ANALYSIS OF HYPER-SINGULAR, FRACTIONAL, AND ORDER-ZERO SINGULAR INTEGRAL OPERATORS

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ABSTRACT. In this article, we conduct a study of integral operators defined in terms of non-convolution type kernels with singularities of various degrees. The operators that fall within our scope of research include fractional integrals, fractional derivatives, pseudodifferential operators, Calderón-Zygmund operators, and many others. The main results of this article are built around the notion of an operator calculus that connects operators with different kernel singularities via vanishing moment conditions and composition with fractional derivative operators. We also provide several boundedness results on weighted and unweighted distribution spaces, including homogeneous Sobolev, Besov, and Triebel-Lizorkin spaces, that are necessary and sufficient for the operator’s vanishing moment properties, as well as certain behaviors for the operator under composition with fractional derivative and integral operators. As applications, we prove $T1$ type theorems for singular integral operators with different singularities, boundedness results for pseudodifferential operators belonging to the forbidden class $S^{0}_{1,1}$, fractional order and hyper-singular paraproduct boundedness, a smooth-oscillating decomposition for singular integrals, sparse domination estimates that quantify regularity and oscillation, and several operator calculus results. It is of particular interest that many of these results do not require $L^2$-boundedness of the operator, and furthermore, we apply our results to some operators that are known not to be $L^2$-bounded.

1. INTRODUCTION

Our primary goal in this article is to develop a theory that connects integral operators of different singularities through composition with fractional derivatives, and to understand how these operators are related through vanishing moment conditions and distribution space boundedness properties. The operators we consider are, formally speaking, $\nu$-order singular integral operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

where $K$ is a kernel that satisfies $|K(x, y)| \leq |x - y|^{-(n+\nu)}$, which will be defined precisely in Section 2 as members of the $\nu$-order Singular Integral Operator class $SIO_{\nu}$. When $\nu > 0$, $T$ is a hyper-singular operator and resembles differentiation in some sense. When $\nu < 0$, $T$ is a fractional order operator and resembles a fractional integral or anti-differentiation operator. The prototypical example for such operators are the $\nu$-order fractional derivatives $|\nabla|^\nu \in SIO_{\nu}$ for $\nu \neq 0$.

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which are defined via the Fourier multiplier $|\xi|^\nu$. Such operators are typically viewed as \(\nu\)-order derivatives when \(\nu > 0\) and \(\nu\)-order fractional integrals when \(\nu < 0\), which agrees with our rough interpretation of \(SIO_\nu\). We will also work with the zero-order, or critical index, class \(SIO_0\), which include several mainstays in harmonic analysis, like the Hilbert transform, Riesz transforms, other Calderón-Zygmund operators, zero-order pseudodifferential operators, and others. Zero-order operators have been studied extensively. Some of the work most closely related to the current article includes \([11, 14, 18, 16, 44, 20, 22, 23]\). There appears to be much less theory developed for the classes \(SIO_\nu\) for \(\nu \neq 0\). The most relevant sources are \([44]\) for \(\nu \neq 0\) and \([10]\) for \(\nu < 0\).

We work with operators in \(SIO_\nu\) in several different capacities, which we will eventually show are all equivalent in some sense. For \(T \in SIO_\nu\), we consider the following questions: Under what conditions on \(T\) and \(s, t \in \mathbb{R}\) does \(|\nabla|^{-s}T|\nabla|^t\) belong to \(SIO_{\nu + t - s}\)? Under what condition can \(T \in SIO_\nu\) be extended to a bounded operator on various distribution spaces, including weighted and unweighted homogeneous Sobolev, Besov, and Triebel-Lizorkin spaces? We will show that the appropriate conditions for \(T\) are a \(\nu\)-order Weak Boundedness Property, which we refer to as \(WBP_\nu\), and vanishing moment conditions of the form \(T^*(x^\alpha) = 0\). Furthermore, we show that, in many situations, both the operator calculus properties and distribution space boundedness properties of an operator \(T \in SIO_\nu\) are sufficient conditions for \(T^*(x^\alpha) = 0\). Thus, we develop many equivalent conditions between vanishing moment properties, distribution space boundedness, and operator composition properties.

The original motivation for this work came from the notion of an operator calculus for different classes of integral operators, some examples of which include classes of linear and bilinear pseudodifferential operators, Calderón-Zygmund operators, and fractional integral operator. It appears to be a consensus that the origins of a pseudodifferential symbolic calculus lie in the theory of singular integral operators, but it is not clear exactly where or when such a calculus first appeared. Some have credited early development of the topic to Bokobza, Calderón, Mihlin, Hörmander, Kohn, Nirenberg, Seeley, Unterberger, and Zygmund; see \([39, 36, 40]\) for more information on the early history of this. An operator calculus for a forbidden class, sometimes also referred to as an exotic class, of pseudodifferential operators was formulated by Bourdaud \([7]\), and the notion of a symbolic calculus for bilinear pseudodifferential operators was introduced by Bényi, Maldonado, Naibo, and Torres \([3]\). Several algebras of Calderón-Zygmund operators have been formulated, for example, by Coifman and Meyer in \([12]\). These subclasses of operators can be defined in terms of almost diagonal operators, which is a notion depending on wavelet decompositions, but they ultimately bare out to be equivalent to vanishing moment conditions for the operator. Finally, the current authors developed a restricted calculus for linear and bilinear fractional integral operators in \([10]\) that in some senses resembles the one we present here. One of the main results of this article is a restricted calculus for \(SIO_\nu\) (see Theorems 4.2 and 4.3), and as an application we also introduce some new operator algebras associated to \(SIO_\nu\) in Section 7.6.

Our first objective is to develop the restricted operator calculus for \(SIO_\nu\) being acted on by \(|\nabla|^s\) for \(s \in \mathbb{R}\). A little more precisely, in Theorem 4.2 we show that if \(T \in SIO_\nu\) satisfies a \(\nu\)-order weak boundedness property and \(T^*(x^\alpha) = 0\) for appropriate multi-indices \(\alpha \in \mathbb{N}_0^n\), then \(|\nabla|^{-s}T|\nabla|^t\) agrees modulo polynomials with an operator in \(SIO_{\nu + t - s}\) for certain ranges of \(s, t \in \mathbb{R}\). In the process of formulating this restricted calculus, we prove some estimates for functions of the form \(\psi_k \ast T(f(x))\) for \(T \in SIO_\nu\), which are of interest and useful on their own right; see Theorem 3.1 and Corollary 3.2.
Our second goal is to show that the same conditions on $T$ mentioned in the previous paragraph are also sufficient for the boundedness of $|\nabla|^{-s} T |\nabla|^r$, and hence of $T$, on certain distribution spaces. We show that $|\nabla|^{-s} T |\nabla|^r$ can be extended, modulo polynomials, to a bounded linear operator from $\dot{W}^{N+t-s,p}$ into $L^p$, which implies $T$ is can be extended to a bounded linear operator from $\dot{W}^{N-t,p}$ into $\dot{W}^{-t,p}$ for appropriate $t > \nu$; see Theorem 4.3 for more on this. We extend the boundedness properties of both $|\nabla|^{-s} T |\nabla|^r$ and $T$ to other functions spaces as well, including weighted Besov and Triebel-Lizorkin spaces, in Theorem 5.1, Theorem 5.4, Corollary 5.5, and Corollary 5.6. A significant feature of all the results mentioned to this point, including the ones in the preceding paragraph, is that no a priori boundedness of neither $T$ nor $\nabla$ is required. Indeed, if $T \in SIO_0$, one need not require even $L^2$-boundedness for $T$ to apply these results. Furthermore, it is even possible to apply these results to operators that are not $L^2$-bounded, and we provide some examples of such operators. This notion will be explored in more detail in the applications provided in Section 7, specifically in Sections 7.2-7.5.

Our third objective is to establish several sufficient conditions for vanishing moment properties of the form $T^\alpha(x^\alpha) = 0$. We show that under slightly stronger initial assumptions on $T$, the results pertaining to $|\nabla|^{-s} T |\nabla|^r$ and boundedness of $T$ discussed in the last two paragraphs are not only necessary for $T^\alpha(x^\alpha) = 0$ conditions, but also sufficient. Hence combining the results obtained in the direction of our second and third goals, we prove $T1$-type theorems that provide necessary and sufficient conditions for boundedness on many classes of distribution spaces. Furthermore, this verifies that the vanishing moment conditions $T^\alpha(x^\alpha) = 0$ are necessary and sufficient to well-define $|\nabla|^{-s} T |\nabla|^r$ as a singular integral operator. The results pertaining to sufficiency conditions for $T^\alpha(x^\alpha) = 0$ are in Theorem 6.3, and $T1$-type theorems are discussed in Section 7.1.

There are some general insights about sufficient conditions for vanishing moments of the form $T^\alpha(x^\alpha) = 0$ that can be gained for the results associated to our third goal. It is well known that if $T$ is a Calderón-Zygmund operator that can be extended to a bounded operator on the weighted Hardy spaces $H^p$ for $p \leq 1$, then $T^\alpha(x^\alpha) = 0$ for appropriate $\alpha \in \mathbb{N}_0^d$ depending on the size of $p$. We refer to $p$ here as a Lebesgue parameter or Lebesgue index since $H^p$ is defined in terms of an $L^p$ norm. Thus, if $T$ is bounded on distribution spaces with small enough Lebesgue index $p$, then $T^\alpha$ must vanish on polynomials up to some degree. This is one way to formulate sufficient conditions for $T^\alpha(x^\alpha) = 0$. We show that if $T$ is bounded on spaces $\dot{W}^{-t,p}$ for certain ranges of $t > 0$ and $1 < p < \infty$, then $T^\alpha(x^\alpha) = 0$; the same holds for boundedness for Triebel-Lizorkin and Besov spaces. This shows that boundedness on negative smoothness index spaces provide another way to formulate sufficient conditions for $T^\alpha(x^\alpha) = 0$. Working formally, it then also follow by duality that if $T$ is bounded on positive index smoothness spaces, then $T(x^\alpha) = 0$. These two ways to formulate sufficient conditions for vanishing moments are well-understood, for example some results along these lines can be found in [1, 12, 22]. We provide two other types of sufficient conditions for $T^\alpha(x^\alpha) = 0$. One is to require $T$ to be bounded on weighted distribution spaces where the weights are outside the natural weight class for the Lebesgue index of the space. For instance, if $T$ is bounded on $H^2_w$ for all Muckenhoupt weights $w \in A_\infty$, then $T^\alpha(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^d$. Similar results hold for other Triebel-Lizorkin spaces, for Lebesgue indices other than 2, and for $A_q$ in place of $A_\infty$. The final way we formulate sufficient conditions for $T^\alpha(x^\alpha) = 0$ is by requiring the operator $|\nabla|^{-s} T |\nabla|^r$ to agree modulo polynomials with operators in $SIO_{q+s-t}$, which of course are closely related to the boundedness of $T$ on distribution spaces, but nonetheless provide another
sufficient condition for $T^*(x^\alpha) = 0$. All of these approaches to formulating sufficient conditions for $T^*(x^\alpha) = 0$ are illustrated in Theorem 6.3.

Finally, in Section 7 we will provide several applications. Our first application, in Section 7.1, is a $T1$ type theorem that extends the boundedness of a Calderón-Zygmund operator outside of the realm of Lebesgue spaces and for operators of arbitrary order $\nu \in \mathbb{R}$. In Corollary 7.1, we impose a little more on a given operator a priori, that $T$ belong to CZO$_\nu$ rather than just SIO$_\nu$, and in doing so we obtain necessary and sufficient conditions using some of the results from Sections 4-6.

In Section 7.2, we verify that pseudodifferential operators with symbols in the forbidden class $S^{0,1}_{1,1}$ can be treated with our results. We show that such operators that satisfy $T(x^\alpha) = 0$ type conditions can be extended to bounded operators on several smooth distribution spaces. In particular, we apply our result to typical examples of operators in this forbidden class that are not $L^2$-bounded, as well as their transposes.

In Section 7.3, we construct some paraproduct operators that belong to SIO$_\nu$, for any given $\nu \in \mathbb{R}$, to which we can apply our operator calculus and boundedness results. Furthermore, we construct paraproducts in SIO$_0$ that are not $L^2$-bounded, but are bounded on homogeneous Sobolev spaces $W^{-t,2}$ for all $t > 0$, as well as other negative smoothness indexed spaces. Hence we are outside of the class of Calderón-Zygmund operators, but still obtain several boundedness results. The paraproducts we construct are of interest in their own right as well. In form and function, they resemble the Bony paraproduct, however we construct them for any class SIO$_\nu$ with $\nu \in \mathbb{R}$, and we construct them to reproduce higher order moments as opposed to just the typical condition $\Pi_b 1 = b$. See (7.1) for the definition of these paraproducts, as well as Corollary 7.4 and Lemma 7.7 for more information on the relevant properties they satisfy.

In Section 7.4, we provide a decomposition of operators in SIO$_\nu$ into two terms, an oscillation-preserving term and a regularity-preserving term. Roughly speaking, we show in Theorem 7.8 that under some mild moment conditions on $T \in$ SIO$_\nu$, we can write $T = S + O$ where $S, O \in$ SIO$_\nu$, $S$ is bounded on $\dot{W}^{t,p}$ for a range of $t > 0$ and $1 < p < \infty$, and $O$ is bounded on $\dot{W}^{-t,p}$ for all $t > 0$ and $1 < p < \infty$. We actually show that our decomposition satisfies the conditions $S(x^\alpha) = 0$ for several values of $\alpha \in \mathbb{N}_0^d$ and $O^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^d$. Then by our results in Section 5, we obtain boundedness results for both $S$ and $O$ on different classes of spaces. In some senses, Theorem 7.8 describes how any operator $T \in$ SIO$_\nu$ can be decomposed $T = S + O$, where $S$ behaves like a convolution operator with respect to smoothness properties and $O$ behaves like a convolution operator with respect to oscillatory properties. See Section 7.4 for more information on this. It should be noted that this decomposition is valid for operators $T \in$ SIO$_\nu$ that, once again, are not bounded from $\dot{W}^{\nu,2}$ into $L^2$.

In Section 7.5, we prove smooth and oscillatory sparse domination principles for operators in SIO$_\nu$. This application is included to demonstrate the following notion. We expend a lot of effort to provide conditions for an operator $T \in$ SIO$_\nu$ that imply $|\nabla|^{-(\nu+t)}T|\nabla|^t$ is a Calderón-Zygmund operator (modulo polynomials). Hence we can obtain new results for any such $T$ by applying existing results from Calderón-Zygmund theory to $|\nabla|^{-(\nu+t)}T|\nabla|^t$. In a way, our restricted operator calculus allows us to translate CZO$_0$ theory to SIO$_\nu$ theory for $\nu \neq 0$. We demonstrate this principle through the sparse domination CZO$_0$ to SIO$_\nu$ in Corollary 7.10.

In Section 7.6, we develop some new operator calculus results. It appears that this is the first operator algebra that includes operators of non-convolution type containing hypersingular and fractional integral operators. Furthermore, in Theorem 7.12 we describe several operator algebras that
include operators of different singularities. Some are made up of differential operators, fractional integral operators, and/or order-zero operators in various combinations.

This article is organized as follows. In Section 2, we provide several definitions, notation, and preliminary results. Section 3 contains the bulk of the work of truncating and approximating singular integral operators, and provides crucial support for the results in the sections that follow. In Section 4, we formulate our restricted operator calculus by verifying that singular integral operators, and provides crucial support for the results in the sections that follow. In Section 5, we are dedicated to proving several boundedness results for operators $T \in SIO_v$ that satisfy $T^*(x^\alpha) = 0$ vanishing moment conditions, and Section 6 provides sufficient conditions for $T^*(x^\alpha) = 0$. Finally, we present several applications in Section 7.

2. Preliminaries, Definitions, and Notation

Let $\mathcal{S}$ be the Schwartz class, $\mathcal{S}_p$ be the subspace of $\mathcal{S}$ made up of functions with vanishing moments of all order up to $P$, and $\mathcal{S}_\infty$ be the intersection of $\mathcal{S}_p$ for all $P \in \mathbb{N}$. We give $\mathcal{S}$ the standard Schwartz semi-norm topology defined via

$$\rho_{\alpha, \beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \cdot D^\beta f(x)|.$$ 

It is easy to verify that $\mathcal{S}_p$ for $P \in \mathbb{N}_0$ and $\mathcal{S}_\infty$ are closed subspaces of $\mathcal{S}$. Hence we can give $\mathcal{S}_p$ and $\mathcal{S}_\infty$ the Frechét topology endowed by the Schwartz semi-norms for $\mathcal{S}$. Let $\mathcal{S}'$, $\mathcal{S}'_p$, and $\mathcal{S}'_\infty$ be the dual spaces of $\mathcal{S}$, $\mathcal{S}_p$, and $\mathcal{S}_\infty$ respectively, which we refer to as tempered distributions, tempered distributions modulo polynomials of degree $P$, and tempered distributions modulo polynomials, respectively. Let $\mathcal{D} = C_0^\infty$ be the space of smooth compactly supported functions, and define $\mathcal{D}_p$ to be the subspace of $\mathcal{D}$ made up of all function with vanishing moments up to order $P$. We endow $\mathcal{D}$ with the topology characterized by the following sequential convergence: for $f_j, f \in \mathcal{D}$, we say $f_j \to f$ in $\mathcal{D}$ if there exists a compact set $K \subset \mathbb{R}^n$ such that $\text{supp}(f), \text{supp}(f_j) \subset K$ for all $j$ and $D^\alpha f_j \to D^\alpha f$ uniformly as $j \to \infty$ for all $\alpha \in \mathbb{N}_0^\infty$. It follows that $\mathcal{D}_p$ is a closed subspace of $\mathcal{D}$ for any $P \in \mathbb{N}_0$, and hence we endow $\mathcal{D}_p$ with the topology inherited from $\mathcal{D}$. Let $\mathcal{D}'$ and $\mathcal{D}'_p$ be the dual spaces of $\mathcal{D}$ and $\mathcal{D}_p$, respectively, which we refer to as distributions and distributions modulo polynomials of degree at most $P$.

For $1 < p < \infty$, a non-negative locally integrable function $w$ belongs to the Muckenhoupt weight class $A_p$ if

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right)^p \left( \frac{1}{|Q|} \int_Q w(x)^{-p'/p}dx \right)^{p'/p} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and $w$ belongs to $A_1$ if there exists a constant $C > 0$ such that $Mw(x) \leq Cw(x)$, where $M$ is the Hardy-Littlewood maximal operator. Also define $A_\infty$ to be the union of all $A_p$ for $1 \leq p < \infty$.

Let $\psi \in \mathcal{S}_\infty$ such that $\psi$ is supported in the annulus $1/2 < |\xi| < 2$ and $\widehat{\psi}(\xi) \geq c > 0$ for some $c > 0$ and all $3/5 < |\xi| < 5/3$. For $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, define $\psi_k(x) = 2^{kn}\psi(2^kx)$. For $w \in A_\infty$, define
$F^{s,q}_{p,w}$ to be the collection of $f \in \mathcal{S}_0$ such that
\[ ||f||_{F^{s,q}_{p,w}} = \left( \sum_{k \in \mathbb{Z}} (2^{sk} ||\psi_k \ast f||_{L^p})^q \right)^{1/q} < \infty \]
for $0 < p, q < \infty$ and $s \in \mathbb{R}$, and define $B^{s,q}_{p,w}$ to be the collection of all $f \in \mathcal{S}_0$ such that
\[ ||f||_{B^{s,q}_{p,w}} = \left( \sum_{k \in \mathbb{Z}} (2^{sk} ||\psi_k \ast f||_{L^p})^q \right)^{1/q} < \infty \]
for $0 < p \leq \infty, 0 < q < \infty$, and $s \in \mathbb{R}$. Also define $\hat{F}^{s,q}_{p,w}$ to be the collection of all $f \in \mathcal{S}_0$ such that
\[ ||f||_{\hat{F}^{s,q}_{p,w}} = \sup_Q \left( \frac{1}{w(Q)} \int_Q \sum_{k \in \mathbb{Z}} (2^{sk} ||\psi_k \ast f(x)||_q)^q \right)^{1/q} < \infty, \]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes and $\ell(Q)$ denotes the side length of $Q$. Finally define $\hat{B}^{s,\infty}_{w} = \hat{B}^{s,\infty}_w$ for $s \in \mathbb{R}$ to be the collection of $f \in \mathcal{S}_0$ such that
\[ ||f||_{\hat{B}^{s,\infty}_w} = \sup_{k \in \mathbb{Z}} 2^{sk} ||\psi_k \ast f||_{L^\infty} < \infty. \]

Taking these spaces modulo polynomials makes them Banach spaces for $1 \leq p, q < \infty$ and $s \in \mathbb{R}$ and quasi-Banach spaces when $0 < p, q < \infty$ and $s \in \mathbb{R}$. Note that by the work in [17] for the unweighted setting and [8] for the weighted setting, it follows that $\mathcal{S}_0$ is dense in $\hat{F}^{s,q}_{p,w}$ for all $0 < p, q < \infty, s \in \mathbb{R}$, and $w \in A_{\infty}$. It also follows that $H^p_w = L^p_w = F^{0,2}_{p,w}$ for all $1 < p < \infty$ and $w \in A_p$, $H^p_w = F^{0,2}_{p,w}$ for all $0 < p < \infty$ and $w \in A_{\infty}$, and $W^p_w = F^{2,2}_{p,w}$ for all $1 < p < \infty$ and $w \in A_p$. Here $L^p_w$ denote weighted Lebesgue spaces, $H^p_w$ denote weighted Hardy spaces, and $W^p_w$ denote weighted homogeneous Sobolev spaces. Even more, $B^{s,\infty}_w = \mathcal{A}_s$ is the (homogeneous) space of Lipschitz functions when $s > 0$ is not an integer. When $s \in \mathbb{N}$, $B^{s,\infty}_w$ is the Zygmund class of smooth functions, which strictly contain $\mathcal{A}_s$; see [46] for more on Zygmund’s smooth functions. For $s = 0$, the space $\hat{B}^{0,\infty}_w$ is sometimes referred to as the Bloch space, and it is closely related to certain Bergman spaces; see for example [12] for more on this.

Finally, we note also that $\hat{F}^{s,2}_{\infty,w} = \hat{F}^{s,2}_{\infty,w} = I_s(BMO)$ for all $s > 0$ and $w \in A_{\infty}$, which was proved in [23]. Here $I_s(BMO)$ are Sobolev-BMO spaces for $s > 0$, and we take the convention $I_0(BMO) = BMO$. See [37, 41, 42, 23] for more information on these spaces.

