Julia Robinson numbers and arithmetical
dynamic of quadratic polynomials

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Abstract

For the ring of integers of a totally real field of algebraic numbers, J. Robinson defined a set which is always \{+∞\} or of the form \([\lambda, +\infty)\) or \((\lambda, +\infty)\) for some real number \(\lambda \geq 4\). All known examples give either \{+∞\} or \([4, +\infty)\). In this paper, we construct infinitely many fields such that the set is an interval, but not equal to \([4, +\infty)\).\(^1\)

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1 Introduction

Motivated by a problem in Logic, for any given field \(K\) of totally real algebraic numbers, Julia Robinson [6] considered the following set \(A(O_K)\):

\[ \{t \in \mathbb{R} \cup \{+\infty\} : \text{there are infinitely many } r \in O_K \text{ such that } 0 \ll r \ll t\}, \]

where \(0 \ll r \ll t\) means that every conjugate of \(r\) lies strictly between \(0\) and \(t\), and \(O_K\) is the ring of integers of \(K\). The set \(A(O_K)\) is either the set \{+∞\}, or an interval of the form \([\lambda, +\infty)\) or \((\lambda, +\infty)\). We define the JR number of \(O_K\) to be \(+\infty\) when \(A(O_K) = \{+\infty\}\), and \(\lambda\) when it is an interval.

The study of the distribution of algebraic integers is a problem that goes back to Kronecker [2]. He proved that the only totally real integers which lie, together with their conjugates, in the interval \([-2, 2]\), are \(2\cos(2\pi r)\), with \(r \in \mathbb{Q}\). From this one can deduce that we always have \(\lambda \geq 4\) — see [6], or [1] for a more

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detailed account. In all of Julia Robinson’s examples, \( A(\mathcal{O}_K) \) is either \( \{ +\infty \} \) (if \( K \) is a totally real number field, or if \( K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots) \)), or \( [4, +\infty) \) (if \( K \) is the ring of integers of the field of all totally real algebraic integers). She asks whether there is any \( K \) such that \( A(\mathcal{O}_K) \) is an open interval.

Note that, by a result of Pólya, any interval of size \( < 4 \) contains finitely many sets of conjugates of totally real algebraic integers, whereas the intervals of length \( > 4 \) contain infinitely many of them, as proved in [7] by Raphael Robinson. If we restrict ourselves to smaller rings of integers, as Julia Robinson does, it is unclear what the relevant length is.

The JR number of a ring is relevant for decision problems in Logic, as discovered by Julia Robinson, and it is also connected to the Northcott property of sets of algebraic numbers — see [11] and [14]. Note that in [10], it is proved that there are subrings \( R \) of rings of the form \( \mathcal{O}_K \) such that \( A(R) \) is an open interval and \( \lambda \) is different from 4 and \( +\infty \). It is not known whether any of these rings is an \( \mathcal{O}_K \). In this paper, we explore this problem for 2-towers of nested roots.

An immediate corollary of our main result is the following.

**Theorem 1.1.** There are infinitely many fields \( K \) with \( A(\mathcal{O}_K) \) distinct from \( \{ +\infty \} \) and \( [4, +\infty) \).

Our fields \( K \) are the ones that were already considered in [10], namely, fields of nested square roots defined in the following way. Let \( \nu \) be a non-square integer \( > 4 \). Let \( x_1 = \sqrt{\nu} \), and for each \( n \geq 1 \), \( x_{n+1} = \sqrt{\nu + x_n} \). Note that for each \( n \), the field \( K_n = \mathbb{Q}(x_n) \) is totally real, and that since \( \nu \) is not a square, the tower increases at each step — apply [9, Cor. 1.3] to the iterations of \( f(t) = t^2 - \nu \).

Write \( \mathcal{O}^\nu = \bigcup_n \mathcal{O}_{K_n} \).

Notice that \( \mathcal{O}^\nu \) is the ring of integers of \( \bigcup_n K_n \).

