HYPERBOLICITY VIA ADMISSIBILITY IN DELAY EQUATIONS

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Abstract. We show that the notion of an exponential dichotomy for a nonautonomous linear delay equation can be characterized in terms of an admissibility property. After introducing Banach spaces where we take the perturbations of the linear equation and where we look for the solutions, we show that the existence of an exponential dichotomy is equivalent to the admissibility of this pair of spaces. We consider the general cases of delay equations satisfying the Carathéodory conditions and of noninvertible evolution families defined by the linear equations. Moreover, we consider delay equations on the line and on the half-line.

1. Introduction

We show in this paper that the notion of an exponential dichotomy for a nonautonomous linear delay equation can be characterized in terms of an admissibility property. After introducing Banach spaces where we take the perturbations of the linear equation and where we look for the solutions, we show that the existence of an exponential dichotomy is equivalent to the admissibility of this pair of spaces (see Section 1.1 for details).

We consider the general cases of delay equations satisfying the Carathéodory conditions in the theory of ordinary differential equations—and so also solutions in the sense of Carathéodory—and of noninvertible evolution families defined by the linear equations. This lack of invertibility other than along the unstable spaces creates a major difficulty in showing that if the admissibility property is satisfied, then there exist subspaces along which the dynamics is invertible. Moreover, we consider both delay equations on the line and on the half-line.

1.1. Hyperbolicity via admissibility. The notion of admissibility, essentially introduced by Perron in [18], referred originally (in the context of ordinary differential equations) to the existence and uniqueness of bounded solutions of the perturbations

\[ v' = A(t)v + g(t) \] (1)

of a linear equation

\[ v' = A(t)v \] (2)

for any bounded continuous function \( g \). This property can be used to deduce the stability or the conditional stability of a given linear equation under sufficiently small linear or nonlinear perturbations. Incidentally, with [18]
Perron not only contributed to prepare the ground for the notion of an exponential dichotomy, but also for the stable manifold theorem.

For our purposes it suffices to recall a particular result from [18] (or, more precisely, a simple consequence of one of the results). Let $A(t)$ be $n \times n$ matrices varying continuously with $t \in \mathbb{R}$.

**Theorem 1.** If equation (1) has at least one bounded solution on $\mathbb{R}_0^+$ for each bounded continuous function $g$, then each bounded solution of the linear equation (2) tends to zero when $t \to +\infty$.

The assumption in Theorem 1 is called the *admissibility* of the pair of spaces in which we take the perturbations and look for the solutions. The theorem is probably the first place in the literature where one can see a relation between admissibility and stability.

We note that one can also consider the admissibility of other pairs of spaces, such as the pair $(L^p, L^q)$ for $p, q \in [1, \infty)$ with $p \geq q$. In this case the (strong) admissibility property corresponds to require that for each $g \in L^q$ there exists a unique $v \in L^p$ that is absolutely continuous on each compact interval and satisfies identity (1) for Lebesgue-almost every $t$. In addition, one can consider a (weak) admissibility property, which is defined in terms of the existence of mild solutions for the corresponding integral equation.

There is an extensive literature on the relation between admissibility and stability, also in infinite-dimensional spaces. For some of the most relevant early contributions in the area we refer to the books by Massera and Schäffer [13] (building on their former work [12]) and by Dalec’kiǐ and Kreǐn [3]. We also refer to [9] for some early results on infinite-dimensional spaces. For details and references we refer the reader to the book [1] (see also [7, 14, 15, 16, 20, 22] and the references therein). See [2, 19] for related results for skew-product semiflows over a compact base.

In a related direction, one can also use a Fredholm alternative to obtain a criterion for the existence of an exponential dichotomy, sometimes in connection with admissibility. For the linear differential equation (2) this consists of giving a description of the existence of solutions of equation (1) in terms of those of the adjoint equation $w' = -A(t)^*w$. A slightly different approach starts by writing equation (1) in the form $Bv = g$, where $B$ is the linear operator defined by

$$(Bv)(t) = v'(t) - A(t)v(t)$$

in some appropriate space. It turns out that the admissibility of certain pairs of spaces is related to the invertibility or the Fredholm properties of $B$. Related work is due to Palmer [17] for ordinary differential equations and to Lin [10] and Mallet-Paret [11] for functional differential equations.

The two approaches described above, respectively in terms of an admissibility property and in terms of a Fredholm alternative, can be used to characterize the existence of an exponential dichotomy for a linear ordinary differential equation. Roughly speaking, the approaches can be described, respectively, as direct and indirect since the first considers only the solutions themselves while the second considers properties of certain related operators.

We note that there exists related work for functional differential equations using the second approach (see [10, 11]).
In the present paper we develop a version of the first approach for delay equations, which to our best knowledge had not been considered before in the context of functional differential equations. To some extent our arguments are inspired by those in the works mentioned above for ordinary differential equations, although even the appropriate spaces had to be carefully chosen so that we could obtain an equivalence between admissibility and hyperbolicity. On the other hand, the fact that the variation of constants formula for delay equations is not the usual one required a special care with the extension of the dynamics to some discontinuous initial functions, which has no correspondence for ordinary differential equations. The most technical aspect was to establish exponential bounds along stable and unstable directions (see the proofs of Lemmas 6 and 7), which as always in the area depends substantially on each particular equation.

1.2. Basic notions. We consider a linear delay equation
\[ v' = L(t)v, \]
where \( L(t): C \to \mathbb{R}^n \), for \( t \in \mathbb{R} \), are bounded linear operators on the Banach space \( C = C([-r,0],\mathbb{R}^n) \) of all continuous functions \( \phi: [-r,0] \to \mathbb{R}^n \) equipped with the supremum norm (for some given \( r > 0 \), with \( v_t(\theta) = v(t + \theta) \) for \( \theta \in [-r,0] \)). We always assume that \( t \mapsto L(t) \phi \) is measurable for each \( \phi \in C \) and that \( t \mapsto \|L(t)\| \) is locally integrable, with
\[ \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|L(\tau)\| \, d\tau < +\infty. \]
Under these assumptions, equation (3) determines an evolution family \( T(t,s) \) on \( C \) defined by
\[ T(t,s)\phi = v_t \]
for \( t \geq s \) and \( \phi \in C \), where \( v \) is the unique solution (in the sense of Carathéodory) of the equation on \( [s-r, +\infty) \) with \( v_s = \phi \) (see Section 2 for details). Given an interval \( I \subset \mathbb{R} \), we say that equation (3) has an exponential dichotomy on \( I \) if:
1. there exist projections \( P(t): C \to C \), for \( t \in I \), such that
\[ P(t)T(t,s)P(s) = T(t,s)P(s) \quad \text{for } t \geq s; \]
2. the linear operator
\[ \overline{T}(t,s) = T(t,s)\ker P(s): \ker P(s) \to \ker P(t) \]
is invertible for \( t \geq s; \)
3. there exist \( \lambda, D > 0 \) such that for every \( t,s \in I \) with \( t \geq s \) we have
\[ \|T(t,s)P(s)\| \leq De^{-\lambda(t-s)}, \quad \|\overline{T}(s,t)Q(t)\| \leq De^{-\lambda(t-s)}, \]
where \( \overline{T}(s,t) = \overline{T}(t,s)^{-1} \) and \( Q(t) = \text{Id} - P(t) \).

We refer the reader to the books [5, 6] for comprehensive introductions to the theory of delay equations.
1.3. Formulation of the results. Finally we introduce spaces where we take the perturbations and look for the solutions, and we formulate our results on the line and on the half-line.