Let $X$ be a closed subspace of $\mathcal{S}(\mathbb{R}^n)$. We say that a linear operator $T$ mapping $X$ into $\mathcal{S}'(\mathbb{R}^n)$ is continuous if there exists a distribution kernel $\mathcal{K} \in \mathcal{S}'(\mathbb{R}^{2n})$ such that
\[ \langle Tf, g \rangle = \langle \mathcal{K}, g \otimes f \rangle = \int_{\mathbb{R}^{2n}} \mathcal{K}(x,y)g(x)f(y)dydx \]
for all $f \in X$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Here and throughout this article, any integral that has $\mathcal{K}(x,y)$ in the integrand should be interpreted as a dual pairing between $\mathcal{S}'(\mathbb{R}^{2n})$ and $\mathcal{S}(\mathbb{R}^{2n})$. We will use this notion of continuity when $X$ is $\mathcal{S}$ and $\mathcal{S}_p$ for $P \in \mathbb{N}_0$ at various points throughout the article. It is obvious that continuity from $\mathcal{S}$ into $\mathcal{S}'$ implies continuity from $\mathcal{S}_p$ into $\mathcal{S}'$ for $P \in \mathbb{N}_0$, which implies continuity from $\mathcal{S}_{P+1}$ into $\mathcal{S}'$ and from $\mathcal{S}_\infty$ into $\mathcal{S}'$. 
We consider operators that are continuous from $\mathcal{S}_p$ into $\mathcal{S}'$ since it makes it easier in some situations to initially define and work with operators. For example, consider the negative index derivative operator $\nabla^{-\nu}f$ defined via the Fourier multiplier $|\xi|^{-\nu}$ for $\nu > 0$. For $0 < \nu < n$, it is easy to define $|\nabla|^{-\nu}f$ for $f \in \mathcal{S}$ since $|\xi|^{-\nu}$ is locally integrable for such $\nu$, but it is a little more tedious to define $|\nabla|^{-\nu}f$ when $\nu \geq n$. Since we allow for our operators to be defined a priori only on $\mathcal{S}_p$ for some $P \in \mathbb{N}$, it is much easier to work with such operators. For any $\nu > 0$, we choose $P \geq \nu$, and it follows that $|\nabla|^{-\nu}f(\xi)$ is uniformly bounded for $f \in \mathcal{S}_p$. So $|\nabla|^{-\nu}$ trivially defines a continuous operator from $\mathcal{S}_p$ into $\mathcal{S}'$ as long as $P \geq \nu$. This type of issue is less severe for the fractional derivative operator $|\nabla|^\nu$, defined in the same way via the Fourier multiplier $|\xi|^\nu$, but using $\mathcal{S}_p$ rather than $\mathcal{S}$ may still be of value. This is because $|\xi|^\nu$ is not smooth at the origin (for certain $\nu > 0$), and requiring $f \in \mathcal{S}_p$ for some $P \in \mathbb{N}$ smooths this non-regularity, at least to some degree.

Assuming that $T$ is continuous from $\mathcal{S}_p$ into $\mathcal{S}'$ makes defining the transpose of $T$ a little tricky. In this paper we will impose on a given operator $T$ that both $T$ and $T^*$ are continuous from $\mathcal{S}_p$ into $\mathcal{S}'$ for some $P \in \mathbb{N}$. By this we mean that $T$ is continuous from $\mathcal{S}_p$ into $\mathcal{S}'$, and there exists another operator $S$ that is also continuous from $\mathcal{S}_p$ into $\mathcal{S}'$ such that $\langle Tf, g \rangle = \langle Sg, f \rangle$ for all $f, g \in \mathcal{S}_p$. We call this operator $S = T^*$ the transpose of $T$. It should be noted that we assume that $T^*$ is continuous from $\mathcal{S}_p$ into $\mathcal{S}'$. This does not necessarily follow from the continuity of $T$ from $\mathcal{S}_p$ into $\mathcal{S}'$.

**Definition 2.1.** Let $\nu \in \mathbb{R}$, $M \geq 0$ be an integer, and $0 < \gamma \leq 1$. A linear operator $T$ is in the class of $\nu$-order Singular Integral Operators, denoted $T \in SIO_\nu(M + \gamma)$, if $T$ and $T^*$ are continuous from $\mathcal{S}_p$ into $\mathcal{S}'$ from some $P \in \mathbb{N}$, there is a kernel function $K(x, y)$ such that

$$\langle Tf, g \rangle = \int_{\mathbb{R}^{2n}} K(x, y)f(y)g(x)dydx$$

for any pair $(f, g) \in D_p \times C^\infty_0$ or $(f, g) \in C^\infty \times D_p$ with disjoint support,

$$|D_\alpha D_\beta K(x, y)| \lesssim \frac{1}{|x-y|^{n+\nu+|\alpha|+|\beta|}}$$

for all $x \neq y$ and $\alpha, \beta \in \mathbb{N}^n_0$ with $|\alpha| + |\beta| \leq M$, and

$$|D_\alpha D_\beta K(x+h, y) - D_\alpha D_\beta K(x, y)| + |D_\alpha D_\beta K(x, y) - D_\alpha D_\beta K(x, y+h)| \lesssim \frac{|h|^\gamma}{|x-y|^{n+\nu+M+\gamma}}$$

for all $x, y, h \in \mathbb{R}^n$ satisfying $|h| < |x-y|/2$ and $\alpha, \beta \in \mathbb{N}^n_0$ satisfying $|\alpha| + |\beta| = M$. We will refer to $SIO_\nu$ as the union of all $SIO_\nu(M + \gamma)$ for $M \in \mathbb{N}^n_0$ and $0 < \gamma \leq 1$ and $SIO_\nu(\infty)$ as the intersection of $SIO_\nu(M + \gamma)$ for $M \in \mathbb{N}^n_0$ and $0 < \gamma \leq 1$.

Note that the kernel representation imposed on $T \in SIO_\nu(M + \gamma)$ implies that $Tf$ can be realized as a function for $f \in D_p$ and $x \notin \text{supp}(f)$. Indeed, for such $f$, $Tf \in \mathcal{S}'$ by assumption. Then by taking appropriate kernel functions $\varphi_k \in C^\infty_0$ that generate an approximation to the identity, the kernel representation of $T$ and dominated convergence imply that $\varphi_k \ast Tf(x) = \langle Tf, \varphi_k^\ast \rangle$ converges as $k \to \infty$ as long as $x \notin \text{supp}(f)$. Here and throughout this article, we use the notation $\varphi_k^\ast(y) = \varphi_k(x - y)$. Hence we can realize the distribution $Tf \in \mathcal{S}'$ pointwise as

$$Tf(x) = \lim_{k \to \infty} \varphi_k \ast Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy$$
when \( f \in \mathcal{D}_p \) and \( x \notin \text{supp}(f) \).

**Definition 2.2.** For \( \nu \in \mathbb{R} \), \( M \in \mathbb{N}_0 \), and \( 0 < \gamma \leq 1 \), an operator \( T \in \text{SIO}_\nu(M+\gamma) \) is a \( \nu \)-order Calderón-Zygmund Operator, denoted \( T \in \text{CZO}_\nu(M+\gamma) \), if \( T \) can be continuously extended to an operator from \( W^{\nu,p} \) into \( L^p \) for all \( 1 < p < \infty \). We will also refer to \( \text{CZO}_\nu \) as the union of all \( \text{CZO}_\nu(M+\gamma) \) for \( M \in \mathbb{N}_0 \) and \( 0 < \gamma \leq 1 \) and \( \text{CZO}_\nu(\infty) \) as the intersection of all \( \text{CZO}_\nu(M+\gamma) \) for \( M \in \mathbb{N}_0 \) and \( 0 < \gamma \leq 1 \).

**Definition 2.3.** An operator \( T \in \text{SIO}_\nu \) satisfies the \( \nu \)-order Weak Boundedness Property (WBP\( _\nu \)) if there are integers \( M,N \geq 0 \) and a constant \( C > 0 \) such that

\[
|\langle T\psi, \varphi \rangle| + |\langle T^*\psi, \varphi \rangle| \leq C|B|^{1-\nu/n} \tag{2.1}
\]

for any ball \( B \subset \mathbb{R}^n \), \( \psi \in \mathcal{D}_M \) and \( \varphi \in C^0_0 \) with \( \text{supp}(\psi) \cup \text{supp}(\varphi) \subset B \) and \( \|D^\alpha\psi\|_{L^\infty}, \|D^\alpha\varphi\|_{L^\infty} \leq |B|^{-|\alpha|/n} \) for \( |\alpha| \leq N \).

**Remark 2.4.** Note that the definition of \( \text{SIO}_\nu \) and \( \text{WBP}_\nu \) are both symmetric under \( T \) and \( T^* \), but this is not always the case for \( \text{CZO}_\nu \). When \( \nu \neq 0 \), \( T \in \text{CZO}_\nu \) and \( T^* \in \text{CZO}_\nu \) are not equivalent conditions. On the other hand when \( \nu = 0 \), \( \text{CZO}_0 \) is closed under transposes and actually collapses to the traditional definition of a Calderón-Zygmund operator.

For a function \( F \) on \( \mathbb{R}^n \), \( x_0 \in \mathbb{R}^n \), and an integer \( L \geq 0 \), we define the Taylor polynomial (sometimes also called the jet) centered at \( x_0 \) by

\[
J_{x_0}^L[F](x) = \sum_{|\alpha| \leq L} \frac{D^\alpha F(x_0)}{\alpha!}(x-x_0)^\alpha.
\]

**Definition 2.5.** Let \( \nu \in \mathbb{R} \), \( M \geq 0 \) be an integer, \( 0 < \gamma \leq 1 \) and \( T \in \text{SIO}_\nu(M+\gamma) \). Let \( P \in \mathbb{N}_0 \) be the integer specified for the kernel representation in the definition of \( T \in \text{SIO}_\nu(M+\gamma) \), and without loss of generality assume \( P \geq M + |\nu| \). Let \( \eta \in \mathcal{D}_{2P} \) such that \( \eta = 1 \) on \( B(0,1) \), and define \( \eta_R(x) = \eta(x/R) \) for \( R > 0 \) and \( x \in \mathbb{R}^n \). For \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| < M + \nu + \gamma \), define \( T^*(x^\alpha) \in \mathcal{D}_{2P}^* \) by

\[
\langle T(x^\alpha), \psi \rangle = \lim_{R \to \infty} \langle T(x^\alpha \cdot \eta_R), \psi \rangle = \lim_{R \to \infty} \int_{\mathbb{R}^{2n}} \mathcal{K}(x,y) y^\alpha \eta_R(y) \psi(x) dy \, dx
\]

for \( \psi \in \mathcal{D}_P \). Here \( \mathcal{K} \in \mathcal{S}'(\mathbb{R}^{2n}) \) denotes the distributional kernel of \( T \), and the integrals above should be interpreted as the dual pairing between \( \mathcal{S}'(\mathbb{R}^{2n}) \) and \( \mathcal{S}(\mathbb{R}^{2n}) \). Note that if \( M + \nu + \gamma < 0 \), then we do not define \( T(x^\alpha) \) for any \( \alpha \in \mathbb{N}_0 \). Furthermore, we will simply state that \( T(x^\alpha) = 0 \) to mean that there exists an integer \( P \geq 0 \) so that \( T(x^\alpha) = 0 \) in \( \mathcal{D}_P^* \).

It is a somewhat standard argument by now to show that \( T(x^\alpha) \) is well-defined. We provide a brief sketch of this since our definition is slightly different than others that have appeared; compare, for example, with the corresponding definitions in [18, 16, 44, 21, 22, 23].

**Proof.** Let \( \eta_R \in \mathcal{D}_{2P} \) be as in Definition 2.5. Let \( \psi \in \mathcal{D}_P \) with \( \text{supp}(\psi) \subset B(0,R_0/8) \). For \( |\alpha| < M + \nu + \gamma \) and \( R > 0 \),

\[
\langle T(x^\alpha \cdot \eta_R), \psi \rangle = \langle T(x^\alpha \cdot \eta_{R_0}), \psi \rangle + \lim_{R \to \infty} \int_{\mathbb{R}^{2n}} (K(x,y) - J^M_0[K(\cdot,y)])(x) y^\alpha (\eta_R(y) - \eta_{R_0}(y)) \psi(x) dy \, dx
\]
\[ = \langle T(x^\alpha \cdot \eta_{R_0}), \psi \rangle + \lim_{R \to \infty} \int_{\mathbb{R}^{2n}} (K(x,y) - J_0^M[K(\cdot,y)](x)) y^\alpha (1 - \eta_{R_0}(y)) \psi(x) dy dx. \]

The first term is well-defined since \( T \) is maps \( \mathcal{S} \) into \( \mathcal{S}' \) and \( \eta \in \mathcal{D}_{2D} \) implies \( x^\alpha \cdot \eta \in \mathcal{D}_D \) for \( |\alpha| < M + v + \gamma \). The second term is also well-defined by the support properties of \( \eta_R(x) - \eta_{R_0}(x) \), the kernel representation of \( T \), and the vanishing moment properties of \( \psi \). In fact, dominated convergence can be applied to the second term since

\[
\left| (K(x,y) - J_0^M[K(\cdot,y)](x)) y^\alpha (\eta_R(y) - \eta_{R_0}(y)) \right| \lesssim \frac{R_0^{M+\gamma}}{(R_0 + |y|)^{n+v+M+\gamma-|\alpha|}}
\]

for \( x \in \text{supp}(\psi) \). Therefore \( T(x^\alpha) \) is also well defined. Furthermore, it is not hard to see that the definition of \( T(x^\alpha) \) does not depend on the particular function \( \eta_R \in \mathcal{D}_{2D} \) chosen in Definition 2.5.

Throughout, we will use the notation \( \Phi^N_k(x) = 2^{kn}(1 + 2^k|x|)^{-N} \) for \( N > 0, k \in \mathbb{Z}, \) and \( x \in \mathbb{R}^n \). It is well known that \( \Phi^N_k * |f|(x) \lesssim Mf(x) \) for any locally integrable function \( f \) and \( N > n \), where the constant may depend on \( N \) and \( M \) is the Hardy-Littlewood maximal operator. It is also well known that \( \Phi^N_k * \Phi^M_j(x) \lesssim \Phi^{\min(M,N)}_{\min(j,k)}(x) \) for any \( j,k \in \mathbb{Z}, M,N > 0 \) such that \( M,N > n \), and the constant depends on \( M,N \), but not on \( j,k,x \).

The following lemmas are also well-known. More information can be found for example in [17].

\textbf{Lemma 2.6.} Let \( P \geq 0 \) be an integer. There exist functions \( \psi \in \mathcal{D}_0 \) and \( \tilde{\psi} \in \mathcal{S}_\infty \) such that

\[
f(x) = \sum_{k \in \mathbb{Z}} \psi_k * \tilde{\psi} \ast f(x)
\]

in \( \mathcal{S}_\infty \) for any \( f \in \mathcal{S}_\infty \). Furthermore, \( \psi \) and \( \tilde{\psi} \) can be chosen to be radial.

\textbf{Lemma 2.7.} Let \( P \geq 0 \) be an integer. There exist functions \( \phi \in \mathcal{D}_0 \) and \( \tilde{\phi} \in \mathcal{S}_\infty \) and an integer \( N_0 \in \mathbb{Z} \) such that

\[
f(x) = \sum_{k \in \mathbb{Z} \cap Q: \ell(Q) = 2^{-(k+N_0)}} \tilde{\phi}_k * f(c_Q) \phi_k(x - c_Q)
\]

in \( \mathcal{S}_\infty \) for any \( f \in \mathcal{S}_\infty \). The sum in \( Q \) here is over all dyadic cubes of side length \( 2^{-(k+N_0)} \) and \( c_Q \) denotes the lower-left hand corner of \( Q \). Furthermore, \( \tilde{\phi} \) and \( \phi \) can be chosen to be radial.

Throughout this article, we will choose convolution kernel functions \( \psi, \tilde{\psi}, \phi, \) and \( \tilde{\phi} \) to be radial so that they are self-transpose; that is, so that \( \langle \psi_j * f, g \rangle = \langle \psi_j * g, f \rangle \). We will make the same convention when working with approximation to the identity operators and functions \( \psi \) used to define Besov and Triebel-Lizorkin spaces. Of course, this simplification is not necessary, but it eases the need for complicated notation.

Define

\[
M^r_j f(x) = \left\{ M \left[ \left( \sum_{Q: \ell(Q) = 2^{-(j+N_0)}} f(c_Q) \chi_Q \right)^r \right] (x) \right\}^{1/r}.
\]
Lemma 2.8. Let \( f : \mathbb{R}^n \rightarrow \mathbb{C} \) be a non-negative continuous function, \( \mu > 0 \), and \( \frac{n}{n+\mu} < r \leq 1 \). Then
\[
\sum_{Q: \ell(Q)=2^{-(j+N_0)}} |Q|\Phi^{n+\mu}_{\min(j,k)}(x-c_Q)f(c_Q) \lesssim 2^{r\max(0,j-k)\mu}M^*_jf(x)
\]
for all \( x \in \mathbb{R}^n \).

The properties of \( M^*_j \) in Lemma 2.8 can, at least in part, be attributed to Frazier and Jawerth, but can be found in several other places as well. A proof of it, as stated here, can also be found in [22, Proposition 2.2].

Lemma 2.9. For any \( 0 < p, q < \infty \), \( 0 < r < \min(p,q) \), \( t \in \mathbb{R} \), \( w \in A_{p/r} \) and \( f \in F^t,q_{p,w} \), we have
\[
\left\| \sum_{j \in \mathbb{Z}} 2^j M^*_j(\tilde{\phi}_j * f) \right\|_{L^p_w} \lesssim ||f||_{F^t,q_{p,w}},
\]
where \( \tilde{\phi} \) is chosen as in Lemma 2.7.

Lemma 2.9 is implicit in the work of Bui in [8]. Indeed Lemma 2.9 can be proved with a standard argument using the weighted version of the Fefferman-Stein vector-valued maximal inequality proved in [2] and the discrete Littlewood-Paley characterization of \( \dot{B}^0 \) proved in [8]. See also [32], where they prove this result in the setting of weighted Hardy spaces (i.e. for \( t = 0 \) and \( q = 2 \)).

3. Smoothly Truncated Kernel Estimates

In this section, we start to work with operators \( T \in SIO_{\nu}(M+\gamma) \). A priori, we are very limited in what we can do with such operators. We will impose a little more structure on \( T \) through the \( WBP_{\nu} \) and \( T^*(x^\alpha) = 0 \) conditions to help gain some traction, which we will see later are necessary conditions in many situations. The main purpose of these results is to find representation formulas for \( T \) that make it easier to work with. Our first result provides an integral representation and estimates for \( \psi_k \ast Tf \), which will be useful for working with boundedness properties of \( T \) on Besov and Triebel-Lizorkin spaces. These estimates are similar to ones proved in [22]. Though it should be noted that the corresponding result in [22] is only for order-zero operators and it assumes that the operator is \( L^2 \)-bounded, whereas we address \( \nu \) order operators and only assume Weak Boundedness Properties here.

Theorem 3.1. Let \( \nu \in \mathbb{R} \), \( M, L \geq 0 \) be integers satisfying \( \nu \leq L \leq M+\nu \), \( 0 < \gamma \leq 1 \), and \( T \in SIO_{\nu}(M+\gamma) \). Assume that \( T \) satisfies the \( WBP_{\nu} \) and that \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \). Then for any \( \psi \in D_p \), with \( p \) sufficiently large, there is a kernel function \( \theta_k(x,y) \) such that
\[
\psi_k \ast Tf(x) = \int_{\mathbb{R}^n} \theta_k(x,y)f(y)dy
\]
for \( f \in S_p \), where \( \theta_k(x,y) \) satisfies
\[
\begin{align*}
(3.1) \quad |D^\alpha_y \theta_k(x,y)| & \lesssim 2^{(\nu + |\alpha|)k} \Phi^{n+\nu+M+\gamma}(x-y) & \text{for } |\alpha| \leq |L - \nu| \\
(3.2) \quad |D^\alpha_y \theta_k(x,y+h) - D^\alpha_y \theta_k(x,y)| & \lesssim 2^{(\nu + |\alpha|)k} (2^k |h|)^{\gamma'} \Phi^{n+\nu+M+\gamma}(x-y) & \text{for } |\alpha| = |L - \nu| 
\end{align*}
\]
for all \( x, y, h \in \mathbb{R}^n \) with \( |h| < (2^{-k} + |x-y|)/2 \) and any \( 0 < \gamma' < \gamma \).
Proof. Fix $P \in \mathbb{N}_0$ large enough so that $P \geq M$. $T$ satisfies $\mathcal{WB}_P$ with parameters $M = N = P$, the kernel representation for $T$ is valid for $f, g \in \mathcal{D}_P$ with disjoint support, and $T^*(x^\alpha) = 0$ in $\mathcal{D}_P$ for all $|\alpha| \leq L$. Let $\psi \in \mathcal{D}_P$, and assume without loss of generality that $\text{supp}(\psi) \subset B(0, 1)$.

Since $T$ is continuous from $\mathcal{S}_P$ into $\mathcal{S}'$, we have $T^*\psi^k \in \mathcal{S}'$ for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, where $\psi^k(x) = \psi_k(x - y)$. This is the distribution kernel of $\psi^k \ast T f$ in the sense that

$$
\psi_k \ast T f(x) = \langle T f, \psi_k^x \rangle = \langle T^* \psi_k^x, f \rangle
$$

for $f \in \mathcal{S}_P$. So we would like to define $\theta_k(x, y) = T^* \psi_k^x(y)$, but this expression may not be well-defined pointwise since $T^* \psi_k^x$ is a priori only a distribution (not necessarily a function in $y$). However, we will show that this is merely a technicality, and that $T^* \psi_k^x$, as a distribution, agrees with integration against a function $T^* \psi_k^x(y)$.

Let $\varphi \in C^\infty_0$ with integral 1 such that $\overline{\varphi}(x) = 2^n \varphi(2x) - \varphi(x)$ and $\overline{\psi} \in \mathcal{D}_P$. Define $P_k f = \varphi_k \ast f$ and $Q_k f = \psi_k \ast f$. Note that $P_N f \to f$ as $N \to \infty$ in $\mathcal{S}$ for $f \in \mathcal{S}$, and so $P_N T^* \psi_k^x \to T^* \psi_k^x$ as $N \to \infty$ in $\mathcal{S}'$. Furthermore, $P_N T^* \psi_k^x(y) = \langle T^* \psi_k^x, \varphi_N \rangle$ is a function in $y$ for all $x \in \mathbb{R}^n$ and $k, N \in \mathbb{Z}$ by the definition of distributional convolution (in fact, it is a $C^\infty$ function of tempered growth since $T^* \psi_k^x \in \mathcal{S}'$). Define

$$
\theta_k(x, y) = \lim_{N \to \infty} P_N T^* \psi_k^x(y).
$$

It should be noted that from what we have shown so far, it is not clear yet that the limit in (3.3) exists for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We will show that this is indeed the case and that the kernel function defined in (3.3) satisfies estimates (3.1) and (3.2). Assuming for the moment that (3.3) holds pointwise, that $|P_N T^* \psi_k^x(y) f(y)|$ is bounded uniformly in $N$ by an integrable function of $y$, and that $\theta_k$ satisfies estimates (3.1) and (3.2) (all of which we will prove), by the $\mathcal{S}'$ convergence $P_N T^* \psi_k^x \to T^* \psi_k^x$ and by dominated convergence, we have

$$
\psi_k \ast T f(x) = \langle T^* \psi_k^x, f \rangle = \lim_{N \to \infty} \int_{\mathbb{R}^n} P_N T^* \psi_k^x(y) f(y) dy = \int_{\mathbb{R}^n} \theta_k(x, y) f(y) dy
$$

for all $f \in \mathcal{S}_P$. So to complete the proof, we must show that the limit in (3.3) exists for each $x, y, k$ and that $\theta_k$ satisfies (3.1) and (3.2).