We prove:

**Theorem 1.2.** Suppose that \( \nu = 2^m \mu \), with \( m \geq 1 \), \( \mu \geq 3 \) odd and not a quadratic residue modulo any Fermat prime greater than 3. The JR number of \( \mathcal{O}^\nu \) is either strictly between 4 and \( +\infty \), or it is 4 and it is not a minimum.

In Section 5 we prove that 3 and 7 are non-squares modulo any Fermat prime greater than 3. So for example, for any odd integer \( k \) and for any \( m \geq 1 \), \( \nu = 2^m k^2 \cdot 3 \) and \( \nu = 2^m k^2 \cdot 7 \) satisfy the hypothesis of Theorem 1.2.

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## 2 Sketch of the proof of Theorem 1.1

The fact that the JR number is not \( \{ +\infty \} \) is an immediate consequence of the fact that \( \mathcal{O}^\nu \) has a subring with JR number not \( \{ +\infty \} \) — the subring in
question is $\bigcup_n \mathbb{Z}[x_n]$ and its JR number is $[\alpha] + \alpha$, where $\alpha = (1 + \sqrt{1 + 4\nu})/2$. This is proven in [10, Thm. 1.4]. The hypothesis of this theorem that $\nu$ must be congruent to 2 or 3 modulo 4 is not satisfied for our choice of $\nu$, but this hypothesis was only there to ensure that the tower increases at each step.

If $m$ is an integer, we write $\zeta_m$ for a primitive $m$-th root of unity.

The JR number of $\mathcal{O}^{\nu}$ is 4 and is a minimum if and only if, in $\mathcal{O}^{\nu}$ there exist infinitely many numbers of the form

$$\zeta_m^j + \zeta_m^{-j} = 2 \cos \left(\frac{2\pi j}{m}\right),$$

with $j = 1, \ldots, m - 1$, if and only if in $\mathcal{O}^{\nu}$ there exist infinitely many numbers of the form

$$\zeta_m + \zeta_m^{-1} = 2 \cos \left(\frac{2\pi}{m}\right).$$

The first equivalence is a consequence of a theorem of Kronecker, see [4, Thm. 2.5].

Since the fraction field of $\mathcal{O}^{(\nu,x_0)}$ is a 2-tower, for each $m$, $\zeta_m + \zeta_m^{-1}$ is constructible with ruler and compass, and we have the following equivalence: $\zeta_m + \zeta_m^{-1}$ is constructible with ruler and compass, if and only if $\zeta_m + \zeta_m^{-1}$ is constructible with ruler and compass, if and only if $m = 2^d p_1 \cdots p_k$, where $d \geq 0$ and $p_i$ are distinct Fermat Primes (by the Gauss-Wantzel theorem). Thus, the strategy consists in finding $\nu$ such that $\mathcal{O}^{\nu}$ has only finitely many numbers $\zeta_m + \zeta_m^{-1}$ with $m$ of the form $2^d p_1 \cdots p_k$.

The proof is then done in two steps. In Section 4, we will prove the following proposition.

**Proposition 2.1.** Assume that $K_n$ has degree $2^n$ over $\mathbb{Q}$. Suppose that

1. for every Fermat prime $p > 3$, $\nu$ is not a square modulo $p$, and
2. $\sqrt{2}$ is not in $\mathcal{O}^{\nu}$.

The JR number of $\mathcal{O}^{\nu}$ is either strictly between 4 and $+\infty$, or it is 4 and it is not a minimum.

In Section 6 we prove that if $\nu = 2^m \mu$, with $m \geq 1$ and $\mu \geq 3$ odd and square-free, then $\sqrt{2}$ is not in $\mathcal{O}^{\nu}$.

Putting everything together, this proves Theorem 1.2. Note that if there are only finitely many Fermat primes, then item 1 of Proposition 2.1 is not relevant for our purposes, because they would contribute only to finitely many elements of the form $\zeta_m + \zeta_m^{-1}$.