We start with the case when \( I = \mathbb{R} \). Let \( C_b = C_b(\mathbb{R}, \mathbb{R}^n) \) be the set of all bounded continuous functions \( v: \mathbb{R} \rightarrow \mathbb{R}^n \) and let \( M = M(\mathbb{R}, \mathbb{R}^n) \) be the set of all measurable functions \( g: \mathbb{R} \rightarrow \mathbb{R}^n \) such that

\[
\sup_{t \in \mathbb{R}} \int_{t}^{t+1} |g(\tau)| \, d\tau < +\infty,
\]

identified if they are equal almost everywhere. Now we consider the equation

\[
v' = L(t)v + g(t)
\]

with the operators \( L(t) \) as in Section 1.2 and with \( g \in M \). The following result is a combination of Theorems 5 and 6 (see Section 4). Again we refer to Section 2 for the notion of a solution (in the sense of Carathéodory).

**Theorem 2.** Equation (3) has an exponential dichotomy on \( \mathbb{R} \) if and only if for each \( g \in M \) there exists a unique solution \( v \in C_b \) of equation (6) on \( \mathbb{R} \).

Now we consider the case of the interval \( I = \mathbb{R}_0^+ \), which requires a slightly different notion of admissibility. Let \( C_b' = C_b([-r, +\infty), \mathbb{R}^n) \) be the set of all bounded continuous functions \( v: [-r, +\infty) \rightarrow \mathbb{R}^n \) and given a closed subspace \( B \subset C \), let

\[
C_b', B = \{ v \in C_b': v_0 \in B \}.
\]

Finally, let \( M' = M([-r, +\infty), \mathbb{R}^n) \) be the set of all measurable functions \( g: [-r, +\infty) \rightarrow \mathbb{R}^n \) such that

\[
\sup_{t \geq -r} \int_{t}^{t+1} |g(\tau)| \, d\tau < +\infty,
\]

identified if they are equal almost everywhere. The following result is obtained in Section 5.

**Theorem 3.** Equation (3) has an exponential dichotomy on \( \mathbb{R}_0^+ \) if and only if there exists a closed subspace \( B \subset C \) such that for each \( g \in M' \) there exists a unique solution \( v \in C_b', B \) of equation (6) on \( \mathbb{R}_0^+ \).

The space \( B \) turns out to be the unstable space at time 0.

1.4. Solutions with values in a Banach space. Here we discuss briefly whether our results can be extended to equations whose solutions take values in a Banach space. It turns out that this is the case for reflexive Banach spaces, which includes all Hilbert spaces and all \( L^p \) spaces with \( 1 < p < \infty \).

One can show that \( v: [s - r, a) \rightarrow \mathbb{R}^n \) is a solution of equation (6) (in the sense of Carathéodory) if and only if

\[
v(t) = v(s) + \int_{s}^{t} (L(\tau)v_{\tau} + g(\tau)) \, d\tau
\]

for \( t \in [s, a) \). We emphasize that this equivalence does not hold in general for equations whose solutions have values in a Banach space \( X \). More precisely, assume that \( L(t): C \rightarrow X \), for \( t \in \mathbb{R} \), are bounded linear operators such that \( t \mapsto L(t)\phi \) is Bochner measurable for each \( \phi \in C \) and that \( g: \mathbb{R} \rightarrow X \)
is a Bochner measurable function. Then any function \( v: [s - r, a) \rightarrow X \) satisfying (7), considering Bochner integrals, is absolutely continuous on \([s, a)\) and satisfies (6) for almost every \( t \in [s, a)\). However, the converse does not hold in general, although it holds when \( X \) is a reflexive Banach space (see [8]). Moreover:

1. Each operator \( L(t): C \rightarrow X \) can be written as a Riemann–Stieltjes integral (see (12)) for some measurable map \( \eta: \mathbb{R} \times [-r, 0] \rightarrow \mathcal{L}(X), \) where \( \mathcal{L}(X) \) is the set of all bounded linear operators from \( X \) into itself, such that \( \theta \mapsto \eta(t, \theta) \) has bounded variation and is left-continuous for each \( t \in \mathbb{R} \) (see [21]). This allows us to extend \( L(t) \) to certain discontinuous functions and so use the variation of constants formula for delay equations.

2. The Lebesgue differentiation theorem holds for functions with values in a Banach space (see for example [4]). The theorem is used in the proof of Theorem 6 to show that a certain linear operator \( R \) (see (24)) is well defined. Somehow this is at the heart of our approach: once \( R \) is shown to be well defined the admissibility hypothesis corresponds to its invertibility.

It turns out that these two somewhat technical properties suffice to ensure that all the results in our paper hold for equations whose solutions take values in a reflexive Banach space, provided that we consider Bochner integrals. The necessary changes, although of somewhat simple nature, are also technical and so we refrain from complicating the text.

2. Preliminaries

2.1. Basic theory. In this section we recall a few basic notions and results. Let \( |\cdot| \) be the norm on \( \mathbb{R}^n \). Given \( r > 0 \), we denote by \( C = C([-r, 0], \mathbb{R}^n) \) the Banach space of all continuous functions \( \phi: [-r, 0] \rightarrow \mathbb{R}^n \) equipped with the norm

\[
\|\phi\| = \sup\{|\phi(\theta)|: -r \leq \theta \leq 0\}.
\]

We consider perturbations of a linear delay equation

\[
v' = L(t)v_t
\]

of the form

\[
v' = L(t)v_t + g(t),
\]

where \( L(t): C \rightarrow \mathbb{R}^n \), for \( t \in \mathbb{R} \), are bounded linear operators and \( g: \mathbb{R} \rightarrow \mathbb{R}^n \) is a measurable function such that:

1. \( t \mapsto L(t)\phi \) is measurable for each \( \phi \in C; \)

2. \[
\sup_{t \in \mathbb{R}} \int_t^{t+1} \|L(\tau)\| d\tau < +\infty \quad \text{and} \quad \sup_{t \in \mathbb{R}} \int_t^{t+1} |g(\tau)| d\tau < +\infty.
\]

A continuous function \( v: [s - r, a) \rightarrow \mathbb{R}^n \) with \( a \leq +\infty \) is called a solution of equation (9) (in the sense of Carathéodory) if \( v \) is absolutely continuous on \([s, a)\) and (9) holds for almost every \( t \in [s, a)\).

Under the former assumptions on the operators \( L(t) \), it follows from standard results on the existence and uniqueness of solutions of a delay equation
that equation (8) has unique solutions on \([s-r, +\infty)\) and so it determines an evolution family \(T(t,s) : C \to C\), for \(t \geq s\), defined by

\[
T(t,s)\phi = v_t
\]

for \(\phi \in C\), where \(v\) is the unique solution of equation (8) on \([s-r, +\infty)\) with \(v_s = \phi\) (see [5]).

We note that the linear operators \(T(t,s)\) in (11) can be extended to a certain space of discontinuous functions. Let \(C'\) be the set of all functions \(\phi : [-r,0] \to \mathbb{R}^n\) that are continuous on \([-r,0)\) for which the limit

\[
\phi(0^-) = \lim_{\theta \to 0^-} \phi(\theta)
\]

exists. This is a Banach space with the supremum norm. Now we write the linear operator \(L(t) : C \to \mathbb{R}^n\) as a Riemann–Stieltjes integral

\[
L(t)\phi = \int_{-r}^{0} d\eta(t,\theta)\phi(\theta) = \int_{-r}^{t} d\eta(t,\theta)\phi(\theta)
\]

for some measurable map \(\eta : \mathbb{R} \times [-r,0] \to M_n\), where \(M_n\) is the set of all \(n \times n\) matrices, such that \(\theta \mapsto \eta(t,\theta)\) has bounded variation and is left-continuous for each \(t \in \mathbb{R}\). Since each function \(\phi \in C'\) is right-continuous, it is Riemann–Stieltjes integrable with respect to \(\eta(t,\cdot)\) for each \(t \in \mathbb{R}\). Hence, one can extend each linear operator \(L(t)\) to \(C'\) using the integral in (12).

We continue to denote the extension by \(L(t)\) and we note that

\[
\Pi(t) := \|L(t)|C\| = \|L(t)|C'\|.
\]

Given \(t,s \in \mathbb{R}\) with \(t \geq s\), we define a linear operator \(T_0(t,s)\) on \(C'\) by

\[
T_0(t,s)\phi = v_t
\]

for \(\phi \in C'\), where \(v\) is the unique solution of equation (8) on \([s-r, +\infty)\) with \(v_s = \phi\) (see [5] for details).

