}(2n)\) are isomorphic. Under this isomorphism, partitions with \(\ell\) open blocks are mapped to partitions with \(2\ell\) singletons, and partitions with \(k\) closed blocks are mapped to partitions with \(k\) pairs at even depth. In particular, the Möbius function on this poset is
\[
\mu((\hat{0}, \hat{0}), (\pi, S)) = \begin{cases} 
(-1)^{|n - |S||}, & \pi \in \text{Int}(n), \forall U \in \pi \setminus S : |U| = 1, \\
0, & \text{otherwise}.
\end{cases}
\]
For this poset, \(\tau_{s(1), \ldots, s(k)}\) is given by the same expression as for \(\mathcal{I}P\).
\end{proposition}

\begin{proof}
We exhibit an bijection between \(\mathcal{I}NC(n)\) and \(\mathcal{I}NC_{1,2}(\{1, \tilde{1}, \ldots, n, \tilde{n}\})\). A closed block \((i_1 < i_2 < \ldots < i_k)\) is replaced by blocks \((i_1, \tilde{i}_k), (\tilde{i}_1, i_2), \ldots, (\tilde{i}_{k-1}, i_k)\), while the open block with these elements is replaced by \((\hat{i}_1), (\hat{i}_k), (\hat{i}_1, \hat{i}_2), \ldots, (\hat{i}_{k-1}, \hat{i}_k)\). This is easily seen to be a bijection. Moreover, combining two (outer) open blocks \((i_1 < i_2 < \ldots < i_k)\) and \((j_1 < j_2 < \ldots < j_\ell)\) with \(i_k < j_1\) corresponds to pairing off \(i_k\) and \(j_1\), while closing the open block \((i_1 < i_2 < \ldots < i_k)\) corresponds to pairing off \(i_1\) and \(\tilde{i}_k\). So this map is a poset isomorphism. Finally, each open block of \(\pi \in \mathcal{I}NC(n)\) produces two singletons in the image. The statement about closed blocks follows from a recursive argument.

Next, for \(\sigma \in \mathcal{I}NC_{1,2}(2n)\), from Proposition 6
\[
\mu_{\mathcal{I}NC_{1,2}}(\hat{0}, \sigma) = \begin{cases} 
(-1)^{|Pair(\sigma)|}, & \sigma \in \text{Int}(2n), \\
0, & \text{otherwise}.
\end{cases}
\]
Clearly the bijection above maps incomplete interval matchings onto the set of incomplete interval partitions all of whose closed blocks are singletons. It remains to note that if \(\sigma\) is mapped to \((\pi, S)\), then \(|Pair(\sigma)| = \frac{1}{2}(2n - |Sing(\sigma)|) = n - |S|\). The formula for \(\tau_{s(1), \ldots, s(k)}\) follows from the definition of the order.
\end{proof}

\begin{definition}
Denote by \(\mathcal{I}PRM(n)\), the incomplete permutations (sometimes called partial permutations [BRR89]), although this term has also been used for different objects), the collection of maps
\[
\mathcal{I}PRM(n) = \{((\Lambda, f)) : \Lambda \subset [n], f : \Lambda \to [n] \text{ injective}\}.
\]
 Equip \(\mathcal{I}PRM(n)\) with the following poset structure:
\[
((\Lambda, f)) \preceq (\Omega, g) \text{ if } \Lambda \subset \Omega \text{ and } g|_\Lambda = f.
\]
Proposition 11. $\mathcal{IPRM}(n)$ is isomorphic as a poset to pairs $(\pi, S)$, where $\pi$ is a partition of $[n]$ with an order on each block of the partition, $S$ is a collection of some blocks of this partition, and the order on the blocks in $\pi \setminus S$ is defined only up to a cyclic permutation. Equivalently, these are collections of words in $[n]$, where each letter appears exactly once, and some of the words are defined only up to cyclic order. In the poset structure, $(\pi, S) \leq (\sigma, T)$ if $U \in \pi \setminus S \Rightarrow U \in \sigma \setminus T$, the restriction of partitions $\pi|_{\pi \setminus S} \leq \sigma|_{\pi \setminus S}$, and the words corresponding to blocks of $\sigma$ combined out of blocks of $\pi$ are obtained by concatenating the words corresponding to these blocks of $\pi$, in some order (and the combined word possibly cyclically rotated if the block of $\sigma$ is closed).

Proof. Given $(\Lambda, f)$ as in the definition of $\mathcal{IPRM}(n)$, we define the partition $\pi'$ of $\Lambda \cup f(\Lambda)$ to be the partition into orbits of $f$, that is, largest subsets $\{w_1, \ldots, w_\ell\}$ such that $w_{i+1} = f(w_i)$. The block is in $\pi' \setminus S$ if $w_\ell \in \Lambda$, so that $f(w_\ell) = w_1$, and the block is in $S$ if $w_\ell \in f(\Lambda) \setminus \Lambda$. Complete $\pi'$ to a partition $\pi$ of $[n]$ by letting each elements in $[n]\setminus(\Lambda \cup f(\Lambda))$ be a singleton block in $S$. Note that each block in $S$ has an order structure $w_1 < w_2 < \ldots < w_\ell$, and each block in $\pi' \setminus S$ has a cyclic order. Conversely, we can recover $(\Lambda, f)$ from $(\pi, S)$ by setting $[n]\setminus \Lambda$ to consist of the largest elements (according to the block order) of blocks in $S$, and $f$ defined by the order on the blocks.

Next, let $(\Lambda, f) \leftrightarrow (\pi, S), (\Omega, g) \leftrightarrow (\sigma, T)$, and $(\Lambda, f) \leq (\Omega, g)$. The orbits of $g$, ordered according to the mapping structure of $g$, have the form either

\[
\{w_1 < w_2 < \ldots < w_\ell\} \in \pi \setminus S,
\]

or

\[
\{v_{1,0} < \ldots < v_{k(0),0} < w_{1,1} < \ldots < w_{\ell(1),1} < v_{1,1} < \ldots < v_{k(1),1} < w_{1,2} < \ldots < w_{\ell(2),2} < \ldots\},
\]

where all $v_{i,j} \in (\Omega \cup g(\Omega)) \setminus (\Lambda \cup f(\Lambda))$ are singletons in $S$ and each $\{w_{1,j} < \ldots w_{\ell(j),j}\} \in S$ is an orbit of $f$ in $S$. The description above follows. \hfill \Box

Example 12. For $\Lambda = \{2, 3, 5, 7, 8, 9\} \subset [9]$ and

\[
f(2) = 5, f(3) = 8, f(5) = 4, f(7) = 3, f(8) = 7, f(9) = 9,
\]

the corresponding incomplete partition with ordered blocks is

\[
\pi = \{(1), (2 < 5 < 4), (3 < 8 < 7 < 3), (6), (9)\}, \quad S = \{(1), (2 < 5 < 4), (6)\}.
\]

Proposition 13. The Möbius function on $\mathcal{IPRM}(n)$ is

\[
\mu((\hat{0}, \hat{0}), (\pi, S)) = (-1)^{n-|S|}.
\]

$(\Lambda, f)|_{s(1), \ldots, s(k)} = (\Omega, g)$, where

\[
\Omega = \bigcup_{i=1}^{k} \{x \in \Lambda \cap J_i : f(x) \in J_i\}
\]

and $g = f|_{\Omega}$. In particular, $(\Lambda, f)|_{s(1), \ldots, s(k)}$ equals the minimal element of $\mathcal{IPRM}(n)$ if for each $i$ and $x \in \Lambda \cap J_i$, $f(x) \notin J_i$. We will call this final family incomplete derangements and denote it by $\mathcal{ID}(s(1), \ldots, s(k))$. 
Proof. The blocks of $\pi$ are simply the orbits of $f$, with elements of $[n]\setminus \Lambda$ included as open singletons. To compute the Möbius function, it suffices to assume that $f$ has a single orbit. Elements smaller than $(\Lambda, f)$ are in an ordered bijection with subsets of $\Lambda$, and the Möbius function $(-1)^{|\Lambda|}$. It remains to note that $\Lambda = [n]$ if the corresponding block is closed, and $|\Lambda| = n - 1$ if the corresponding block is open. Formulas for $(\Lambda, f)_{s(1),\ldots,s(k)}$ follow from the definition of the order. \qed

3. Multiplication of Wick Products

Let $\mathcal{M}, \Gamma(\mathcal{M})$ be as in the beginning of the introduction.

Notation 14. Order the blocks of a partition according to the order of the largest elements of the blocks. For an ordered index set $\Lambda$, denote $a_\Lambda = \prod_{i \in \Lambda} a_i$. Finally, write $J_i = \{u(1), \ldots, u_i(s(i))\}$, so that

\[
\{1, 2, \ldots, n\} = (u(1), \ldots, u_1(s(1)), u_2(1), \ldots, u_2(s(2)), \ldots, u_k(1), \ldots, u_k(s(k))).
\]

Proposition 15. Define $W_{\mathcal{P}_{1,2}}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ recursively by

\[
W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes a_n) X(a_{n+1})
- \sum_{i=1}^n W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n) \langle a_i a_{n+1}\rangle.
\]

Then

\[
X(a_1) \ldots X(a_n) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} \prod_{U \in \pi: |U| = 2} \langle a_U \rangle W_{\mathcal{P}_{1,2}} \left( \bigotimes_{V \in \pi: |V| = 1} a_V \right),
\]

and

\[
W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes a_n) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{|\pi|-|\text{Sing}(\pi)|} \prod_{U \in \pi: |U| = 2} \langle a_U \rangle \prod_{V \in \pi: |V| = 1} X(a_V),
\]

and

\[
\prod_{i=1}^k W_{\mathcal{P}_{1,2}}(a_{ui(1)} \otimes a_{ui(2)} \otimes \ldots \otimes a_{ui(s(i))})
= \sum_{\pi \in \mathcal{P}_{1,2}(n)} \prod_{U \in \pi: |U| = 2} \langle a_U \rangle W_{\mathcal{P}_{1,2}} \left( \bigotimes_{V \in \pi: |V| = 1} a_V \right).
\]

Proof. Equation (6) is well known, see for example Theorem 2.1 in [EP03] for $q = 1$. It implies that for $\pi \in \mathcal{P}_{1,2}(n)$,

\[
\prod_{U \in \pi: |U| = 2} \langle a_U \rangle \prod_{V \in \pi: |V| = 1} X(a_V) = \sum_{\sigma \in \mathcal{P}_{1,2}(n)} \prod_{U \in \sigma: |U| = 2} \langle a_U \rangle W_{\mathcal{P}_{1,2}} \left( \bigotimes_{V \in \sigma: |V| = 1} a_V \right).
\]

Denoting the left-hand-side of this equation by $F(\bar{\pi})$ and each term in the sum on the right-hand-side by $G(\sigma)$, we see that these functions satisfy the relation (2), and $F$ satisfies the multiplicative property (3). So Theorem 2 and Proposition 5 imply the results. \qed
Proposition 16. Define $W_{\mathcal{I}P}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ recursively by

$$W_{\mathcal{I}P}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W_{\mathcal{I}P}(a_1 \otimes \ldots \otimes a_n)X(a_{n+1}) - W_{\mathcal{I}P}(a_1 \otimes \ldots \otimes a_n)\langle a_{n+1} \rangle$$

$$- \sum_{i=1}^{n} W_{\mathcal{I}P}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_ia_{n+1})$$

$$- \sum_{i=1}^{n} W_{\mathcal{I}P}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n)\langle a_ia_{n+1} \rangle.$$ 