3 The discriminant of $x_n$.

For each $n$, let $P_n(t) = f^{(n)}(t)$, where $f(t) = t^2 - \nu$. By [9, Corollary 1.3], each polynomial $P_n$ is the minimal polynomial of $x_n$. In this section we prove the following proposition.
**Proposition 3.1.** Assume that $K_n$ has degree $2^n$ over $\mathbb{Q}$. We have
\[
\text{disc}(x_1) = 4\nu,
\]
and for $n \geq 2$ we have
\[
\text{disc}(x_n) = (\text{disc}(x_{n-1}))^2 \cdot 2^{2^n} P_n(0).
\]

The field $K_n$ has basis
\[
B_n := \{1, x_n, x_n^2, \ldots, x_n^{2^n-1}\}
\]
over $\mathbb{Q}$. Note that the field extension $K_n/K_m$ has degree $2^{n-m}$. We will denote by $\text{disc}_{n-1}^n(x_n)$ the discriminant of the basis $(1, x_n)$ from $K_n$ to $K_{n-1}$. Hence, for $n \geq 1$, we have
\[
\text{disc}_{n-1}^n(x_n) = \left| \begin{array}{cc} 1 & x_n \\ 1 & -x_n \end{array} \right|^2 = 4(x_n)^2 = 4(\nu + x_{n-1}).
\]

**Notation 3.2.** For $n \geq 1$, we denote by $N_n$ the norm from $K_n$ to $\mathbb{Q}$ of $\text{disc}_{n+1}^n(x_{n+1})$, and by $N_0$ the discriminant of $x_1$ from $K_1$ to $\mathbb{Q}$.

**Proposition 3.3.** We have

1. $N_0 = 2^{2\nu}$, and
2. $N_n = 2^{2^{n+1}} P_{n+1}(0)$ for any $n \geq 1$.

**Proof.** Item 1 is immediate from our above computation, so we prove item 2. Let $\ell_1 = \nu^2$ and $\ell_n = ((\ell_{n-1}) - \nu)^2$ for $n \geq 2$. Let $n \geq 1$. We have
\[
N_n = \text{Norm}_{\mathbb{Q}}^{K_n}(\text{disc}_{n+1}^n(x_{n+1}))
= \text{Norm}_{\mathbb{Q}}^{K_n}(4(\nu + x_n))
= (2^2)^{2^n} \text{Norm}_{\mathbb{Q}}^{K_n}(\nu + x_n)
= 2^{2^{n+1}} \prod_{i=1}^{2^n} (\nu + x_n^{\sigma_i^n}),
\]
where the $\sigma_i^n$ are the $2^n$ embeddings from $K_n$ to $\mathbb{C}$.

**Fact.** For all $t \in \{0, \ldots, n\}$ we have
\[
\prod_{i=1}^{2^n} \left( \nu + x_n^{\sigma_i^n-1} \right) = \prod_{i=1}^{2^{n-t}} \left( \ell_i - (\nu + x_{n-t})^{\sigma_i^{n-t}} \right).
\]

We prove the fact by induction on $t$. Assume it is true for $t - 1$, namely,
\[
\prod_{i=1}^{2^n} \left( \nu + x_n^{\sigma_i^n-1} \right) = \prod_{i=1}^{2^{n-(t-1)}} \left( \ell_{t-1} - (\nu + x_{n-(t-1)})^{\sigma_i^{n-(t-1)}} \right),
\]
we have
\[
\prod_{i=1}^{2^n} \left( \nu + x_n^{\sigma_{n-1}'} \right) = \prod_{i=1}^{2^{n-t+1}} \left( \ell_t - (\nu - x_{n-t+1})\sigma_{n-t+1}^{\sigma_{n-t+1}'} \right) \\
= \prod_{i=1}^{2^{n-t+1}} \left( \ell_t - \nu + x_{n-t+1}^{\sigma_{n-t+1}'} \right) \\
= \prod_{i=1}^{2^{n-t}} \left( (\ell_t - \nu) - x_{n-t+1}^{\sigma_{n-t}} \right) \left( (\ell_t - \nu) + x_{n-t+1}^{\sigma_{n-t}} \right) \\
= \prod_{i=1}^{2^{n-t}} (\ell_t - \nu)^2 - (x_{n-t+1}^{\sigma_{n-t}})^2 \\
= \prod_{i=1}^{2^{n-t}} (\ell_t - (\nu + x_{n-t+1})\sigma_{n-t}^{\sigma_{n-t}'}). \\
\]
This proves the fact.
Hence, taking \( t = n \) in the Fact above, we obtain
\[
\prod_{i=1}^{2^n} \left( \nu + x_n^{\sigma_{n-1}'} \right) = (\ell_n - \nu) = P_{n+1}(0).
\]

We need the following proposition — see [3, Chap. 2, Exercise 23, p. 43].