By the variation of constants formula for delay equations, the unique solution \(v\) of equation (9) with \(v_s = \phi\) \(\in C\) satisfies

\[
v_t = T(t,s)\phi + \int_{s}^{t} T_0(t,\tau)X_0g(\tau) \, d\tau
\]

for all \(t \geq s\), where \(X_0 : \mathbb{R}^n \to C'\) is the linear operator defined by

\[
(X_0p)(\theta) = \begin{cases} 
0 & \text{if } -r \leq \theta < 0, \\
p & \text{if } \theta = 0
\end{cases}
\]

for each \(p \in \mathbb{R}^n\). This means that \(v_s = \phi\) and

\[
v(t + \theta) = (T(t,s)\phi)(\theta) + \int_{s}^{t+\theta} (T_0(t,\tau)X_0g(\tau))(\theta) \, d\tau
\]

for all \(t \geq s\) and \(\theta \in [-r,0]\) with \(t + \theta \geq s\).
2.2. Extension of the hyperbolicity to $C'$. Now we assume that the linear equation (8) has an exponential dichotomy on an interval $I$ that is either $\mathbb{R}$ or $\mathbb{R}_0^+$ (see Section 1.1). Then the spaces

$$E(t) = P(t)C \quad \text{and} \quad F(t) = Q(t)C$$

are called, respectively, stable and unstable spaces at time $t$. For each $t \in I$ we define linear operators $P_0(t), Q_0(t) : \mathbb{R}^n \rightarrow C'$ by

$$P_0(t) = X_0 - Q_0(t)$$

and

$$Q_0(t) = \mathcal{T}(t, t + r)Q(t + r)T_0(t + r, t)X_0.$$ 

One can show that $P_0(t) p \in C' \setminus C$ and $Q_0(t) p \in C$ for all $p \in \mathbb{R}^n$.

Although the following result can be considered well known (see for example [5]), for completeness we give a simple direct proof. Essentially it extends the exponential bounds of an exponential dichotomy to the space $C'$ in terms of the operators $\mathcal{T}(t, s)$ and $T_0(t, s)$ in (4) and (13).

**Proposition 4.** Assume that condition (10) holds and that equation (8) has an exponential dichotomy on $I$. Then there exist constants $\lambda, N > 0$ such that

$$\|T_0(t, s)P_0(s)\| \leq Ne^{-\lambda(t-s)}, \quad \|\mathcal{T}(s, t)Q_0(t)\| \leq Ne^{-\lambda(t-s)}$$

(14)

for every $t, s \in I$ with $t \geq s$.

**Proof.** The unique solution $v$ of equation (8) on $[s-r, +\infty)$ with $v_s = \phi \in C'$ satisfies

$$v(t) = \phi(0) + \int_s^t L(\tau)v_\tau \, d\tau \quad \text{for} \ t \geq s. \quad (15)$$

In particular, for $t \in [s, s + r]$ it follows from (15) that

$$\|v_t\| \leq \max \left\{ \|v_s\|, \max_{\tau \in [s, t]} |v(\tau)| \right\}$$

$$\leq \max \left\{ \|\phi\|, |\phi(0)| + \int_s^t \|L(\tau)\| \cdot \|v_\tau\| \, d\tau \right\}$$

$$\leq \|\phi\| + \int_s^t \|L(\tau)\| \cdot \|v_\tau\| \, d\tau.$$ 

On the other hand, for $t \geq s + r$, we have

$$|v(t + \theta)| \leq |\phi(0)| + \int_s^{t+\theta} \|L(\tau)\| \cdot \|v_\tau\| \, d\tau \quad (16)$$

for $\theta \in [-r, 0]$. Thus,

$$\|v_t\| \leq \|\phi\| + \int_s^t \Pi(\tau)\|v_\tau\| \, d\tau \quad \text{for} \ t \geq s$$

and it follows from Gronwall’s lemma that

$$\|v_t\| \leq \|\phi\| \exp \left( \int_s^t \Pi(\tau) \, d\tau \right) \quad \text{for} \ t \geq s.$$

Hence,

$$\|T_0(t, s)\| \leq \exp \left( \int_s^t \Pi(\tau) \, d\tau \right) \quad \text{for} \ t \geq s.$$
and so, by (10), there exist constants \( \omega, K > 0 \) such that
\[
\| T_0(t, s) \| \leq Ke^{\omega(t-s)} \quad \text{for } t \geq s. \tag{17}
\]

Now take \( p \in \mathbb{R}^n \). Then
\[
\| \overline{T}(s, t)Q_0(t)p \| = \| \overline{T}(s, t+r)Q(t+r)T_0(t+r, t)X_0p \|
\leq De^{-\lambda(t+r-s)}\|T_0(t+r, t)X_0p\|
\text{for } t \geq s. \quad \text{On the other hand, by (17), we have}
\| T_0(t+r, t)X_0p \| \leq Ke^{\omega r}\|X_0p\| = K\|p\|e^{\omega r} \tag{18}
\text{and so}
\| \overline{T}(s, t)Q_0(t)p \| \leq De^{-\lambda(t+r-s)}K\|p\|e^{\omega r}. \tag{19}
\]

This establishes the second inequality in (14).

For the other inequality, note that taking \( t = s \) in (19) we obtain
\[
\| Q_0(t) \| \leq DK e^{(\omega-\lambda)r} \quad \text{for } t \in I.
\]

Hence, for \( t \in [s, s + r] \), by (17) we have
\[
\| T_0(t, s)P_0(s) \| \leq Ke^{\omega r}\|X_0 - Q_0(s)\| \leq Ke^{\omega r}(1 + N'), \tag{20}
\]
where \( N' = DK e^{(\omega-\lambda)r} \). Now take \( t \geq s + r \). Then
\[
T_0(t, s)P_0(s) = T_0(t, s)(X_0 - Q_0(s))
= T(t, s + r)T_0(s + r, s)(X_0 - Q_0(s))
= T(t, s + r)(T_0(s + r, s)X_0 - Q(s + r)T_0(s + r, s)X_0)
= T(t, s + r)P(s + r)T_0(s + r, s)X_0.
\]

By (5) and (18), we obtain
\[
\| T_0(t, s)P_0(s) \| \leq De^{-\lambda(t-s-r)}Ke^{\omega r}. \tag{21}
\]

The first inequality in (14) follows now readily from (20) and (21). \( \square \)

3. Admissible spaces

In this section we introduce appropriate admissible spaces, that is, appropriate spaces in which we look for the solutions and in which we take the perturbations. We start with the real line, that is, with the case when \( I = \mathbb{R} \). Let \( C_b = C_b(\mathbb{R}, \mathbb{R}^n) \) be the set of all continuous functions \( v: \mathbb{R} \to \mathbb{R}^n \) such that
\[
|v|_\infty := \sup_{t \in \mathbb{R}}|v(t)| < +\infty. \tag{22}
\]
Note that \( C_b \) is a Banach space when equipped with the norm \( |. |_\infty \). Moreover, let \( M = M(\mathbb{R}, \mathbb{R}^n) \) be the set of all measurable functions \( g: \mathbb{R} \to \mathbb{R}^n \) such that
\[
|g|_M := \sup_{t \in \mathbb{R}} \int_t^{t+1} |g(\tau)| \, d\tau < +\infty,
\]
identified if they are equal almost everywhere. It is known that \( M \) is a Banach space when equipped with the norm \( |. |_M \) (see [13, 23.F]).
Now we introduce the notion of admissibility for a linear delay equation on the line: we say that the pair of spaces \((C_b, M)\) is \textit{admissible} for equation (8) if for each \(g \in M\) there exists a unique \(v \in C_b\) such that

\[ v_t = T(t, s) v_s + \int_s^t T_0(t, \tau) X_0 g(\tau) \, d\tau \]  

(23)

for all \(t, s \in \mathbb{R}\) with \(t \geq s\).