Then

$$(7) \quad X(a_1) \ldots X(a_n) = \sum_{(\pi, S) \in \mathcal{I}P(n)} \prod_{U \in \pi \setminus S} \langle a_U \rangle W_{\mathcal{I}P} \left( \bigotimes_{V \in S} a_V \right).$$

If $\mathcal{M}$ is commutative, then

$$W_{\mathcal{I}P}(a_1 \otimes \ldots \otimes a_n) = \sum_{(\pi, S) \in \mathcal{I}P(n)} (-1)^{|S|-1}|S| \prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} (|V| - 1)!X(a_V),$$

and

$$\prod_{i=1}^{k} W_{\mathcal{I}P}(a_{u_1(1)} \otimes a_{u_1(2)} \otimes \ldots \otimes a_{u_1(s(i))}) = \sum_{(\pi, S) \in \mathcal{I}P(n)} \prod_{\pi \wedge \{1_{s(1)}, \ldots, 1_{s(k)}\} = 0_n} \prod_{S \in \text{Sing}(\pi) \cap S} \langle a_U \rangle W_{\mathcal{I}P} \left( \bigotimes_{V \in S} a_V \right).$$

Proof. Equation (7) is known, see for example Proposition 2.7(a) in [Ans04a]. For commutative $\mathcal{M}$, it implies that for $(\pi, S) \in \mathcal{I}P(n)$,

$$\prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} X(a_V) = \sum_{(\sigma, T) \in \mathcal{I}P(n)} \prod_{U \in \sigma \setminus T} \langle a_U \rangle W_{\mathcal{I}P} \left( \bigotimes_{V \in T} a_V \right).$$

So Theorem 2 and Proposition 8 imply the results. \qed
Theorem 17 (Cf. Section 4 in [Śni00]). Define $W_{\mathcal{IPRM}}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ recursively by

$$W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle$$

$$- \sum_{i=1}^{n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_i a_{n+1})$$

$$- \sum_{i=1}^{n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_{n+1} a_i)$$

$$- \sum_{1 \leq i < j \leq n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes \hat{a}_j \otimes \ldots \otimes a_n \otimes a_i a_{n+1} a_j)$$

$$- \sum_{1 \leq i < j \leq n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes \hat{a}_j \otimes \ldots \otimes a_n \otimes a_i a_{n+1} a_j).$$

For $(\Lambda, f) \in \mathcal{IPRM}(n)$, let $(\pi, S) \in \mathcal{INC}(n)$ be the corresponding orbit decomposition. Order each block $\{w_1, w_2, \ldots, w_\ell\}$ of $\pi$ so that $w_{i+1} = f(w_i)$ and, in case the block is closed, so that $w_\ell$ is the numerically largest element in the block. Denote

$$W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)} = \prod_{U \in \pi \setminus S} \langle a_U \rangle W_{\mathcal{IPRM}} \left( \bigotimes_{V \in S} a_V \right)$$

and

$$M(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)} = \prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} X(a_V),$$

where on each block of $\pi$ we use the order described above. Then

$$(8) \quad X(a_1) \ldots X(a_n) = \sum_{(\Lambda, f) \in \mathcal{IPRM}(n)} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)}.$$

Also,

$$W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n) = \sum_{(\Lambda, f) \in \mathcal{IPRM}(n)} (-1)^{n - |S|} M(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)},$$

and

$$\prod_{i=1}^{k} W_{\mathcal{IPRM}}(a_{u(1)} \otimes a_{u(2)} \otimes \ldots \otimes a_{u(s(i))})$$

$$= \sum_{(\Lambda, f) \in \mathcal{ID}(s(1), \ldots, s(k))} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)}.$$

Proof. We prove equation (8) by induction. $X(a) = W_{\mathcal{IPRM}}(a) + \langle a \rangle$. Denoting

$$S = \{V_1 < V_2 < \ldots < V_{|S|}\}$$
and using the inductive hypothesis,

\[
X(a_1) \ldots X(a_n)X(a_{n+1}) = \sum_{(\Lambda, f) \in IPRM(n)} W_{\text{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)} X(a_{n+1}) \\
= \sum_{(\Lambda, f) \in IPRM(n)} \prod_{U \in \pi \setminus S} \langle a_U \rangle \\
\left( W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{n+1} \right) \\
+ W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{|S|}} \right) \langle a_{n+1} \rangle \\
+ \mathbf{1}_{|S| \geq 1} \sum_{i=1}^{|S|} W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{n+1} a_{V_i} \right) \\
+ \mathbf{1}_{|S| \geq 1} \sum_{i=1}^{|S|} W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{V_i} a_{n+1} a_{V_i} \right) \\
+ \mathbf{1}_{|S| \geq 2} \sum_{1 \leq i < j \leq |S|} W \left( a_{V_1} \otimes \ldots \otimes a_{V_i} \otimes \ldots \otimes a_{V_j} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{V_i} a_{n+1} a_{V_j} \right) \\
+ \mathbf{1}_{|S| \geq 2} \sum_{1 \leq i < j \leq |S|} W \left( a_{V_1} \otimes \ldots \otimes a_{V_i} \otimes \ldots \otimes a_{V_j} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{V_i} a_{n+1} a_{V_j} \right) \right).
\]

The first term produces all the partitions in $IPRM(n + 1)$ in which $n + 1$ is an open singleton. The second term produces all the partitions in which $n + 1$ is a closed singleton. The third term produces all the partitions in which $n + 1$ is a final letter in an open word of length at least 2. The fourth term produces all the partitions in which $n + 1$ is the initial letter in an open word of length at least 2. The fifth term produces all the partitions in which $n + 1$ is contained in a closed word of length at least 2. The sixth term produces all the partitions in which $n + 1$ is contained in an open word of length at least 3, is neither the initial nor the final letter in it, and the largest letter preceding it is smaller than the largest letter following it. The seventh term produces all the partitions in which $n + 1$ is contained in an open word of length at least 3, is neither the initial nor the final letter in it, and the largest letter preceding it is larger than the largest letter following it. These seven classes are disjoint and exhaust $IPRM(n + 1)$.

It follows that for $(\Lambda, f) \in IPRM(n),

\[
M(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)} = \sum_{(\Omega, g) \in IPRM(n)} W_{\text{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Omega, g)}.
\]

So Theorem 2 and Proposition 13 imply the results. \qed
Remark 18. Assume additionally that \( \langle \cdot \rangle \) is a trace. By applying the functional \( \varphi_{\text{IPRM}} \) to (8), we obtain the moment formula for \( \{X(a_i)\} \), which implies that the cumulants of \( \varphi_{\text{IPRM}} \) are

\[
K^{\varphi_{\text{IPRM}}}[X(a_1), \ldots, X(a_n)] = \frac{1}{n} \sum_{\alpha \in \text{Sym}(n)} \langle a_{\alpha(1)} \ldots a_{\alpha(n)} \rangle.
\]

Remark 19. Note that in Propositions 15 and 16, we do not assume that \( W \) is symmetric in its arguments, and in Theorem 17, we do not assume that \( \mathcal{M} \) is commutative. The results in Proposition 15 are known by direct methods, see Theorems 3.1 and 3.3 in [EP03]. The results in Proposition 16 are stated in Proposition 2.7 of [Ans04a].

If \( \mathcal{M} \) is commutative, \( W_{\text{IPRM}}(a_1 \otimes \ldots \otimes a_n) \) depends only on the underlying incomplete partition, so we may re-write the expansions in Theorem 17 as

\[
X(a_1) \ldots X(a_n) = \sum_{\{\pi, S\} \in \text{IP}(n)} \prod_{U \in \pi \setminus S} (|U| - 1)! \langle a_U \rangle \prod_{V \in S} (|V|)! W_{\text{IPRM}} \left( \bigotimes_{V \in S} a_V \right),
\]

and

\[
W_{\text{IPRM}}(a_1 \otimes \ldots \otimes a_n) = \sum_{\{\pi, S\} \in \text{IP}(n)} (-1)^{n-|S|} \prod_{U \in \pi \setminus S} (|U| - 1)! \langle a_U \rangle \prod_{V \in S} (|V|)! X(a_V),
\]

Note that this additional assumption does not imply that \( \Gamma(\mathcal{M}) \) is commutative; it is however natural to assume such commutativity to have a non-degenerate representation, see Section 6.

Remark 20. Let \( (\Lambda, f) \in \text{IPRM}(n) \). For \( w \in [n] \), we say that it is

- A valley if \( w \notin \Lambda \cup f(\Lambda) \), or \( w \in \Lambda \setminus f(\Lambda) \) and \( w < f(w) \), or \( w \notin f(\Lambda) \setminus \Lambda \) and \( f^{-1}(w) > w \), or \( w \notin \Lambda \cup f(\Lambda) \) and \( f^{-1}(w) > w < f(w) \).
- A closed singleton if \( f(w) = w \).
- A double rise if \( f^{-1}(w) > w \) and either \( w > f(w) \) or \( w \notin \Lambda \).
- A double fall if \( w < f(w) \) and either \( f^{-1}(w) < w \) or \( w \notin f(\Lambda) \).
- A cycle max if \( w \) is the (numerically) largest element in a closed word of length at least 2.
- A peak if \( f^{-1}(w) < w > f(w) \) and it is not a cycle max.

Clearly each letter in \([n]\) belongs to one of these six types. Then a slight extension of the argument in the previous proposition shows that if we define

\[
W(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - \alpha W(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle
- \sum_{i=1}^{n} \beta_1 W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_i a_{n+1}) - \sum_{i=1}^{n} \beta_2 W(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1} a_i)
- \sum_{i=1}^{n} \gamma W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_i a_{n+1} a_j)
- \sum_{1 < i < j \leq n} \gamma W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_j a_{n+1} a_i),
\]
then
\[
X(a_1) \ldots X(a_n) = \sum_{(\Lambda, f) \in \mathcal{F}(M(n))} W_{\mathcal{F}(M)}(a_1 \otimes \ldots \otimes a_n)^{\Lambda, f} \times \alpha \# \text{closed singletons} \beta_1 \# \text{double rises} \beta_2 \# \text{double falls} \beta_4 \# \text{cycle max} \gamma \# \text{peaks}.
\]

See [Bia93, SS94, CSZ97, KZ01] for related results. As in most of these references, there is a natural way of including a \(q\) parameter in this expansion, based on the values of the \(i, j\) indices from the Wick product recursion. However, our technique for obtaining inversion and product formulas does not apply to this extension, and based on the results in Section 5, it is unclear what these formulas should be.

**Remark 21.** Let \(\mathcal{D}\) be a unital star-subalgebra, \(\mathcal{M}\) be a complex star-algebra which is also a \(\mathcal{D}\)-bimodule such that for \(d_1, d_2 \in \mathcal{D}\) and \(a \in \mathcal{M}\),

\[
(9) \quad d_1(ad_2) = (d_1a)d_2,
\]

and \(\langle \cdot, \cdot \rangle: \mathcal{M} \to \mathcal{D}\) a star-linear \(\mathcal{D}\)-bimodule map. (We do not assume that \(\mathcal{D} \subset \mathcal{M}\) since \(\mathcal{M}\) may not be unital.) Let \(\Gamma(\mathcal{M})\) be the complex unital star-algebra generated by non-commuting symbols \(\{X(a): a \in \mathcal{M}\}\) and \(\mathcal{D}\), subject to the linearity relations

\[
X(d_1ad_2 + d_3bd_4) = d_1X(a)d_2 + d_3X(b)d_4, \quad d_1, d_2, d_3, d_4 \in \mathcal{D}.
\]

The star-operation on it is determined by the requirement that all \(X(a^*) = X(a)^*\). Thus

\[
\Gamma(\mathcal{M}) \simeq \bigoplus_{n=0}^{\infty} \mathcal{M}^{\otimes \mathcal{D}^n}.
\]

We denote \(M(a_1 \otimes \ldots \otimes a_n) = X(a_1) \ldots X(a_n)\), and note that \(M\) may be extended to a \(\mathcal{D}\)-bimodule map on \(\mathcal{M}^{\otimes \mathcal{D}^n}\).