For $|x - y| > 2^{3-k}$, it follows from the kernel representation of $T$ that $T^* \psi_k^x$ is a continuous function on a sufficiently small neighborhood of $y$ for any fixed $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, and that $P_N T^* \psi_k^x(y) \to T^* \psi_k^x(y)$ pointwise as $N \to \infty$. In particular, the limit in (3.3) exists when $|x - y| > 2^{3-k}$. Furthermore, still assuming that $|x - y| > 2^{3-k}$, by the kernel representation for $T$ and the support properties of $\psi_k$, it follows that $T^* \psi_k^x$ is $M$-times differentiable on a neighborhood of $y$. For $|\alpha| \leq [L - v]$, we have

$$
|D_y^\alpha \theta_k(x, y)| = |D_y^\alpha T^* \psi_k^x(y)| = \left| \int_{\mathbb{R}^n} \left( D_y^\alpha K(u, y) - D_y^{|\alpha|} M^{-|\alpha|} D_y^\alpha K(\cdot, y)(u) \right) \psi_k(u - x) du \right|
$$

$$
\lesssim \int_{\mathbb{R}^n} \frac{|x - u|^{-|\alpha| + v + M + \gamma}}{|x - y|^{v + M + \gamma}} |\psi_k(u - x)| du \lesssim 2^{(v + |\alpha|)k} \Phi_{k}^{n+v+M+\gamma}(x - y).
$$

If $|h| \geq 2^{-k}$, then it trivially follows that

$$
|D_y^\alpha \theta_k(x, y + h) - D_y^\alpha \theta_k(x, y)| \lesssim 2^{(v + |\alpha|)k} \left[ \Phi_{k}^{n+v+M+\gamma}(x - y) + \Phi_{k}^{n+v+M+\gamma}(x - y - h) \right]
$$
\[
\lesssim 2^{(\nu + |\alpha|)k} (2^k |h|)^\gamma \left[ \Phi_k^{n+V+M+\gamma}(x-y) + \Phi_k^{n+V+M+\gamma}(x-y-h) \right].
\]

Otherwise we assume that \(|h| < 2^{-k}\), and it follows that \(|x-y-h| \geq |x-y| - |h| > |x-y|/2 \geq 2^{1-k}\).

In this situation, we consider two cases: if \(|\alpha| = M\) and if \(|\alpha| < M\). When \(|\alpha| = M\), we have

\[
|D_y^\alpha \theta_k(x,y+h) - D_y^\alpha \theta_k(x,y)| = \left| \int_{\mathbb{R}^n} \left( D_y^\alpha K(u,y+h) - D_y^\alpha K(u,y) \right) \psi_k(u-x)du \right|
\]

\[
\lesssim \int_{\mathbb{R}^n} |h|^\gamma |u-y|^{n+V+|\alpha|+\gamma} |\psi_k(u-x)|du
\]

\[
\lesssim 2^{(\nu + |\alpha|)k} (2^k |h|)^\gamma \Phi_k^{n+V+M+\gamma}(x-y).
\]

Now assume that \(|\alpha| < M\). Then

\[
|D_y^\alpha \theta_k(x,y+h) - D_y^\alpha \theta_k(x,y)| = \left| \int_{\mathbb{R}^n} \left( D_y^\alpha K(u,y+h) - D_y^\alpha K(u,y) \right) \psi_k(u-x)du \right|
\]

\[
= \left| \int_{\mathbb{R}^n} \left( (D_y^\alpha K(u,y+h) - D_y^\alpha K(u,y)) - 2^{M-1-|\alpha|} [D_y^\alpha K(\cdot, y+h) - D_y^\alpha K(\cdot, y)](u) \right) \psi_k(u-x)du \right|
\]

\[
\lesssim \sum_{|\beta| = M-|\alpha|} |D_0^{\beta} D_y^{\alpha} K(\xi, y+h) - D_0^{\beta} D_y^{\alpha} K(\xi, y)| |u-x|^{M-|\alpha|} |\psi_k(u-x)|du
\]

for some \(\xi = cx + (1-c)u\) with \(0 < c < 1\)

\[
\lesssim \int_{\mathbb{R}^n} |h|^\gamma |x-u|^{M-|\alpha|} |\psi_k(u-x)|du \lesssim 2^{(\nu + |\alpha|)k} (2^k |h|)^\gamma \Phi_k^{n+V+M+\gamma}(x-y).
\]

So \(\theta_k(x,y)\) is well-defined and satisfies (3.1) and (3.2) when \(x\) and \(y\) are far apart.

Note that for \(N \in \mathbb{N}\) and when \(|x-y| > 2^{3-k}\), using the estimates just proved and that \(T^* \psi_k^\ell(y) = \theta_k(x,y)\) for such \(x\) and \(y\), it follows that

\[
|P_N T^* \psi_k^\ell(y)f(y)| \lesssim |f(y)| \int_{\mathbb{R}^n} |\psi_N(u-y)\theta_k(x,u)|du \lesssim 2^{vk} |f(y)| M \left( \Phi_k^{n+V+M+\gamma} \right)(x-y).
\]

Since \(\Phi_k^{n+V+M+\gamma} \in L^2\) we impose \(f \in \mathcal{S} \subset L^2\), it follows that \(f \cdot M \left( \Phi_k^{n+V+M+\gamma} \right)(x-\cdot) \in L^1(\mathbb{R}^n)\), which verifies that \(|P_N T^* \psi_k^\ell(y)f(y)|\) is bounded uniformly in \(N\) by an \(L^1(\mathbb{R}^n)\) function when \(|x-y| > 2^{3-k}\).

When \(|x-y| \leq 2^{3-k}\), we decompose \(P_N T^* \psi_k^\ell\) further,

\[
P_N T^* \psi_k^\ell(y) = \sum_{\ell=k}^{N-1} Q_{\ell} T^* \psi_k^\ell(y) + P_k T^* \psi_k^\ell(y).
\]

(3.4)

Recall that \(\Phi \in C^\infty_0\) was chosen so that \(\bar{\Phi}_{\ell} = \Phi_{\ell+1} - \Phi_{\ell} \in D_p\). For the remainder of the proof, we drop the tilde on top of \(\tilde{\Phi}\) for the sake of simplifying notation. This causes an overlap in notation between \(\tilde{\Psi}_{\ell}\) and \(\psi_k\), which is ultimately harmless, but the distinction can still be recovered at any point in the remainder of the proof by identifying the subscripts, \(\ell\) versus \(k\).

Let \(\alpha \in \mathbb{N}_0^n\) with \(|\alpha| \leq |L - \nu|\). Using the hypothesis \(T^*(x^\mu) = 0\) for \(|\mu| \leq L\) we write

\[
|D_y^\alpha (T^* \theta_k^\ell, \psi_k^\ell)| \leq |A_{\ell,k}(x,y)| + |B_{\ell,k}(x,y)|, \text{ where}
\]

\[
A_{\ell,k}(x,y) = |D_y^\alpha \theta_k(x,y)| + |D_y^\alpha \theta_k(x,y+h)|,
\]

\[
B_{\ell,k}(x,y) = \left| \int_{\mathbb{R}^n} \left( D_y^\alpha K(u,y+h) - D_y^\alpha K(u,y) \right) \psi_k(u-x)du \right|.
\]
\[
A_{\ell,k}(x,y) = 2^{\ell|\alpha|} \int_{\mathbb{R}^n} T\left((D^\alpha \psi)^{Y}_{\ell}\right)(u) \left(\psi_k^x(u) - J_y^z[\psi_k^x](u)\right) \eta_{2^\ell}(u-y) du,
\]
\[
B_{\ell,k}(x,y) = \lim_{R \to \infty} 2^{\ell|\alpha|} \int_{\mathbb{R}^n} T\left((D^\alpha \psi)^{Y}_{\ell}\right)(u) \left(\psi_k^x(u) - J_y^z[\psi_k^x](u)\right) \left(\eta_R(u) - \eta_{2^\ell}(u-y)\right) du,
\]
where \(\eta_R \in \mathcal{D}_P\) with \(\eta_R(x) = \eta(x/R)\), \(\text{supp}(\eta) \subset B(0,2)\), and \(\eta = 1\) on \(B(0,1)\). We apply the WBP\(v\) property to estimate \(A_{\ell,k}\) as follows,

\[
|A_{\ell,k}(x,y)| \leq 2^{\ell|\alpha|} 2^{(\ell+k)\nu_2(L+\nu)(k-\ell)} \left| T\left(\frac{(D^\alpha \psi)^{Y}_{\ell}}{2^n}, \frac{\psi_k^x - J_y^z[\psi_k^x]}{2^{kn}2(L+\nu)(k-\ell)} \eta_{2^\ell}(\cdot-y)\right)\right|
\]

\[
\lesssim 2^{\ell|\alpha|} 2^{(\ell+2\nu_2(L+\nu)(k-\ell))} 2^{kn} = 2^{k(\nu+|\alpha|)2(L+\nu-|\alpha|)(k-\ell)} 2^{kn}.
\]

Note that \(\text{supp}(2^{-\ell n}(D^\alpha \psi)^{Y}_{\ell}) \subset B(y,2^{3-\ell})\) and \(||D^\mu(2^{-\ell n}(D^\alpha \psi)^{Y}_{\ell})||_{L^n} \lesssim 2^{\mu|\ell|}\) for all \(\mu \in \mathbb{N}_0^n\), where the associated constants are independent of \(x,y,\ell,k\). Similarly, we have \(\text{supp}(\eta_{2^\ell}(\cdot-y)) \subset B(y,2^{3-\ell})\) and \(\left| D^\mu\left(\frac{\psi_k^x(u) - J_y^z[\psi_k^x](u)}{2^{kn}2(L+\nu)(k-\ell)} \eta_{2^\ell}(u-y)\right)\right| \lesssim 2^{\ell|\mu|}\)

for all \(\mu \in \mathbb{N}_0^n\) as long as \(k \leq \ell\), where the associated constant does not depend on \(u,x,y,k,\ell\). Hence it is an appropriate function to apply WBP\(v\) here.

Let \(\gamma', \gamma'' > 0\) such that \(\gamma' < \gamma'' < \gamma\). The \(B_{\ell,k}\) term is bounded using the kernel representation of \(T\) in the following way

\[
|B_{\ell,k}(x,y)| \leq 2^{\ell|\alpha|} \limsup_{R \to \infty} \int_{|u-y| > 2^{1-\ell}} \int_{\mathbb{R}^n} |K(u,v) - J_y^M[K(u,\cdot)](v)| \left| (D^\alpha \psi)^{Y}_{\ell}(v)\right| dv \times |\psi_k^x(u) - J_y^z[\psi_k^x](u)| \eta_R(u) du
\]

\[
\lesssim 2^{\ell|\alpha|} \sum_{m=1}^{\infty} \int_{2^{m-\ell} < |u-y| \leq 2^{m+1-\ell}} \left( \int_{\mathbb{R}^n} \frac{2^{-\ell}(M+\gamma)^\ell}{2^{(n+\nu+M+\gamma)(m-\ell)}} \left| (D^\alpha \psi)^{Y}_{\ell}(v)\right| dv \right) 2^{kn} 2^{k(2m-\ell)L+\gamma''} du
\]

\[
\lesssim 2^{k(n+\nu+|\alpha|)2(L+\gamma''-\nu-|\alpha|)(k-\ell)} \sum_{m=1}^{\infty} 2^{-(\nu+M+\gamma-L-\gamma''m)}
\]

\[
\lesssim 2^{k(n+\nu+|\alpha|)2(L+\gamma''-\nu-|\alpha|)(k-\ell)}.
\]

Here we used that \(\nu + M + \gamma > L + \gamma''\) since \(L \leq M + \nu\) and \(\gamma'' < \gamma\). It is not crucial here that we took \(\gamma' < \gamma'' < \gamma\), but this estimate will be used again later where our choice of \(\gamma' < \gamma''\) will be important. It follows that

\[
\sum_{\ell \geq k} 2^{\ell|\alpha|} \left| T\left((D^\alpha \psi)^{Y}_{\ell}\right), \psi_k^x\right| \lesssim 2^{k(n+\nu+|\alpha|)} \lesssim 2^{k(\nu+|\alpha|)} \Phi_k^{n+\nu+M+\gamma}(x-y).
\]

Here we use that \(\alpha \in \mathbb{N}_0^n\) must satisfy \(|\alpha| \leq L-\nu\), which implies that \(\nu + |\alpha| < L + \gamma''\). This verifies that the limit of (3.4) as \(N \to \infty\) exists for \(|x-y| \leq 2^{3-k}\) as well, and that the first term on the right hand side of (3.4) satisfies (3.1). Hence \(\theta_k\) is well-defined by (3.3). Since \(T\) satisfies the WBP\(v\), it
also follows that
\[ |D^\alpha y P_k T^* \psi^\alpha_k(y)| = 2^{\alpha |y|} 2^{2kn} \left| \left< T^* \left( \frac{\psi^\alpha_k(u)}{2kn} \right), \frac{(D^\alpha \Phi)_k(\cdot - y)}{2kn} \right> \right| \leq 2^{(n + v + |\alpha|)k}. \]

Then second term on the right hand side of (3.4) also satisfies (3.1). Therefore \( \theta_k \) satisfies (3.1).

We also verify the \( \gamma' \)-Hölder regularity estimate (3.2): let \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| = L \). It trivially follows from the estimates already proved that
\[
\sum_{\ell \geq k: 2^{-\ell} < |h|} \left| \left< D^\alpha_T (\psi^\alpha_{\ell} - \psi^\alpha_k), \psi^\alpha_k \right> \right| \leq 2^{k(v + |\alpha|)} (2^k |h|) \gamma' \left( \Phi^{n+v+M+\gamma}_{k}(x-y) + \Phi^{n+v+M+\gamma}_{k}(x-y-h) \right) \sum_{\ell \geq k: 2^{-\ell} < |h|} 2^{(L+\gamma''-\gamma'-v-|\alpha|)(k-\ell)} \left| \left< D^\alpha_T (\psi^\alpha_{\ell} - \psi^\alpha_k), \psi^\alpha_k \right> \right| \leq 2^{k(v + |\alpha|)} (2^k |h|) \gamma' \left( \Phi^{n+v+M+\gamma}_{k}(x-y) + \Phi^{n+v+M+\gamma}_{k}(x-y-h) \right).
\]

Note that \( v + |\alpha| < L + \gamma'' - \gamma' \) since we chose \( \gamma' < \gamma'' \). On the other hand, for the situation where \( |h| \leq 2^{-\ell} \), we consider
\[
\sum_{\ell \geq k: 2^{-\ell} \geq |h|} \left| \left< D^\alpha_T (\psi^\alpha_{\ell} - \psi^\alpha_k), \psi^\alpha_k \right> \right| \leq |A_{\ell,k}(x,y,h)| + |B_{\ell,k}(x,y,h)|,
\]
where
\[
A_{\ell,k}(x,y,h) = 2^{\ell |\alpha|} \int_{\mathbb{R}^n} T((D^\alpha \psi)_{\ell}^{y+h} - (D^\alpha \psi)_{\ell}^y)(u) (\psi^\alpha_k(u) - J^L_{\gamma}(\psi^\alpha_k)(u)) \eta_{2^{-\ell}}(u-y) du \quad \text{and}
\]
\[
B_{\ell,k}(x,y,h) = \lim_{R \to \infty} 2^{\ell |\alpha|} \int_{\mathbb{R}^n} T((D^\alpha \psi)_{\ell}^{y+h} - (D^\alpha \psi)_{\ell}^y)(u) (\psi^\alpha_k(u) - J^L_{\gamma}(\psi^\alpha_k)(u)) \times (\eta_R(u) - \eta_{2^{-\ell}}(u-y)) du.
\]
Note that when \( |h| \leq 2^{-\ell} \), the function \( 2^{-\ell n}(2^\ell |h|)^{-\gamma'} \left[ (D^\alpha \psi)_{\ell}^{y+h} - (D^\alpha \psi)_{\ell}^y \right] \in \mathcal{D}_p \) is supported in
\[
B(y,2^{-\ell}) \cup B(y+h,2^{-\ell}) \subset B(y,|h|+2^{-\ell}) \subset B(y,2^{1-\ell})
\]
with
\[
\left| 2^{-\ell n}(2^\ell |h|)^{-\gamma'} D^\mu \left[ (D^\alpha \psi)_{\ell}^{y+h} - (D^\alpha \psi)_{\ell}^y \right] \right|_{L^\infty} \leq 2^{\ell |\mu|}
\]
for \( |\mu| \leq N \), where the constant does not depend on \( y, h, \) or \( \ell \); here \( N \) is the integer specified in the \( WBP_v \) condition for \( T \). Then using the \( WBP_v \) for \( T \), we have have
\[
|A_{\ell,k}(x,y,h)| \leq 2^{\ell |\alpha|}(2^\ell |h|)^{\gamma'} 2^{kn} 2^{(k-\ell)(L+\gamma)} 2^{n}
\]
\[
\times \left| \left< T \left( \frac{(D^\alpha \psi)_{\ell}^{y+h} - (D^\alpha \psi)_{\ell}^y}{2^{\ell n}(2^\ell |h|)^{\gamma'}} \right), \psi^\alpha_k - J^L_{\gamma}(\psi^\alpha_k) \right> \right| 2^{(k-\ell)(L+\gamma)} \eta_{2^{-\ell}}(\cdot - y) \right| \leq 2^{(v + |\alpha|)}(2^\ell |h|)^{\gamma'} 2^{kn} 2^{(L+\gamma)(k-\ell)} = 2^{k(n+v+|\alpha|)2(\gamma'-\gamma-v)(k-\ell)}(2^k |h|)^{\gamma'}.
\]
Recall the selection of $\gamma''$ such that $0 < \gamma' < \gamma'' < \gamma$. The $B_{\ell,k}$ term is bounded using the kernel representation of $T$

$$|B_{\ell,k}(x, y, h)| \leq 2^{\ell|\alpha|} \int_{|u-y|>2^{1-\ell}} \int_{\mathbb{R}^n} |K(u, v) - J^M_y[K(u, \cdot)](v)|$$

$$\times |(D^\alpha \psi)^{\gamma+h}_\ell(v) - (D^\alpha \psi)^{\gamma}_\ell(v)| |(\psi^\ell_y(u) - J^M_y[\psi^\ell_y])(u)| du dv$$

$$\lesssim 2^{\ell|\alpha|} \sum_{m=1}^\infty \int_{2^{m-\ell}<|u-y|\leq 2^{m+1-\ell}} \int_{\mathbb{R}^n} \frac{2^{-(M+\gamma)\ell}}{2^{(n+\nu+M+\gamma)(m-\ell)}} (2^\ell |h|)\gamma'$$

$$\times (\Phi^{\gamma+1}_\ell(y-v) + \Phi^{\gamma+1}_\ell(y+h-v)) dv 2^{kn}(2^k |u-y|)^{L+\gamma''} du$$

$$\lesssim 2^{k(n+\nu+|\alpha|)}(2^k |h|)^{\gamma'} 2^{(L-\nu-|\alpha|+\gamma''-\gamma')(k-\ell)} \sum_{m=1}^\infty 2^{-(\nu+M+\gamma-L-\gamma'')m}$$

Again we use that $\nu + M + \gamma > L + \gamma'$. Then it follows that

$$\sum_{\ell=1}^\infty |A_{\ell,k}(x, y, y')| + |B_{\ell,k}(x, y, y')| \lesssim 2^{k(n+\nu+|\alpha|)}(2^k |h|)^{\gamma'}$$

since $|\alpha| + \nu + \gamma' < L + \gamma''$. This completes the estimate in (3.2) for the first term on the right hand side of (3.4). Now we check that $P_k T^* \psi^\ell_y(y)$, the second term from the right hand side of (3.4), also satisfies the $\gamma'$-Hölder estimate. The estimate is trivial when $|h| \geq 2^{-k}$. Since $(2^k |h|)^{\gamma'} (\psi^{\gamma+h}_k - \psi^\gamma_k) \in \mathcal{D}_P$ with the appropriate derivative estimates. When $|h| \leq 2^{-k}$, it follows from the WBP of $T$

$$|D^\alpha_y P_k T^* (\psi^{\gamma+h}_k - \psi^\gamma_k)(y)| = (2^k |h|)^{\gamma'} 2^{k|\lambda|} 2^{kn} \left| \left( T^* \left( \frac{(\psi^{\gamma+h}_k - \psi^\gamma_k)}{2^{kn}(2^k |h|)^{\gamma'}} \right) , \frac{(D^\alpha \varphi^\gamma_k)}{2^{kn}} \right) \right|$$

$$\lesssim (2^k |h|)^{\gamma'} 2^{(n+\nu+|\alpha|)k}.$$ 

Hence both terms on the right hand side of (3.4) satisfy the appropriate estimates, and hence so does $\theta_k(x, y)$.

Finally, note that using the decomposition in (3.4) and the proof above, it follows that

$$|P_k T^* \psi^\ell_y(y) f(y)| \lesssim 2^{kn} \Phi^{n+\nu+M+\gamma}_k(x-y)$$

for $|x-y| < 2^{3-k}$. That is, combined with the uniform estimates proved above, this verifies that $|P_k T^* \psi^\ell_y(y) f(y)|$ is bounded uniformly in $N$ by an $L^1$ function. This completes the proof. \(\square\)

**Corollary 3.2.** Let $\nu, M, L \geq 0$ be integers satisfying $\nu \leq L \leq M + \nu$, $0 < \gamma \leq 1$, and $T \in SIO_{\nu}(M+\gamma)$. Assume that $T^* (x^\alpha) = 0$ for $|\alpha| \leq L$ and that $T \in \mathcal{WBP}_\nu$. Fix $\psi, \tilde{\psi} \in \mathcal{D}_P$ for $P$ sufficiently large, and define $\lambda_{j,k}(x, y)$, for $j, k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n$. Then

$$|D^\alpha_x D^\beta_y \lambda_{j,k}(x, y)| \lesssim 2^{(n+|\alpha|)j + |\beta|k} (2^{L+\gamma} \min(0, j-k)) \Phi^{n+\nu+M+\gamma}_{\min(j,k)}(x-y)$$

$$|D^\alpha_x D^\beta_y \lambda_{j,k}(x+h, y) - D^\alpha_x D^\beta_y \lambda_{j,k}(x, y)| \lesssim 2^{(n+|\alpha|)j + |\beta|} (2^k |h|)^{\gamma'-\delta} 2^{\min(0, j-k)(L+\delta)} \Phi^{n+\nu+M+\gamma}_{\min(j,k)}(x-y).$$
for all $\alpha, \beta \in \mathbb{N}_0$, $0 \leq \delta \leq \gamma' < \gamma$ and $x, y, h \in \mathbb{R}^n$ satisfying $|h| < (2^{-\min(j,k)} + |x - y|)/2$, where $\widetilde{L} = [L - \nu]$.