We also introduce corresponding Banach spaces for the half line, that is, when \(I = \mathbb{R}^+_0\). Let \(C'_b = C_b([-r, +\infty), \mathbb{R}^n)\) be the set of all continuous functions \(v: [-r, +\infty) \to \mathbb{R}^n\) such that

\[ |v|'_\infty := \sup_{t \geq -r} |v(t)| < +\infty. \]

Then \(C'_b = (C_b, |\cdot|'_\infty)\) is a Banach space. Moreover, given a closed subspace \(B \subset C\), we denote by \(C'_{b, B}\) the set of all functions \(v \in C'_b\) such that \(v_0 \in B\). Clearly, \(C'_{b, B}\) is a closed subspace of \(C'_b\). Finally, let \(M' = M([-r, +\infty), \mathbb{R}^n)\) be the set of all measurable functions \(g: [-r, +\infty) \to \mathbb{R}^n\) such that

\[ |g|'_M := \sup_{t \geq -r} \int_t^{t+1} |g(\tau)| \, d\tau < +\infty, \]

identified if they are equal almost everywhere. Note that \(M' = (M', |\cdot|'_M)\) can be identified with a closed subspace of \(M(\mathbb{R}, \mathbb{R}^n)\) (by extending each function in \(M'\) by 0 to the whole \(\mathbb{R}\)). Hence, \(M'\) is also a Banach space.

4. Admissibility on the line

In this section we show that the admissibility of the pair of spaces

\( (C_b, M) = (C_b(\mathbb{R}, \mathbb{R}^n), M(\mathbb{R}, \mathbb{R}^n)) \)

for equation (8) is equivalent to the existence of an exponential dichotomy for this equation (see Section 3 for the definition of the Banach spaces \(C_b\) and \(M\)). We start with the simplest direction: the admissibility property is a consequence of the existence of an exponential dichotomy.

**Theorem 5.** If equation (8) has an exponential dichotomy on \(\mathbb{R}\), then the pair of spaces \((C_b, M)\) is admissible for this equation.

**Proof.** Take a function \(g \in M\). For each \(t \in \mathbb{R}\) we define \(v_t = x_t + y_t\), where

\[ x_t = \int_{-\infty}^t T_0(t, \tau) P_0(\tau) g(\tau) \, d\tau \]

and

\[ y_t = - \int_t^{+\infty} T(t, \tau) Q_0(\tau) g(\tau) \, d\tau. \]
It follows from Proposition 4 that
\[
\int_{-\infty}^{t} \|T_0(t, \tau)P_0(\tau)g(\tau)\| \, d\tau \leq \int_{-\infty}^{t} Ne^{-\lambda(t-\tau)}|g(\tau)| \, d\tau \\
= \sum_{m=0}^{\infty} \int_{t-m-1}^{t-m} Ne^{-\lambda(t-\tau)}|g(\tau)| \, d\tau \\
\leq \sum_{m=0}^{\infty} Ne^{-\lambda m} \int_{t-m-1}^{t-m} |g(\tau)| \, d\tau \\
\leq \frac{N}{1 - e^{-\lambda}} |g|_M
\]
and, similarly,
\[
\int_{t}^{+\infty} \|T(t, \tau)Q_0(\tau)g(\tau)\| \, d\tau \leq \frac{N}{1 - e^{-\lambda}} |g|_M,
\]
for all \( t \in \mathbb{R} \). Hence, the function \( v: \mathbb{R} \to \mathbb{R}^n \) is well defined, is continuous and satisfies (22). In other words, \( v \in C_b \). Moreover, for \( t \geq s \) we have
\[
v_t = \int_{s}^{t} T_0(t, \tau)X_0g(\tau) \, d\tau - \int_{s}^{t} T_0(t, \tau)P_0(\tau)g(\tau) \, d\tau \\
- \int_{s}^{t} T(t, \tau)Q_0(\tau)g(\tau) \, d\tau + \int_{-\infty}^{t} T_0(t, \tau)P_0(\tau)g(\tau) \, d\tau \\
- \int_{t}^{+\infty} T(t, \tau)Q_0(\tau)g(\tau) \, d\tau \\
= \int_{s}^{t} T_0(t, \tau)X_0g(\tau) \, d\tau + \int_{-\infty}^{s} T_0(t, \tau)P_0(\tau)g(\tau) \, d\tau \\
- \int_{s}^{+\infty} T(t, \tau)Q_0(\tau)g(\tau) \, d\tau \\
= \int_{s}^{t} T_0(t, \tau)X_0g(\tau) \, d\tau + T(t, s)v_s
\]
and so (23) holds.

It remains to establish the uniqueness of the solution \( v \). We note that it suffices to show that if \( v_t = T(t, s)v_s \) for \( t \geq s \) with \( v \in C_b \), then \( v = 0 \). Let
\[
x_t = P(t)v_t \quad \text{and} \quad y_t = Q(t)v_t.
\]
Then \( v_t = x_t + y_t \) and it follows from (10) that
\[
x_t = T(t, s)x_s \quad \text{and} \quad y_t = T(t, s)y_s
\]
for \( t \geq s \). Since \( x_t = T(t, t-s)x_{t-s} \) for \( s \geq 0 \), we obtain
\[
\|x_t\| = \|T(t, t-\tau)x_{t-\tau}\| \\
= \|T(t, t-s)P(t-s)v_{t-s}\| \\
\leq Ne^{-\lambda s}\|v_{t-s}\| \\
\leq Ne^{-\lambda s}\|v\|_\infty.
\]
Letting \( s \to +\infty \) we find that \( x_t = 0 \) for all \( t \in \mathbb{R} \). One can show in a similar manner that \( y_t = 0 \) for all \( t \in \mathbb{R} \) and so \( v = 0 \). \( \square \)
Now we establish the converse of Theorem 5: the admissibility property implies the existence of an exponential dichotomy.

**Theorem 6.** If the pair of spaces \((C_b, M)\) is admissible for equation (8), then the equation has an exponential dichotomy on \(\mathbb{R}\).

**Proof.** We separate the proof into steps.

**Step 1. The operator \(R\).** We define a linear operator \(R\) by
\[
R: D(R) \to M, \quad Rv = g
\]
in the domain \(D(R)\) of all \(v \in C_b\) for which there exists \(g \in M\) satisfying (23) for all \(t, s \in \mathbb{R}\) with \(t \geq s\). We first show that \(R\) is well defined. Take \(g, h \in M\) such that
\[
v_t = T(t, s)v_s + \int_s^t T_0(t, \tau)X_0g(\tau)\,d\tau
\]
and
\[
v_t = T(t, s)v_s + \int_s^t T_0(t, \tau)X_0h(\tau)\,d\tau
\]
for \(t \geq s\). Then
\[
\frac{1}{t-s} \int_s^t T_0(t, \tau)X_0g(\tau)\,d\tau = \frac{1}{t-s} \int_s^t T_0(t, \tau)X_0h(\tau)\,d\tau.
\]
Since the maps \(\tau \mapsto T_0(t, \tau)X_0g(\tau)\) and \(\tau \mapsto T_0(t, \tau)X_0h(\tau)\) are locally integrable, letting \(s \to t\) it follows from the Lebesgue differentiation theorem that \(g(t) = h(t)\) for almost all \(t \in \mathbb{R}\).