Let \(\pi \in \mathcal{NC}(n)\). We will define a bimodule map \(\langle \cdot, \cdot \rangle^\pi\) on \(\mathcal{M}^{\otimes \mathcal{D}^n}\) recursively as follows. First,

\[
\langle d_0a_1d_1 \otimes \ldots \otimes a_n d_n \rangle^\pi_1 = d_0 \langle a_1d_1 \ldots a_n \rangle d_n.
\]

Next let

\[
\text{Out}(\pi) = \{V_1 < V_2 < \ldots < V_\ell\}, \quad V_i = \{v(i, 1) < \ldots < v(i, t(i))\}.
\]

Denote \(I_{ij} = [v(i, j) + 1, \ldots, v(i, j + 1) - 1]\) for \(1 \leq i \leq \ell, 1 \leq j \leq t(i) - 1\), and \(\pi_{i,j} = \pi|_{I_{ij}}\). Note that an interval may be empty. Then we recursively define

\[
\langle d_0a_1d_1 \otimes \ldots \otimes a_n d_n \rangle^\pi = \prod_{i=1}^{\ell} \left( a^{v(i,j)}_v d^v_i \langle a_v d_v : v \in I_{i,j} \rangle^{\pi_{i,j}} a_n \right) d_n.
\]

Note that this is the not the same definition as that in [Spe98] or Section 3 in [ABFN13], although it is related to them and may be expressed in terms of them as long as \(\pi\) is appropriately transformed.

Next, let \(F\) be a \(\mathcal{D}\)-bimodule map on \(\mathcal{M}^{\otimes \mathcal{D}^n}\) (in our examples, either \(M\) or \(W\)). Let \((\pi, S) \in \mathcal{I}\mathcal{NC}(n)\), and this denote

\[
S = \{V_1 < V_2 < \ldots < V_\ell\}, \quad V_i = \{v(i, 1) < \ldots < v(i, t(i))\}.
\]

Let \(I_{ij}, \pi_{ij}\) be as before, and define additionally \(v(\ell + 1, 1) = n + 1, I_0 = [1, \ldots, v(1, 1) - 1], \quad I_0 = [1, \ldots, v(1, 1) - 1].\)
and the corresponding $\pi_{ij}$, $\pi_0$. Denote
\[
F(d_0a_1d_1 \otimes \ldots \otimes a_n d_n)^{\pi, S} = d_0 F \left( \langle a_v d_v : v \in I_0 \rangle^{\pi_0} \otimes \prod_{i=1}^{t(i)} \langle a_{v(i,j)} d_{v(i,j)} : v \in I_{i,j} \rangle^{\pi_{ij}} \right).
\]

In the scalar-valued case, the following results are known, see Theorem 3.3 in [EP03] and Proposition 29 in [Ans04b] for $q = 0$.

**Proposition 22.** Define $W_{\mathcal{I}NC_{1,2}} (a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ recursively by
\[
W_{\mathcal{I}NC_{1,2}} (a_1 \otimes \ldots \otimes a_n) = \sum_{\pi \in I NC_{1,2}(n)} W_{\mathcal{I}NC_{1,2}} (a_1 \otimes \ldots \otimes a_n)^{\pi, Sing(\pi)},
\]
and
\[
W_{\mathcal{I}NC_{1,2}} (a_1 \otimes \ldots \otimes a_n) = \sum_{\pi \in Int_{1,2}(n)} (-1)^{|\pi| - |Sing(\pi)|} W_{\mathcal{I}NC_{1,2}} (a_1 \otimes \ldots \otimes a_n)^{\pi, Sing(\pi)},
\]
and
\[
\prod_{i=1}^{k} W_{\mathcal{I}NC_{1,2}} (a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}) = \sum_{\pi \in I NC_{1,2}(n)} W_{\mathcal{I}NC_{1,2}} (a_1 \otimes \ldots \otimes a_n)^{\pi, Sing(\pi)}.
\]

The proof is similar to and simpler than that of the next proposition, so we omit it.

**Proposition 23.** Define $W_{\mathcal{I}NC} (a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ recursively by
\[
W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_n) X (a_{n+1}) - W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_{n+1}) - W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_{n+1}) X (a_{n+1}).
\]

Then
\[
X (a_1) \ldots X (a_n) = \sum_{(\pi, S) \in I NC(n)} W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_n)^{\pi, S},
\]
and
\[
W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_n) = \sum_{(\pi, S) \in I NC(n)} (-1)^{|\pi| - |Sing(\pi)|} \langle M (a_1 \otimes \ldots \otimes a_n)^{\pi, S} \rangle,
\]
and
\[
\prod_{i=1}^{k} W_{\mathcal{I}NC} (a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}) = \sum_{\pi \in I NC(n)} W_{\mathcal{I}NC} (a_1 \otimes \ldots \otimes a_n)^{\pi, S}. \]
Proof. The proof of equation (11) is similar to the argument in Theorem 17 above or Theorem 28 below, so we only outline it. It is based on the observation that the four terms in the recursion relation for $W_{\mathcal{NC}}$ correspond to the decomposition of $\mathcal{NC}(n)$ as a disjoint union of four sets: those where $n + 1$ is an open singleton, a closed singleton, those where it belongs to a larger open block, and those where it belongs to a larger closed block. Equation (11) implies that for $(\pi, S) \in \mathcal{NC}(n)$,

$$M(a_1 \otimes \ldots \otimes a_n)^{(\pi, S)} = \sum_{(\sigma, T) \in \mathcal{NC}(n) \mid (\sigma, T) \geq (\pi, S)} W_{\mathcal{NC}}(a_1 \otimes \ldots \otimes a_n)^{(\pi, S)}.$$ 

So Theorem 2 and Proposition 9 imply the results.

4. EXPANSIONS FOR FREE MEIXNER WICK PRODUCTS

Notation 24. A covered partition is a partition $\pi \in \mathcal{NC}(\Lambda)$ with a single outer block, or equivalently such that $\min(\Lambda) \leq \max(\Lambda)$; their set is denoted by $\mathcal{NC}^c(\Lambda)$. We define an additional order on $\mathcal{NC}(\Lambda)$: $\pi \ll \sigma$ if $\pi \leq \sigma$ and in addition, for each block $U \in \mathcal{E}$, $\pi | U \in \mathcal{NC}(U)$. See [BN08, Nic10] for more details.

For $(\pi, S) \in \mathcal{NC}(n)$ and $\pi \ll \pi$, we say that a block of $\pi$ is open if it contains the smallest element of an open block of $\pi$; their collection is denoted $S'(\pi, S)$. In particular, each open singleton block of $\pi$ consists of the smallest element of some open block of $\pi$.

Definition 25. For $D$, $M$, $\Gamma(M)$, and $X(a)$ as in Remark 21, define the free Meixner-Kailath-Segall polynomials by the recursion

$$W(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W(a_1 \otimes \ldots \otimes a_n)X(a_{n+1}) - \alpha W(a_1 \otimes \ldots \otimes a_n)\langle a_{n+1} \rangle$$

$$- \beta W(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_{n+1}) - t W(a_1 \otimes \ldots \otimes a_{n-1})\langle a_{n+1} \rangle$$

$$- \gamma W(a_1 \otimes \ldots \otimes a_{n-2} \otimes a_{n-1}a_{n+1})$$

and in particular

$$W(a_1 \otimes a_2) = W(a_1)X(a_2) - \alpha W(a_1)\langle a_2 \rangle - \beta W(a_1a_2) - t \langle a_1a_2 \rangle$$

and $W(a_1) = X(a_1) - \alpha \langle a_1 \rangle$. Compare with Section 7 in [Sni00].

Notation 26. For $(\pi, S) \in \mathcal{NC}(n)$, let

$$C_{\alpha, \beta, t, \gamma}^{(\pi, S)} = \sum_{\sigma \in \pi, U \in \pi \setminus \sigma, S \supseteq \sigma, \pi \in \mathcal{NC}(U)} \alpha^{|\text{Sing}(\pi \setminus S)|} \beta^{n - 2|\pi| + |S| + |\text{Sing}(\pi | S)|} t^{\pi | S | - | \text{Sing}(\pi \setminus S) |} \gamma^{|\pi| - |\pi|}.$$

In particular,

$$C_{\alpha, \beta, t, \gamma}^{\pi} = C_{\alpha, \beta, t, \gamma}^{(\pi, \emptyset)} = \sum_{\sigma \in \pi, \text{Sing}(\pi) \subseteq \text{Sing}(\pi)} \alpha^{|\text{Sing}(\pi)|} \beta^{n - 2|\pi| + |\text{Sing}(\pi)|} t^{\pi | \pi | - | \text{Sing}(\pi) |} \gamma^{|\pi| - |\pi|}.$$

Lemma 27. Denote $M_n(\beta, \gamma)$ a particular case of the Jacobi-Rogers polynomials, the sum over Motzkin paths of length $n$ with flat steps given weight $\beta$ and down steps given weight $\gamma$. Then

$$C_{\alpha, \beta, t, \gamma}^{(\pi, S)} = \prod_{U \in \pi | S} K_{\alpha, \beta, t, \gamma}^{U} \prod_{V \in \pi} \omega_{\alpha, \beta, t, \gamma}^{V}.$$
where
\[ \omega_{\alpha, \beta, t, \gamma}^n = M_{n-1}(\beta, \gamma), \quad \kappa_{\alpha, \beta, t, \gamma}^1 = \alpha, \quad \kappa_{\alpha, \beta, t, \gamma}^n = tM_{n-2}(\beta, \gamma). \]

Proof. We note that
\[ \kappa_{\alpha, \beta, t, \gamma}^1 = \alpha, \quad \kappa_{\alpha, \beta, t, \gamma}^n = t \sum_{\tau \in NC_{\geq 2}(n)} \beta^{n-2|\tau|+1} |\tau|^{n-1} = tM_{n-2}(\beta, \gamma), \]
and
\[ \omega_{\alpha, \beta, t, \gamma}^1 = 1, \quad \omega_{\alpha, \beta, t, \gamma}^n = \sum_{\tau \in NC(n), \Sing(\tau) \subset \{1\}} \beta^{n-2|\tau|+1} |\tau|^{n-1} = \sum_{\tau \in NC_{\geq 2}(n+1)} \beta^{n-2|\tau|+1} |\tau|^{n-1} = M_{n-1}(\beta, \gamma). \]

We formulate and prove the following theorem for the case \( D = C \) to simplify notation, but the result carries over verbatim for general \( D \).