**Proposition 3.4.** Let \( K \subseteq L \subseteq M \) be number fields, \([L: K] = n\), \([M: L] = m\), and let \( \{\alpha_1, \ldots, \alpha_n\} \) and \( \{\beta_1, \ldots, \beta_m\} \) be bases for \( L \) over \( K \) and \( M \) over \( L \), respectively. We have
\[
\text{disc}_K^M (\alpha_1\beta_1, \ldots, \alpha_n\beta_m) = \left( \text{disc}_K^L (\alpha_1, \ldots, \alpha_n) \right)^m \cdot \text{Norm}_{K/L}^M \left( \text{disc}_L^M (\beta_1, \ldots, \beta_m) \right).
\]

Proposition 3.1 follows from Propositions 3.3 and 3.4 in the following way. Take
\[ K = \mathbb{Q}, \quad L = K_{n-1} \quad \text{and} \quad M = K_n. \]

The degree of \( L \) over \( K \) is \( 2^{n-1} \) and \( L \) has basis
\[ \{1, x_{n-1}, x_n^2, \ldots, x_{n-1}^{2^{n-1}-1} \} \]
over \( K \), while the degree of \( M \) over \( L \) is 2 and \( M \) has basis \( \{1, x_n\} \) over \( L \). The set \( \{\alpha_1\beta_1, \ldots, \alpha_n\beta_m\} \) in Proposition 3.4 corresponds to the set
\[ B' = \left\{1, x_{n-1}, x_n^2, \ldots, x_{n-1}^{2^{n-1}-1}, x_n, x_{n-1}x_n, x_{n-1}^2x_n, \ldots, x_{n-1}^{2^{n-1}-1}x_n \right\}. \]
This set $B'$ is a basis for $M$ over $K$. Indeed, we have

$$|B'| = 2 \left(2^n - 1 \right) + 2 = 2^n = |B_n|,$$

and since $x^2_n = \nu + x_{n-1}$, each element of $B_n$ can be written as a $\mathbb{Z}$-linear combination of elements of $B'$. Similarly, each element of $B'$ is a $\mathbb{Z}$-linear combination of elements of $B_n$. Since the base change matrices from $B_n$ to $B'$ and from $B'$ to $B_n$ have an integral determinant and because the discriminants are also integers, we deduce

$$\text{disc}_K^M(B') = \text{disc}_K^M(B_n) = \text{disc}_K^M(x_n).$$

One obtains the formula in Proposition 3.1 by using in Proposition 3.4 the formulas from Proposition 3.3.

4 Proof of Proposition 2.1

We will need the following proposition.

**Proposition 4.1** (Prop. 2.13, [4]). Let $\theta$ be an algebraic integer. We have

$$\text{disc}(\theta) = m^2 \text{disc}(\mathbb{Q}(\theta)),$$

where $m$ is the index in $\mathcal{O}_{\mathbb{Q}(\theta)}$ of the $\mathbb{Z}$-module $\mathbb{Z}[\theta]$.

The following remark shows that it is sufficient to consider $\zeta_m + \zeta_m^{-1}$ where $m \in \{2^d : d \geq 2 \} \cup \{p : p$ is a Fermat prime$\}$.

**Remark 4.2.** Let $m_1$ and $m_2$ be positive coprime integers, and write $m = m_1 m_2$. The field $\mathbb{Q}(\zeta_{m_1} \zeta_{m_2})$ is the compositum of $\mathbb{Q}(\zeta_{m_1})$ and $\mathbb{Q}(\zeta_{m_2})$.