**Proof.** This follows immediately by applying Theorem 3.1 with $(D^\alpha \psi)_j * T f$ in place of $\psi_k * T f$. When $k \geq j$, we have

$$|D_\alpha x D_\beta y \lambda_{j,k}(x,y)| \leq 2^{|\alpha| + |\beta|} \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x} \left( \theta_j(x,u) - \frac{\partial^2}{\partial y^2} \right) \right| (D^\beta \psi)^\gamma_k(u) \, du \lesssim 2^{|\alpha| + |\beta|} \int_{\mathbb{R}^n} 2^{\gamma' j} |u - y|^{L + \gamma'} \left( \Phi_j^{n + \nu + M + \gamma}(x-u) + \Phi_j^{n + \nu + M + \gamma}(x-y) \right) |(D^\beta \psi)^\gamma_k(u) \, du \lesssim 2^{\gamma' j} 2^{|\alpha| + |\beta|} 2^{(L + \gamma')(j-k)} \Phi_{\min(j,k)}^\gamma(x-y).$$

When $k < j$, we have

$$|D_\alpha x D_\beta y \lambda_{j,k}(x,y)| \leq 2^{|\alpha| + |\beta|} \int_{\mathbb{R}^n} |T^* (D^\alpha \psi)_j \psi_k \psi_k^\gamma u (u) |(D^\beta \psi)^\gamma_k(u) | du \lesssim 2^{|\alpha| + |\beta|} \int_{\mathbb{R}^n} 2^{\gamma' j} |u - y|^{L + \gamma'} |(D^\beta \psi)^\gamma_k(u) | \times \left( \Phi_j^{n + \nu + M + \gamma}(x-u) + \Phi_j^{n + \nu + M + \gamma}(x-y) \right) \, du \lesssim 2^{(\nu + |\alpha|) + |\beta|} 2^{(L + \gamma') \min(0,j-k)} \Phi_{\min(j,k)}^\gamma(x-y).$$

Combining this with the first estimate yields the second one. \qed

## 4. A Restricted Operator Calculus

In this section, we prove two operator calculus type results, in Theorems 4.2 and 4.3. The first provides conditions on $T \in SIO_\psi$ of the form $T^*(x^\alpha) = 0$ so that $|\nabla|^{-s} T |\nabla|^t \in SIO_{\psi^t - s}$ (technically, this holds modulo polynomials), and the second provides conditions so that $|\nabla|^{-s} T |\nabla|^t \in CZO_{\psi^t - s}$. It is of particular interest to note that in Theorem 4.3, we can actually conclude that $|\nabla|^{-s} T |\nabla|^t \in CZO_{\psi^t - s}$ while $T$ only belongs to $SIO_\psi$. We provide some example later that show there are operators $T$ in $SIO_\psi$ but not $CZO_\psi$ such that $|\nabla|^{-s} T |\nabla|^t \in CZO_{\psi^t - s}$. In fact, we construct two classes of such example, pseudodifferential operators with symbols in the forbidden class $S^0_{1,1}$ and a variant of the Bony paraproduct.

**Lemma 4.1.** For any $M > -n$ such that $N > n + M$ and $x \neq 0$

$$\sum_{j \in \mathbb{Z}} 2^{jM} \Phi_j^N(x) \lesssim \frac{1}{|x|^{n+M}}.$$

**Proof.** This estimate is straightforward to prove. For $M$ and $N$ as above and $x \neq 0$,

$$\sum_{j \in \mathbb{Z}} 2^{jM} \Phi_j^N(x) \leq \sum_{j \in \mathbb{Z}, 2^j \leq |x|^{-1}} 2^{j(n+M)} + \frac{1}{|x|^N} \sum_{j \in \mathbb{Z}, 2^j > |x|^{-1}} 2^{-j(N-n-M)} \lesssim \frac{1}{|x|^{n+M}}.$$ \qed
Theorem 4.2. Let $v \in \mathbb{R}$, $M, L \geq 0$ be integers satisfying $v \leq L \leq M + v$, $0 < \gamma \leq 1$, and $T \in SIO_v(M + \gamma)$. Assume that $T^*(x^\alpha) = 0$ for $|\alpha| \leq L$ and that $T \in WB_P^v$. Also fix $s, t \in \mathbb{R}$ satisfying $t < [L - v] + \gamma$, $s > v$, and $t - s < n + M + \gamma$. Then there exists $T_{s,t} \in SIO_{v+s+t}^{\gamma'}$ for $0 < \gamma' < \gamma$ such that $\langle |\nabla|^{-s}T|\nabla|^t f, g \rangle = \langle T_{s,t}f, g \rangle$ for all $f, g \in \mathcal{S}_\infty$. The kernel $K_{s,t}(x, y)$ of $T_{s,t}$ satisfies

$$|D^\alpha D^\beta K_{s,t}(x, y)| \lesssim \frac{1}{|x - y|^{|n + v + t - s| + |\alpha| + |\beta|}}$$

for all $x \neq y$ and $\alpha, \beta \in \mathbb{N}_0^n$ satisfying $|\alpha| < s - v$, $|\beta| < [L - \nu] + \gamma - t$, and $|\alpha| + |\beta| < n + M + \gamma + s - t$, and

$$|D^\alpha D^\beta K_{s,t}(x + h, y) - D^\alpha D^\beta K_{s,t}(x, y)| \lesssim \frac{|h|^{\gamma'}}{|x - y|^{|n + v + t - s| + |\alpha| + |\beta| + \gamma'}}$$

for all $x, y, h \in \mathbb{R}^n$ with $|h| < |x - y|/2$, $0 < \gamma' < \gamma$, and $\alpha, \beta \in \mathbb{N}_0^n$ satisfying $|\alpha| < s - (v + \gamma')$, $|\beta| < [L - v] - t + \gamma - \gamma'$, and $|\alpha| + |\beta| < n + M + \gamma + s - (t + \gamma')$. Furthermore, $T_{s,t}$ and $T_{s,t}^*$ are continuous from $\mathcal{S}_p$ into $\mathcal{S}_p'$ for $p$ sufficiently larger, and can be defined

$$\langle T_{s,t}f, g \rangle = \sum_{j,k \in \mathbb{Z}} 2^{tk-sj} \int_{\mathbb{R}^{2n}} \langle T^*\psi_j^*, \psi_k^* \rangle \langle |\nabla|^t \tilde{\psi}_j, \psi_k \rangle f(y) \langle |\nabla|^{-s} \tilde{\psi}_j, g(y) \rangle dx dy,$$

where $\psi$ and $\tilde{\psi}$ as chosen as in Lemma 2.6.

Proof. Let $\psi \in \mathcal{D}_P$ and $\tilde{\psi} \in \mathcal{S}_\infty$ be as in Lemma 2.6. Define $\Lambda_{j,k} = Q_j T Q_j$, whose kernel is given by $\lambda_{j,k}(x, y) = \langle T^*\psi_j^*, \psi_k^* \rangle$ for $j, k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n$. For any $f, g \in \mathcal{S}_\infty$, it follows that

$$\langle |\nabla|^{-s}T|\nabla|^t f, g \rangle = \sum_{j,k \in \mathbb{Z}} \langle |\nabla|^{-s}Q_j f, Q_j T Q_k \tilde{\psi}_j \psi_k \rangle f(y) \langle |\nabla|^t \tilde{\psi}_j, g(y) \rangle dx dy$$

$$= \sum_{j,k \in \mathbb{Z}} 2^{tk-sj} \int_{\mathbb{R}^n} Q_j T Q_k \tilde{\psi}_j (u) Q_j \tilde{\psi}_j^*(u) g(u) du$$

$$= \sum_{j,k \in \mathbb{Z}} 2^{tk-sj} \int_{\mathbb{R}^{2n}} \left( \int_{\mathbb{R}^n} \lambda_{j,k}(u, v) \tilde{\psi}_j^*(v - y) \psi_k(u - x) dv du \right) g(x) f(y) dx dy.$$

Here we denote $\tilde{Q}_j f = \langle |\nabla|^t \psi_j \rangle f$ and likewise for $\tilde{Q}_j^{-s}$. For $x \neq y$, define

$$K_{s,t}(x, y) = \sum_{j,k \in \mathbb{Z}} 2^{tk-sj} \int_{\mathbb{R}^n} \lambda_{j,k}(u, v) \tilde{\psi}_j^*(v - y) \psi_k(u - x) dv du.$$

Let $0 < \gamma' < \gamma$ and $\alpha, \beta \in \mathbb{N}_0^n$ satisfying $|\alpha| < s - v$, $|\beta| < L + \gamma - t$, and $|\alpha| + |\beta| < n + M + \gamma + s - t$, where $L = [L - v]$. Then for all $x \neq y$, using Corollary 3.2, we have

$$\sum_{j,k \in \mathbb{Z}} 2^{tk-sj} \left| D^\alpha D^\beta \int_{\mathbb{R}^n} \lambda_{j,k}(u, v) \tilde{\psi}_j^*(v - y) \psi_k(u - x) dv du \right|$$

$$\leq \sum_{j,k \in \mathbb{Z}} 2^{(|\alpha| - s)j + (|\beta| + s)k} \int_{\mathbb{R}^n} |\lambda_{j,k}(u, v)| |D^\beta \psi_k(v - y)| |D^\alpha \psi_j|^s(u - x)| dv du.$$
\[
\leq \sum_{j,k \in \mathbb{Z}} 2^{(v+|\alpha|-s)j+(t+|\beta|)k} 2(\bar{L}+\gamma) \min(0,j-k) \\
\cdot \int_{\mathbb{R}^{2n}} \Phi_n^{n+\gamma+M+\gamma} (u-v) \Phi_{n+\gamma+M+\gamma} (v-y) (x-u) dv du \\
\leq \sum_{j,k \in \mathbb{Z}} 2^{(v+|\alpha|-s)j+(t+|\beta|)k} 2(\bar{L}+\gamma) \min(0,j-k) \Phi_{n+\gamma+M+\gamma} (x-y) \\
\lesssim \sum_{j,k \in \mathbb{Z}, j \leq k} 2^{(v+|\alpha|-s)j+(t+|\gamma|)k} \Phi_j^{n+\gamma+M+\gamma} (x-y) \\
\quad + \sum_{j,k \in \mathbb{Z}, j > k} 2^{(v+|\alpha|-s)j+(t+|\gamma|)k} \Phi_k^{n+\gamma+M+\gamma} (x-y) \\
\lesssim \sum_{j \in \mathbb{Z}} 2^{(v+|\alpha|+|\beta|+t-s)j} \Phi_j^{n+\gamma+M} (x-y) + \sum_{k \in \mathbb{Z}} 2^{(v+|\alpha|+|\beta|+t-s)k} \Phi_k^{n+\gamma+M} (x-y) \\
\leq \frac{1}{|x-y|^{n+\gamma+|\alpha|+|\beta|+t-s}}.
\]

Here we use that \( t + |\beta| < \bar{L} + \gamma \) and \( |\alpha| < s - v \) to assure that the summations above converge and Lemma 4.1 to justify the last inequality. It follows that for \( x \neq y \), we have

\[
|D_x^\alpha D_y^\beta K_{s,t}(x,y)| \leq \sum_{j,k \in \mathbb{Z}} 2^{(v+|\alpha|-s)j+(t+|\beta|)k} \left| \int_{\mathbb{R}^{2n}} \lambda_{j,k}(u,v) \tilde{\psi}_j^S (v-y) \tilde{\psi}_j^{-S} (u-x) dv dv \right| \\
\lesssim \frac{1}{|x-y|^{n+\gamma+|\alpha|+|\beta|+t-s}}.
\]

Now suppose \( \alpha, \beta \in \mathbb{N}_0^n \) with \( |\alpha| < s - (v+\gamma) \), \( |\beta| < \bar{L} + \gamma - \gamma' - t \), and \( |\alpha| + |\beta| < n + M + \gamma + s - (t + \gamma') \). For any \( x, y, h \in \mathbb{R}^n \) satisfying \( |h| < |x-y|/2 \), we have

\[
\sum_{j,k \in \mathbb{Z}} 2^{(v+|\alpha|-s+\gamma')j+(t+|\beta|)k} \left| \int_{\mathbb{R}^{2n}} \lambda_{j,k}(u,v) \tilde{\psi}_j^S (v-y) \tilde{\psi}_j^{-S} (u-x-h) dv dv \right| \\
\leq |h|^{\gamma'} \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^{2n}} \Phi_{n+\gamma+M} (u-v) \Phi_{n+\gamma+M} (v-y) (x-u) dv dv \right| \\
\cdot \int_{\mathbb{R}^{2n}} \Phi_{n+\gamma+M+\gamma} (u-v) \Phi_{n+\gamma+M+\gamma} (v-y) (x-u) dv dv \\
\leq |h|^{\gamma'} \sum_{j \in \mathbb{Z}} 2^{(v+|\alpha|+|\beta|+\gamma'+t-s)j} \Phi_j^{n+\gamma+M+\gamma} (x-y) \\
\quad + |h|^{\gamma'} \sum_{k \in \mathbb{Z}} 2^{(v+|\alpha|+|\beta|+\gamma'+t-s)k} \Phi_k^{n+\gamma+M+\gamma} (x-y) \\
\leq \frac{|h|^{\gamma'}}{|x-y|^{n+\gamma+|\alpha|+|\beta|+\gamma'+t-s}}.
\]

Here we use that \( |\beta| < \bar{L} + \gamma - \gamma' - t \) and \( |\alpha| < s - (v+\gamma') \) so that the summations above converge, and Lemma 4.1 for the last line. Fix \( \gamma' < \gamma'' < \gamma \) such that \( |\beta| < \bar{L} + \gamma'' - \gamma' - t \). By a similar
argument, but applying Corollary 3.2 with \( \gamma'' \) in place of \( \gamma' \), it follows that
\[
\sum_{j,k \in \mathbb{Z}} 2^{t-k-s} \left| \sum_{j,k \in \mathbb{Z}} \lambda_{j,k}(u,v) (\tilde{\psi}_j'(v-y) - \tilde{\psi}_k'(v-y-h)) \tilde{\psi}_j^{-s}(u-x) dv du \right|
\lesssim |h|^{\gamma'} \sum_{j,k \in \mathbb{Z}: j < k} 2^{(L+\gamma''+\nu+|\alpha|+s)} 2^{(t+|\beta|+\nu-L-\gamma')} k \Phi_j^{n+\nu+M+\gamma} (x-y)
+ |h|^{\gamma'} \sum_{j,k \in \mathbb{Z}: j > k} 2^{(\nu+|\alpha|+s) + (t+|\beta|+\nu+\gamma') k \Phi_k^{n+\nu+M+\gamma} (x-y)}
\lesssim \frac{|h|^{\gamma'}}{|x-y|^{n+\nu+\max(|\alpha|,|\beta|)+\gamma'+t-s}},
\]
for which we use that \( |\beta| < L + \gamma'' - \gamma' - t \) and \( |\alpha| < s - v \). \( \square \)

**Theorem 4.3.** Let \( v \in \mathbb{R}, L \) be an integer with \( L \geq |v|, (L-v)_* < \gamma \leq 1 \), and \( M \geq \max(L,L-v) \). If \( T \in SIO_\nu(M+\gamma) \) satisfies \( WBP \) and \( T^s(x^0) = 0 \) for all \( |\alpha| \leq L \), then for each \( v < s < v + |L-v| + \gamma \) and \( 0 < t < |L-v| + \gamma \), there exists \( T_{s,t} \in CZO_{V_t-s} \) such that \( |\nabla|^{-s} T |\nabla|^t f - T_{s,t}f \) is a polynomial for all \( f \in \mathcal{S}_\infty \).

**Proof.** Assume that \( T \in SIO_\nu(M+\gamma) \) satisfies \( WBP \) and that there is an integer \( P \geq M \) such that 
\( T^s(x^0) = 0 \) in \( D_p \) for all \( |\alpha| \leq L \). Fix \( \psi \in D_p \) and \( \psi \in \mathcal{S}_\infty \) as in Lemma 2.6. Define \( \Lambda_{j,k} f = Q_j T Q_k \), whose kernel is given by \( \lambda_{j,k}(x) = \langle T^s \psi_j^x, \psi_k^x \rangle \). Fix \( s, t \in \mathbb{R} \) so that \( 0 < s - v, t < L + \gamma \), and define the operator \( T_{s,t} \), which is continuous from \( \mathcal{S} \) into \( \mathcal{S}' \), by
\[
T_{s,t} f(x) = \sum_{j,k \in \mathbb{Z}} 2^{t-k-s} \tilde{Q}_j^{-s} \Lambda_{j,k} \tilde{Q}_k^t f(x).
\]
By Theorem 4.2, it follows that \( T_{s,t} \in SIO_{V_t-s}(\gamma') \) for all \( 0 < \gamma' < \gamma \) and that \( |\nabla|^{-s} T |\nabla|^t f - T_{s,t} f \) is a polynomial for all \( f \in \mathcal{S}_\infty \). By Corollary 3.2, it follows that
\[
|Q_j \tilde{Q}_j^{-s} \Lambda_{j,k} \tilde{Q}_k^t f(x)| \lesssim 2^{\nu} 2^{-K|\ell-j|/2} (L+\gamma')^{|(0,j-k)|} |f(x)|,
\]
where we choose \( \gamma' \) so that \( \max(0, s-v-\bar{L}, t-\bar{L}) < \gamma' < \gamma \) and \( K > 0 \). Then for all \( f \in \mathcal{S}_P \) and \( 1 < p, q < \infty \), it follows that
\[
\left\| T_{s,t} f \right\|_{L^{p,q}} \leq \left\| \sum_{j,k \in \mathbb{Z}} 2^{t-k-s} |Q_j \tilde{Q}_j^{-s} \Lambda_{j,k} \tilde{Q}_k^t f| \right\|_{L^p}^{1/q} \left\| \sum_{j,k \in \mathbb{Z}} 2^{t-k-s} 2^{\nu} 2^{-K|\ell-j|/2} (L+\gamma')^{|(0,j-k)|} |f(x)| \right\|_{L^p}^{1/q}
\]
\[
\lesssim \left\| \sum_{j,k \in \mathbb{Z}} 2^{t-k-s} 2^{\nu} 2^{-K|\ell-j|/2} (L+\gamma')^{|(0,j-k)|} |f(x)| \right\|_{L^p}^{1/q}
\]
\[
\lesssim \left\| \sum_{j,k \in \mathbb{Z}} 2^{q(|s-v|)} 2^{(s-v)(k-j)} 2^{-K|\ell-j|/2} (L+\gamma')^{|(0,j-k)|} |f(x)| \right\|_{L^p}^{1/q}
\]
applying Corollary 3.2. Then it follows that

$$
\|T f\|_{L^p} \lesssim \left( \sum_{k \in \mathbb{Z}} 2^{q(\nu+t-s)k} \left[ M(\tilde{Q}_k f) \right]^q \right)^{1/q} \lesssim ||f||_{F_p^{\nu+t-s,q}}.
$$

Therefore $T_{s,t}$ can be extended to a bounded linear operator from $\dot{F}_p^{\nu+t-s,q}$ into $\dot{F}_p^{0,q}$ by density. In particular taking $q = 2$, $T_{s,t}$ can be extended to a bounded linear operator from $W^{\nu+t-s,p}$ into $L^p$ for all $1 < p < \infty$. Then we have verified that $T_{s,t} \in CZO_{\nu+t-s}(\gamma')$ for all $0 < \gamma' < \gamma$. \hfill \Box

5. BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS

In this section, we show that $T^* (x^a) = 0$ conditions for $T \in SIO_\nu$ are sufficient for several boundedness properties. The first two results, in Theorems 5.1 and 5.4, are on negative smoothness Besov and Triebel-Lizorkin spaces, and they are proved directly with the help of Corollary 3.2 and Theorem 4.2. The last two boundedness results, in Corollaries 5.5 and 5.6, are consequences of Theorems 5.1 and 5.4 by duality. To aid in the discussion of the boundedness of these spaces, we describe some geometric properties of the parameter space related to the Triebel-Lizorkin space estimates for $T$. Another interesting feature of the results in this section is that cancellation properties $T^* (x^a) = 0$ for $T$ allow for us to conclude boundedness on certain weighted Triebel-Lizorkin spaces $\dot{L}^{s,q}_{p,w}$, where the weight $w$ does not belong to $A_p$. This type of behavior for operators was observed in [32, 23] in relation to Hardy spaces. We should also note that the boundedness results proved in this section are also sufficient for $T^* (x^a) = 0$ conditions. This is addressed in the next section.

**Theorem 5.1.** Let $\nu \in \mathbb{R}$, $L$ be an integer with $L \geq |\nu|, M \geq \max(L, L-\nu), (L-\nu)_+ < \gamma \leq 1$, and $T \in SIO_\nu(M + \gamma)$ satisfy WBQ. If $T^* (x^a) = 0$ for all $|a| \leq L$, then $T$ can be extended to a bounded operator from $B_{p,w}^{-L,q}$ into $B_{p,w}^{-\gamma,q}$ for all $1 < p < \infty, 0 < q < \infty, w \in A_p$, and $\nu < t < \nu + |L-\nu| + \gamma$.