**Theorem 28.** We have expansions of monomials
\[
X(a_1)X(a_2) \ldots X(a_n) = \sum_{(\pi, S) \in LNC(n)} C_{\alpha, \beta, t, \gamma}^{(\pi, S)} \prod_{U \in \pi \setminus S} \langle a_U \rangle W \left( \bigotimes_{V \in S} a_V \right).
\]

Proof. By induction
\[
\prod_{i=1}^n X(a_i)X(a_{i+1})
\]
\[
= \sum_{(\pi, S) \in LNC(n)} C_{\alpha, \beta, t, \gamma}^{(\pi, S)} \prod_{U \in \pi \setminus S} \langle a_U \rangle W \left( \bigotimes_{V \in S} a_V \right) X(a_{n+1})
\]
\[
= \sum_{(\pi, S) \in LNC(n)} \sum_{U \in \pi \setminus S} \alpha^{|\Sing(\pi \setminus S)|} \beta^{n-2|\tau|+|\tau|+1} |\tau|^{n-1} \prod_{U \in \pi \setminus S} \langle a_U \rangle
\]
\[
\left( W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_j} \otimes a_{n+1} : \{V_1 < V_2 < \ldots < V_j\} = S \right) + \alpha W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_j} : \{V_1 < V_2 < \ldots < V_j\} = S \right) \langle a_{n+1} \rangle \right)
\]
\[
+ \beta 1_{|S| \geq 1} W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_j} a_{n+1} : \{V_1 < V_2 < \ldots < V_j\} = S \right) \langle a_{V_j}a_{n+1} \rangle
\]
\[
+ t \beta 1_{|S| \geq 1} W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{j-1}} : \{V_1 < V_2 < \ldots < V_j\} = S \right) \langle a_{V_j}a_{n+1} \rangle
\]
\[
+ \gamma 1_{|S| \geq 2} W \left( a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{j-1}} a_{V_j} a_{n+1} : \{V_1 < V_2 < \ldots < V_j\} = S \right) \right).
\]

For fixed \((\pi', S, \sigma)\), the first term produces all the triples \((\pi', S', \sigma')\) with
\((\pi', S') \in LNC(n+1), \sigma' \leq \pi', U \in \pi' \setminus S' \Rightarrow \sigma'|U| \in NC'(U), \Sing(\sigma') \subset \Sing(\pi') \cup S'(\sigma', S')\)
in which \(n + 1\) is an open singleton in \(\pi'\) (and so in \(\sigma'\)). Since \(n, |\pi|, |S|\), and \(|\sigma|\) are all incremented by \(1\), \(C_{\alpha, \beta, t, \gamma}^{(\pi, S)}\) does not change. The second term produces all triples in which \(n + 1\) is a closed singleton in \(\pi'\) (and so in \(\sigma'\)). Since \(n, |\pi|, |\pi \setminus S|, |\Sing(\pi \setminus S)|\), and \(|\sigma|\) are all incremented by \(1\),
%

\[ C^{(\pi, S)} \] is multiplied by \( \alpha \). The third term produces all triples in which \( n + 1 \) belongs to an open block in both \( \sigma' \) and \( \pi' \), each of size at least 2, by adjoining it to the largest open block of \( \pi \) and the corresponding open block of \( \sigma \). Since only \( n \) is incremented, \( C^{(\pi, S)} \) is multiplied by \( \beta \). The fourth term produces all triples in which \( n + 1 \) belongs to a closed non-singleton block of \( \pi' \) (and so also of \( \sigma' \)), by adjoining it to the largest open block of \( \pi \) and the corresponding open block of \( \sigma \), and closing them both. Since \( n \) is incremented by 1 and \( |S| \) decreases by 1, \( C^{(\pi, S)} \) is multiplied by \( t \).

The fifth term produces all triples in which \( n + 1 \) belongs to an open block of \( \pi' \) but a closed block of \( \sigma' \), by adjoining \( n + 1 \) to the second largest open block of \( \pi \) and the corresponding open block of \( \sigma \), combining the two largest open blocks of \( \pi \), and closing the open block of \( \sigma \) which belonged to the largest open block of \( \pi \). Since \( n \) is incremented by 1 while \( |\pi| \) and \( |S| \) are decreased by 1, \( C^{(\pi, S)} \) is multiplied by \( \gamma \). These five classes are disjoint and exhaust the triples \((\pi', S', \sigma')\) above. \( \square \)

**Corollary 29.** For the state corresponding to the Wick products from Definition 25, the joint moments are

\[ \varphi [X(a_1)X(a_2) \ldots X(a_n)] = \sum_{\pi \in NC(n)} C_{\pi}^{\alpha, \beta, t, \gamma} (a_1 \otimes \ldots \otimes a_n)^{\pi}. \]

**Remark 30.** Using Lemma 27, we may re-write formula (13) as

\[ \varphi [X(a_1)X(a_2) \ldots X(a_n)] = \sum_{\pi \in NC(n)} \left( \prod_{U \in \pi} \kappa_{\alpha, \beta, t, \gamma}^{[U]} \right) (a_1 \otimes \ldots \otimes a_n)^{\pi}. \]

Therefore by definition, the joint free cumulants of \( \{X(a_i)\} \) are

\[ R[X(a_1)] = \kappa_{\alpha, \beta, t, \gamma}^{[1]} (a_1) = \alpha \langle a_1 \rangle, \]

\[ R[X(a_1), \ldots, X(a_n)] = \kappa_{\alpha, \beta, t, \gamma}^{[n]} (a_1 \ldots a_n) = t^{n-2} \beta, \gamma \langle a_1 \ldots a_n \rangle. \]

Compare with Theorem 8 in [Śni00]. Note that \( M_{n-2}(1, 1) = M_{n-2} \), the Motzkin number. On the other hand,

\[ M_{n-2}(2, 1) = \sum_{\pi \in NC_2(n)} 2^{n-2}|\pi| = |NC'(n)| = c_{n-1}, \]

the Catalan number. In general \( M_n(\beta, \gamma) \) are the moments of a semicircular distribution with mean \( \beta \) and variance \( \gamma \). Cf. Theorem 2 in [Ans07].

The following proposition is, roughly speaking, taken as the definition in [Śni00].

**Proposition 31.**

\[ \varphi \left[ W(a_1 \otimes \ldots \otimes a_n) W(b_k \otimes \ldots \otimes b_1) \right] = \delta_{n=k} \sum_{\pi \in lat(n)} t^{|U|} \gamma^{|\pi|} \langle a_1 \otimes \ldots \otimes a_n \otimes b_k \otimes \ldots \otimes b_1 \rangle^{\pi \cup \{U \cup (2n+1-U) : U \in \pi \}}. \]
Proof. For \( n = 0 \) and arbitrary \( k \), the result follows from the definition of \( \varphi \). So it suffices to show that the result for \((u,v) \leq (n-1,k+1)\) implies the result for \((n,k)\). For \( n \geq 1 \),

\[
\varphi \left[ W (a_1 \otimes \ldots \otimes a_n) W (b_k \otimes \ldots \otimes b_1) \right] \\
= \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) X(a_n) W (b_k \otimes \ldots \otimes b_1) \right] \\
\quad - \alpha \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n \rangle W (b_k \otimes \ldots \otimes b_1) \right] \\
\quad - \beta \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-2} a_{n-1} a_n) W (b_k \otimes \ldots \otimes b_1) \right] \\
\quad - t \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-2}) \langle a_{n-1} a_n \rangle W (b_k \otimes \ldots \otimes b_1) \right] \\
\quad - \gamma \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-3} a_{n-2} a_{n-1} a_n) W (b_k \otimes \ldots \otimes b_1) \right].
\]

Applying the recursion in Definition 25 to the first term (and using the adjoint symmetry in Proposition 37 below), this term equals

\[
\varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) W (a_n b_k \otimes \ldots \otimes b_1) \right] \\
+ \alpha \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n \rangle W (a_n b_k \otimes \ldots \otimes b_1) \right] \\
+ \beta \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) W (a_n b_k \otimes \ldots \otimes b_1) \right] \\
+ t \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n b_k \rangle W (b_{k-1} \otimes \ldots \otimes b_1) \right] \\
+ \gamma \varphi \left[ W (a_1 \otimes \ldots \otimes a_{n-1}) W (a_n b_k b_{k-1} \otimes b_{k-2} \ldots \otimes b_1) \right].
\]

This the expression (14) equals

\[
\delta_{n-k+1} \sum_{\pi \in \text{Int}(n-1)} t^{\|\gamma_{n-k}\|} \langle a_1 \otimes \ldots \otimes a_n b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-1-U);U\in\pi)} \\
+ \beta \delta_{n-k} \sum_{\pi \in \text{Int}(n-1)} t^{\|\gamma_{n-k}\|} \langle a_1 \otimes \ldots \otimes a_{n-1} a_n b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-1-U);U\in\pi)} \\
+ t \langle a_n b_k \rangle \delta_{n-k} \sum_{\pi \in \text{Int}(n-1)} t^{\|\gamma_{n-k}\|} \langle a_1 \otimes \ldots \otimes a_{n-1} b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-1-U);U\in\pi)} \\
+ \gamma \delta_{n-k} \sum_{\pi \in \text{Int}(n-1)} t^{\|\gamma_{n-k}\|} \langle a_1 \otimes \ldots \otimes a_{n-1} a_n b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-1-U);U\in\pi)} \\
\quad - \beta \delta_{n-k} \sum_{\pi \in \text{Int}(n-1)} t^{\|\gamma_{n-k}\|} \langle a_1 \otimes \ldots \otimes a_{n-1} a_n b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-1-U);U\in\pi)} \\
\quad - t \langle a_{n-1} a_n \rangle \delta_{n-2-k} \sum_{\pi \in \text{Int}(n-2)} t^{\|\gamma_{n-2-k}\|} \langle a_1 \otimes \ldots \otimes a_{n-2} b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-3-U);U\in\pi)} \\
\quad - \gamma \delta_{n-2-k} \sum_{\pi \in \text{Int}(n-2)} t^{\|\gamma_{n-2-k}\|} \langle a_1 \otimes \ldots \otimes a_{n-2} a_{n-1} a_n b_k \otimes \ldots \otimes b_1 \rangle^{(U \cup (2n-3-U);U\in\pi)}
\]

The second and fifth sums cancel term-by-term. Next, suppose \( n = k + 2 \). If in a partition \( \pi \in \text{Int}(n-1) \) in the first sum, \((n-1)\) is a singleton, the term corresponding to this partition cancels with the corresponding term in the sixth sum. If \((n-1)\) is not a singleton, the term corresponding to this partition cancels with the corresponding term in the seventh sum. The sum of the remaining sums (third and fourth), for \( n = k \), equals

\[
\delta_{n-k} \sum_{\pi \in \text{Int}(n)} t^{\|\gamma_{n-k}\|} \langle a_n^* \otimes \ldots \otimes a_1^* b_1 \otimes \ldots \otimes b_n \rangle^{(U \cup (2n+1-U);U\in\pi)}
\]
by the same decomposition.
Theorem 32. We may expand

\[(15) \quad W(a_1 \otimes \ldots \otimes a_n) = \sum_{\substack{\pi \in \text{Int}(n+1) \\ S \subset \pi}} (-1)^{n+1-|S|} \prod_{U \in \pi \setminus S} c_U \langle a_U \rangle \prod_{V \in S} q_V X(a_V), \]

where

\[c_k = \alpha_{k-1} - t \alpha_{k-2}.\]

Case I: \(\gamma = 0\). Then

\[o_k = \beta^{k-1}, \quad c_k = (\alpha \beta - t) \beta^{k-2} \]

For \(\gamma \neq 0\), factor \(1 - \beta z + \gamma z^2 = (1 - uz)(1 - vz)\).