**Lemma 1.** The operator \(R\) in (24) is closed.

**Proof of the lemma.** Let \((v_m)_{m \in \mathbb{N}}\) be a sequence in \(D(R)\) converging to \(v \in C_b\) such that \(Rv_m = g_m\) converges to \(g \in M\). For each \(s \in \mathbb{R}\), we have
\[
v_t - T(t, s)v_s = \lim_{m \to \infty} (v_t^m - T(t, s)v_s^m)
\]
for \(t \geq s\). Moreover, by (16),
\[
\left\| \int_s^t T_0(t, \tau)X_0g_m(\tau)\,d\tau - \int_s^t T_0(t, \tau)X_0g(\tau)\,d\tau \right\|
\leq \int_s^t Ke^{\omega(t-\tau)}\|g_m(\tau) - g(\tau)\|\,d\tau
\leq Ke^{\omega(t-s)}(t-s + 1)|g_m - g|_M.
\]
Since \(g_m\) converges to \(g\) in \(M\), we conclude that
\[
\lim_{m \to \infty} \int_s^t T_0(t, \tau)X_0g_m(\tau)\,d\tau = \int_s^t T_0(t, \tau)X_0g(\tau)\,d\tau
\]
and it follows from (25) that identity (23) holds for all \(t, s \in \mathbb{R}\) with \(t \geq s\). Hence, \(Rv = g\) and \(v \in D(R)\). □
By the admissibility hypothesis in the theorem, the operator \( R \) in (24) has an inverse \( S : M \to C_b \). Moreover, in view of Lemma 1, it follows from the closed graph theorem that \( R \) and so also \( S \) are bounded.

**Step 2. Invariant subspaces.** For each \( s \in \mathbb{R} \), let

\[
E(s) = \left\{ \phi \in C : \sup_{t \geq s} ||T(t, s)\phi|| < +\infty \right\}
\]

and

\[
F(s) = \left\{ \phi \in C : \text{there exists } v : (-\infty, s] \to \mathbb{R}^n \text{ continuous with} \right.
\]

\[
v_s = \phi, \sup_{t \leq s} |v(t)| < +\infty \text{ and } v_t = T(t, \tau)v_{\tau} \text{ for } s \geq t \geq \tau \right\}.
\]

Note that \( E(s) \) and \( F(s) \) are subspaces of \( C \).

**Lemma 2.** For each \( s \in \mathbb{R} \) we have

\[
C = E(s) \oplus F(s).
\]

**Proof of the lemma.** Take \( s \in \mathbb{R} \) and \( \phi \in C \). We define \( \psi : [s-r, +\infty) \to \mathbb{R}^n \) by \( \psi_s = \phi \) and \( \psi(t) = \phi(0) \) for \( t \geq s \). Now we consider the function \( g : \mathbb{R} \to \mathbb{R}^n \) given by

\[
g(t) = L(t)\psi_t = \int_{-r}^{0} d\eta(t, \theta)\psi(t + \theta)
\]

for \( t \geq s \) and \( g(t) = 0 \) for \( t < s \). In view of (10) we have

\[
|g|_M = \sup_{t \in \mathbb{R}} \int_{t-1}^{t+1} |g(\tau)| d\tau 
\leq \sup_{t \in \mathbb{R}} \int_{t-1}^{t+1} \Pi(\tau) d\tau ||\phi|| < +\infty
\]

and so \( g \in M \). Hence, there exists \( v \in D(R) \) such that \( Rv = g \), with \( R \) as in (24). The function \( u = v + \psi \) is a solution of equation (8) with \( u_s = v_s + \phi \), that is,

\[
u_t = T(t, s)(v_s + \phi) \quad \text{for } t \geq s.
\]

Moreover,

\[
\sup_{t \geq s} |u(t)| \leq \sup_{t \geq s} |v(t)| + \sup_{t \geq s} |\psi(t)| < +\infty
\]

since \( v \in C_b \) and \( |\psi(t)| \leq ||\phi|| \). This implies that \( v_s + \phi \in E(s) \).

Now we show that \( v_t = T(t, \tau)v_{\tau} \) for \( s \geq t \geq \tau \). Indeed, by (23) we have

\[
v_t = T(t, \tau)v_{\tau} + \int_{\tau}^{t} T_0(t, \sigma)X_0g(\sigma) d\sigma
\]

and since \( g(t) = 0 \) for \( t < s \), the desired property holds. Hence, \( v_s \in F(s) \) and so

\[
\phi = (v_s + \phi) - v_s \in E(s) + F(s).
\]

It remains to show that \( E(s) \cap F(s) = \{0\} \). Take \( \phi \in E(s) \cap F(s) \). Then there exists a continuous function \( v : (-\infty, s] \to \mathbb{R}^n \) with \( v_s = \phi \) such that
sup_{t \leq s} |v(t)| < +\infty and \( v_t = T(t, \tau)v_{\tau} \) for \( s \geq t \geq \tau \). We define a function \( u: \mathbb{R} \rightarrow \mathbb{R}^n \) by
\[
u_t = \begin{cases} T(t, s)\phi & \text{if } t \geq s, \\
v_t & \text{if } t \leq s.\end{cases}\]

Note that \( u \) is continuous and \( \sup_{t \in \mathbb{R}} |u(t)| < +\infty \) (since \( \phi \in E(s) \)). Moreover, one can easily verify that \( u_t = T(t, \tau)u_{\tau} \) for \( t \geq \tau \). Hence, \( Ru = 0 \) and \( u \in D(R) \). Since \( R \) is invertible, we conclude that \( u = 0 \) and thus \( \phi = v_s = u_s = 0. \)

\textbf{Step 3. Projections on the line.}\n
Let \( P(s), Q(s): C \rightarrow C \) be the projections associated with the splitting in (28), with \( E(s) \) and \( F(s) \) as in (26) and (27).

\textbf{Lemma 3.} For each \( t \geq s \), the linear operator
\[
T(t, s)\mid F(s): F(s) \rightarrow F(t)
\]
is onto and invertible.

\textit{Proof of the lemma.} Assume that \( T(t, s)\phi = 0 \) for some \( \phi \in F(s) \). Since \( \phi \in F(s) \), there exists a continuous function \( v: (-\infty, s] \rightarrow \mathbb{R}^n \) with \( v_s = \phi \) such that \( \sup_{\tau \leq s} |v(\tau)| < +\infty \) and \( v_{\tau} = T(\tau, s)v_s \) for \( s \geq \tau \geq s \). We define a map \( u: \mathbb{R} \rightarrow \mathbb{R}^n \) by
\[
u_{\tau} = \begin{cases} T(\tau, s)\phi & \text{if } \tau \geq s, \\
v_{\tau} & \text{if } \tau \leq s.\end{cases}\]

Note that \( u \) is continuous and \( \sup_{\tau \in \mathbb{R}} |u(\tau)| < +\infty \) (since \( u_{\tau} = 0 \) for \( \tau \geq t \)). Moreover, \( u_{\tau} = T(\tau, s)u_s \) for \( \tau \geq s \). Hence, \( Ru = 0 \) and \( u \in D(R) \), again with \( R \) as in (24). Since \( R \) is invertible, we conclude that \( u = 0 \) and thus \( \phi = v_s = u_s = 0 \). This shows that the map in (31) is one-to-one.

Now take \( \phi \in F(t) \). Then there exists a continuous function \( v: (-\infty, t] \rightarrow \mathbb{R}^n \) with \( v_t = \phi \) such that \( \sup_{t \leq s} |v(\tau)| < +\infty \) and \( v_{\tau} = T(\tau, s)v_s \) for \( t \geq \tau \geq s \). In particular,
\[
\phi = v_t = T(t, s)v_s
\]
and since \( v_s \in F(s) \), we conclude that the map in (31) is also onto. \( \square \)

\textbf{Lemma 4.} We have \( P(t)T(t, s) = T(t, s)P(s) \) for \( t \geq s \).