We need the following result.

**Proposition 4.3.**

1. ([13], p. 15) The field $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ is the maximal totally real subfield of $\mathbb{Q}(\zeta_m)$. The extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ is of degree 2.

2. ([13], Ex. 2.1, p. 17) Let $p$ be a prime number. The field $\mathbb{Q}(\zeta_p)$ contains the field $\mathbb{Q}(\sqrt{p})$ if $p \equiv 1 \pmod{4}$ and contains $\mathbb{Q}(\sqrt{-p})$ if $p \equiv 3 \pmod{4}$.

3. Let $K$ be a number field. The number $p$ is ramified in $K$ if and only if $p$ divides $\text{disc } K$.

We prove the following proposition.

**Proposition 4.4.** Let $p > 3$ be a Fermat prime. If $\zeta_p + \zeta_p^{-1} \in \mathcal{O}_p$, then there exists $n \geq 1$ such that $p$ divides $\text{disc } K_n$. 

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Proof. Let \( p = 2^{2m} + 1 > 3 \) be a Fermat prime. Note that, since \( m \geq 1 \), \( p \) is congruent to 1 modulo 4. Hence, by Proposition 4.3, we have

\[
\mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(\zeta_p + \zeta_p^{-1}),
\]

hence \( \sqrt{p} \in \mathcal{O}^\nu \) by hypothesis, so in particular \( \sqrt{p} \) lies in \( K_n \) for some \( n \geq 1 \). Therefore, \( p = (\sqrt{p})^2 \) is ramified in \( K_n \), so \( p \) divides \( \text{disc} K_n \) by Proposition 4.3.

**Lemma 4.5.** Let \( p \) be an odd prime. If \( p \) divides \( \text{disc} (x_n) \), then \( p \) divides the product \( P_1(0) \cdots P_n(0) \).

*Proof.* By induction on \( n \). For \( n = 1 \) we have \( \text{disc} (x_1) = 4\nu = -4P_1(0) \).

If it is true for \( n \), then it is true for \( n + 1 \) by Proposition 3.1, since we have

\[
\text{disc} (x_{n+1}) = (\text{disc} (x_n))^2 \cdot 2^{2n+1} P_{n+1}(0).
\]

**Corollary 4.6.** Let \( p \) be an odd prime. If \( p \) divides \( \text{disc} (K_n) \) for some \( n \geq 1 \), then \( p \) divides the product \( P_1(0) \cdots P_n(0) \).

*Proof.* This is an immediate consequence of Lemma 4.5, because we know by Proposition 4.1 that the discriminant of \( K_n \) divides the discriminant of \( x_n \).

**Proposition 4.7.** Let \( p > 3 \) be a Fermat prime. If \( \nu \) is not a square modulo \( p \) (so in particular \( p \) does not divide \( \nu \)), then for each \( n \geq 1 \), \( p \) does not divide \( \text{disc} K_n \).

*Proof.* We use induction on \( n \). For \( n = 1 \) we have that

\[
\text{disc } \mathbb{Q}(x_1) = \begin{cases} 
\nu, & \text{if } \nu \equiv 1 \mod 4 \\
4\nu, & \text{if } \nu \equiv 2, 3 \mod 4
\end{cases}
\]

In both cases, since \( p \) does not divide \( \nu \), we have that \( p \) does not divide \( \text{disc } \mathbb{Q}(x_1) \).

Assume by contradiction that \( p \) divides the discriminant of \( K_n \), so that \( p \) divides \( P_j(0) \) for some \( j \in \{1, \ldots, n\} \) by Corollary 4.6. If \( j = 1 \), then \( p \) divides \( \nu \), which contradicts our hypothesis. Assume \( j > 1 \). Recall that \( P_n(t) = f^n(t) \), where \( f(t) = t^2 - \nu \). Therefore, we have

\[
P_j(0) = P_{j-1}(0)^2 - \nu,
\]

which contradicts the hypothesis that \( \nu \) is not a square modulo \( p \).