Case II: \(\gamma \neq 0, \beta^2 \neq 4\gamma\), so that \(u \neq v\). Then

\[o_k = \frac{1}{u - v} (u^k - v^k), \quad c_k = \frac{1}{u - v} (\alpha (u^k - v^k) - t(u^{k-1} - v^{k-1})).\]

Case II': if in addition, \(\alpha^2 - \alpha \beta t + \gamma t^2 = 0\), so that \(v = t / \alpha\), then

\[o_k = \frac{1}{\beta - 2t / \alpha} ((\beta - t / \alpha)^k - (t / \alpha)^k), \quad c_k = \alpha(\beta - t / \alpha)^{k-1}.\]

Case III: \(\gamma \neq 0, \beta^2 = 4\gamma\), so that \(u = v = \beta / 2\). Then

\[o_k = k(\beta / 2)^{k-1}, \quad c_k = (\alpha k(\beta / 2) - t(k - 1))(\beta / 2)^{k-2}.\]

Case III': if in addition, \(\alpha \beta = 2t\), so that \(u = v = t / \alpha\), then

\[o_k = k(\beta / 2)^{k-1}, \quad c_k = \alpha(\beta / 2)^{k-1}.\]

Proof. Write \(W(a_1 \otimes \ldots \otimes a_n)\) in the form (15); we will show that this is possible by exhibiting coefficients in this expansion. Plugging in this expansion into the recursion in Definition 25, we obtain

\begin{align*}
\sum_{\substack{\pi \in \text{Int}(n+1) \\ S \subset \pi}} (-1)^{n+1-|S|} &= \sum_{\substack{\pi \in \text{Int}(n) \\ S \subset \pi}} (-1)^{n-|S|} X(a_{n+1}) - \alpha \sum_{\substack{\pi \in \text{Int}(n) \\ S \subset \pi}} (-1)^{n-|S|} \langle a_{n+1} \rangle \\
&\quad - \beta \sum_{\substack{\pi \in \text{Int}(n+1) \\ S \subset \pi}} (-1)^{n-|S|} \langle a_{n+1} \rangle \\
&\quad - t \sum_{\substack{\pi \in \text{Int}(n+1) \\ S \subset \pi}} (-1)^{n-|S|} \langle a_1 a_{n+1} \rangle - \gamma \sum_{\substack{\pi \in \text{Int}(n+1) \\ S \subset \pi}} (-1)^{n-1-|S|},
\end{align*}

where in each term we sum the expression

\[\prod_{U \in \pi \setminus S} c_U \langle a_U \rangle \prod_{V \in S} q_V X(a_V).\]

Now compare the factors corresponding to the block \(B\) containing \(n + 1\) on the left-hand-side. If \(B\) is an open singleton, it matches with a term in the first sum, with the same coefficient (since the number of open blocks on the left is one more than on the right). Thus \(o_1 = 1\). For the remaining terms, the size of \(S\) does not change, so we omit it from the coefficients. If \(B\) is a closed singleton, it matches with a term from the second sum, and the coefficients are \((-1)^{n+1} c_1 = (-1)^n (-\alpha),\)
Then the following is Definition 4.9 from [Ans04a]. Here \( M/D_4/A_1 \) the fourth sums, and the coefficients are \((-1)^{n+1} o_2 = (-1)^{n}(-\beta) o_1 \), so \( o_2 = \beta o_1 \). If \( B \) is a closed pair, it matches with terms in the third and the fifth sums, and the coefficients are \((-1)^{n+1} c_2 = (-1)^{n}(-\beta)c_1 + (-1)^{n+1}(-t) \), so \( c_2 = \beta c_1 - t \). If \( B \) is a larger block, it matches with terms in the third and the fifth sums, and the coefficients are \((-1)^{n+1} o_k = (-1)^{n}(-\beta) o_{k-1} + (-1)^{n+1}(-\gamma) o_{k-2} \) (and the corresponding expression for \( c_k \)), so that

\[
o_k = \beta o_{k-1} - \gamma o_{k-2}, \quad c_k = \beta c_{k-1} - \gamma c_{k-2}
\]

for \( k \geq 3 \). Let

\[
O(z) = \sum_{k=1}^{\infty} o_k z^{k-1}, \quad C(z) = \sum_{k=1}^{\infty} c_k z^{k-1}.
\]

Then

\[
O(z) = 1 + \beta z O(z) - \gamma z^2 O(z), \quad C(z) = \alpha - tz + \beta z C(z) - \gamma z^2 C(z),
\]

so

\[
O(z) = \frac{1}{1 - \beta z + \gamma z^2}, \quad C(z) = \frac{\alpha - tz}{1 - \beta z + \gamma z^2} = \alpha O(z) - tz O(z).
\]

The specific cases follow.

5. Counterexample

The following is Definition 4.9 from [Ans04a]. Here \( M, \Gamma(M) \) are as in the introduction.

**Definition 33.** For \( a_i \in M^a \), define the \( q \)-Kailath-Segall polynomials by \( W_q(a) = X(a) - \langle a \rangle \) and

\[
W_q(a, a_1, a_2, \ldots, a_n) = X(a) W_q(a_1, a_2, \ldots, a_n) - \sum_{i=1}^{n} q^{i-1} \langle a a_i \rangle W_q(a_1, \ldots, \hat{a}_i, \ldots, a_n)
\]

(16)

\[-\sum_{i=1}^{n} q^{i-1} W_q(a a_i, \ldots, \hat{a}_i, \ldots, a_n) - \langle a \rangle W_q(a_1, a_2, \ldots, a_n).
\]

This map has a \( \mathbb{C} \)-linear extension, so that each \( W \) is really a multi-linear map from \( M \) to \( \Gamma(M) \).

**Example 34.** According to Corollary 4.13 from [Ans04a],

\[
\varphi_q \left[ W_q(a_0) \ W_q(a_1, a_2, a_3) \ W_q(a_4) \right] = 0.
\]

However a direct calculation shows that in fact

\[
\varphi_q \left[ W_q(a_0) \ W_q(a_1, a_2, a_3) \ W_q(a_4) \right] = (q - q^2)(\langle a_0 a_2 \rangle \langle a_1 a_3 a_4 \rangle - \langle a_0 a_2 a_4 \rangle \langle a_1 a_3 \rangle).
\]

To be completely explicit, we consider the case where \( a_0 = a_2 = 1_I, a_1 = a_3 = a_4 = 1_J, I \cap J = \emptyset \), and the state is the Lebesgue measure. Then we get

\[
\varphi_q \left[ W_q(a_0) \ W_q(a_1, a_2, a_3) \ W_q(a_4) \right] = (q - q^2) |I| \cdot |J|.
\]

Thus Corollary 4.13, and so also Theorem 4.11 part (c) in [Ans04a], are false.
The formula in Theorem 4.11(c) is true if the arguments of each $W$ are orthogonal; however this does not imply the general result since $\varphi_q$ is not tracial. See Remark 42. There are many particular cases when 4.11(c) is true. For the case $q = 1$ (classical), and $q = 0$ (free), the proof provided in [Ans04a] still works. For the $q$-Gaussian case, this is Theorem 3.3 in [EP03]. Finally, for univariate polynomials obtained for equal idempotent $a$ and general $q$, the linearization formulas in Corollary 4.13 also hold [KSZ06, IKZ13].

6. Representations and completions

Let $\mathcal{M}$ and $\mathcal{B}$ be $\mathcal{D}$-bimodules with the actions satisfying (9). For a linear $\mathcal{D}$-bimodule map $F : \mathcal{M} \to \mathcal{B}$, define the map $\mathcal{F}(F) : \mathcal{M}^{\otimes n} \to \mathcal{B}^{\otimes n}$ by

$$\mathcal{F}(F)[d_0a_1d_1 \otimes \cdots \otimes a_n d_n] = F(d_0 a_1 d_1) \otimes \cdots \otimes F(a_n d_n),$$

and the map $\Gamma(F) : \Gamma(\mathcal{M}) \to \Gamma(\mathcal{B})$ by $\Gamma(F)[d] = d$ for $d \in \mathcal{D}$ and

$$\Gamma(F)[W_{\mathcal{I}N\mathcal{C}}(a)] = W_{\mathcal{I}N\mathcal{C}}(\mathcal{F}(F)[a]).$$

**Definition 35.** Let $\mathcal{M}$ be a star-algebra and $\mathcal{B}$ a star-subalgebra. An **algebraic conditional expectation** is a star-linear $\mathcal{B}$-bimodule map $F : \mathcal{M} \to \mathcal{B}$ such that $F^2 = F$. If $\mathcal{M}$ is a $\mathcal{D}$-bimodule, $\mathcal{DBD} \subset \mathcal{B}$, and $\tau : \mathcal{M} \to \mathcal{D}$ is a star-linear functional, we say that $F$ preserves $\tau$ if $\tau[F(a)] = \tau[a]$ for $a \in \mathcal{M}$.

**Proposition 36.** Let $\mathcal{D}, \mathcal{B}, \mathcal{M}$ be as in the preceding definition, and $F : \mathcal{M} \to \mathcal{B}$ an algebraic conditional expectation preserving $\langle \cdot \rangle$. Then $\Gamma(F) : \Gamma(\mathcal{M}) \to \Gamma(\mathcal{B})$ is an algebraic conditional expectation preserving $\varphi_{\mathcal{I}N\mathcal{C}}$.

**Proof.** Clearly $\Gamma(F)$ is the identity on $\Gamma(\mathcal{B})$. For $a \in \mathcal{B}^{\otimes n}, b \in \mathcal{M}^{\otimes k}, c \in \mathcal{B}^{\otimes \ell}$,

$$\Gamma(F)[W_{\mathcal{I}N\mathcal{C}}(a) W_{\mathcal{I}N\mathcal{C}}(b) W_{\mathcal{I}N\mathcal{C}}(c)] = \sum_{\{\pi, S\} \in \mathcal{I}N\mathcal{C}[n+k+\ell]} \Gamma(F)[W_{\mathcal{I}N\mathcal{C}}(a \otimes b \otimes c)^{\pi, S}]$$

$$= \sum_{\pi \wedge (1_n, 1_k, 1_\ell) = 0_{n+k+\ell}}\quad \sum_{\pi \wedge (1_n, 1_k, 1_\ell) = 0_{n+k+\ell}} W_{\mathcal{I}N\mathcal{C}}(a \otimes \mathcal{F}(F)b \otimes c)^{\pi, S}$$

$$= W_{\mathcal{I}N\mathcal{C}}(a) \Gamma(F)[W_{\mathcal{I}N\mathcal{C}}(b)] W_{\mathcal{I}N\mathcal{C}}(c),$$

where we have used the inhomogeneity of the partitions, the bimodule property of $F$ for open blocks, and both properties of $F$ for closed blocks. The final property is clear.$\square$