**Proof.** We denote $\bar{L} = |L-\nu|$. Fix $\nu < t < \nu + \bar{L} + \gamma, 1 < p < \infty$, and $0 < \gamma' < \gamma$ such that $t < \nu + \bar{L} + \gamma'$. Let $T_t$ be as in Theorem 4.2 such that $|\nabla|^{-t} T |\nabla|^t f - T_t f$ is a polynomial for all $f \in S_\infty$. It follows that

$$
\tilde{Q}_t T_t f(x) = \sum_{j,k} 2^{(k-j)} \tilde{Q}_j \tilde{Q}_k \Lambda_{j,k} \tilde{Q}_k f(x).
$$

For any $K > 0$ and $\varepsilon > 0$ sufficiently small, it follows that

$$
|\tilde{Q}_t \tilde{Q}_j^{-s} \Lambda_{j,k} f(x)| \lesssim 2^{\nu/2} 2^{-K(j-k)} M f(x)
$$

and

$$
|\tilde{Q}_t \tilde{Q}_j^{-s} \Lambda_{j,k} f(x)| \lesssim 2^\nu 2^{(L+\gamma'+\varepsilon) \min(0, j-k)} M f(x).
$$

The first estimate here is a standard almost orthogonality estimate, and the second is obtained by applying Corollary 3.2. Then it follows that

$$
|\tilde{Q}_t \tilde{Q}_j^{-s} \Lambda_{j,k} f(x)| \lesssim 2^{\nu(j-k)} 2^{-K(j-k)} 2^{(L+\gamma'+\varepsilon) \min(0, j-k)} 2^{\frac{k}{2} \nu} M f(x)
$$

for some $K > 0$. Using that $\nu < t < \nu + \bar{L} + \gamma'$, for $w \in A_p$ it follows that

$$
||T_t f||_{B_{p,w}^{0,q}} \lesssim \left( \sum_{\ell \in \mathbb{Z}} \left( \sum_{j,k \in \mathbb{Z}, j \leq k} 2^{-K(j-k)} 2^{(L+\gamma'+\nu-t)(j-k)} 2^{\frac{k}{2} \nu} ||M(\tilde{Q}_k f)||_{L_p^q} \right)^q \right)^{1/q}.
$$
Then for all \( f \)
\[
+ \left( \sum_{\ell \in \mathbb{Z}} \left[ \sum_{j, k \in \mathbb{Z} : j > k} 2^{-\mathcal{K} \ell - j} 2^{(t - \nu) (k - j)} 2^{\nu k} \| \mathcal{M}(\mathcal{Q}_k f) \|_{L^p_w} \right]^{q} \right)^{1/q}
\]
\[
\leq \left( \sum_{\ell, j, k \in \mathbb{Z} : j \leq k} 2^{-\mathcal{K} \ell - j} 2^{(\mathcal{L} + \gamma - t)(j - k)} \left[ 2^{\nu k} \| \mathcal{M}(\mathcal{Q}_k f) \|_{L^p_w} \right]^{q} \right)^{1/q}
\]
\[
+ \left( \sum_{\ell, j, k \in \mathbb{Z} : j > k} 2^{-\mathcal{K} \ell - j} 2^{(t - \nu)(k - j)} \left[ 2^{\nu k} \| \mathcal{M}(\mathcal{Q}_k f) \|_{L^p_w} \right]^{q} \right)^{1/q}
\]
\[
\leq \left( \sum_{k \in \mathbb{Z}} \left[ 2^{\nu k} \| \mathcal{M}(\mathcal{Q}_k f) \|_{L^p_w} \right]^{q} \right)^{1/q}
\]
\[
\approx \| f \|_{\dot{B}^{\nu,q}_p}.
\]

Therefore \( T_i \) is bounded from \( \dot{B}^{\nu,q}_{p,w} \) into \( \dot{B}^{\nu-1,q}_{p,w} \) for all \( 1 < p, q < \infty, w \in A_p \), and \( \nu < t < \nu + \mathcal{L} + \gamma \). Then for all \( f \in \mathcal{S} \), it follows that
\[
\| T f \|_{\dot{B}^{\nu-1,q}_{p,w}} = \| \nabla^{-t} T \nabla^\ell (|\nabla|^{-t} f) \|_{\dot{B}^{\nu,q}_{p,w}} = \| T_i (|\nabla|^{-t} f) \|_{\dot{B}^{\nu,q}_{p,w}} \leq \| |\nabla|^{-t} f \|_{\dot{B}^{\nu-1,q}_{p,w}}.
\]

Therefore \( T \) can be extended to a bounded linear operator on \( \dot{B}^{\nu-1,q}_{p,w} \) for all \( 1 < p, q < \infty, w \in A_p \), and \( \nu < t < \nu + \mathcal{L} + \gamma \).

A similar argument to the one above can be made to show that \( T \) is bounded from \( \dot{F}^{\nu,q}_{p,w} \) into \( \dot{F}^{\nu-1,q}_{p,w} \) under the same assumptions on \( T \) and for the same ranges of parameters (in particular imposing that \( w \in A_p \)). However, we do not pursue this argument since we can prove something stronger for Triebel-Lizorkin spaces, where the weight is allowed to range outside of the \( A_p \) class corresponding to the Lebesgue space parameter \( p \). Prior to stating and proving this stronger results, which is in Theorem 5.4, we prove the following lemma that will be used in the proof Theorem 5.4.

**Proposition 5.2.** Let \( T \) be an operator, \( s, t \in \mathbb{R} \), \( 0 < p_0, q < \infty \), and \( \lambda \geq 1/p_0 \). Assume that \( T \) is bounded from \( \dot{F}^{s,q}_{p,w} \) into \( \dot{F}^{s,q}_{p,w} \), and there is an increasing function \( N : \mathbb{R} \to [1, \infty) \) such that
\[
\| T f \|_{\dot{F}^{s,q}_{p,w}} \leq N(\| w \|_{A_{p_0}}) \| f \|_{\dot{F}^{s,q}_{p,w}}
\]
for all \( w \in A_{p_0} \). Then \( T \) is bounded from \( \dot{F}^{s,q}_{p,w} \) into \( \dot{F}^{s,q}_{p,w} \) for all \( 1/\lambda < p < \infty \) and \( w \in A_{\lambda p} \).

**Proof.** Define
\[
(F, G) = \left( \sum_{k \in \mathbb{Z}} \left( 2^{sk} |Q_k f| \right)^q \right)^{\frac{1}{q}}, \left( \sum_{k \in \mathbb{Z}} \left( 2^{sk} |Q_k T f| \right)^q \right)^{\frac{1}{q}}
\]

For all \( w \in A_{\lambda p_0} \)
\[
\| F \|_{L^q_{w_0}} = \| T f \|_{L^q_{w_0}}^{1/\lambda} \leq N(\| w \|_{A_{\lambda p_0}})^{1/\lambda} \| f \|_{L^q_{w_0}}^{1/\lambda} = N(\| w \|_{A_{\lambda p_0}})^{1/\lambda} \| G \|_{L^q_{w_0}}^{1/\lambda}.
\]
We apply extrapolation to the pairs of functions \((F, G)\) indexed by \(f \in \mathcal{S}_\infty\). Note that \(1 \leq \lambda p_0 < \infty\). Then, by extrapolation, it follows that

\[
\|G\|_{L_w^p} \leq K(w)^{1/\lambda} \|F\|_{L_w^p}
\]

for all \(1 < r < \infty\) and \(w \in A_r\), where \(K(w)\) is specified in [15]. Therefore

\[
\|T f\|_{p^{r,q}_{x/\lambda,w}} = \|G\|_{L_w^p}^{1/\lambda} \|F\|_{L_w^p}^{1/\lambda} = K(w)\|f\|_{p^{r,q}_{x/\lambda,w}}
\]

for all \(1 < r < \infty\) and \(w \in A_r\). Now we simply shift notation to \(p = r/\lambda\), and it follows that

\[
\|T f\|_{p^{r,q}_{x/\lambda,w}} \leq K(w)\|f\|_{p^{r,q}_{x/\lambda,w}}
\]

for all \(1/\lambda < p < \infty\), \(w \in A_{\lambda p}\), and \(f \in \mathcal{S}_\infty\). By density, \(T\) is bounded from \(\mathcal{F}^{r,q}_{p,w}\) into \(\mathcal{F}^{r,q}_{x/\lambda,w}\) for the same range of indices.

\[\square\]

**Remark 5.3.** Though Proposition 5.2 is a relatively trivial application of Rubio de Francia’s extrapolation, there are some interesting subtleties that can be observed in this result. It demonstrates a way to “trade” the seemingly unnatural weighted estimates on \(L_w^p\) for \(w \in A_r\) when \(r > p\) for the ability to move the index \(p\) below 1. In particular, this provides a way to avoid a typical difficulty that arises in Hardy space theory for indices smaller than 1. Suppose you’d like to prove that a given operator \(T\) is bounded on \(H^{p_0}\) for some \(1/2 < p_0 < 1\). For such \(p_0\), the duality theory of \(H^{p_0}\) can be cumbersome. However, by Proposition 5.2 it is sufficient to prove that \(T\) is bounded on \(H_w^2 = \mathcal{F}_2^{0,2}\) for all \(w \in A_{2p_0}\) (note that \(H_w^2 \neq L_w^2\) for all such \(w\) since \(p_0 > 1/2\)). However, \(H_w^2\) may be easier to work with since, for example, it is a Banach space rather than only a quasi-Banach space like \(H^{p_0}\). Being able to “bump up” the index from \(p_0\) to 2 may also make it possible to use duality arguments that may not be viable for quasi-Banach spaces.

**Theorem 5.4.** Let \(v \in \mathbb{R}\), \(L\) be an integer with \(L \geq |v|\), \(M \geq \max(L, L - v)\), \((L - v)_+ < \gamma \leq 1\), and \(T \in SIO_v(M + \gamma)\). Further assume that \(T\) satisfies WBP\(_v\). If \(T^*(x^\alpha) = 0\) for all \(|\alpha| \leq L\), then \(T\) can be extended to a bounded operator from \(\mathcal{F}^{r-t,q}_{p,w}\) into \(\mathcal{F}^{r-t,q}_{p,w}\) for all \(v < t < v + |L - v| + \gamma\). Furthermore, there is an increasing function \(N : [1, \infty) \to (0, \infty)\) (possibly depending on \(v, p, t,\) and \(q\)) such that, for the same range of indices, we have

\[
\|T f\|_{p^{r-t,q}_{x/\lambda,w}} \leq N([w]_{A_p})\|f\|_{p^{r-t,q}_{x/\lambda,w}}.
\]

**Proof.** The approach to this proof is to first prove Theorem 5.4 for \(p \leq \min(1, q)\), and then apply Proposition 5.2 to extend to the full range of indices. Fix \(v < t < v + |L + \gamma|\), \(1/\lambda < p \leq 1\), \(p < q < \infty\), and \(w \in A_{\lambda p}\). Here we denote \(\tilde{L} = |L - v|\) and \(\lambda = \frac{n + v + |L - v| + \gamma - t}{n}\). Then there exists \(1/\lambda < r < p\) such that \(w \in A_{\lambda r}\). Also let \(0 < \gamma' < \gamma\) and \(0 < \mu < v + \tilde{L} + \gamma' - t\) so that \(t < v + \tilde{L} + \gamma'\) and \(1/\lambda < \mu < p < r < p\). Let \(T_f\) be defined as in Theorem 4.2. Now we fix \(\tilde{\phi}, \tilde{\psi} \in \mathcal{S}\) as in Lemma 2.7. Then it follows that

\[
\tilde{Q}_t T_f(x) = \sum_{j,k \in \mathbb{Z}} 2^{t(k-j)} \tilde{Q}_j \tilde{Q}_{-j}^t \Lambda_{j,k} \tilde{Q}_{k} f(x) = \sum_{j,k,m \in \mathbb{Z}} \sum_{Q : t(Q) = 2^{-m}} 2^{t(k-j)} \tilde{Q}_j \tilde{Q}_{-j}^t \Lambda_{j,k} \tilde{Q}_{k} \tilde{\psi}_m^Q (x) \tilde{\phi}_m * f(c_Q).
\]
For any any $K > 0$ and $\varepsilon > 0$ sufficiently small, it follows that
\[
|\tilde{Q}_t \tilde{Q}_j^\perp A_{j,k} Q_m^C \phi_m^C(x)| \lesssim 2^{\nu j} 2^{-K[\ell-j]} \Phi_{\min(\ell,j,k,m)}^{n+\gamma}(x - c_Q),
\]
\[
|\tilde{Q}_t \tilde{Q}_j^\perp A_{j,k} Q_m^C \phi_m^C(x)| \lesssim 2^{\nu j} 2^{-K[k-m]} \Phi_{\min(\ell,j,k,m)}^{n+\gamma}(x - c_Q),
\]
\[
|\tilde{Q}_t \tilde{Q}_j^\perp A_{j,k} Q_m^C \phi_m^C(x)| \lesssim 2^{\nu j} 2^{(\tilde{L} + \gamma' + \varepsilon) \min(0,j-k)} \Phi_{\min(\ell,j,k,m)}^{n+\gamma}(x - c_Q).
\]
As before, the first two lines here follow from standard almost orthogonality estimates, and the third follows from Corollary 3.2. Combining these estimates, it also follows that
\[
|Q_t Q_j^\perp A_{j,k} Q_m^C \phi_m^C(x)| \lesssim 2^{\nu j} 2^{-K[\ell-j]} 2^{-K[k-m]} 2^{(L+\gamma') \min(0,j-k)} \Phi_{\min(\ell,j,k,m)}^{n+\gamma}(x - c_Q)
\]
for some $\tilde{K} > 2\mu + |t|$, as long as $K$ is selected sufficiently large, depending on $L$, $\gamma$, and $v$. Then using Lemma 2.8, it follows that
\[
\sum_{Q_t \in \mathcal{Q}} |Q_t Q_j^\perp A_{j,k} Q_m^C \phi_m^C(x) | \lesssim 2^{\nu j} 2^{-K[\ell-j]} 2^{-K[k-m]} 2^{(L+\gamma') \min(0,j-k)} 2^{\varepsilon \max(0,m-j,m-k,m-\ell)} M_{m}^{\epsilon}(\Phi_m^C * f)
\]
Using that $\tilde{K} > 2\mu + |t|$ and that $v < t < v + \tilde{L} + \gamma' - \mu$, we also have
\[
\sum_{\ell \in \mathcal{Z}} |Q_t T_{f}^{q}|^q \lesssim \sum_{\ell \in \mathcal{Z}} \left[ \sum_{j,k,m \in \mathcal{Z}, j \leq m} 2^{-(K-\mu)(\ell-j)} 2^{-(K-2\mu-|t|)(j-k)} 2^{(v+L+\gamma'-t-\mu)(j-m)} 2^{\nu m} M_m^{\epsilon}(\Phi_m^C * f) \right]^q
\]
and hence that
\[
\|T_{f}\|_{F_{p,w}^{0,q}} = \left( \sum_{\ell \in \mathcal{Z}} |Q_t T_{f}^{q}|^q \right)^{1/q} \lesssim \left( \sum_{m \in \mathcal{Z}} 2^{\nu m} M_m^{\epsilon}(\Phi_m^C * f) \right)^{1/q} \lesssim \|f\|_{F_{p,w}^{0,q}},
\]
where we use Lemma 2.9 in the last inequality. Therefore $T_{f}$ is bounded from $F_{p,w}^{\nu,q}$ into $F_{p,w}^{0,q}$ for all $\nu < t < v + \tilde{L} + \gamma$, $1/\lambda < p \leq \min(1,q)$, and $w \in A_{1,p}$. Then for the same range of parameters and for all $f \in \mathcal{S}_\infty$, it follows that
\[
\|T_{f}\|_{F_{p,w}^{\nu-t,q}} = \|T_{f}^{\nu-t}|\|_{L_{w}^{p}} \lesssim \|T_{f}^{\nu-t}|\|_{F_{p,w}^{0,q}} \lesssim \|f\|_{L_{w}^{p}} \|T_{f}^{\nu-t}|\|_{F_{p,w}^{0,q}} = \|f\|_{F_{p,w}^{0,q}}.
\]
Therefore $T$ can be extended to a bounded linear operator form $F_{p,w}^{\nu-t,q}$ into $F_{p,w}^{\nu-t,q}$ for all $\nu < t < \nu + \tilde{L} + \gamma$, $1/\lambda < p \leq \min(1,q)$, and $w \in A_{\lambda,p}$. It is not hard to note that the dependence on $w \in A_{\lambda,p}$ from the argument about yields an estimate depending on a positive power of $[w]_{A_{p}}$, and hence $N$ can be taken as such (in particular, such a function exists). This completes the proof under the additional restriction that $p \leq \min(1,q)$.

For the general setting, fix an arbitrary $1/\lambda < q < \infty$. Then choose a $p_{0}$ satisfying $1/\lambda < p_{0} < \min(1,q)$. The first part of this proof verifies that $T$ is bounded from $F_{p_{0},w}^{\nu-t,q}$ into $F_{p_{0},w}^{\nu-t,q}$ for all $w \in A_{\lambda,p_{0}}$ and satisfies the estimate in the hypothesis of Proposition 5.2. Hence by Proposition 5.2, it follows that $T$ is bounded from $F_{p_{0},w}^{\nu-t,q}$ into $F_{p_{0},w}^{\nu-t,q}$ for all $1/\lambda < p < \infty$ and $w \in A_{\lambda,p}$. This completes the proof of Theorem 5.4 for the full range of indices $1/\lambda < p, q < \infty$. □

Below we represent $F_{p}^{t,2}$ in parameter space $(t, \frac{1}{p})$ for $t \in \mathbb{R}$ and $0 < p \leq \infty$. Most of the discussion here will apply for $F_{p}^{t,2}$ when $q \neq 2$, but for simplicity we restrict to only $q = 2$. However, we caution that some of the discussion here and below may not apply when $0 < q \leq 1$ since the duality of $F_{p}^{t,2}$ does not behave the same as the $q = 2$ case.

With our parameter identification defined, we note that the vertical axis $(t, \frac{1}{p}) = (0, \frac{1}{p})$ represents the Hardy spaces $F_{p}^{0,2} = H^{p}$ for $0 < p < \infty$ and Lebesgue spaces $F_{p}^{0,2} = L^{p}$ for $1 < p < \infty$. The horizontal lines given by $(t, \frac{1}{p})$ when $1 < p < \infty$ describe the homogeneous Sobolev spaces $F_{p}^{t,2} = W^{t,p}$. We will always identify the origin $(t, \frac{1}{p}) = (0,0)$ as $F_{p}^{0,2} = \text{BMO}$, but for $(t, \frac{1}{p}) = (t,0)$ our notation is somewhat inconsistent. Sometimes these locations will represent Sobolev-\text{BMO} via $F_{t}^{\infty,2} = l_{1}(\text{BMO})$ for $t > 0$, and other times they should be interpreted as the Besov-Lipschitz spaces $B_{t}^{\infty,\infty}$ when $t > 0$. Recall that $B_{t}^{\infty,\infty}$ is a Lipschitz space only when $t > 0$ is not an integer; otherwise it is the Zygmund class of smooth functions; see [46].

Let us first consider boundedness results for $T \in \text{SI0}_{\nu}(M + \gamma)$ when $\nu = 0$. The graph on the left in Figure 1 is a depiction of the boundedness properties of $T$ provided in Theorem 5.4 (restricted to the unweighted version). That is, if $T^{*}(\chi^{\alpha}) = 0$ for $|\alpha| \leq L$, then $T$ is bounded on $F_{p}^{t,2}$ when $(t, \frac{1}{p})$ lies in the blue shaded region (excluding the boundary) in the left picture. If we were to assume in addition that $T$ is $L^{2}$-bounded, then it also follows that $T$ is bounded on $F_{p}^{0,2} = H^{p}$ when $(t, \frac{1}{p}) = (0, \frac{1}{p})$ satisfies $\frac{n}{N+L+\gamma} < p < \infty$. This does not follow from what we’ve proved here, but these boundedness properties are classical in the Lebesgue space setting when $1 < p < \infty$ and were proved in [22] for $0 < p \leq 1$.\(^1\)

The middle picture of Figure 1 restricts the boundedness region for $T$ to where $1 < p < \infty$, where the spaces $F_{p}^{t,2}$ are reflexive. In this situation, we have $(F_{p}^{t,2})^{*} = F_{p'}^{-t,2}$ for $t \in \mathbb{R}$ and $1 < p < \infty$. Geometrically, the dual of $F_{p}^{t,2}$ can be found by reflecting over the vertical line $t = 0$ and the horizontal line $p = 2$, as pictured. Then by duality, $T^{*}(\chi^{\alpha}) = 0$ for $|\alpha| \leq L$ implies that $T^{*}$ is bounded on $F_{p}^{t,2}$ for all $1 < p < \infty$ and $0 < t < L + \gamma$. This boundedness result for $T^{*}$ is shown in the green shaded region in Figure 1.

\(^1\)Plots appearing in this article were generated using the Desmos.com online graphing tool and Matlab.
In the picture on the right in Figure 1, we describe the dual boundedness implications for $T^*$ when $0 < p < 1$, which are more delicate than the ones already discussed. We consider two situations: where $p = 1$ and where $0 < p < 1$. It was proved by Frazier and Jawerth \cite{Frazier86} that $(\mathcal{F}_{p,1}^{t,2})^* = \mathcal{F}_{\infty}^{-t,2}$. Then the boundedness of $T$ on $\mathcal{F}_{p,1}^{t,2}$ for indices on the horizontal line segment given by $(t, \frac{1}{p}) = (t, 1)$ with $-(L+\gamma) < t < 0$ implies that $T^*$ is bounded on the Sobolev-BMO spaces $\mathcal{F}_{\infty}^{t,2} = L^1(\text{BMO})$ for $0 < t < L+\gamma$. Geometrically, this summarizes boundedness for $T^*$ on the horizontal line $(t, \frac{1}{p}) = (t, 0)$ with $0 \leq t < L+\gamma$, where we make the Triebel-Lizorkin space identification $\mathcal{F}_{\infty}^{t,2}$. This duality still obeys the geometric rule of reflecting over the lines $t = 0$ and $p = 2$ to obtain the appropriate indices for dual spaces.

Now we turn our attention to the picture on the right in Figure 1 when $0 < p < 1$. The remaining portion of the red region lying to the left of the axis is where $-(L+\gamma) < t < 0$ and $\frac{n}{n+L+\gamma-1} < p < 1$. In this situation, we invoke a duality result of Jawerth \cite{Jawerth85} that says $(\mathcal{F}_{p,1}^{t,2})^* = \mathcal{B}_{\infty}^{-t+n(1/p-1),\infty}$ for $0 < p < 1$ and $t \in \mathbb{R}$. Note that the duals of $\mathcal{F}_{p,1}^{t,2}$ coincide for several values of $t$ and $p$ here. In particular, for any $t \in \mathbb{R}$ and $0 < p < 1$ with $-t+n(1/p-1) = s$ satisfies $(\mathcal{F}_{p,1}^{t,2})^* = \mathcal{B}_{\infty}^{-s,\infty}$. This is depicted above by the highlighted red line, and the associated red x on the horizontal axis located at $(t, \frac{1}{p}) = (1, 0)$. Then by duality $T^*(x^\alpha) = 0$ for $|\alpha| \leq L$ implies that $T^*$ is bounded on $\mathcal{B}_{\infty}^{-s,\infty}$ for $0 < t < L+\gamma$. This conclusion can be made by duality from the boundedness of $T$ at any point along the appropriate line. Geometrically, this deviates slightly from the previous cases. In particular, when $0 < p < 1$ and $t \in \mathbb{R}$, in order to obtain the appropriate indices for the dual of $\mathcal{F}_{p,1}^{t,2}$, first project $(t, \frac{1}{p})$ along a line with slope $1/n$ onto the line $p = 1$, then reflect over $t = 0$ and $p = 2$. Making these geometric manipulations lands the dual indices on the horizontal axis, overlapping with the previous case where $p = 1$. We emphasize that when $0 < p < 1$, the appropriate interpretation of the pictures above is that $(\mathcal{F}_{p,1}^{t,2})^* = \mathcal{B}_{\infty}^{-t+n(1/p-1),\infty}$, with this Besov space in place of $\mathcal{F}_{\infty}^{-t,2}$. Hence the distinction $0 < p < 1$ versus $p = 1$ determines when we interpret the horizontal axis as $\mathcal{F}_{\infty}^{t,2}$ versus $\mathcal{B}_{\infty}^{t,\infty}$.