*Proof of Proposition 2.1.* We follow the strategy described in the introduction. Let \( p \) be a Fermat Prime greater than 3. By Proposition 4.7, if \( \nu \) is not a square modulo \( p \), then \( p \) does not divide \( \text{disc} K_n \), so \( \zeta_p + \zeta_p^{-1} \) does not lie in \( \mathcal{O}^\nu \) by Proposition 4.4.
Let \( s_1 = \sqrt{2} \) and \( s_n = \sqrt{2} + s_{n-1} \). Since 
\[
\zeta_{2^d} + \zeta_{2^d}^{-1} = \begin{cases} 
-2 & \text{if } d = 1 \\
 s_{d-1} & \text{if } d \geq 2, 
\end{cases}
\]
and \( \sqrt{2} \) is not in \( \mathcal{O} \) by hypothesis, \( \zeta_{2^d} + \zeta_{2^d}^{-1} \) does not lie in \( \mathcal{O} \) for any \( d \geq 2 \). Remark 4.2 allows us to conclude the proof. \( \square \)

5 Some non-squares modulo all Fermat primes greater than 3

We start with an easy lemma.

**Lemma 5.1.** For all \( n \geq 1 \), we have
\[
2^{2^n} + 1 \equiv \begin{cases} 
3 & \text{if } n \text{ is even} \\
5 & \text{if } n \text{ is odd}, 
\end{cases} \quad (\text{mod } 7)
\]
and
\[
2^{2^n} + 1 \equiv 2 \quad (\text{mod } 3). 
\]

**Proof.** Since \( 2^n \) is congruent to \((-1)^n \) modulo 3, we have \( 2^n = 1 + 3k \) for some odd \( k \) when \( n \) is even, and \( 2^n = 2 + 3k \) for some even \( k \) when \( n \) is odd. Therefore, we have
\[
2^{2^n} = \begin{cases} 
2^{1+3k} = 2 \cdot 8^k \equiv 2 \quad (\text{mod } 7) & \text{if } n \text{ is even} \\
2^{2+3k} = 4 \cdot 8^k \equiv 4 \quad (\text{mod } 7) & \text{if } n \text{ is odd}. 
\end{cases}
\]
and
\[
2^{2^n} = \begin{cases} 
2^{1+3k} \equiv 2 \cdot (-1)^k \equiv 1 \quad (\text{mod } 3) & \text{if } n \text{ is even} \\
2^{2+3k} \equiv 4 \cdot (-1)^k \equiv 1 \quad (\text{mod } 3) & \text{if } n \text{ is odd}. 
\end{cases}
\]

**Proposition 5.2.** The numbers 3 and 7 are not squares modulo any Fermat prime greater than 3.

**Proof.** Let \( p = 2^{2^n} + 1 \) be a Fermat prime greater than 3. By the quadratic reciprocity law, since \( p \neq 7 \), we have
\[
\left( \frac{7}{p} \right) \left( \frac{p}{7} \right) = (-1)^{\frac{2^{2^n} - 6}{2}} = (-1)^{2^{2^n-1} \cdot 3} = 1,
\]
hence, 7 is a square modulo \( p \) if and only if \( p \) is a square modulo 7. Since the set of squares modulo 7 is \( \{0, 1, 2, 4\} \), we deduce by Lemma 5.1 that \( p \) is not a square modulo 7.

Similarly, we have \( \left( \frac{3}{p} \right) \left( \frac{7}{3} \right) = 1 \), so we can proceed as above. \( \square \)
6 Galois Group of $K_n$.

In this section, we assume that $\nu$ is not a square.

Let $C_2$ be the cyclic group of order 2, and denote by $[C_2]^n$ the $n$-fold wreath product of $C_2$ — for basic facts about the wreath product, we refer the reader to [8].

Let $L_n$ be the Galois closure of $K_n$, and $\text{Gal}(L_n)$ be its Galois group. The following is a particular case of a theorem by M. Stoll [9, Section 3, p. 243].