**Proposition 37.** In all six examples above,

$$W\left( a_1 \otimes a_2 \otimes \cdots \otimes a_n \right)^* = W\left( a_1^* \otimes a_2^* \otimes \cdots \otimes a_n^* \right),$$

where for $\varphi_{\mathcal{I}P}$ we additionally assume that $\mathcal{M}$ is commutative. If $\mathcal{D} = \mathbb{C}$ and the linear functional $\langle \cdot \rangle$ on $\mathcal{M}$ is tracial, all six linear functionals $\varphi$ are tracial. If $\mathcal{D}$ is a unital $C^*$-algebra, and $\langle \cdot \rangle$ is positive, the functionals $\varphi$ are positive, where for $\varphi_{\mathcal{I}P}$ and $\varphi_{\mathcal{I}P,\mathcal{R},\mathcal{M}}$ we additionally assume that $\mathcal{M}$ is commutative.
Proof. The trace and adjoint properties follow from the moment formulas and expansions of Wick products in terms of monomials, since in all cases the coefficients in the expansions depend only on the size of the blocks. For positivity,
\[
\varphi_{INC_{1,2}} \left[ W_{INC_{1,2}} (a_1 \otimes \ldots \otimes a_n)^* W_{INC_{1,2}} (b_1 \otimes \ldots \otimes b_k) \right] = \varphi_{INC} \left[ W_{INC} (a_1 \otimes \ldots \otimes a_n)^* W_{INC} (b_1 \otimes \ldots \otimes b_k) \right] = \delta_{n=k} \langle a_n^* \otimes \ldots \otimes a_1^* \otimes b_1 \otimes \ldots \otimes b_n \rangle^{(1,2n),(2,2n-1),\ldots,(n,n+1)} ,
\]
The proof of positivity of this inner product on \(\mathcal{M}^{\otimes \mathbb{N}}\) (which we denote \(\langle \cdot, \cdot \rangle_n\)) is almost verbatim the argument in Theorem 3.5.6 of [Spe98]. Also, by Proposition 31,
\[
\varphi \left[ W (a_1 \otimes \ldots \otimes a_n)^* W (b_1 \otimes \ldots \otimes b_k) \right] = \delta_{n=k} \sum_{\pi \in \text{Int}(n)} t^{\|\pi\|_1} \gamma^{n-|\pi|} \langle a_n^* \otimes \ldots \otimes a_1^* \otimes b_1 \otimes \ldots \otimes b_n \rangle^{U \cup (2n+1-U); \pi(\pi)}
\]
(17)
and so this inner product is also positive. For commutative \(\mathcal{M}\),
\[
\varphi_{IP_{1,2}} \left[ W_{IP_{1,2}} (a_1 \otimes \ldots \otimes a_n)^* W_{IP_{1,2}} (b_1 \otimes \ldots \otimes b_k) \right] = \varphi_{IP} \left[ W_{IP} (a_1 \otimes \ldots \otimes a_n)^* W_{IP} (b_1 \otimes \ldots \otimes b_k) \right] = \delta_{n=k} \sum_{\alpha \in \text{Sym}(n)} \langle a_\alpha^*(1) b_1 \rangle \ldots \langle a_\alpha(n) b_n \rangle ,
\]
This inner product on \(\mathcal{M}^{\otimes \mathbb{N}}\) is well known to be positive semi-definite. Finally,
\[
\varphi_{IP_{RM}} \left[ W_{IP_{RM}} (a_1 \otimes \ldots \otimes a_n)^* W_{IP_{RM}} (b_1 \otimes \ldots \otimes b_k) \right] = \delta_{n=k} \sum_{\alpha, \beta \in \text{Sym}(n)} \prod_{\pi \in \pi(\beta)} \langle \prod_{i \in U} (a_\alpha(i)^* b_i) \rangle ,
\]
where \(\pi(\beta)\) is the orbit decomposition of \(\beta\) and the order in each \(U \in \pi(\beta)\) is according to \(\beta\) as in Theorem 17. As observed in Section 4 of [Snı00], this inner product is in general not positive. If \(\mathcal{M}\) is commutative, we may re-write
\[
\varphi_{IP_{RM}} \left[ W_{IP_{RM}} (a_1 \otimes \ldots \otimes a_n)^* W_{IP_{RM}} (a_1 \otimes \ldots \otimes a_k) \right] = \delta_{n=k} \sum_{\alpha \in \text{Sym}(n)} \prod_{\pi \in \pi(\alpha)} (|U| - 1)! \left\langle \left( \prod_{i \in U} a_\alpha(i) \right)^* \left( \prod_{i \in U} a_i \right) \right\rangle
\]
\[
= \delta_{n=k} \frac{1}{n!} \sum_{\pi \in \pi(\alpha)} \prod_{\alpha, \beta \in \text{Sym}(n) \alpha, \beta \in \pi(\beta)} (|U| - 1)! \left\langle \left( \prod_{i \in U} a_\alpha(i) \right)^* \left( \prod_{i \in U} a_\beta(i) \right) \right\rangle
\]
\[
= \delta_{n=k} \frac{1}{n!} \sum_{\pi \in \pi(\alpha)} \prod_{\alpha, \beta \in \text{Sym}(n) \alpha, \beta \in \pi(\beta)} (|U| - 1)! \left\langle \left( \prod_{i \in U} (P_n a_i) \right)^* \left( \prod_{i \in U} (P_n a_i) \right) \right\rangle \geq 0 ,
\]
where in the next-to-last term we replaced \(\alpha \circ \beta\) with \(\alpha\), and \(P_n\) is the symmetrization operator. \(\square\)

Remark 38. For \(\mathcal{D} = \mathbb{C}\), \(\langle \cdot, \cdot \rangle_n\) is essentially the induced inner product on a tensor product of Hilbert spaces, and so is non-degenerate if \(\langle \cdot \rangle\) is faithful. In general, \(\langle \cdot, \cdot \rangle_n\), and so \(\varphi_{INC}\),
is rarely faithful. For example, let \( D = M, \langle a \rangle = a, \) and \( p \in M \) be a idempotent. Then \( \langle (1 - p) \otimes p, (1 - p) \otimes p \rangle_2 = 0. \)

**Notation 39.** For \( a \in M^{\otimes n} \) and \( (\pi, S) \in \mathcal{IR}(n), \) define the contraction \( C^{(\pi,S)}(a) \) by a linear extension of

\[
C^{(\pi,S)}(a_1 \otimes \ldots \otimes a_n) = \prod_{U \in \pi \setminus S} \langle a_U \rangle \otimes a_V.
\]

Note that in all our examples with \( D = C, \)

\[
W(a)^{(\pi,S)} = W(C^{(\pi,S)}(a)).
\]

**Proposition 40.** Assume \( \langle \cdot \rangle \) is a faithful state such that in its representation on \( L^2(M, \langle \cdot \rangle) \), \( M \) is represented by bounded operators. Let \( a \in M^{\otimes n} \) and \( b \in M^{\otimes k}. \) Denote \( \|a\|_2 = \sqrt{\langle a, a \rangle} \) and

\[
W(a)\|_\phi = \sqrt{\phi \left[ W(a)^* W(a) \right]}.
\]

Denote

\[
Z_{n,k} = \{ (\pi, S) \in \mathcal{IR}(n + k) : \pi \wedge (\hat{1}_n, \hat{1}_k) = 0_{n + k}, \text{Sing}(\pi) \subset S \}.
\]

For \( (\pi, S) \in Z_{n,k}, \) the map

\[
b \mapsto W_{\mathcal{IR}}(a \otimes b)^{(\pi,S)}
\]

is bounded as a map from \( L^2(M, \langle \cdot \rangle)^{\otimes k} \) to \( L^2(M, \langle \cdot \rangle)^{\otimes |S|} \), as is the map

\[
b \mapsto W_{\mathcal{IR}}(a) W_{\mathcal{IR}}(b).
\]

Therefore the definitions of \( C^{(\pi,S)}(a \otimes b) \) and \( W_{\mathcal{IR}}(a \otimes b)^{(\pi,S)} \) and the identity

\[
W_{\mathcal{IR}}(a) W_{\mathcal{IR}}(b) = \sum_{(\pi, S) \in Z_{n,k}} W_{\mathcal{IR}}(C^{(\pi,S)}(a \otimes b)) = \sum_{(\pi, S) \in Z_{n,k}} W_{\mathcal{IR}}(a \otimes b)^{(\pi,S)}
\]

extend to \( a \in M^{\otimes n} \) (algebraic tensor product) and \( b \in L^2(M, \langle \cdot \rangle)^{\otimes k} \) (Hilbert space tensor product). If \( \langle \cdot \rangle \) is tracial, we may switch \( a \) and \( b. \)

**Proof.** Since

\[
\left| W_{\mathcal{IR}}(a \otimes b)^{(\pi,S)} \right|_{\phi^{\mathcal{IR}}} = \|C^{(\pi,S)}(a \otimes b)\|_2,
\]

it suffices to consider the map \( b \mapsto C^{(\pi,S)}(a \otimes b). \) Denote

\[
\pi_\ell = \{(i) : 1 \leq i \leq n - \ell, n + \ell + 1 \leq i \leq n + k; (n - j + 1, n + j) : 1 \leq j \leq \ell \},
\]

and

\[
S_\ell = \{(i) : 1 \leq i \leq n - \ell, n + \ell + 1 \leq i \leq n + k \}, \quad S'_\ell = S_\ell \cup \{(n - \ell + 1, n + \ell)\}.
\]

Note \( |S_\ell| = n + k - 2\ell, |S'_\ell| = n + k - 2\ell + 1, \) and \( |Z_{n,k}| = 2 \min(n, k). \) Then

\[
Z_{n,k} = \{ (\pi_\ell, S_\ell), (\pi_\ell, S'_\ell) : 0 \leq \ell \leq \min(n - k) \}.\]
For $a = a_1 \otimes \ldots \otimes a_n$ and $b = \sum b_{i1} \otimes \ldots \otimes b_{ik},$

$$\left\| C^{(\pi_{\ell}, S\ell)}(a \otimes b) \right\|_2^2 = \sum_{i, j, r, s} \left\| a_{n-r+1} b_{i, r} \otimes \ldots \otimes a_{n-\ell} b_{i, \ell} \otimes \ldots \otimes b_{ik} \right\|_2^2 \leq \left| a_{n-\ell+1} \right|^2 \left\| C^{(\pi_{\ell-1}, S\ell-1)}(a \otimes b) \right\|_2^2$$

Also, the statement

$$\left\| C^{(\pi_{\ell}, S\ell)}(a \otimes b) \right\|_2 \leq \|a\|_2 \|b\|_2$$

is about Hilbert spaces and not algebras, and as such is well known, see for example Proposition 5.3.3 in [BS98] (one may identify the Hilbert space with the space of square-integrable functions on a measure space, and apply coordinate-wise Cauchy-Schwarz inequality). The boundedness of the first map follows.

Next, note that $W_{\mathcal{I}NC} \left( C^{(\pi, S)}(a \otimes b) \right)$ are orthogonal for different $|S|$. Thus

$$\left\| W_{\mathcal{I}NC}(a) W_{\mathcal{I}NC}(b) \right\|_{\mathcal{I}NC}^2 = \sum_{(\pi, S) \in Z_{n, k}} \left\| W_{\mathcal{I}NC} \left( C^{(\pi, S)}(a \otimes b) \right) \right\|_{\mathcal{I}NC}^2$$

The results follow. \hfill \Box

**Example 41.** Let $f(x, y) = 1_{[0,1]}(y)1_{[0, y^{-1/4}]}(x)$ and $g(y) = y^{-1/4}$. Then $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ (and so in $L^2(\mathbb{R}^2)$) and $g \in L^2(\mathbb{R})$, but

$$C^{(\{(1), (2, 3)\}, \{1\}, (2, 3))}(f \otimes g)(x, y) = f(x, y)g(y)$$

is not in $L^2(\mathbb{R}^2)$. Cf. Remark 3.3 in [BP14].

**Remark 42.** In stochastic analysis, see for example [PT11] or [BS98], it is usual to prove product formulas

$$W(a_1 \otimes \ldots \otimes a_n) W(b_1 \otimes \ldots \otimes b_k) = \sum W$$

for all $a_i$’s, and separately all $b_j$’s, orthogonal to each other. One can then conclude using the Itô isometry that the same formula holds for general $a_i, b_j$. Some, but not all, of the ingredients of this approach generalize to the Wick product setting.