\textit{Proof of the lemma.} It follows readily from the definitions that
\[
T(t, s)E(s) \subset E(t) \quad \text{for } t \geq s,
\]
with \( E(s) \) as in (26). Indeed, if \( \phi \in E(s) \), then
\[
\sup_{\tau \geq s} \|T(\tau, s)\phi\| < +\infty
\]
and so
\[
\sup_{\tau \geq t} \|T(\tau, t)T(t, s)\phi\| \leq \sup_{\tau \geq s} \|T(\tau, s)\phi\| < +\infty.
\]
Hence, \( T(t, s)\phi \in E(t) \). On the other hand, by Lemma 3 we have
\[
T(t, s)F(s) = F(t),
\]
with $F(s)$ as in (27). This implies that
\[ T(t, s)P(s)\phi \in E(t) \quad \text{and} \quad T(t, s)Q(s)\phi \in F(t) \]
for each $\phi \in C$. Since
\[ T(t, s) = T(t, s)P(s) + T(t, s)Q(s) \]
and
\[ T(t, s) = P(t)T(t, s) + Q(t)T(t, s), \]
we conclude that
\[ P(t)T(t, s)\phi = T(t, s)P(s)\phi \quad \text{and} \quad Q(t)T(t, s)\phi = T(t, s)Q(s)\phi. \]
This yields the desired property. □

We also show that the projections in (30) are uniformly bounded.

**Lemma 5.** There exists $K' > 0$ such that
\[ \|P(s)\| \leq K' \quad \text{for} \quad s \in \mathbb{R}. \] 

**Proof of the lemma.** Take $s \in \mathbb{R}$ and $\phi \in C$. Moreover, let $\psi$, $g$ and $v$ be as in the proof of Lemma 2. Then $P(s)\phi = v_s + \phi$ and so
\[ \|P(s)\phi\| = \|v_s + \phi\| \leq |v|_\infty + \|\phi\| = |Sg|_\infty + \|\phi\|, \]
where $S$ is the inverse of the operator $R$ in (24). Moreover, by (29), we have
\[ |Sg|_\infty \leq \|S\| \cdot |g|_M \leq \|S\| \beta \|\phi\|, \] 
(33)
where
\[ \beta = \sup_{t \in \mathbb{R}} \int_t^{t+1} \Pi(\tau) \, d\tau. \]
Hence,
\[ \|P(s)\phi\| \leq (\|S\| \beta + 1) \|\phi\| \]
and property (32) holds taking $K' = \|S\| \beta + 1$. □

**Step 4. Exponential bounds along the spaces $E(s)$ in (26).** We continue to denote by $T(t, s)$ the linear operator in (11).

**Lemma 6.** There exist constants $\lambda, D > 0$ such that
\[ \|T(t, s)P(s)\| \leq De^{-\lambda(t-s)} \quad \text{for} \quad t \geq s. \]

**Proof of the lemma.** Given $s \in \mathbb{R}$ and $\phi \in E(s)$, let $\psi$ and $g$ be as in the proof of Lemma 2. Letting $u_t = T(t, s)\phi$ for $t \geq s$, the continuous function $v: \mathbb{R} \to \mathbb{R}^n$ defined by
\[ v(t) = \begin{cases} u(t) - \psi(t) & \text{if} \ t \geq s - r, \\ 0 & \text{if} \ t < s - r \end{cases} \]
for $t \geq 0$.
is in $C_b$ (because $\phi \in E(s)$). Moreover, $v \in D(R)$ and $Rv = g$, with $R$ as in (24). Indeed, for $t \geq s$ we have

$$v(t) = u(t) - \psi(t)$$

$$= \phi(0) + \int_s^t L(\tau)u_\tau \, d\tau - \phi(0)$$

$$= \int_s^t (v_\tau + \psi_\tau) \, d\tau$$

$$= \int_s^t (L(\tau)v_\tau + g(\tau)) \, d\tau$$

and since $v(s) = \phi(0) - \psi(s) = 0$, we obtain

$$v(t) = v(s) + \int_s^t (L(\tau)v_\tau + g(\tau)) \, d\tau.$$  

This shows that $v$ is a solution of the equation

$$v' = L(t)v_t + g(t)$$

on $[s - r, +\infty)$ and so also on $\mathbb{R}$ (since $v_t = 0$ and $g(t) = 0$ for $t < s$). Thus, $v \in R(D)$ and $Rv = g$. Hence, by (33),

$$\sup\{\|u_t\| : t \in [s, +\infty)\} \leq \sup\{\|v_t\| : t \in [s, +\infty)\} + \|\phi\|$$

$$\leq |v|_\infty + \|\phi\| = |Sg|_\infty + \|\phi\|$$

$$\leq (\|S\|\beta + 1)\|\phi\|,$$

where $S$ is the inverse of the operator $R$ in (24), which shows that

$$\|u_t\| \leq (\|S\|\beta + 1)\|\phi\| \quad \text{for } t \geq s.$$

Now we prove that there exists $\ell \in \mathbb{N}$ (independent of $s$ and $\phi$) such that

$$\|u_t\| \leq \frac{1}{2}\|\phi\| \quad \text{for } t - s \geq \ell.$$  

(34)

Take $t_0 > s$ with $\|u_{t_0}\| > \|\phi\|/2$ (there is nothing to show if $t_0$ does not exist). It follows from (34) and this inequality with $t$, $s$ and $\phi$ replaced, respectively, by $t_0$, $\tau$ and $u_\tau$ that

$$\frac{1}{2(\|S\|\beta + 1)}\|\phi\| < \|u_\tau\| \leq (\|S\|\beta + 1)\|\phi\|$$  

(36)

for $s \leq \tau \leq t_0$. Let

$$w(t) = u(t) \int_{-\infty}^t \chi_{[s,t_0]}(\tau)\|u_\tau\|^{-1} \, d\tau$$

and

$$h(t) = \int_{-r}^0 d\eta(t, \theta) \left( u(t + \theta) \int_{t+\theta}^t \chi_{[s,t_0]}(\tau)\|u_\tau\|^{-1} \, d\tau \right) + \chi_{[s,t_0]}(t)u(t)\|u_t\|^{-1}$$

for $t \in \mathbb{R}$. Clearly, $w$ is continuous and since

$$|w(t)| \leq |u(t)| \int_s^{t_0} \|u_\tau\|^{-1} \, d\tau$$

$$\leq (\|S\|\beta + 1)\|\phi\| \int_s^{t_0} \|u_\tau\|^{-1} \, d\tau,$$
we have \( w \in C_b \). Moreover,

\[
|h(t)| \leq \Pi(t) \sup_{\theta \in [-r,0]} \left| u(t + \theta) \int_{t+\theta}^t \|u_r\|^{-1} \, d\tau \right| + 1.
\]

For \( \tau \in [t + \theta, t] \) we have \( t + \theta \in [\tau - r, \tau] \) and so \( \|u_r\| \geq |u(t + \theta)| \). Hence,

\[
|h(t)| \leq \Pi(t)r + 1
\]

and so \( h \in M \). Finally, we show that \( w \) is a solution of the equation

\[
w' = L(t)w_t + h(t).
\]

Let \( \alpha(t) = \chi_{[s,t_0]}(t)\|u_t\|^{-1} \). For \( t \geq s \) we have

\[
\begin{align*}
\int_s^t (L(\tau)w_r + h(\tau)) \, d\tau &= \int_s^t \int_{-r}^0 d\eta(\tau, \theta) \left( w(\tau + \theta) + u(\tau + \theta) \int_{\tau + \theta}^\tau \alpha(\sigma) \, d\sigma \right) \, d\tau \\
&= \int_s^t \int_{-r}^0 d\eta(\tau, \theta) u(\tau + \theta) \int_{\tau + \theta}^\tau \alpha(\sigma) \, d\sigma \, d\tau \\
&= \int_s^t (L(\tau)u_r) \int_{-\infty}^\tau \alpha(\sigma) \, d\sigma \, d\tau \\
&= \left( \int_s^t L(\sigma)u_\sigma \, d\sigma \int_{-\infty}^\tau \alpha(\sigma) \, d\sigma \right) \bigg|_{\tau = s}^{\tau = t} - \int_s^t \int_s^t L(\sigma)u_\sigma \, d\sigma \, d\alpha(\tau) \, d\tau \\
&= \int_s^t L(\sigma)u_\sigma \, d\sigma \int_{-\infty}^1 \alpha(\sigma) \, d\sigma - \int_s^t (u(\tau) - u(s))\alpha(\tau) \, d\tau,
\end{align*}
\]

which implies that

\[
\begin{align*}
\int_s^t (L(\tau)w_r + h(\tau)) \, d\tau &= \int_s^t L(\sigma)u_\sigma \, d\sigma \int_{-\infty}^t \alpha(\sigma) \, d\sigma + \int_s^t \phi(0) \, d\tau \\
&= \left( \int_s^t L(\sigma)u_\sigma \, d\sigma \int_{-\infty}^t \alpha(\sigma) \, d\sigma \right) + \int_s^t \phi(0) \, d\tau \\
&= u(t) \int_{-\infty}^t \alpha(\sigma) \, d\sigma = w(t).
\end{align*}
\]