**Theorem 6.1.** If $\nu$ is a multiple of 4, then $\text{Gal}(L_n) \cong [C_2]^n$.

In order to show that $\sqrt{2}$ is not in $K_n$, we will show that it is not in $L_n$. For this we will use a counting argument. First we will show that there are exactly $2^n - 1$ quadratic subfields of $L_n$. Then we will construct $2^n - 1$ quadratic subfields, none of which is $\mathbb{Q}(\sqrt{2})$.

**Lemma 6.2.** There are $2^n - 1$ subfields of $L_n$ which are quadratic extensions of $\mathbb{Q}$.

**Proof.** We will give two different proofs. By the Galois correspondence, we need to count how many subgroups $H$ of $[C_2]^n$ are such that the quotient $[C_2]^n/H$ is isomorphic to $C_2$.

**Proof 1.** We prove that $[C_2]^n$ has $2^n - 1$ subgroups of index 2 (they are maximal subgroups). Let $M$ be the set of maximal subgroups of $[C_2]^n$. Since $[C_2]^n$ has order $2^{2^n-1}$, it is a 2-group. The groups in $M$ have index 2, so they are normal. The intersection of all the maximal subgroups of $[C_2]^n$ is called the *Frattini subgroup* of $[C_2]^n$ and is denoted by $\phi([C_2]^n)$. By [8, Th 5.48], the group $\phi = \phi([C_2]^n)$ is normal, and the quotient $[C_2]^n/\phi$ is an $F_2$-vector space. Let $d$ be the dimension of this vector space.

For every $H \in M$, since $\phi \leq H \leq [C_2]^n$, the quotient $H/\phi$ is a subspace of $[C_2]^n/\phi$, and every subspace of $[C_2]^n/\phi$ corresponds to a maximal subgroup $H$. It is easy to see that the number of non-trivial subspaces of a vector space of dimension $d$ over $F_2$ is $2^d - 1$. So we have $2^d - 1$ maximal subgroups.

On the other hand, by Burnside’s basis Theorem [8, Th 5.50], all minimal systems of generators of $[C_2]^n$ have the same cardinal, and this cardinal is $d$. However, the cardinality of a minimal set of generators for wreath products of cyclic groups has been computed by Woryna — see the comments after Theorem 1.1 in [12]. In our case, we get $d = n$.

**Proof 2.** We use the following well-known results from Group theory. Let $G$ be a group and $D(G)$ the commutator subgroup of $G$. Let $H$ be any subgroup of $G$. The following are true:

1. $D(G)$ is contained in $H$ if and only if $H$ is a normal subgroup of $G$ and $G/H$ is abelian.

2. If $D(G)$ is contained in $H$, then $(G/D(G))/(H/D(G))$ is isomorphic to $G/H$. 

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Moreover, we need the fact that the quotient $[C_2]^n/D([C_2]^n)$ is isomorphic to $C_2^n$ — see [9, Proof of Lemma 1.5].

Suppose that $H$ is a subgroup of $[C_2]^n$ with $[C_2]^n/H$ isomorphic to $C_2$. By item 1 above, we deduce that $D([C_2]^n)$ is contained in $H$. By item 2 and by Lagrange theorem, we have

$$|H/D([C_2]^n)| = \frac{2^n}{2} = 2^{n-1}.$$  

Furthermore, the subgroups containing $D([C_2]^n)$ correspond bijectively to subgroups of $[C_2]^n/D([C_2]^n)$. As in the first proof, the group $C_n^2$ is a vector space of dimension $n$ over $F_2$, and every subgroup of order $2^{n-1}$ corresponds bijectively to a subspace of dimension $n-1$. This number is well known to be $2^{n-1}$. \qed

For $n \geq q$, let $c_n = P_n(0)$ be the constant term of the minimal polynomial of $x_n$, and let $c_1 = \nu = -P_1(0)$.

**Lemma 6.3.** Let $p$ be a prime that divides some $c_n$. Let $m = \min\{n \geq 1: p$ divides $c_n\}$ and $e$ be the order of $c_m$ at $p$. For every $n$, $p$ divides $c_n$ if and only if $p^e$ divides $c_n$ if and only if $m$ divides $n$.