- In the case of $W_{\mathcal{I}NC}$, $W$, and $W_q$, we have isometries between $\Gamma(\mathcal{M})$ and $\mathcal{M} \otimes^{\infty} 0$ with, respectively, the usual inner product induced by $\langle \cdot, \cdot \rangle$, the inner product (17), and the appropriate $q$-inner product (equation 4.73 in [Ans04a]). So in all these cases, one may extend the definition of $W$ to the appropriate closure, which however are different in all three cases.

- Instead of starting with general simple tensors, we could have started with the analog of functions supported away from diagonals. As noted in Lemma 43 below, in the infinite-dimensional setting such elements are still dense with respect to the usual inner product. However they are clearly not dense for the inner product (17). For example, in the natural
commutative setting of \((\mathcal{M}, \langle \cdot, \cdot \rangle) = ((L^1 \cap L^\infty(\mathbb{R}), dx),\) the inner product on functions of \(n\) arguments is
\[
\sum_{\pi \in \text{Int}(n)} \overline{f(x)} g(x) \, d\mu_\pi(x),
\]
where \(\mu_\pi\) is a multiple of the \(|\pi|\)-dimensional Lebesgue measure on the diagonal set
\[
\{ x \in \mathbb{R}^n : x_i = x_j \iff i \sim j \}.
\]
This is the reason why the formulas in Theorem 28 take a considerably simpler form if the arguments have orthogonal components.

Finally, to extend the product relation (18), we need the product map to be continuous, at least when one of the arguments is in the algebraic tensor product and the other is bounded in two-norm. If the state \(\varphi\) is not tracial, this need not be the case. Since \(\varphi_q\) is not tracial, it is natural to expect a counterexample in Section 5.

It is well-known that for non-atomic measures, functions supported away from diagonals are dense in the product space of all square integrable functions. The next lemma (applied to \(L^2(\mathcal{M}, \langle \cdot, \cdot \rangle)\)) shows that this results remains true for non-commutative algebras, in fact with no assumptions on the state other than faithfulness. The result is surely known, but we could not find it in the literature.

**Lemma 43.** Let \(H\) be a Hilbert space. In the Hilbert space tensor product \(H \otimes H\), consider the span \(S\) of tensors of the form \(f \otimes g\) with \(\langle f, g \rangle = 0\). This span is dense if and only if \(H\) is infinite dimensional.

**Proof.** Choose an orthonormal basis \(\{e_i\}\) for \(H\). Identify \(H \otimes H\) with the space of Hilbert-Schmidt operators \(\text{HS}(H)\),
\[
f \otimes g \mapsto f \langle \cdot, g \rangle.
\]
Then for any \(f, g\), we have
\[
\text{tr}(f \otimes g) = \sum_i \langle f, e_i \rangle \langle e_i, g \rangle = \langle f, g \rangle.
\]
In particular, if \(\langle f, g \rangle = 0\), \(\text{tr}(f \otimes g) = 0\). So if \(\dim H < \infty\), all operators in \(S\) have trace zero, and so \(S\) is not dense.
Now suppose that \(\dim H = \infty\). Then \(H\) is isomorphic to \(L^2([0, 1], dx)\), in which case the result is well-known (it is also not hard to give a direct argument in terms of Hilbert-Schmidt operators; it is left to the interested reader). \(\square\)

**Appendix A. Combinatorial Corollaries**

**Proposition 44.** All five examples of incomplete posets in Section 2 are graded by the number of open blocks. In addition:

(a) The number of incomplete partitions (analog of Bell numbers) is
\[
|\mathcal{IP}(n)| = \sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i},
\]
sequence A001861 in [OEIS17]. The incomplete Stirling numbers of the second kind are
\[ S_{n,k,\ell} = \{((\pi, S) \in \mathcal{IP}(n) : |\pi\backslash S| = k, |S| = \ell) \} = \binom{k + \ell}{\ell} S_{n,k,\ell}, \]
and the number of elements of rank \( \ell \) is \( \sum_k \binom{k + \ell}{\ell} S_{n,k,\ell} \), sequence A049020.

(b) The number of incomplete non-crossing partitions (analog of Catalan numbers) is
\[ |\mathcal{IN}(n)| = \binom{2n}{n}, \]
the central binomial coefficients, sequence A000984. Define the incomplete Narayana numbers
\[ N_{n,k,\ell} = \{((\pi, S) \in \mathcal{IN}(n) : |\pi\backslash S| = k, |S| = \ell) \}. \]
Then denoting
\[ F(t, x, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} N_{n,k,\ell} t^k x^\ell z^n \]
their generating function and
\[ \tilde{F}(t, z) = F(t, 0, z) = \frac{1 + z(t - 1) - \sqrt{1 - 2z(t + 1) + z^2(t - 1)^2}}{2tz} \]
the generating function of the regular Narayana numbers,
\[ F(t, x, z) = \frac{1 - z\tilde{F}(t, z)}{1 - z(t + x + \tilde{F}(t, z))}. \]
The rank generating function is
\[ F(1, x, z) = \frac{1 + \sqrt{1 - 4z}}{1 - 2z(x + 1) + \sqrt{1 - 4z}}, \]
and the number of elements of rank \( \ell \) is \( \frac{2^{\ell+1}}{n+\ell+1} \binom{2n}{n-\ell} \), sequence A039599.

(c) The number of partial permutations is \( |\mathcal{IPRM}(n)| = \sum_{\ell=0}^{n} \binom{n}{\ell}^2 \ell! \), sequence A002720. The incomplete Stirling numbers of the first kind
\[ s_{n,k,\ell} = \{((\pi, S) \in \mathcal{IPRM}(n) : |\pi\backslash S| = k, |S| = \ell) \}
\]
have the generating function
\[ \sum_{k=0}^{n} s_{n,k,\ell} t^k = \binom{n}{\ell} (t + \ell) \ldots (t + n - 1), \]
and the number of elements of rank \( \ell \) is \( \binom{n}{\ell}^2 \ell! \).

Proof. The formula for the incomplete Stirling numbers of the second kind is obvious. Then using for example the solved Exercise 1.32 in [Aig07],
\[ |\mathcal{IP}(n)| = \sum_{k,\ell=0}^{n} \binom{k + \ell}{\ell} S_{n,k,\ell} = \sum_{k,\ell=0}^{n} \sum_{i=\ell-k}^{n-k} \binom{n}{i} S_{i,\ell} S_{n-i,k} = \sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i}. \]
The incomplete Narayana numbers satisfy the recursion relation

\[ N_{n+1,k,\ell} = N_{n,k-1,\ell} + N_{n,k,\ell-1} + \sum_{i=1}^{n} \sum_{j=0}^{k} N_{i,j,\ell} N_{n-i,j-1,0}. \]

It follows that

\[ F(t, x, z) = 1 + z t F(t, x, z) + z x F(t, x, z) + z (F(t, x, z) - 1) \tilde{F}(t, z). \]

For \( t = 1 \) this relation is easily solved to give the rank generating function, while setting additionally \( x = 1 \), we see that the generating function for \( |IN\mathcal{C}(n)| \) is

\[ F(1, 1, z) = \frac{1}{\sqrt{1 - 4z}}. \]

Using the bijection from Proposition 9,

\[ |\{(\pi, S) \in IN\mathcal{C}(n) : |S| = \ell\}| = |\{\pi \in IN\mathcal{C}_{1,2}(2n) : |\text{Sing}(\pi)| = 2\ell\}|. \]

The latter number is clearly the same as the number of lattice paths with \( W \) and \( N \) steps which go from \((0, 0)\) to \((n + \ell, n - \ell)\) and do not cross the main diagonal. Using the reflection principle, this number is \( \binom{2n}{n+\ell} - \binom{2n}{n+\ell+1} \).

The formula for \( |ITPR\mathcal{M}(n)| \) is obvious. The formula for the incomplete Stirling numbers of the first kind follows from the recursion relation

\[ s_{n+1,k,\ell} = s_{n,k-1,\ell} + s_{n,k,\ell-1} + (n + \ell)s_{n,k,\ell}, \]

obtained in the usual way by adjoining \( n + 1 \) to an incomplete permutation of \( n \); note that in a closed work of length \( u \), \( n + 1 \) can be inserted in \( u \) places, while in an open word it can be inserted in \( u + 1 \) spaces. \( \square \)

Remark 45. For completeness, we include combinatorial corollaries of Proposition 16 and Theorem 17. Take \( a \) to a projection, so that \( a^2 = a \) and \( \langle a \rangle = t \). Denote \( X(a) = x \). Then

\[ x W_{ITP_{1,2}}(a^{\otimes n}) = W_{ITP_{1,2}}(a^{\otimes n+1}) + tn W_{ITP_{1,2}}(a^{\otimes n-1}), \]

\[ x W_{ITP}(a^{\otimes n}) = W_{ITP}(a^{\otimes n+1}) + (t + n) W_{ITP}(a^{\otimes n}) + tn W_{ITP}(a^{\otimes n-1}), \]

and

\[ x W_{ITPR\mathcal{M}}(a^{\otimes n}) = W_{ITPR\mathcal{M}}(a^{\otimes n+1}) + (t + n) W_{ITPR\mathcal{M}}(a^{\otimes n}) + (tn + n(n - 1)) W_{ITPR\mathcal{M}}(a^{\otimes n-1}). \]

Thus \( W_{ITP_{1,2}}(a^{\otimes n}) = H_n(x, t) \), the Hermite polynomial; \( W_{ITP}(a^{\otimes n}) = C_n(x, t) \), the Charlier polynomial; and \( W_{ITPR\mathcal{M}}(a^{\otimes n}) = L_{n-1}^{(1)}(x) \), the Laguerre polynomial. We thus get (mostly known) inversion, moment, product, and linearization formulas for these polynomials. For example, for the Laguerre case

\[ x^n = \sum_{|\pi, S| \in ITPR\mathcal{M}(n)} t^{|\pi, S|} L_{n-1}^{(1)}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} (t + \ell) \ldots (t + n - 1) L_{|\pi, S|}^{(\ell)}(x), \]

\[ L_n^{(1)}(x) = \sum_{|\pi, S| \in ITPR\mathcal{M}(n)} (-1)^{n-|\pi|} t^{|\pi, S|} x^{|\pi|} = \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} (t + \ell) \ldots (t + n - 1)x^\ell. \]
and
\[ \prod_{i=1}^{k} f_{s(i)}^{(l-1)}(x) = \sum_{(\pi,S) \in \mathcal{I}(n)} t^{\text{cyc} \pi} L_{s|S|}^{(l-1)}(x). \]

Extending the moment and linearization formulas [FZ88]
\[ \int x^n d\mu(x) = \sum_{\pi \in \text{Sym}(n)} t^{\text{cyc} \pi} = t(1 + 1) \cdots (t + n - 1), \]
\[ \int \prod_{i=1}^{k} L_{s(i)}^{(l-1)}(x) d\mu(x) = \sum_{\pi \in \mathcal{D}(s(1)) \cdots s(k)} t^{\text{cyc} \pi}. \]

Similarly, since
\[ \sum_{(\pi,S) \in \mathcal{I}(n), U \in \pi \setminus S} \prod \lambda(V) = \left( \begin{array}{c} n \\ k \end{array} \right) \sum_{\pi \in \mathcal{P}(n-k)} \prod (|V| - 1)! = \left( \begin{array}{c} n \\ k \end{array} \right) s_{n-k, \ell}, \]

we obtain the familiar result that the Charlier polynomials are
\[ C_n(x, t) = \sum_{k, \ell=0}^{n} (-1)^{n-k} \left( \begin{array}{c} n \\ k \end{array} \right) s_{n-k, \ell} t^k x^\ell = \sum_{k=0}^{n} (-1)^{k} \frac{n!}{k!} t^k \left( \begin{array}{c} x \\ n-k \end{array} \right). \]

Finally, since
\[ \left| \{(\sigma,S) \in \mathcal{NC}(n) \mid \sigma \in \text{Int}(n), V \in \pi \setminus S \Rightarrow |V| = 1, |\pi \setminus S| = k, |S| = \ell \} \right| = \left( \begin{array}{c} n-k \\ \ell-1 \end{array} \right) \left( \begin{array}{c} k + \ell \\ \ell \end{array} \right), \]

the free Charlier polynomials are
\[ P_n(x, t) = \sum_{k, \ell} (-1)^{n-k} \left( \begin{array}{c} n-k \\ \ell-1 \end{array} \right) t^k x^\ell. \]

See, for example, Chapter 7 in [Aig07] for many related combinatorial results.