Moreover, \( w_t = 0 \) and \( h(t) = 0 \) for \( t < s \). Hence, \( w \) is a solution of equation (38) and so \( w \in D(R) \) and \( Rw = h \). By (37) we have

\[
|w|_\infty = |Sh|_\infty \leq \|S\| \cdot |h|_M \leq \|S\|(\beta r + 1).
\]

Hence, it follows from (36) that

\[
(\beta r + 1)\|S\| \geq |w(t_0)|
\]

\[
\geq |u(t_0)| \int_s^{t_0} \|u_r\|^{-1} \, d\tau
\]

\[
\geq \|u_0\| \cdot \frac{t_0 - s}{K e^{\omega r}} \cdot \frac{t_0 - s}{(\|S\|\beta + 1)\|\phi\|}
\]

\[
\geq \frac{1}{2Ke^{\omega r}(\|S\|\beta + 1)^2} (t_0 - s)
\]
and so property (35) holds taking
\[
\ell > 2Ke^{\omega r}(\beta r + 1)\|S\|(\|S\|\beta + 1)^2.
\tag{39}
\]
Finally, given \( t \geq s \), write \( t - s = k\ell + \tau \), with \( k \in \mathbb{N} \) and \( 0 \leq \tau < \ell \). By (34) and (35), we obtain
\[
\|T(t, s)P(s)\phi\| = \|T(s + k\ell + \tau, s)P(s)\phi\|
\leq \frac{1}{2^k} \|T(s + \tau, s)P(s)\phi\|
\leq \frac{\|S\|\beta + 1}{2^k} \|P(s)\phi\|
\leq 2(\|S\|\beta + 1)Me^{-(t-s)\log 2/\ell} \|\phi\|
\]
for \( \phi \in C \). This completes the proof of the lemma. \( \square \)

**Step 5. Exponential bounds along the spaces \( F(s) \) in (27).** Let \( T(s, t) \) be the inverse of the linear operator in (11).

**Lemma 7.** There exist constants \( \lambda, D > 0 \) such that
\[
\|T(t, s)Q(s)\| \leq De^{-\lambda(s-t)} \quad \text{for} \quad t \leq s.
\]

**Proof of the lemma.** Take \( s \in \mathbb{R}, \phi \in F(s) \) and consider the continuous function \( \psi: (-\infty, s] \to \mathbb{R}^n \) defined by \( \psi_s = \phi \) and \( \psi(t) = \phi(-r) \) for \( t \leq s - r \). Moreover, we define \( g: \mathbb{R} \to \mathbb{R}^n \) by
\[
g(t) = L(t)\psi_t = \int_{-r}^{0} d\eta(t, \theta)\psi(t + \theta)
\]
for \( t \leq s \) and \( g(t) = 0 \) for \( t > s \). In view of (10) we have
\[
|g|_M = \sup_{t \in \mathbb{R}} \int_{t}^{t+1} |g(\tau)| d\tau
\leq \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \Pi(\tau) d\tau \|\phi\| < +\infty
\]
and so \( g \in M \). Now we define \( u_t = T(t, s)\phi \) for \( t \leq s \). Proceeding as in the proof of Lemma 6, one can show that the continuous function \( v: \mathbb{R} \to \mathbb{R}^n \) defined by
\[
v(t) = \begin{cases} u(t) - \psi(t) & \text{if} \ t \leq s, \\ 0 & \text{if} \ t > s \end{cases}
\]
is in \( D(R) \) and satisfies \( Rv = g \), again with the operator \( R \) as in (24). Hence, by (33),
\[
\sup\{\|u_t\| : t \in (-\infty, s]\} \leq \sup\{\|v_t\| : t \in (-\infty, s]\} + \|\phi\|
\leq |v|_\infty + \|\phi\| = |Sg|_\infty + \|\phi\|
\leq (\|S\|\beta + 1)\|\phi\|,
\]
where \( S \) is the inverse of \( R \), which shows that
\[
\|u_t\| \leq (\|S\|\beta + 1)\|\phi\| \quad \text{for} \quad t \leq s. \tag{40}
\]
Now we prove that there exists \( \ell \in \mathbb{N} \) (independent of \( s \) and \( \phi \)) such that
\[
\|u_t\| \leq \frac{1}{2} \|\phi\| \quad \text{for} \ s - t \geq \ell. \tag{41}
\]
Take $t_0 < s$ with $\|u_{t_0}\| > \|\phi\|/2$. It follows from (40) and this inequality with $t$, $s$ and $\phi$ replaced, respectively, by $t_0$, $\tau$ and $u_\tau$ that
\[
\frac{1}{2(\|S\|\beta + 1)}\|\phi\| < \|u_\tau\| \leq (\|S\|\beta + 1)\|\phi\|
\] (42)
for $s \geq \tau \geq t_0$. Let
\[
w(t) = u(t) \int_t^{+\infty} \chi_{[t_0,s]}(\tau)\|u_\tau\|^{-1} d\tau
\]
and
\[
h(t) = - \int_{-\infty}^0 d\eta(t,\theta) \left( u(t + \theta) \int_{t+\theta}^t \chi_{[t_0,s]}(\tau)\|u_\tau\|^{-1} d\tau \right)
\]
\[- \chi_{[t_0,s]}(t)u(t)\|u_t\|^{-1}
\]
for $t \in \mathbb{R}$. One can show in a similar manner to that in the proof of Lemma 6 that $w \in C_b$ and $h \in M$ with $|h|_M \leq \beta r + 1$. Moreover, $w \in D(R)$ and $Rw = h$. Therefore,
\[
|w|_\infty = |Sh|_\infty \leq \|S\| \cdot |h|_M \leq \|S\| (\beta r + 1).
\]
Hence, it follows from (42) that
\[
(\beta r + 1)\|S\| \geq |w(t_0)|
\]
\[
\geq |u(t_0)| \int_{t_0}^s \|u_\tau\|^{-1} d\tau
\]
\[
\geq \frac{\|u_{t_0}\|}{Ke^{\omega r}} \cdot \frac{s - t_0}{(\|S\|\beta + 1)\|\phi\|}
\]
\[
\geq \frac{1}{2K e^{\omega r}(\|S\|\beta + 1)^2} (s - t_0)
\]
and so property (41) holds taking $\ell$ as in (39).

Finally, given $t \leq s$, write $s - t = k\ell + \tau$, with $k \in \mathbb{N}$ and $0 \leq \tau < \ell$. By (40) and (41), we obtain
\[
\|T(t,s)Q(s)\phi\| = \|T(s - k\ell - \tau, s)Q(s)\phi\|
\]
\[
\leq \frac{1}{2^k}\|T(s - \tau, s)Q(s)\phi\|
\]
\[
\leq \frac{\|S\|\beta + 1}{2^k} \|Q(s)\phi\|
\]
\[
\leq 2(\|S\|\beta + 1)Me^{-(s-t)\log 2/\ell}\|\phi\|
\]
for $\phi \in C$. This completes the proof of the lemma. \qed

Combining the former lemmas we conclude that equation (8) has an exponential dichotomy on $\mathbb{R}$. This completes the proof of the theorem. \qed

5. Admissibility on the half-line

In this section we obtain corresponding results on the half-line leading to a characterization of the notion of an exponential dichotomy in terms of an admissibility property. Let
\[
C'_b = C_b(I_+, \mathbb{R}^n), \quad C'_{b,B} = \{v \in C'_b : v_0 \in B\} \quad \text{and} \quad M' = M(I_+, \mathbb{R}^n),
\]
where \( I_r = [-r, +\infty) \), be the spaces introduced in Section 3.