The estimates for $T$ on $\mathcal{F}_{p,1}^{t,2}$ indicated in the blue shaded region in the left picture of Figure 1 describes only the the unweighted estimates proved in Theorem 5.4, but we can extend this representation to weighted estimates as well. In order to do so, consider the parameter space made up of ordered triples of the form $(t, \frac{1}{p}, \frac{1}{q})$ for which $T$ is bounded on $\mathcal{F}_{p,w}^{t,2}$ when $w \in A_q$. Theorem 5.4 says that the triple $(t, \frac{1}{p}, \frac{1}{q})$ represents where $T$ is bounded on the weighted spaces $\mathcal{F}_{p,w}^{t,2}$ for all
$w \in A_q$ when $\frac{n}{n+L+\gamma+t} \frac{1}{p} \leq \frac{1}{q} \leq 1$. This defines a solid in $\mathbb{R}^3$ lying under the blue shaded region on the left picture in Figure 1, which is shown in Figure 2.

![Figure 2](image)

**Figure 2.** Parameter space $(t, \frac{1}{p}, \frac{1}{q})$ for boundedness properties of $T$ on $\dot{F}^{t,2}_{p,w}$ with $w \in A_q$, pictured with $n = 2$, $L = 1$, and $\gamma = 1$. The plot on the left depicts the same region as the plot on the left of Figure 1, which coincides with the $q = 1$ cross-section of the middle and right plots.

All of these duality results are made precise in the following corollaries, which we state for a general $\nu \in \mathbb{R}$. Similar geometric depictions of the boundedness results above for $\nu \neq 0$ can be made by tracking the original and/or terminal indices $(t, \frac{1}{p})$ of the boundedness properties of $T$ from $\dot{F}^{t,2}_{p,v}$ into $\dot{F}^{t-v,2}_{p,v}$ for $\nu < -t < \nu + \tilde{L} + \gamma$ and $\frac{n}{n+\nu+L+\gamma+t} < p < \infty$. Figure 3 briefly demonstrates how the boundedness of $T$ and $T^*$ is expressed in parameter space when $\nu \neq 0$.

![Figure 3](image)

**Figure 3.** Parameter space for boundedness properties of $T$ from $\dot{F}^{\nu+t,2}_{p,v}$ into $\dot{F}^{t,2}_{p,v}$ and for $T^*$ from $\dot{F}^{t,2}_{p,v}$ into $\dot{F}^{t-v,2}_{p,v}$, pictured with $n = 2$, $L = 3$, and $\gamma = \frac{1}{2}$. The left, middle, and right plots correspond to $T$ belonging to $SIO_{\nu}$ for $\nu = -\frac{3}{2}, 0, \frac{3}{2}$ respectively, which yields $\tilde{L} = 5, 3, \frac{5}{2}$ respectively.

**Corollary 5.5.** Let $\nu \in \mathbb{R}$, $L$ be an integer with $L \geq |\nu|$, $M \geq \max(L, L - \nu)$, $(L - \nu)_* < \gamma \leq 1$, and $T \in SIO_{\nu}(M+\gamma)$. Further assume that $T$ satisfies $WBP_{\nu}$. If $T^*(\lambda^\alpha) = 0$ for all $|\alpha| \leq L$, then $T^*$ can be extended to a bounded operator from $\dot{F}^{t,q}_{p,v}$ into $\dot{F}^{t-v,q}_{p,v}$ and from $\dot{B}^{t,q}_{p,v}$ into $\dot{B}^{t-v,q}_{p,v}$ for all $1 < p, q < \infty$ and $\nu < t < \nu + [L - \nu] + \gamma$.

This corollary is immediate given Theorems 5.1 and 5.4 applied only in the unweighted and $1 < p, q < \infty$ situation.
Corollary 5.6. Let $\nu \in \mathbb{R}$, $L$ be an integer with $L \geq |\nu|$, $M \geq \max(L, L - \nu)$, $(L - \nu)_+ < \gamma \leq 1$, and $T \in SIO_{\nu}(M + \gamma)$. Further assume that $T$ satisfies WBPP. If $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$, then $T^*$ is bounded from $\dot{F}_0^{L, q}$ into $\dot{F}^{-L, q}_0$ and from $\dot{B}_0^{\infty, \nu}_q$ into $\dot{B}^{-\nu, \infty}_0$ for all $\nu < t < \nu + \bar{L} + \gamma$ and $1 < q < \infty$.

Proof. For $0 < t < \nu + \bar{L} + \gamma$ and $1 < q < \infty$, Theorem 5.4 implies that $T$ is bounded from $\dot{F}_1^{\nu - t, q}$ into $\dot{F}_1^{-\nu, q}$. Then by duality (see for example the Frazier and Jawerth article [17, Theorem 5.13]), it follows that $T^*$ is bounded from $\dot{F}_0^{L, q}$ into $\dot{F}^{-L, q}_0$ for $\nu < t < \nu + \bar{L} + \gamma$ and $1 < q < \infty$. For $\nu \leq t < \nu + \bar{L} + \gamma$, choose $\nu < s < L + \gamma$ and $n + \nu + L + \gamma - s < p < 1$ such that $s + n(1/p - 1) = t$. Then it follows that $T$ is bounded from $\dot{F}_p^{\nu - s, 2}$ into $\dot{F}_p^{-s, 2}$. So by duality, it follows that $T^*$ is bounded from $\dot{B}_0^{s + n(1/p - 1), \infty}$ into $\dot{B}^{-s - \nu + n(1/p - 1), \infty}$. That is, $T^*$ is bounded from $\dot{B}^{\infty, \nu}_0$ into $\dot{B}^{-\nu, \infty}_0$. Here we use the duality result of Jawerth [26, Theorem 4.2].

6. Necessity of Vanishing Moment Conditions

In this section, we establish the necessity of the $T^*(x^\alpha) = 0$ condition for many boundedness results. In fact, this provides a type of Theorem that characterizes necessary and sufficient conditions for $T$ (or $T^*$) to be bounded based on Weak Boundedness Properties and cancellation conditions. It is interesting to note that $T^*(x^\alpha) = 0$ is necessary and sufficient cancellation for many of these results, while we need not require anything on $T(x^\alpha)$. Some of these implications come from Proposition 5.2 and Lemma 6.1, both of which are interesting in their own right.

Lemma 6.1. Let $L \geq 0$ be an integer and $\gamma > 0$. If $f \in \dot{B}_p^{t, \infty} \cap L^{1}(1 + |x|^{L + \gamma})$ for all $0 < t < L + \gamma$ and $1 < p < \infty$, then

$$\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0$$

for all $|\alpha| \leq L$.

Consequently, for any $0 < q < \infty$, the same conclusion holds if $f \in \dot{F}_p^{t, q} \cap L^{1}(1 + |x|^{L + \gamma})$ or $f \in \dot{B}_p^{t, q} \cap L^{1}(1 + |x|^{L + \gamma})$ for all $0 < t < L + \gamma$ and $1 < p < \infty$.

Proof. We proceed by induction. First assume that $L = 0$. Let $0 < t < \gamma$ and $1 < p < \infty$ be small enough so that $n/p' < t < n/p + \gamma$. Assume $f \in \dot{B}_p^{t, \infty} \cap L^{1}(1 + |x|^\gamma)$. Then for any $\psi \in \mathcal{S}_c$, we have

$$||f||_{\dot{B}^{-t, \infty}_p} \geq \sup_{k<0} 2^{n(p'-t)k} ||\psi||_{L^p} \left|\int_{\mathbb{R}^n} f(y) dy\right|$$

$$- \sup_{k<0} 2^{-tk} \left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (\psi_k(x-y) - \psi_k(x)) f(y) dy\right|^p dx\right]^{1/p}.$$

The second term above is bounded since we have

$$\sup_{k<0} 2^{(\gamma-t)k} \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |y|^\gamma (\Phi_k^N(x-y) + \Phi_k^N(x)) |f(y)| dy\right)^p dx\right]^{1/p}$$

$$\leq \sup_{k<0} 2^{(\gamma-t)k} \left[\int_{\mathbb{R}^n} \Phi_k^N * (|y|^\gamma |f|)(x) dx\right]^{1/p} + ||f||_{L^{1}(|x|^\gamma)} \sup_{k<0} 2^{(\gamma-t)k} \left[\int_{\mathbb{R}^n} \Phi_k^N(x) dx\right]^{1/p}$$
Note that we chose $t$ so that $n/p' < t < n/p' + \gamma$, which implies $2^{(\gamma-t)k}||\Phi_k^N||_{L^p}$ is bounded uniformly in $k$ (as long as $N > n/p$) and that $2^{(n/p'-t)k}$ is unbounded for $k < 0$. Then it follows that $f$ must have integral zero. Now assume that Lemma 6.1 holds for all $M \leq L - 1$. Let $0 < t < L + \gamma$ and $1 < p < \infty$ be small enough so that $n/p' + L < t < n/p' + L + \gamma$. Assume $f \in B_p^{-t,\infty} \cap L^1(1 + |x|^{L+\gamma})$. Then for any $\psi \in \mathcal{L}_\infty$

$$||f||_{B_p^{-t,\infty}} \geq \sup_{k < 0} 2^{-tk} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_x^L[\psi_k](y)f(y)dy \left| \frac{p}{dx} \right|^p \right]^{1/p} - \sup_{k < 0} 2^{-tk} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\psi_k(x-y) - J_x^L[\psi_k](y))f(y)dy \left| \frac{p}{dx} \right|^p \right]^{1/p}.$$  

Again the second term is bounded above since

$$\sup_{k < 0} 2^{-tk} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\psi_k(x-y) - J_x^L[\psi_k](y))f(y)dy \left| \frac{p}{dx} \right|^p \right]^{1/p} \lesssim \sup_{k < 0} 2^{-tk} \left[ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} (2^k|y|^{L+\gamma}(\Phi_k^N(x-y) + \Phi_k^N(x)))|f(y)|dy \right)^p dx \right]^{1/p} \lesssim \sup_{k < 0} 2^{(L+\gamma+n/p'-t)k} ||f||_{L^1(|x|^{L+\gamma})} \lesssim ||f||_{L^1(|x|^{L+\gamma})}.$$  

We also have, by the inductive hypothesis, that

$$2^{-tk} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_x^L[\psi_k](y)f(y)dy \left| \frac{p}{dx} \right|^p \right]^{1/p} = 2^{(L+n/p'-t)k} \left[ \int_{\mathbb{R}^n} \left| \sum_{|\alpha| = L} \frac{D^\alpha \psi(x)}{\alpha!} \int_{\mathbb{R}^n} f(y)y^{\alpha}dy \right| \frac{p}{dx} \right]^{1/p}.$$  

Since $2^{(L+n/p'-t)k}$ is unbounded for $k < 0$, it follows that

$$\int_{\mathbb{R}^n} \left| \sum_{|\alpha| = L} \frac{D^\alpha \psi(x)}{\alpha!} \int_{\mathbb{R}^n} f(y)y^{\alpha}dy \right| \frac{p}{dx} = 0$$  

for all $\psi \in \mathcal{L}_\infty$, and hence that

$$\int_{\mathbb{R}^n} f(y)y^{\alpha}dy = 0$$  

for all $|\alpha| = L$. By induction, this completes the proof when $f \in B_p^{-t,\infty}$. Note that the remaining properties trivially follow since $\dot{F}_p^{-t,q} \subset B_p^{-t,\infty}$ and $\dot{B}_p^{-t,q} \subset B_p^{-t,\infty}$ for all $0 < q < \infty$, $1 < p < \infty$, and $t \in \mathbb{R}$. 

\begin{remark}
It is known that $H^p$ quantifies vanishing moment properties for its members (see e.g. [19] by Grafakos and He on weak Hardy spaces), and by Lemma 6.1 we have vanishing moment properties for negative smoothness index Triebel-Lizorkin and Besov spaces. In particular, Lemma 6.1 should be interpreted as follows. The spaces $\dot{F}_p^{-t,q}$ for $L \leq t + n(1/p - 1) < L + 1$ quantify vanishing moment properties for order $|\alpha| = L$ for its members, as described in Lemma 6.1. \hfill \Box
\end{remark}
Theorem 6.3. Let $v \in \mathbb{R}$, $L$ be an integer with $L \geq |v|$, $M \geq \max(L, L-v)$, $(L-v)_+ < \gamma \leq 1$, and $T \in \text{CZO}_v(M+\gamma)$. If any one of the conditions hold, then $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$.

1. For every $1 < p < \infty$ and $v < t < v + \tilde{L} + \gamma$, there exists $0 < q \leq \infty$ such that $T$ is bounded from $F_p^{\tilde{L}-q}$ into $F_p^{L-\tilde{q}}$.
2. For every $1 < p < \infty$ and $v < t < v + \tilde{L} + \gamma$, there exists $1 < q < \infty$ such that $T^*$ is bounded from $F_p^{M-q}$ into $F_p^{L-M}$.
3. For every $1 < p < \infty$ and $v < t < v + \tilde{L} + \gamma$, there exists $0 < q \leq \infty$ such that $T$ is bounded from $B_p^{\tilde{L}-q}$ into $B_p^{L-\tilde{q}}$.
4. For every $1 < p < \infty$ and $v < t < v + \tilde{L} + \gamma$, there exists $1 < q < \infty$ such that $T^*$ is bounded from $B_p^{M-q}$ into $B_p^{L-M}$.
5. For each $v < t < v + \tilde{L} + \gamma$, there exist $0 < q \leq \infty$ and $1/\lambda < p < \infty$ such that $T$ is bounded from $F_p^{L-q}$ into $F_p^{L-\tilde{q}}$.
6. For each $v < t < v + \tilde{L} + \gamma$ and $0 < t < |L-v| + \gamma$ there exists $T_{s,t} \in \text{CZO}_{v+s-t}$ such that $|\nabla|^{-s} T |f| f - T_{s,t} f$ is a polynomial for all $f \in \mathcal{H}_v$.

Proof. Assume that (1) holds, and let $\psi \in D_p$, for $p \in \mathbb{N}_0$ sufficiently large, with supp$(\psi) \subset B(0,R_0/4)$ for some $R_0 > 1$. Note that $T\psi$ is locally integrable by the $T \in \text{CZO}_v$ assumption. Also if $x \notin B(0,R_0)$, then it follows that

$$|T \psi(x)| = \left| \int_{\mathbb{R}^n} (K(x,y) - J^M_0 \{K(x,\cdot)(y)\}) \psi(y) dy \right| \lesssim \int_{\mathbb{R}^n} \frac{|y|^{M+\gamma} |\psi(y)| dy}{|x|^{n+L+\gamma}}.$$

Then it follows that $T \psi \in L^1(1 + |x|^{L+\tilde{\gamma}})$ for any $0 < \gamma' < \gamma$ since

$$\int_{\mathbb{R}^n} |T \psi(x)|(1 + |x|^{L+\gamma'}) dx \lesssim (1 + R_0^{L+\gamma}) \int_{|x| \leq R_0} |T \psi(x)| dx + R_0^{L+\gamma'}.$$

This, in addition to (1), says that $T \psi \in \dot{B}_{\infty,\infty}^{L+\gamma} \cap L^1(1 + |x|^{L+\tilde{\gamma}})$ for all $0 < t < L + \gamma'$ and $1 < p < \infty$. So by Lemma 6.1, it follows that $T \psi$ has vanishing moments up to order $L$. But this means exactly that $T^*(x^\alpha) = 0$ for the same $\alpha$’s since

$$\langle T^* x^\alpha, \psi \rangle = \lim_{R \to \infty} \int_{\mathbb{R}^n} T \psi(x) \eta_R(x) x^\alpha dx = \int_{\mathbb{R}^n} T \psi(x) x^\alpha dx = 0,$$

where we use dominated convergence and that $T \psi \in L^1(1 + |x|^{L+\tilde{\gamma}})$ to handle the limit in $R$. Therefore $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$. 


By exactly the same argument, it follows that condition (3) also implies \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \). Furthermore, by duality (2) implies (1) and (4) implies (3). Hence we have shown that any one of the conditions (1)–(4) implies \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \).

Assume that (5) holds. Then by Proposition 5.2 it follows that \( T \) is bounded from \( \tilde{F}^{\nu-t,q}_{p,w} \) into \( F^{-t,q}_{p,w} \) for all \( 0 < t < \nu + \tilde{L} + \gamma \), \( 1/\lambda < p < \infty \), and \( w \in A_{\lambda,p} \), where \( \lambda = \frac{n + \nu + \tilde{L} + \gamma - t}{n} \). In particular, (5) implies (1) which in turn implies that \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \).

Assume that (6) holds. Let \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq L \). Note that \( (L - \nu)_* < \gamma \) implies that \( L < \nu + \tilde{L} + \gamma \). Then there is a \( t \notin \mathbb{Z} \) such that \( \max(\nu, |\alpha|) < t < \nu + \tilde{L} + \gamma \). Also let \( 0 < p < 1 \) such that \( n(1/p - 1) = t - \nu \). Then for \( \psi \in \mathcal{D}_{2,p} \), with \( P \) sufficiently large, we have

\[
|\langle T^*(x^\alpha), \psi \rangle| \lesssim \limsup_{R \to \infty} \|\eta R x^\alpha\|_{B^\nu_1} \|\psi\|_{H^p} \lesssim \limsup_{R \to \infty} \|R^{|\alpha|-t}\|\psi\|_{H^p} = 0. 
\]

Here, we simply note that when \( t > 0 \) is a non-integer, \( \dot{B}^\nu_{p,q} \) is the class of \( t \)-Lipschitz functions, and it easily follows that \( \|\tilde{\phi}\|_{B^\nu_1} \lesssim R^{-t} \) for any \( \phi \in \mathcal{C}_0^\infty \), where again \( t_\ast \) is the decimal part of \( t \). Therefore \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \).

Assume that (7) holds. Note that \( \dot{F}^{\nu-t,q}_1 \subset \dot{B}^{\nu-t,q}_1 \) and \( \|f\|_{B^{-\nu-t,q}_1} \leq \|f\|_{F^{\nu-t,q}} \) for any \( \nu < t < \nu + \tilde{L} + \gamma \). Then we argue as we did in the previous case. For \( |\alpha| \leq L \), let \( \max(\nu, |\alpha|) < t < \nu + \tilde{L} + \gamma \) and \( \psi \in \mathcal{D}_{2,p} \). Then

\[
|\langle T^*(x^\alpha), \psi \rangle| \lesssim \limsup_{R \to \infty} \|\eta R x^\alpha\|_{F^{\nu-t,q}_1} \|\psi\|_{F^{\nu-t,q}_1} \leq \limsup_{R \to \infty} \|\eta R x^\alpha\|_{B^{-\nu-t,q}_1} \|\psi\|_{B^{-\nu-t,q}_1} = 0
\]

Therefore \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \).

By duality (8) implies (7) and by density (9) implies (1). Hence both (8) and (9) also imply \( T^*(x^\alpha) = 0 \) for \( |\alpha| \leq L \).

\[\square\]

7. Applications

7.1. Necessary and Sufficient Conditions for Classes of Singular Integral Operators. In this section we collect and summarize the results in the preceding three sections to form several equivalent conditions for cancellation and boundedness of operators \( T \in \mathcal{CZO}_\nu \). This result is essentially a combination of a few of the operator calculus results from Section 4, the boundedness results from Section 5, and the sufficiency for vanishing moments from Section 6. As a result we obtain the following \( T \) type necessary and sufficient condition for several boundedness results for \( \mathcal{SIO}_\nu \).

Corollary 7.1. Let \( \nu \in \mathbb{R}, L \geq |\nu| \) be an integer, \( (L - \nu)_* < \gamma \leq 1, \tilde{L} = |L - \nu|, \) and \( T \in \mathcal{CZO}_\nu(L + \gamma) \). Then the following are equivalent.

1. \( T^*(x^\alpha) = 0 \) for all \( |\alpha| \leq L \)
2. For every \( \nu < s < \nu + \tilde{L} + \gamma \) and \( 0 < t < \nu + \tilde{L} + \gamma \), there exists \( T_{s,t} \in \mathcal{CZO}_{\nu+t-s}(\gamma') \) for \( 0 < \gamma < \gamma \) such that \( |\gamma|^{-s}T f \) is a polynomial for all \( f \in \mathcal{S}_\infty \)
3. For all \( \nu < t < \nu + \tilde{L} + \gamma, 1/\lambda < p < \infty \), and \( \min(1,p) \leq q < \infty \), \( T \) is bounded from \( \tilde{F}^{\nu-t,q}_{p,w} \) into \( \tilde{F}^{\nu-t,q}_{p,w} \) and there is an increasing function \( N : [1,\infty) \to (0,\infty) \) that does not depend on \( w \) such that

\[
\|T f\|_{\tilde{F}^{\nu-t,q}_{p,w}} \leq N([w]_{A_{\lambda,p}}) \|f\|_{\tilde{F}^{\nu-t,q}_{p,w}}
\]
for all \( w \in A_{\lambda p} \), where \( \lambda = \frac{n + v + \tilde{L} + \gamma - t}{n} \).

This corollary follows immediately from Theorems 4.3, 5.4, and 6.3. We should also note that one could obtain many other equivalent conditions to put on this list by turning to Theorems 5.4 and 6.3, as well as Corollaries 5.5 and 5.6.

There is a long history of results along the lines of Corollary 7.1. Several boundedness results for \( v = 0 \) similar to the ones proved in Theorems 5.1 and 5.4 (as well as Corollaries 5.5 and 5.6) can be found for example in [1, 18, 44, 16, 20, 12, 31, 22, 23] as well as several of the references therein. However, we note that there do not seem to be many results like Theorem 5.4 in the sense that we obtain boundedness on for Triebel-Lizorkin spaces with weights in a Muckenhoupt \( A_{\lambda p} \) for \( \lambda > 1 \) class that exceed the Lebesgue space parameter \( p \). The only articles we are aware of where such estimates are proved are [32, 23], where the results are limited to \( v = 0 \) order operator acting on Hardy spaces. Along these lines, when \( v = 0 \) one can add more equivalent conditions than the ones already mentioned by involving weighted and unweighted Hardy space boundedness properties; see [1, 22, 23] for more information on this.

The class of operators \( CZO_{v} \) for \( v \neq 0 \) have been studied to much lesser extent than the \( v = 0 \) order operators that fall within scope of traditional zero-order Calderón-Zygmund theory. The most relevant resource in the literature for non-zero-order operator results of this type is [44], where several sufficient conditions for an operator in \( SIO_{v} \) to be bounded on homogeneous Besov and Triebel-Lizorkin spaces. However, conditions of the form \( T^{*}(x^{\alpha}) = 0 \) and \( T_{1} = 0 \) were assumed in order to prove such boundedness results. Here we remove the assumption on \( T_{1} = 0 \), show that such boundedness properties are also sufficient for \( T^{*}(x^{\alpha}) = 0 \) conditions, and include several other equivalent conditions involving weighted estimates, endpoint Besov and Triebel-Lizorkin spaces, and our restricted operator calculus. One can also compare the following corollary for \( v < 0 \) to the results in [10], but the results here do not imply the ones in [10], nor vice versa.