**Remark 46.** Let \( \mu_{\alpha, \beta, t, \gamma} \) be the measure of orthogonality of the free Meixner polynomials, with the Jacobi-Szegő parameters
\[ \left( \begin{array}{c} \alpha, \alpha + \beta, \alpha + \beta, \ldots \\ t, \ t + \gamma, \ t + \gamma, \ldots \end{array} \right). \]

Then from the Viennot-Flajolet theorem, the \( n \)'th moment of this measure is
\[ \sum_{\sigma \in \mathcal{NC}(n)} \prod_{V \in \text{Out}(\sigma)} \alpha + \beta \prod_{V \in \text{Out}(\sigma)} t \prod_{V \in \text{Out}(\sigma)} (t + \gamma) = \sum_{\sigma \in \mathcal{NC}(n)} \alpha^{\text{Sing}(\sigma)} \beta^{n-2} + \alpha^{\text{Sing}(\sigma)} \gamma^{|\text{Out}(\sigma) \setminus \text{Sing}(\sigma)|} \alpha^{t + \gamma} \beta^{j-2} = \sum_{\sigma \in \mathcal{NC}(n)} \alpha^{\text{Sing}(\sigma)} \beta^{n-2} \gamma^{|\text{Out}(\sigma) \setminus \text{Sing}(\sigma)|} \alpha^{t + \gamma} \beta^{j-2}. \]

We may interpret this as saying that the two-state free cumulants of the pair of free Meixner and free Poisson measures \( (\mu_{\alpha, \beta, t, \gamma}, \mu_{\alpha, \beta, 0, t}) \) are
\[ R_1 = r_1 = \alpha, \quad R_j = (t + \gamma) \beta^{j-2}, \quad r_j = t \beta^{j-2}. \]

Cf. Proposition 10 in [Ans09].
Various classical combinatorial sequences appearing as moments of these measures are listed in Section 7.4 of [Aig07]. These include Catalan, Motzkin, and Schröder numbers. Expansions (13) and (19) then give us various combinatorial identities. For example, for $\alpha = t = \gamma = 1$ and $\beta = 2$, the free cumulants are Catalan numbers while the moments are the (shifted) large Schröder numbers, and we obtain the relations

$$\text{Sch}_{n-1} = \sum_{\pi \in \text{NC}(n)} \prod_{U \in \pi} c_{|U|-1} = \sum_{\sigma \in \text{NC}(n)} 2^{n-|\sigma| - |\text{Out}(\sigma)|} |\text{Sing}(\sigma)|,$$

For the first relation, cf. Corollary 8.4 in [Dyk07]. If $\beta = t = \gamma = 1$, and $\alpha = 0$ the free cumulants are Motzkin numbers, and the moments are

$$\sum_{\pi \in \text{NC}_{\geq 2}(n)} \prod_{U \in \pi} M_{|U|-2} = \sum_{\sigma \in \text{NC}_{\geq 2}(n)} 2^{n-|\sigma| - |\text{Out}(\sigma)|}.$$

Either for $\alpha = 1$ or $\alpha = 0$ this moment sequence does not appear in [OEIS17].

**Remark 47.** In this remark we compute the sum of the coefficients in the expansion (12). According to Lemma 27, this sum is

$$T_n = \sum_{(\pi,S) \in \text{LNC}(n)} \prod_{U \in \pi \setminus \text{Out}(\pi)} (\omega_{\alpha,\beta,t,\gamma}^{[V]} + \kappa_{\alpha,\beta,t,\gamma}^{[V]}) \prod_{V \in S} \kappa_{\alpha,\beta,t,\gamma}^{[V]} \sum_{\pi \in \text{NC}(n)} \prod_{U \in \pi \setminus \text{Out}(\pi)} (\omega_{\alpha,\beta,t,\gamma}^{[V]} + \kappa_{\alpha,\beta,t,\gamma}^{[V]}).$$

Using the same lemma,

$$r(z) = \sum_{n=1}^{\infty} \kappa_{\alpha,\beta,t,\gamma}^{n} z^{n-1} = \alpha + t \sum_{n=2}^{\infty} M_{n-2}(\beta,\gamma)z^{n-1} = \alpha + tz F_{\beta,\gamma}(z)$$

and

$$R(z) = \sum_{n=1}^{\infty} (\omega_{\alpha,\beta,t,\gamma}^{n} + \kappa_{\alpha,\beta,t,\gamma}^{n}) z^{n-1} = \alpha + tz F_{\beta,\gamma}(z) + \sum_{n=1}^{\infty} M_{n-1}(\beta,\gamma)z^{n-1} = \alpha + (tz + 1) F_{\beta,\gamma}(z)$$

where

$$F_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} M_{n}(\beta,\gamma)z^{n} = \frac{1 - \beta z - \sqrt{(\beta z - 1)^2 - 4 \gamma z^2}}{2\gamma z^2}$$

is the generating function of Motzkin polynomials. According to the two-state free probability theory, the moment generating function of $\mu_{\alpha,\beta,t,\gamma}$ from Remark 46 is the solution of

$$\frac{1}{zm_{\alpha,\beta,t,\gamma}(z)} + r(zm_{\alpha,\beta,t,\gamma}(z)) = \frac{1}{z},$$

and the generating function of the desired sequence is

$$M(z) = \sum_{n=0}^{\infty} T_n z^n = \frac{1}{1 - z R(zm_{\alpha,\beta,t,\gamma}(z))}.$$ 

According to Remark 50 below, the most natural choice of the parameters appears to be $\alpha = t = \gamma = 1, \beta = 2$. In this case

$$F(z) = \frac{1 - 2z - \sqrt{1 - 4z}}{2z^2}, \quad R(z) = \frac{1 - z - (z + 1) \sqrt{1 - 4z}}{2z^2},$$

and $m(z)$ is the generating function of shifted large Schröder numbers

$$m(z) = \frac{3 - z - \sqrt{1 - 6z + z^2}}{2}.$$
Using Maple, we compute the first few terms in the sequence $T_n$ to be $1, 2, 7, 30, 140, 684$. This sequence now appears in [OEIS17] as A299296, but has not arisen in other contexts.

**Example 48.** From Theorem 32, we can get a variety of different-looking combinatorial expansions.

For $\alpha = \beta = \gamma = t = 1$, Case II. $u, v = e^{\pm (\pi/3)i}$.

$$o_k = 1, 1, 0, -1, -1, 0, \ldots, \quad c_k = 1, 0, -1, -1, 0, 1, \ldots$$

$$W(a_1 \otimes \ldots \otimes a_n) = \sum_{(\pi, S) \in \pi_{\text{Int}(n)}} (-1)^{|U| - |S|} \prod_{U \in S \mid |U| = 0 \mod 3} \prod_{U \in S \mid |U| = 2 \mod 3} \prod_{U \in S \mid |U| = 1 \mod 2} \prod_{U \in S \mid |U| = 4 \mod 6} \prod_{U \in S \mid |U| = 3 \mod 6} \langle a_U \rangle \prod_{V \in S} X(a_V).$$

For $\alpha = 1, \gamma = t, \beta = t + 1$, Case II'. $u = 1, v = t$.

$$o_k = \frac{1}{1-t} (1-t^k), \quad c_k = 1.$$
moment computations. This approach has not led us to any clarification in the inversion or product formulas.

In place of partitions, we could also (of course) have used colored Motzkin paths. From the point of view of Definition 25, the most natural family are those with a single color for rising steps and flat and falling steps at height zero, two colors for the other falling steps, and three colors for the rest of flat steps. It is not hard to see using the continued fraction form of the generating functions that the number of such paths of length \( n + 1 \) is equal to the number of large \((3, 2)\)-Motzkin paths of length \( n \) in the sense of [CW12] (similar to the above, except their falling and flat steps at height zero are allowed two colors). This number in turn is known to be the (large) Schröder number, see Remark 46.

**Remark 50.** Unlike in the expansions in the five examples in Section 3, the terms on the right hand side of (12) have multiplicities. One can modify Definition 25 to obtain bijective representations. For example, we may define instead

\[
W \left( a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \right) = W \left( a_1 \otimes \ldots \otimes a_n \right) X \left( a_{n+1} \right) - \alpha W \left( a_1 \otimes \ldots \otimes a_n \right) a_{n+1} - \beta W \left( a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n a_{n+1} \right) - t W \left( a_1 \otimes \ldots \otimes a_{n-1} \right) a_n a_{n+1} - \gamma W \left( a_1 \otimes \ldots \otimes a_{n-2} \otimes a_n a_{n+1} a_{n-1} \right).
\]

Note that this definition works only in the scalar-valued and not in the operator-valued case. The corresponding terms are in a bijection with the following collection of incomplete permutations. First, they have no double descents. Second, arrange each closed block so that it ends in its largest element. Then the descent-ascents in each block appear in decreasing order. Finally, split each block into sub-words, ending with the final letter or a descent-ascent, and beginning with the initial letter or right after the preceding descent-ascent. Then the partition into these sub-words is non-crossing.

We may also define

\[
W \left( a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} \right) = W \left( a_1 \otimes \ldots \otimes a_n \right) X \left( a_{n+1} \right) - \alpha W \left( a_1 \otimes \ldots \otimes a_n \right) a_{n+1} - \beta_1 W \left( a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n a_{n+1} \right) - \beta_2 W \left( a_1 \otimes \ldots \otimes a_{n-1} \otimes a_{n+1} a_n \right) - t W \left( a_1 \otimes \ldots \otimes a_{n-1} \right) a_n a_{n+1} - \gamma W \left( a_1 \otimes \ldots \otimes a_{n-2} \otimes a_n a_{n+1} a_{n-1} \right).
\]

The corresponding terms are in a bijection with the following collection of incomplete permutations. Arrange each closed block so that it ends in its largest element. Then the descent-ascents in each block appear in decreasing order. Split each block as above. Then the partition into these sub-words is non-crossing, and on each sub-block, the letters are decreasing and then increasing, with the sub-block maximum at the end.

This description appears related to the work of West [Wes95], who studied permutations avoiding the patterns \((3142, 2413)\) (sometimes called separable permutations). He proved that the cardinality of this set is the Schröder number (see Remark 46), and the argument uses trees reminiscent of the construction above.

**References**


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