**Theorem 7.** If equation (8) has an exponential dichotomy on \( \mathbb{R}_0^+ \), then for each \( g \in M' \) there exists a unique \( v \in C'_{b,B} \) with \( B = \text{Im} \, Q(0) \) satisfying (23) for all \( t \geq s \geq 0 \).

**Proof.** Take a function \( g \in M' \). For each \( t \geq 0 \) we define \( v_t = x_t + y_t \), where

\[
x_t = \int_0^t T_0(t, \tau)P_0(\tau)g(\tau) \, d\tau
\]

and

\[
y_t = -\int_t^{+\infty} T(t, \tau)Q_0(\tau)g(\tau) \, d\tau.
\]

Proceeding as in the proof of Theorem 5, we find that \( v \in C'_{b,B} \) and that (23) holds for \( t \geq s \geq 0 \). Moreover, one can easily verify that \( P(0)v_0 = 0 \) and so \( v_0 \in B \). This shows that \( v \in C'_{b,B} \).

For the uniqueness of the solution \( v \) it suffices to show that if \( v_t = T(t, s)v_s \) for \( t \geq s \geq 0 \) with \( v \in C'_{b,B} \), then \( v = 0 \). It follows from Proposition 4 that

\[
\|Q(0)v_0\| = \|\overline{T}(0, t)Q(t)v_t\| \\
\leq Ne^{-\lambda t}\|v_t\| \\
\leq Ne^{-\lambda t}\|v\|_{\infty}
\]

for \( t \geq 0 \). Hence \( v_0 = Q(0)v_0 = 0 \) and so \( v = 0 \). This completes the proof of the theorem. \( \Box \)

Now we establish a result in the other direction.

**Theorem 8.** If there exists a closed subspace \( B \subset C \) such that for each \( g \in M' \) there exists a unique \( v \in C_{b,B}' \) satisfying (23) for all \( t \geq s \geq 0 \), then equation (8) has an exponential dichotomy on \( \mathbb{R}_0^+ \).

**Proof.** We define a linear operator \( R_B \) by

\[
R_B: D(R_B) \to M', \quad R_Bv = g
\]

in the domain \( D(R_B) \) of all \( v \in C_{b,B}' \) for which there exists \( g \in M' \) such that (23) holds for all \( t \geq s \geq 0 \). One can show as in the proof of Theorem 6 that \( R_B \) is well defined.

By the admissibility hypothesis in the theorem, the operator \( R_B \) in (43) has an inverse \( S_B: M' \to C_{b,B}' \). Moreover, proceeding as in the proof of Lemma 1 we find that \( R_B \) is closed. It follows from the closed graph theorem that \( R_B \) and so also \( S_B \) are bounded.

For each \( s \geq 0 \), let

\[
E(s) = \left\{ \phi \in C : \sup_{t \geq s} \|T(t, s)\phi\| < +\infty \right\} \quad \text{and} \quad F(s) = T(s, 0)B.
\]

Note that \( E(s) \) and \( F(s) \) are subspaces of \( C \).

**Lemma 8.** For each \( s \geq 0 \) we have

\[
C = E(s) \oplus F(s).
\]
Proof of the lemma. Take $s \geq 0$ and $\phi \in C$. We define $\psi: [s-r, +\infty) \to \mathbb{R}^n$ by $\psi_s = \phi$ and $\psi(t) = \phi(0)$ for $t \geq s$. Now we consider the function $g: \mathbb{R}^+_0 \to \mathbb{R}^n$ given by

$$g(t) = L(t)\psi_t = \int_{-r}^0 d\eta(t, \theta)\psi(t + \theta)$$

for $t \geq s$ and $g(t) = 0$ for $t \in [0, s]$. Proceeding as in (29) we find that $g \in M^r$. Hence, there exists $v \in C^r_{b,B}$ such that $R_Bv = g$, with $R_B$ as in (43). Moreover, one can show as in the proof of Lemma 2 that $v_s + \phi \in E(s)$. On the other hand, it follows from (23) that $v_s = T(s, 0)v_0$ (recall that $g(t) = 0$ for $t \in [0, s]$). Since $v \in C^r_{b,B}$, we have $v_s \in F(s)$ and so

$$\phi = (v_s + \phi) - v_s \in E(s) + F(s).$$

Now take $\phi \in E(s) \cap F(s)$ and $\psi \in B$ such that $\phi = T(s, 0)\psi$. We define a map $u: [-r, +\infty) \to \mathbb{R}^n$ by $u_t = T(t, 0)\psi$ for $t \geq 0$. Clearly, $u$ is continuous, $\sup_{t \geq -r} |u(t)| < +\infty$ and $u_0 \in B$. Moreover, one can easily verify that

$$u_t = T(t, \tau)u_\tau \quad \text{for} \quad t \geq \tau \geq 0.$$ 

Hence, $u \in D(R_B)$ and $R_Bu = 0$. Since $R_B$ is invertible, we conclude that $u = 0$ and thus $\phi = u_s = 0$. \hfill $\Box$

Let $P(s), Q(s): C \to C$ be the projections associated with the splitting in (44).

**Lemma 9.** For each $t \geq s \geq 0$, the map

$$T(t, s)|F(s): F(s) \to F(t) \quad (45)$$

is onto and invertible.

Proof of the lemma. Assume that $T(t, s)\phi = 0$ for some $\phi \in F(s)$. Since $\phi \in F(s)$, there exists $\psi \in B$ such that $\phi = T(s, 0)\psi$. Now we define a map $v: [-r, +\infty) \to \mathbb{R}^n$ by

$$v_\tau = T(\tau, 0)\psi \quad \text{for} \quad \tau \geq 0.$$ 

Clearly, $v$ is continuous and $\sup_{\tau \geq -r} |v(\tau)| < +\infty$. Moreover, $v_\tau = T(\tau, \overline{\tau})v_\overline{\tau}$ for $\tau \geq \overline{\tau} \geq 0$. Hence, $v \in D(R_B)$ and $R_Bv = 0$, with $R_B$ as in (43). Since $R_B$ is invertible, we have $v = 0$ and thus $\phi = v_s = 0$. This shows that the map in (45) is one-to-one.

On the other hand, given $\phi \in F(t)$, there exists $\psi \in B$ such that $\phi = T(t, 0)\psi$. Since $T(s, 0)\psi \in F(s)$ and

$$\phi = T(t, s)T(s, 0)\psi,$$

we conclude that the map in (45) is also onto. \hfill $\Box$

Proceeding as in the proofs of Lemmas 4 and 5 we find that

$$P(t)T(t, s) = T(t, s)P(s) \quad \text{for} \quad t \geq s \geq 0$$

and that there exists $K' > 0$ such that

$$\|P(s)\| \leq K' \quad \text{for} \quad s \geq 0.$$
Now let \( \overline{T}(s,t) \) be the inverse of the map in (45). Proceeding as in the proofs of Lemmas 6 and 7 we find that there exist constants \( \lambda, D > 0 \) such that
\[
\| T(t,s)P(s) \| \leq De^{-\lambda(t-s)} \quad \text{and} \quad \| \overline{T}(s,t)Q(t) \| \leq De^{-\lambda(t-s)}
\]
for \( t \geq s \geq 0 \). Combining the former results we conclude that equation (8) has an exponential dichotomy on \( \mathbb{R}_0^+ \).

Combining Theorems 7 and 8, we obtain the following characterization of an exponential dichotomy on \( \mathbb{R}_0^+ \).

**Theorem 9.** Equation (8) has an exponential dichotomy on \( \mathbb{R}_0^+ \) if and only if there exists a closed subspace \( B \subset C \) such that for each \( g \in M' \) there exists a unique \( v \in C'_{b,B} \) satisfying (23) for \( t \geq s \geq 0 \).

**References**


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