### 7.2. Pseudodifferential Operators

In this application we consider the forbidden class of pseudodifferential operators \( OpS^{0}_{1,1} \). They are defined as follows. We say \( \sigma \in S^{0}_{1,1} \) if

\[
|D_{\xi}^{\beta}D_{x}^{\alpha}\sigma(x, \xi)| \lesssim (1 + |\xi|)^{|\beta| - |\alpha|}
\]

for all \( \alpha, \beta \in \mathbb{N}^{n}_{0} \), and \( T_{\sigma} \in OpS^{0}_{1,1} \) is the associated operator defined

\[
T_{\sigma}f(x) = \int_{\mathbb{R}^{n}} \sigma(x, \xi) \hat{f}(\xi) e^{ix\xi} d\xi
\]

for \( f \in \mathcal{S} \). The reason \( OpS^{0}_{1,1} \) is referred to as a forbidden class, or sometimes an exotic class, of operators is because it is not closed under transposes, and they are not necessarily \( L^{2} \)-bounded. However, any \( T_{\sigma} \in OpS^{0}_{1,1} \) has a standard kernel and is bounded on several smooth function spaces. For instance, such \( T_{\sigma} \) is bounded on several classes of inhomogeneous Lipschitz, Sobolev, Besov, and Triebel-Lizorkin spaces; see for example [33, 38, 7, 24, 25, 43, 40, 12]. All of these inhomogeneous space estimates are obtained in the absence of vanishing moment assumptions. On the other hand, Meyer showed that under vanishing moment conditions \( T_{\sigma}(x^{\alpha}) = 0 \) for \( \sigma \in S^{0}_{1,1} \), \( T_{\sigma} \) is also bounded on homogeneous Lipschitz and Sobolev spaces; see [34]. Our next result provides more estimates along the lines of Meyer’s that require vanishing moments for the operator. We also note
that Bourdaud proved a noteworthy result in [7] about the largest sub-algebra of $\text{Ops}_{1,1}^0$. In particular, he showed that the subclass of $\text{Ops}_{1,1}^0$ made up of operators $T_\sigma \in \text{Ops}_{1,1}^0$ so that $T_\sigma^* \in \text{Ops}_{1,1}^0$ is an algebra and that all such operators are $L^2$-bounded. 

**Corollary 7.2.** Let $T_\sigma \in \text{Ops}_{1,1}^0$ and $L \in \mathbb{N}_0$. If $T_\sigma^*(\alpha^k) = 0$ for all $|\alpha| \leq L$, then Theorem 4.3, Theorem 5.1, Theorem 5.4, Corollary 5.5, and Corollary 5.6 can all be applied to $T_\sigma$. If $T_\sigma(x^\alpha) = 0$ for all $|\alpha| \leq L$, then the same results can be applied to $T_\sigma^*$.

Note that Corollary 7.2 does not require, nor imply, that $T_\sigma$ is bounded on $L^2$. In fact, there are standard constructions of operators to which we can apply Corollary 7.2 that are not $L^2$-bounded, as is shown in the next example.

It should also be noted here that even though $\text{Ops}_{1,1}^0$ is not closed under transposes, Corollary 7.2 applies to both $T_\sigma$ and its transpose for any $\sigma \in S_{1,1}^0$. This is because $S_{1,1}^0 \subseteq S\text{IO}_0(\infty)$ and $S\text{IO}_0(\infty)$ is closed under transposes. Hence for any $T_\sigma \in \text{Ops}_{1,1}^0$, both $T_\sigma, T_\sigma^* \in S\text{IO}_0(\infty)$, and so Corollary 7.2 is even capable of concluding operator estimates for operators that do not belong to $\text{Ops}_{1,1}^0$.

**Proof of Corollary 7.2.** It is well known that $\sigma \in S_{1,1}^0$ implies $T_\sigma \in S\text{IO}_0(\infty)$. That is, it is known that such $T_\sigma$ are continuous from $\mathcal{S}$ into $\mathcal{S}'$, and have a standard functional kernel $K(x,y)$. It is also easy to show that $|\langle T_\sigma f, g \rangle| \lesssim \|f\|_{L^1} \|g\|_{L^1}$ for all $f, g \in \mathcal{S}$, and so $T_\sigma$ trivially satisfies $\text{WBP}_0$. Recall that $S\text{IO}_0(\infty)$ and $\text{WBP}_0$ are closed under transposition, and the corollary easily follows.

**Example 7.3.** Let $\psi \in \mathcal{S}_\infty$ be such that $\hat{\psi}$ is supported in an annulus, and define

$$\sigma(x, \xi) = \sum_{k \in \mathbb{Z}} e^{-|2^k \xi|^2} \widehat{\psi}(2^{-k} \xi),$$

as well as the associated pseudodifferential operator $T_\sigma$. It is known that $\sigma \in S_{1,1}^0$ and hence $T_\sigma \in \text{Ops}_{1,1}^0$; see for example [40] for more details. It is easy to verify that $(T_\sigma^*)^*(\alpha^k) = T_\sigma(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$. Then Corollary 7.2 can be applied to $T_\sigma^*$. Furthermore, since $T_\sigma(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$, the restrictions involving $L$ can be removed entirely and one can allow $t > 0$ without bound. So $T_\sigma$ is bounded, for example, on $F^p_{p,w}$ for all $1 < p, q < \infty, t > 0$, and $w \in A_\infty$. Also, for every $s, t > 0$, there is an operator $T_{s,t} \in C\text{Z}_0(-s)$ such that $|\nabla|^{s} T_{s,t}^* |\nabla|^{t} f - T_{s,t} f$ is a polynomial for all $f \in \mathcal{S}_\infty$. In particular, $|\nabla|^{s} T_{s,t}^* |\nabla|^{t} - P_f$ and $|\nabla|^{s} T_{s,t}^* |\nabla|^{t} - \hat{P}_f$ are Calderón-Zygmund operators in $C\text{Z}_0$ for all $t > 0$, where $P_f$ and $\hat{P}_f$ are polynomials depending on $f$ and $t$.

### 7.3. Paraproducts

In this section, we consider a generalization of the Bony paraproduct, constructed originally in [6]. The crucial properties of this operator are, for a given $b \in \text{BMO}$, the Bony paraproduct $\Pi_b$ is a Calderón-Zygmund operator (in particular $L^2$-bounded), $\Pi_b 1 = b$, and $\Pi_b^* 1 = 0$. This operator played a crucial role in the proof of the $T_1$ theorem of David and Journé [14], and it has appear in many other places in various forms.

In this section, we construct paraproducts $\Pi_b^\alpha \in S\text{IO}_\nu$ for $b \in \dot{B}_{\infty,-\nu,\infty}^{|\alpha|^{-\nu}}, \alpha \in \mathbb{N}_0^n$, and $\nu \in \mathbb{R}$. They satisfy prescribed polynomial moment conditions, including $(\Pi_b^\alpha)^*(\alpha^k) = 0$ conditions, and hence are bounded on several negative smoothness distribution spaces. However, they need not (and some in fact do not) belong to $C\text{Z}_0$ or satisfy any continuous mapping properties into Lebesgue...
spaces. See the Corollary 7.4, Lemma 7.7, and the remarks at the end of Sections 7.3 and 7.4 for more on this.

Let \( v \in \mathbb{R}, \alpha \in \mathbb{N}_0^n, \) and \( b \in \mathcal{B}^{|\alpha| - v, \infty}_\infty. \) Also let \( \psi \) and \( \varphi \) be as in Lemma 2.6, and \( \varphi \in \mathcal{S} \) with integral 1. Define the paraproduct operator

\[
\Pi^\alpha_b f(x) = \frac{(-1)^{|\alpha|}}{\alpha!} \sum_{k \in \mathbb{Z}} \widetilde{Q}_k (Q_k b \cdot P_k D^\alpha f)(x).
\]

We can apply our results to these paraproducts as well, as is shown in the next corollary.

**Corollary 7.4.** Let \( v \in \mathbb{R}, \alpha \in \mathbb{N}_0^n, \) and \( b \in \mathcal{B}^{|\alpha| - v, \infty}_\infty. \) Then \( \Pi^\alpha_b \in \mathcal{SIO}_v(\infty) \) satisfies \( WBP_b \) and \( (\Pi^\alpha_b)^*(x^\beta) = 0 \) for all \( \beta \in \mathbb{N}_0^n \) where \( \Pi^\alpha_b \) is as in (7.1). Hence Theorem 4.3, Theorem 5.1, Theorem 5.4, Corollary 5.5, and Corollary 5.6 can all be applied to \( \Pi^\alpha_b \) with any number of vanishing moments.

**Proof.** It is trivial to see that \( \Pi^\alpha_b \) is continuous from \( \mathcal{S}_p \) into \( \mathcal{S}' \) for an appropriately chosen \( p \in \mathbb{N}. \) Indeed, taking \( M \in \mathbb{N} \) to be an even integer larger than \( |v|, \) and \( g \in \mathcal{S}, \) we have

\[
|\langle \Pi^\alpha_b f, g \rangle| \leq \|b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} 2^{(M + v)k} |(|\nabla|^M D^\alpha \varphi)_k| |(|\nabla|^{-M} f)(x)\widetilde{Q}_k g(x)| \, dx
\]

\[
\lesssim \|b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \||\nabla|^{-M} f\|_{L^2} \sum_{k \in \mathbb{Z}} 2^{(M + v)k} \|\widetilde{Q}_k g\|_{L^2} \lesssim \|b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \|f\|_{W^{-M, 2}} \|g\|_{\mathcal{B}^{M + v, 1}_\infty}.
\]

as long as \( M > |v| \) (which assures that \( \mathcal{S} \subset \mathcal{B}^{M + v, 1}_2 \) since \( M + v > 0 \)). For \( f \in \mathcal{S} \) and \( g \in \mathcal{S}_p \)

\[
|\langle \Pi^\alpha_b f, g \rangle| \leq \|b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} 2^{v k} |(D^\alpha \varphi)_k| f(x)\widetilde{Q}_k g(x) \, dx
\]

\[
\lesssim \|b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \|f\|_{L^2} \sum_{k \in \mathbb{Z}} 2^{v k} \|\widetilde{Q}_k g\|_{L^2} \lesssim \|b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \|f\|_{L^2} \|g\|_{\mathcal{B}^{v, 1}_2}.
\]

Here we choose \( p \in \mathbb{N} \) large enough so that \( \mathcal{D}_p \subset \mathcal{W}^{-M, 2} \cap \mathcal{B}^{v, 1}_2. \) Note that this is also sufficient to show that \( \Pi^\alpha_b \) and \( (\Pi^\alpha_b)^* \) both satisfy \( WBP_b. \) The kernel of \( \Pi^\alpha_b \) is

\[
\pi^\alpha_b(x, y) = \frac{(-1)^{|\alpha|}}{\alpha!} \sum_{k \in \mathbb{Z}} 2^{|k| \alpha} \int_{\mathbb{R}^n} \psi_k(x - u)Q_k b(u)(D^\alpha \varphi)_k(u - y) \, du.
\]

For \( \beta, \mu \in \mathbb{N}_0^n \) and \( x \neq y, \) it follows that

\[
|D_x^\beta D_y^\mu \pi^\alpha_b(x, y)| \lesssim \|Q_k b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \sum_{k \in \mathbb{Z}} 2^{(v + |\beta| + |\mu|)k} \Phi_k^{n + v + |\beta| + |\mu| + 1}(x - y)
\]

\[
\lesssim \|Q_k b\|_{\mathcal{B}^{|\alpha| - v, \infty}_\infty} \frac{1}{|x - y|^{n + v + |\beta| + |\mu|}}.
\]

Therefore \( \Pi^\alpha_b \in \mathcal{SIO}_v(\infty). \) It is easy to see that \( (\Pi^\alpha_b)^*(x^\alpha) = 0 \) since \( \widetilde{\psi}_k \in \mathcal{S}_\infty. \)

**Remark 7.5.** It is worth noting that \( \Pi^\alpha_b \) may not belong to \( CZO_v, \) but we still conclude many boundedness results for \( \Pi^\alpha_b. \) In particular, when \( v = 0 \) and \( b \in \mathcal{B}^{0, \infty}_\infty \setminus \mathcal{BMO}, \) the \( T1 \) theorem implies that \( \Pi^0_b \) is not \( L^2 \)-bounded. However, we still conclude boundedness results for \( \Pi^0_b \) on negative smoothness spaces, just not on any Lebesgue spaces. It is likely, by analogy, that \( b \in \mathcal{B}^{0, \infty}_\infty \setminus \mathcal{BMO} \) happens more often than not.\]
\(B^\nu_{\infty} \|_{\nu, \infty} I_{\nu} (BMO)\) implies that \(\Pi^\alpha\) is not bounded from \(W^{\nu, 2}\) into \(L^2\) (and hence \(\Pi^\alpha_p\) would not belong to \(CZO_{\nu}\)), but we don’t pursue this property here.

7.4. Smooth and Oscillating Operator Decompositions. In this application, we decompose a singular integral operator \(T\) into a sum of two terms \(S + O\), one that preserves smoothness and one that preserves oscillatory properties of the input function. We achieve this by constructing several paraproducts which satisfy \(\Pi^* (x^\alpha) = 0\) for all \(\alpha\). A sum of such operators defines \(O\), and hence \(O\) enjoys all of the oscillatory preserving properties associated to the cancellation conditions of the form \(O^* (x^\alpha) = 0\). Furthermore, \(S\) will be constructed so that \(S(x^\alpha) = 0\) for appropriate multi-indices \(\alpha\), which is sufficient for \(S\) to be bounded on many smooth function spaces.

To motivate this, let’s consider for a moment an operator \(T \in SIO_{\nu} (\infty)\) of convolution type. That is, assume there is a distribution kernel \(k \in \mathcal{S}'(\mathbb{R}^n)\) such that \(Tf(x) = \langle k, f(x - \cdot) \rangle\) for \(f \in \mathcal{S}_p\) for some \(P \in \mathbb{N}_0^n\). Such an operator preserves both regularity and oscillation since convolution operators commute. For instance, suppose that \(T\) is bounded from \(X\) into \(Y\), where \(X, Y \subset \mathcal{S}' / \mathcal{S}\) are Banach spaces. It follows that \(T\) is bounded from \(I_s(X)\) into \(I_s(Y)\) for all \(s \in \mathbb{R}\), where \(I_s(X) = \{\|\nabla^s f : f \in X\}\) with the natural norm \(\|f\|_{I_s(X)} = \|\|\nabla^s f\|\|_X\). This is because

\[
\|Tf\|_{I_s(X)} = \|\|\nabla|^s(Tf)\|_X = \|T(\|\nabla|^s f)\|_X \lesssim \|\nabla|^s f\|_Y = \|f\|_{I_s(Y)}.
\]

When \(s > 0\), this says that if \(f\) has \(s\)-order derivatives in \(X\), then \(Tf\) has \(s\)-order derivatives in \(Y\). For \(s < 0\), it says that \(f\) has \(-s\)-order anti-derivatives in \(X\), then \(Tf\) has \(-s\)-order anti-derivatives in \(Y\), which in many situations quantify oscillatory properties of \(f\) and \(Tf\). Hence convolution operators simultaneously preserve both regularity and oscillatory properties of its input functions. This cannot be expected for non-convolution operators, but the main result of this section formulates a decomposition that extends this principle to non-convolution operators in some sense. We show, roughly, that for any operator \(T \in SIO_{\nu}\), we can decompose \(T = S + O\), where \(S\) preserves smoothness and \(O\) preserves oscillation. This is achieved by constructing \(S\) and \(O\) so that \(S|\nabla|^s \approx |\nabla|^s S\) and \(O|\nabla|^{-s} \approx |\nabla|^{-s} O\) for \(s > 0\), in the appropriate sense, so that \(S\) and \(O\) each behave like a convolution operator on one side. Based on our operator calculus from Section 4, to construct \(S\) and \(O\) in this way, it is sufficient to make sure that \(S(x^\alpha) = 0\) and \(O^* (x^\alpha) = 0\) for appropriate \(\alpha\). This is made precise below.

We will have to use the non-convolutional moment for singular integral operators, which were defined for \(SIO_0\) in \([22, 23]\). For \(T \in SIO_{\nu} (M + \gamma)\), \(\alpha \in \mathbb{N}_0^n\) with \(|\alpha| \leq \nu + M\), define \([T]_{\alpha} \in \mathcal{D}_2 P\) by

\[
\langle [T]_{\alpha} \psi, \psi \rangle = \lim_{R \to \infty} \int_{\mathbb{R}^{2n}} K(x, y) (x - y)^{\alpha} \eta_R(y) \psi(x) dy dx.
\]

Following the same argument used to justify the definition of \(T(x^\alpha)\), we can define \([T]_{\alpha} \in \mathcal{D}_2 P\) for the same ranges of indices. See also \([22, 23]\) for more information on this construction.

**Lemma 7.6.** Let \(T \in SIO_{\nu} (M + \gamma)\) and \(L \leq \nu + M\). Then \(T(x^\alpha) = 0\) for all \(|\alpha| \leq L\) if and only if \([T]_{\alpha} = 0\) for all \(|\alpha| \leq L\).

**Proof.** This follows immediately from the following formula, which is just expanding the polynomial \((x - y)^\alpha\). Let \(|\alpha| \leq L\), \(\eta_R \in \mathcal{D}_2 P\) and \(\psi \in \mathcal{D}_p\), for \(P \in \mathbb{N}_0\) sufficiently larger, be as in
the definition of $\langle [T]_{\alpha}, \psi \rangle$, and we have
\[
\langle [T]_{\alpha}, \psi \rangle = \lim_{R \to \infty} \sum_{\beta + \mu = \alpha} \int_{\mathbb{R}^{2n}} \mathcal{K}(x,y)y^\beta \eta_R(y)x^\mu \psi(x)dydx = \sum_{\beta + \mu = \alpha} \int_{\mathbb{R}^{2n}} \mathcal{K}(x,y)y^\beta \eta_R(y)x^\mu \psi(x)dydx.
\]
It immediately follows that $T(x^\alpha)$ vanishes for all $|\alpha| \leq L$ if and only if $\langle [T]_{\alpha} \rangle$ does.

**Lemma 7.7.** Let $v \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}_0$, $b \in B_\infty^{[\alpha] - \nu, \infty}$, and $\Pi_b^\alpha$ be defined as in (7.1). Then $\langle \Pi_b^\alpha \rangle, \beta \in B_\infty^{[\beta] - \nu, \infty}$ for all $\beta \in \mathbb{N}_0$, $\Pi_b^\alpha(x^\beta) = 0$ for all $|\beta| \leq |\alpha|$ with $\beta \neq \alpha$, and $\langle \Pi_b^\alpha \rangle, \alpha = b$.

**Proof.** For any $\beta \in \mathbb{N}_0$ and $\psi \in \mathcal{D}_p$ with $P$ sufficiently large, we have
\[
|\psi_j \ast \langle \Pi_b^\alpha \rangle, \beta(x) | = \lim_{R \to \infty} \left| \int_{\mathbb{R}^{2n}} \pi_b^\alpha(u,y)(u-y)^\beta \eta_R(y)\psi_j(u)dydu \right|
\]
\[
\leq \limsup_{R \to \infty} \sum_{\rho + \mu = \beta} c_{\rho, \mu} \sum_{k \in \mathbb{Z}} 2^{|\alpha|} \left| \int_{\mathbb{R}^{2n}} \mathcal{D}^\alpha \phi_k(v-y)(v-y)^\mu \eta_R(y)dy \right|
\]
\[
\times \mathcal{Q}_k b(v) \mathcal{Q}_k^\alpha u(\nu - v)^\rho \psi_j(u)dydvdu
\]
\[
\leq \limsup_{R \to \infty} \sum_{\rho + \mu = \beta} c_{\rho, \mu} \sum_{k \in \mathbb{Z}} 2^{|\alpha|} \left| \int_{\mathbb{R}^{2n}} \mathcal{D}^\alpha \phi_k(v-y)(v-y)^\mu \eta_R(y)dy \right|
\]
\[
\times |\mathcal{Q}_k b(v)||\mathcal{Q}_k^\alpha u^\nu dvdu
\]
\[
\lesssim \|b\|_{B_\infty^{[\beta] - \nu, \infty}} \sum_{k \in \mathbb{Z}} 2^{-K|j-k|} 2^{|\nu - \beta|} \lesssim 2^{(\nu - |\beta|)j} \|b\|_{B_\infty^{[\beta] - \nu, \infty}},
\]
where $\mathcal{Q}_k^\beta(x) = \mathcal{Q}_k(x)^\beta$. Then $\langle \Pi_b^\alpha \rangle, \beta \in B_\infty^{[\beta] - \nu, \infty}$ for all $\beta \in \mathbb{N}_0$. For $\psi \in \mathcal{D}_p$, with $P \in \mathbb{N}_0$ fixed sufficiently large, we have
\[
\langle \Pi_b^\alpha \rangle, \psi = \lim_{R \to \infty} \int_{\mathbb{R}^{2n}} \pi_b^\alpha(u,y)(u-y)^\alpha \eta_R(y)\psi(x)dydx
\]
\[
= \lim_{R \to \infty} \frac{(-1)^{|\alpha|}}{\alpha!} \sum_{k \in \mathbb{Z}} 2^{|\alpha|} \left( \int_{\mathbb{R}_n} \mathcal{D}^\alpha \phi_k(u-y)(u-y)^\alpha \eta_R(y)dy \right)
\]
\[
\times \mathcal{Q}_k b(u) \mathcal{Q}_k^\alpha \psi(x)du
\]
\[
= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \mathcal{Q}_k(u-x) \mathcal{Q}_k b(u) \psi(x)du
\]
Similarly, for $\psi \in \mathcal{D}_p$ with $P$ sufficiently large and $|\beta| \leq |\alpha|$ such that $\alpha \neq \beta$
\[
\langle \Pi_b^\alpha(x^\beta) \rangle, \psi = \lim_{R \to \infty} \frac{1}{\alpha!} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}_n} \mathcal{Q}_k b(x) \left( \int_{\mathbb{R}_n} \Phi_k(x-y) \mathcal{D}^\alpha \eta_R(y)y^\beta dy \right) \psi(x)dx
\]
\[
= \frac{1}{\alpha!} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}_n} \mathcal{Q}_k b(x) \left( \int_{\mathbb{R}_n} \Phi_k(x-y) \mathcal{D}^\alpha \psi(x)dy \right) \psi(x)dx = 0.
\]
Note that $|\beta| \leq |\alpha|$ and $\beta \neq \alpha$ implies that $\mathcal{D}^\alpha \psi(x) = 0$. \hfill \Box
Theorem 7.8. Let $T \in SIO_{\nu}(M + \gamma)$ for some $M \in \mathbb{N}_0$ and $0 < \gamma \leq 1$. If $[[T]]_{\alpha} \in B^{|\alpha| - \nu, \infty}_0$ for all $|\alpha| \leq M$, then there exist operators $S \in SIO_{\nu}(M + \gamma)$ and $O \in SIO_{\nu}(\infty)$ such that $T = S + O$, $S(x^\alpha) = 0$ for $|\alpha| \leq M$, and $O^*(x^\alpha) = 0$ for $\alpha \in \mathbb{N}_0^n$. Furthermore, if $T$ satisfies WBP, then both $S$ and $O$ satisfy WBP, in which case Theorem 4.3, Theorem 5.1, Theorem 5.4, Corollary 5.5, and Corollary 5.6 can be applied to $S^*$ and $O$.