**Proof.** There is also a simple proof in [9, proof of Lemma 1.1], inspired by Odoni [5]. We give a very elementary proof for the sake of completeness.

Recall that $P_n(t) = f^{\circ n}(t)$, where $f(t) = t^2 - \nu$. If $n = \ell + m$ for some integer $\ell > 0$, then we have

$$c_n = f^{\circ \ell}(f^{\circ m}(0)) = f^{\circ \ell}(c_m) \equiv c_\ell \pmod{c_m^2},$$

hence, if $n$ is a multiple of $m$, then $c_n$ has the same order at $p$ as $c_m$. Conversely, write $n = qm + r$, with $0 \leq r < m$. We have

$$c_n = f^{\circ r}(f^{\circ qm}(0)) \equiv c_r \pmod{c_{qm}^2},$$

hence $c_n$ is congruent to $c_r$ modulo $p$. So, if $p$ divides $c_n$, then it divides $c_r$ with $r < m$, which is a contradiction unless $r = 0$. \qed

We recall that non-zero rational numbers $a_1, \ldots, a_n$ are 2-independent if their residue classes in the $F_2$-vector space $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ are linearly independent.

In [9, Section 1, p. 16], Stoll proves the following theorem.

**Theorem 6.4.** The group $\text{Gal}(L_n)$ is isomorphic to $[C_2]^n$ if and only if $c_1, \ldots, c_n$ are 2-independent.

We also need the following simple observation: $\sqrt{c_1}, \ldots, \sqrt{c_n}$ all lie in $L_n$.

We can now prove our theorem.

**Theorem 6.5.** Suppose the $\nu = 2^m \mu$, with $\mu \geq 3$ odd and square-free and $m \geq 1$. The field $L = \bigcup_n L_n$ does not contain $\sqrt{2}$ — so in particular $O(\nu,0)$ does not contain $\sqrt{2}$.  

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Proof. From Theorem 6.1 and Theorem 6.4 the number $c_1, \ldots, c_n$ are 2-independent. There are
\[
\binom{n}{1} + \cdots + \binom{n}{n} = 2^n - 1
\]
distinct possible products $\sqrt{c_1} \cdots \sqrt{c_k}$. By the observation above, each product corresponds to a distinct quadratic extensions in $L_n$. We conclude that there are no more using Lemma 6.2.

Since $c_1 = \nu$, by Lemma 6.3, $2^{2m}$ is the highest power of 2 which divides $c_n$ for each $n \geq 1$. Hence, in every product of the $\sqrt{c_i}$, an even power of 2 comes out of the square root, and we deduce that $\sqrt{2}$ does not appear in any of the quadratic extensions that we found.

Here is an example. For $\nu = 12$, we have $L_1 = \mathbb{Q}(\sqrt{12}) = \mathbb{Q}(\sqrt{3})$, and
\[
L_2 = L_1 \left( \sqrt{12 + \sqrt{12}}, \sqrt{12 - \sqrt{12}} \right) = \mathbb{Q} \left( \sqrt{3}, \sqrt{12 + \sqrt{12}}, \sqrt{12 - \sqrt{12}} \right).
\]
We have
\[
\sqrt{12 + \sqrt{12}} \sqrt{12 - \sqrt{12}} = \sqrt{12^2 - 12} = \sqrt{12 \cdot 11} = 2\sqrt{33} = \sqrt{c_2}.
\]
Hence, in $L_2$, we have the three following square roots: $\sqrt{3}$, $\sqrt{33}$ and $\sqrt{11}$.

It is still an open problem to characterize the $\nu$ for which $\text{Gal}(L_n)$ is $[C_2]^n$ for every $n$. Note that for $\nu = 3$, the above does not work since $\sqrt{2}$ appears immediately in the tower. Nevertheless, for $\nu = 7$, $\sqrt{2}$ does not appear in the first levels of the tower. This leads to the following question.

**Question 6.6.** Is $\text{Gal}(L_n)$ equal to $[C_2]^n$ when $\nu = 7$?

**References**


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