Note that the results from Sections 4 and 5 can be applied to $O$ regardless of whether $T$ satisfies WBP since $O$ is a sum of paraproducts of the form (7.1) and by Corollary 7.4 the results can be applied to each of these paraproducts.

Proof of Theorem 7.8. Let $T$ be as above. Define $b_0 = [[T]]_0 = T(1) \in \dot{B}^{-\nu, \infty}_0$. For $1 \leq |\alpha| \leq M$, define

$$b_\alpha = [[T]]_\alpha - \sum_{|\beta| < |\alpha|} \left[ \Pi^\beta_{b_\beta} \right]_\alpha \in B^{|\alpha| - \nu, \infty}_0.$$ 

Also define

$$S = T - \sum_{|\alpha| \leq M} \Pi^\alpha_{b_\alpha} \quad \text{and} \quad O = \sum_{|\alpha| \leq M} \Pi^\alpha_{b_\alpha}.$$ 

It immediately follows that $S \in SIO_{\nu}(M + \gamma)$ and $O \in SIO_{\nu}(\infty)$. Using Lemmas 7.6 and 7.7, we also have

$$[[S]]_0 = [[T]]_0 - \sum_{|\alpha| \leq M} \left[ [[\Pi^\alpha_{b_\alpha}]_0 = [[T]]_0 - \left[ [[\Pi^\alpha_{b_\alpha}]_0 = 0,$$

and for $0 < |\alpha| \leq M$ we have

$$[[S]]_\alpha = [[T]]_\alpha - \sum_{|\beta| \leq M} \left[ [[\Pi^\beta_{b_\beta}]_\alpha = [[T]]_\alpha - b_\alpha - \sum_{|\beta| < |\alpha|} \left[ [[\Pi^\beta_{b_\beta}]_\alpha = 0.$$

By Lemma 7.6, it follows that $S(x^\alpha) = 0$ for all $|\alpha| \leq M$. It also follows from Corollary 7.4 that $O^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$. Note also that $O$ always satisfies the WBP, and if $T$ satisfies WBP, then so does $S$.

Remark 7.9. It may seem a little strange that we use the non-convolution moments $[[T]]_{\alpha}$, rather than $T(x^\alpha)$, in Lemma 7.7 and Theorem 7.8. By Lemma 7.7, when using vanishing moment conditions the two are equivalent. However, there is a crucial difference when the moments are not required to vanish, but instead some other conditions as we do in Theorem 7.8. This difference manifests in our setting when computing $[[\Pi^\alpha_{b_\alpha}]_\beta$ versus $\Pi^\alpha_{b_\alpha}(x^\beta)$ for $|\beta| > |\alpha|$. In Lemma 7.7, we showed that $[[\Pi^\alpha_{b_\alpha}]_\beta \in B^{\nu, \infty}_0$, which is the natural condition to expect in this setting. However, it may not be the case that $\Pi^\alpha_{b_\alpha}(x^\beta) \in B^{\nu, \infty}_0$. To demonstrate this, let the dimension $n = 1$, $\nu = 0$, $\alpha = 0$, $b \in \dot{B}^{0, \infty}_0$, and $\beta = 1$. Then for $x \in \mathbb{R}$

$$|\psi_j \ast \Pi^0_{b_\alpha}(x^\beta)(x)| \leq \lim_{R \to \infty} \left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^2} \phi_k(u - y)\eta_R(y)Q_k(b(u)\tilde{\psi}_k \ast \psi_j^R(u)du dy \right|$$

$$\geq |x| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} Q_k(b(u)\tilde{\psi}_k \ast \psi_j^R(u)du - |\psi_j \ast \Pi^0_{b_\alpha}(x)|$$
\[
\geq |x| |\psi_j \ast b(x)| - |\psi_j \ast [\Pi_0^b]|(x)|.
\]
Taking \(j = 0\), we have
\[
\|\Pi_0^b(x)\|_{B_1^{1,\infty}} \geq |x| |\psi \ast b(x)| - \|\Pi_0^b\|_{B_1^{1,\infty}}.
\]
By Lemma 7.7, \(\|\Pi_0^b\|_{B_1^{1,\infty}} < \infty\), and it is not hard to construct \(b \in B_0^{0,1}\) such that \(|x| |\psi \ast b(x)|\) is unbounded (for example, \(b(x) = \sin(x)\) would do). Hence \(\Pi_0^b(x) \notin B_0^{1,\infty}\) for such \(b\). Similar constructions can be done in any dimension and for \(b \in B_0^{[\alpha]-v,\infty}\) to produce the property \(\Pi_0^\alpha(x^\theta) \notin B_0^{[\beta]-v}\).

### 7.5. Sparse Domination

There has been a lot of interest lately in sparse domination results for Calderón-Zygmund operators. The first such result is due to Lerner [30], but there have been many extensions and improvements; see for example [35, 9, 5, 13, 27, 28, 29, 45]. However, there do not appear to be any results that apply to regularity estimates for operators or to hyper-singular extensions and improvements; see for example \([35, 9, 5, 13, 27, 28, 29, 45]\). However, there do not appear to be any results that apply to regularity estimates for operators or to hyper-singular operators. There are some sparse estimates for fractional integral operators, for example in \([35]\).

We will apply the sparse domination result from \([9]\), which can be summarized as follows. For a collection of dyadic cubes \(S\), define the dyadic operator
\[
\mathcal{A}_Sf(x) = \sum_{Q \in S} \langle f \rangle_Q K_Q(x),
\]
where \(\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(x)dx\). The collection of dyadic cubes belonging to the same dyadic grid \(S\) is sparse if for every \(Q \in S\) there exists a measurable subset \(E(Q) \subset Q\) with \(|E(Q)| > |Q|/2\) and \(E(Q) \cap E(Q') = \emptyset\) for ever \(Q' \in S\) with \(Q' \subset Q\). They prove that if \(T \in CZO_0\) and \(f\) is an integrable function supported in a cube \(Q_0\), then there exist sparse dyadic collections \(S_1, \ldots, S_{3^n}\) (possibly associated to different dyadic grids) such that
\[
|Tf(x)| \lesssim \sum_{i=1}^{3^n} \mathcal{A}_{S_i}(|f|)(x)
\]
almost everywhere on \(Q_0\). We use this result to prove our next result.

**Corollary 7.10.** Let \(v \in \mathbb{R}\), \(L\) be an integer with \(L \geq |v|\), \(M \geq \max(L, L-v)\), \((L-v)_+ < \gamma \leq 1\), and \(T \in SIO_v(M + \gamma)\) satisfies \(WBP_v\).

- Assume that \(T^*(x^{\alpha}) = 0\) for all \(|\alpha| \leq L\). Then for any \(1 < p < \infty\), \(0 < t < \tilde{L} + \gamma\), cube \(Q_0 \subset \mathbb{R}^n\), and \(f \in W^{-t,p}\) with \(\text{supp}(|\nabla|^{-t}f) \subset Q_0\), there is a polynomial \(P_f\) and sparse collections of dyadic cubes \(S_1, \ldots, S_{3^n}\) such that
\[
|\nabla|^{-(v+i)}Tf(x) - P_f(x)| \lesssim \sum_{i=1}^{3^n} \mathcal{A}_{S_i}(|\nabla|^{-i}f))(x) \quad \text{a.e. } x \in Q_0.
\]

- Assume that \(T(x^{\alpha}) = 0\) for all \(|\alpha| \leq L\). Then for any \(1 < p < \infty\), \(0 < t < \tilde{L} + \gamma\), cube \(Q_0 \subset \mathbb{R}^n\), and \(f \in W^{v+t,p}\) with \(\text{supp}(|\nabla|^{v+t}f) \subset Q_0\), there is a polynomial \(P_f\) and sparse collections of dyadic cubes \(S_1, \ldots, S_{3^n}\) such that
\[
|\nabla|^tTf(x) - P_f(x)| \lesssim \sum_{i=1}^{3^n} \mathcal{A}_{S_i}(|\nabla|^t f))(x) \quad \text{a.e. } x \in Q_0.
\]
Hence

According to the definition of the operators we defined in Corollary 3.2, which we work with operators in singularities. In order to make our notation and computations a little simpler here, we will only 7.6. Operator Calculus. Throughout this article, we have been working with a restricted operator calculus, where we only consider compositions of the form $|\nabla|^{-\alpha} T |\nabla|^\alpha$. In this application, we construct a true operator calculus (or operator algebra) made up of singular integrals with different singularities. In order to make our notation and computations a little simpler here, we will only work with operators in $\text{SIO}_\nu(\infty)$ that satisfy $T(x^\alpha) = T^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$. These assumptions are necessary for some of the algebras we construct, but not for all. Before we continue, we will need some additional information about the operators we defined in Corollary 3.2, which we provide in the next lemma.

**Lemma 7.11.** Let $\nu \in \mathbb{R}$ and $T \in \text{SIO}_\nu(\infty)$. Assume that $T(x^\alpha) = T^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$ and $T \in \text{WBP}_\nu$. Fix $L \in \mathbb{N}_0$, $\psi, \tilde{\psi} \in \mathcal{D}_P$ for $P$ sufficiently large, and define $\lambda_{j,k}(x,y) = \langle T^* \psi_j^\alpha, \tilde{\psi}_k^\alpha \rangle$ for $j, k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \lambda_{j,k}(x,y)x^\alpha dx = \int_{\mathbb{R}^n} \lambda_{j,k}(x,y)y^\alpha dx = 0$$

for all $|\alpha| \leq L$.

**Proof.** Let $L \in \mathbb{N}_0$, $|\alpha| \leq L$, and $\eta \in \mathcal{D}_P$ with $\eta = 1$ on $B(0,1)$ and $\eta_R(x) = \eta(x/R)$, where $P$ is chosen sufficiently large. By Theorem 3.1, we have

$$\int_{\mathbb{R}^n} \lambda_{j,k}(x,y)y^\alpha dy = \int_{\mathbb{R}^n} \langle T \psi_j^\alpha, \tilde{\psi}_k^\alpha \rangle y^\alpha dy = \lim_{R \to \infty} \langle T \psi_j^\alpha, F_{\alpha,R,k} \rangle,$$

where

$$F_{\alpha,R,k}(u) = \int_{|y|<R} \psi_k(y-u)y^\alpha dy.$$

It follows that $\text{supp}(F_{\alpha,R,k}) \subset B(0, R+2^{k+1}) \setminus B(0, R-2^{k+1})$ and $F_{\alpha,R,k} \in \mathcal{D}_P$. Then as long as $R > 4|x| + 2^{k+4} + 2^{j+4}$, it follows that

$$|\langle T \psi_j^\alpha, F_{\alpha,R,k} \rangle| = \left| \int_{\mathbb{R}^{2n}} (K(u,v) - J^M_x[K(\cdot,v)](u)) \psi_j^\alpha(u) F_{\alpha,R,k}(v) du dv \right|$$
which tends to zero as \( R \to \infty \). Here we take \( M > L + |v| + \gamma \) and \( P > M \); note that \( P \) then depends on \( L \), and so we cannot completely remove the restriction \( |\alpha| \leq L \) in the statement of Lemma 7.11. Therefore the first integral condition in (7.5) holds for \(|\alpha| \leq L \), and by symmetry the same holds for the second one.

**Theorem 7.12.** Let \( V \subset \mathbb{R} \) be a set that is closed under addition. Then the collection of operators

\[
\mathfrak{A}_V = \{ T \in SIO_v(\infty) : v \in V, \, T(x^\alpha) = T^*(x^\alpha) = 0, \, T \in WBP_v \}
\]

is an operator algebra in the sense that it is closed under composition and transpose.

**Proof.** Fix two real numbers \( v_1, v_2 \in V \). In order to show \( S, T \in \mathfrak{A}_V \) implies \( S \circ T \in SIO_{v_1 + v_2}(\infty) \) where \( T \in SIO_{v_1} \) and \( S \in SIO_{v_2} \), we must first show that \( S \circ T \) and \( (S \circ T)^* \) are defined as (or can be extended to) operators from \( \mathcal{S}_P \) into \( \mathcal{S}' \) for some \( P \in \mathbb{N}_0 \) sufficiently large. We first note that the vanishing moment and weak boundedness properties of operators in \( \mathfrak{A}_V \) imply that all members of \( \mathfrak{A}_V \cap SIO_v \) are bounded from \( \dot{W}^{\nu+\gamma} \) into \( \dot{W}^{\nu} \) for all \( s \in \mathbb{R} \) such that \( s < -v \) and for \( s > 0 \) by Theorem 5.4 and Corollary 5.5, respectively. For \( f \in \mathcal{S}_P \) and \( g \in \mathcal{S} \) (with \( P \geq |v_1| + |v_2| \)), we note that for any \( 1 < p < \infty \)

\[
\| S \circ T f, g \| = \| T f, S^* g \| \leq \| T f \|_{\dot{W}^{-\mu,p}} \| S^* g \|_{\dot{W}^{\nu,p'}}
\]

where \( \mu = \max(v_1, -v_2) \). Note that \( T \) is bounded from \( \dot{W}^{\nu_1-\nu,p} \) into \( \dot{W}^{-\mu,p} \) since \( -\mu < -v_1 \), and \( S^* \) is bounded from \( \dot{W}^{\nu_2+\mu,p'} \) into \( \dot{W}^{\nu_2+\mu,p'} \) since \( v_2 + \mu > 0 \). Since \( \mathcal{S}_P \) and \( \mathcal{S} \) embed continuously into \( \dot{W}^{-\mu,p} \) and \( \dot{W}^{\nu_2+\mu,p'} \), respectively, it follows that \( S \circ T \) is continuous from \( \mathcal{S}_P \) into \( \mathcal{S}' \). By symmetry, it follows that \( (S \circ T)^* \) is also continuous from \( \mathcal{S}_P \) into \( \mathcal{S}' \). Furthermore, this inequality, and a similar one for \((S \circ T)^*\), imply that \( S \circ T \) satisfies the \( WBP_{v_1 + v_2} \).

Next we show that \( S \circ T \) has a standard kernel. Let \( \psi \) and \( \tilde{\psi} \) be as in Lemma 2.6, \( Q_k f = \psi_k * f \), and \( \tilde{Q}_k f = \tilde{\psi}_k * f \). For \( f, g \in \mathcal{D}_p \), we have

\[
\langle S \circ T f, g \rangle = \sum_{j,k,\ell,m \in \mathbb{Z}} \langle \tilde{Q}_j \bar{Q}_k T \tilde{Q}_m \bar{Q}_\ell f, g \rangle = \sum_{j,k,\ell,m \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \omega_{j,k,\ell,m}(x,y) f(y) g(x) dy dx,
\]

where

\[
\omega_{j,k,\ell,m}(x,y) = \int_{\mathbb{R}^{2n}} \bar{\psi}_j(u-w) \bar{\omega}_{j,k,\ell,m}(w,\xi) \tilde{\psi}_k(\xi-x) dz d\xi dw du
\]
and

$$\tilde{\omega}_{j,k,\ell,m}(w, \xi) = \int_{\mathbb{R}^n} \lambda^T_{j,\ell,m}(w,z) \tilde{\psi}_m(z-y) \tilde{\psi}_k(u-v) \lambda^N_{\ell,j}(v, \xi) dz dv du.$$ 

For any fixed $K, N \geq 0$, it follows from Corollary 3.2 and Lemma 7.11, as well as similar arguments to those in the proof of Corollary 3.2, that

$$|\tilde{\omega}_{j,k,\ell,m}(w, \xi)| \lesssim 2^{N_1\min(\ell,m)+N_2\min(j,k)} 2^{-K\max(|j-k|,|k-\ell|,|\ell-m|)} \Phi^N_{\min(j,k,\ell,m)}(w - \xi).$$

From this bound, it easily follows that

$$|\omega_{j,k,\ell,m}(x,y)| \lesssim 2^{N_1\min(\ell,m)+N_2\min(j,k)} 2^{-K\max(|j-k|,|k-\ell|,|\ell-m|)} \Phi^N_{\min(j,k,\ell,m)}(x - y).$$

Then for any $\alpha, \beta \in \mathbb{N}_0^n$ and $x \neq y$, we have

$$\sum_{j,k,\ell,m \in \mathbb{Z}} |D^\alpha_y D^\beta_x \omega_{j,k,\ell,m}(x,y)|$$

$$\lesssim \sum_{j,k,\ell,m \in \mathbb{Z}} 2^{(v_1+|\alpha|+|\beta|)j+2} 2^{-K\max(|j-k|,|k-\ell|,|\ell-m|)} \Phi^N_{\min(j,k,\ell,m)}(x - y)$$

$$\lesssim \sum_{j \in \mathbb{Z}} 2^{(v_1+|\alpha|+|\beta|)j} \Phi^N_j(x - y) \lesssim \frac{1}{|x - y|^{n+v_1+v_2+|\alpha|+|\beta|}}.$$ 

These sums converges as long as $K > 4(|v_1| + |v_2| + |\alpha| + |\beta|)$ and $N > n + v_1 + v_2 + |\alpha| + |\beta|$. It follows that

$$K_{S \circ T}(x,y) = \sum_{j,k,\ell,m \in \mathbb{Z}} \omega_{j,k,\ell,m}(x,y)$$

is the kernel of $S \circ T$ and is a $(v_1 + v_2)$-order standard kernel that is $C^\infty$ off the diagonal. Therefore $S \circ T \in SIOV_{v_1+v_2}(\infty)$. Using the estimate for $\omega_{j,k,\ell,m}$ above, for $f, g \in \mathcal{S}_\infty$ and $t \in \mathbb{R}$ we have

$$|\langle S \circ T f, g \rangle| \leq \sum_{j,k,\ell,m \in \mathbb{Z}} 2^{t(m-j)} \left| \langle \tilde{\omega}_{j,k,\ell,m} T_{Q_m} \tilde{Q}_m'(|\nabla|^{-t} f), \tilde{\omega}_{j,k,\ell,m} T_{Q_m} \tilde{Q}_m'(|\nabla|^{-t} g) \rangle \right|$$

$$\lesssim \sum_{j,k,\ell,m \in \mathbb{Z}} 2^{t(m-j)} \int_{\mathbb{R}^{2n}} \left| \tilde{\omega}_{j,k,\ell,m}(x,y) \tilde{Q}_m'(|\nabla|^{-t} f)(y) \tilde{Q}_j'^{-t}(|\nabla|^{-t} g)(x) \right| dy dx$$

$$\lesssim \sum_{j,k,\ell,m \in \mathbb{Z}} 2^{t(v_1+v_2)m} 2^{-K\max(|j-k|,|k-\ell|,|\ell-m|)} \mathcal{M} \left( \tilde{Q}_m'(|\nabla|^{-t} f) \right)(x) \tilde{Q}_j'^{-t}(|\nabla|^{-t} g)(x) dx$$

$$\lesssim \|f\|_{W^{v_1+v_2+t,p}} \|g\|_{W^{-t,p'}}.$$ 

In this estimate, we fix a $P \in \mathbb{N}_0^n$ large enough depending on $t$. Therefore $S \circ T$ can be extended to a bounded linear operator from $W^{v_1+v_2+t,p}$ into $W^{t,p}$ for all $t \in \mathbb{R}$ and $1 < p < \infty$. Taking $t = 0$, we see that $S \circ T \in CZOV_{v_1+v_2}(\infty)$, and we can apply Theorem 6.3, which implies that $(S \circ T)^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$. By symmetry $S \circ T(x^\alpha) = 0$ for all $\alpha \in \mathbb{N}_0^n$ as well. Also $v_1 + v_2 \in V$, and so $S \circ T \in \mathcal{A}_V$ which verifies that $\mathcal{A}_V$ closed under composition as long as $V$ is closed under addition. Since $SIOV(\infty), WBP_V,$ and the condition $T(x^\alpha) = T^*(x^\alpha) = 0$ are all symmetric under transposition, it
is obvious that $\mathcal{A}_V$ is closed under transposes too. Therefore $\mathcal{A}_V$ is an algebra that is closed under composition and transposes.

\[ \square \]

**Remark 7.13.** Theorem 7.12 defines many operator algebras for different classes of singular integral operators. If one takes $V = \{0\}$, then $\mathcal{A}_{\{0\}}$ is a set of Calderón-Zygmund operators and one of the operator algebras discussed by Coifman and Meyer in [12]. Some other interesting examples of algebras $\mathcal{A}_V$ can be constructed by taking $V$ to be, for example, $V = \mathbb{R}$ or $V = \{\mu \nu : \nu \in \mathbb{Z}\}$ for some fixed $\mu \in \mathbb{R}$. One can also modify any of these examples by replacing $V$ with $V \cap (0, \infty)$, $V \cap (-\infty, 0)$, or $V \cap (-\infty, 0]$, which amounts to restricting an algebra to differential or fractional integral operators (strictly or including order-zero operators). Furthermore, one can combine different algebras by defining $\mathcal{A}_V = \mathcal{A}_{V_1} + \mathcal{A}_{V_2}$, where $V \subset \mathbb{R}$ is the smallest set that is closed under addition and contains $V_1 \cup V_2$. Using this convention, we can consider the set $V = \{\mu_1 - \mu_2 \pi : \mu_1, \mu_2 \in \mathbb{Q}, \mu_1 \geq 0, \mu_2 \geq 0, \mu_1 \cdot \mu_2 \neq 0\} \subset \mathbb{R}$, which is closed under addition. Since $\pi$ is transcendental, it follows that $0 \notin V$. This generates a somewhat peculiar example of $\mathcal{A}_V$ since with this selection of $V$, it follows that $\mathcal{A}_V$ is an operator that has both derivative and fractional integral operators, but does not contain any Calderón-Zygmund operators (order zero operators). Of course many other examples can be generated by selecting $V$ in different ways, and even further understand $\mathcal{A}_V$ through the algebraic properties of $V$.

Note that the properties imposed on the operators in $\mathcal{A}_V$ imply that if $\mathcal{A}_V \cap SIO_v \subset CZO_v$. Also note that any algebra $\mathcal{A}_V$ contains all convolutions operators in $CZO_v(\infty)$ as long as $v \in V$. That is

$$\bigcup_{v \in V} \{T \in CZO_v(\infty) : T \text{ is a convolution}\} \subset \mathcal{A}_V \subset \bigcup_{v \in V} CZO_v(\infty).$$

Furthermore, it is not hard to see that both inclusions here are strict for non-empty sets $V \subset \mathbb{R}$.

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