SERRIN’S PROBLEM AND
ALEXANDROV’S SOAP BUBBLE THEOREM:
ENHANCED STABILITY VIA INTEGRAL IDENTITIES

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Abstract. We consider Serrin’s overdetermined problem for the torsional
rigidity and Alexandrov’s Soap Bubble Theorem. We present new integral
identities, that show a strong analogy between the two problems and help to
obtain better (in some cases optimal) quantitative estimates for the radially
symmetric configuration. The estimates for the Soap Bubble Theorem benefit
from those of Serrin’s problem.

1. Introduction

The pioneering symmetry results obtained by A. D. Alexandrov [Al1], [Al2] and
J. Serrin [Se] are now classical but still influential. The former — the well-known
Soap Bubble Theorem — states that a compact hypersurface, embedded in \( \mathbb{R}^N \), that
has constant mean curvature must be a sphere. The latter — Serrin’s symmetry
result — has to do with certain overdetermined problems for partial differential
equations. In its simplest formulation, it states that the overdetermined boundary
value problem

\[
\begin{align*}
\Delta u &= N \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \Gamma, \\
\quad u_\nu &= R \quad \text{on } \Gamma,
\end{align*}
\]

admits a solution for some positive constant \( R \) if and only if \( \Omega \) is a ball of radius
\( R \) and, up to translations, \( u(x) = (|x|^2 - R^2)/2 \). Here, \( \Omega \) denotes a bounded
domain in \( \mathbb{R}^N \), \( N \geq 2 \), with sufficiently smooth boundary \( \Gamma \), say \( C^2 \), and \( u_\nu \) is the
outward normal derivative of \( u \) on \( \Gamma \). This result inaugurated a new and fruitful
field in mathematical research at the confluence of Analysis and Geometry and has
many applications to other areas of mathematics and natural sciences. To be sure,
that same result was actually motivated by two concrete problems in mathematical
physics regarding the torsion of a straight solid bar and the tangential stress of a
fluid on the walls of a rectilinear pipe.

The two problems share several common features. To prove his result, Alexan-
drov introduced his reflection principle, an elegant geometric technique that also
works for other symmetry results concerning curvatures. Serrin’s proof hinges on
his method of moving planes, an adaptation and refinement of the reflection prin-
ciple. That method proves to be a very flexible tool, since it allows to prove radial
symmetry for positive solutions of a far more general class of non-linear equations,
that includes the semi-linear equation

\[
\Delta u = f(u),
\]

where \( f \) is a locally Lipschitz continuous non-linearity.

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torsional rigidity, constant mean curvature, integral identities, stability, quantitative estimates.
Also, alternative proofs of both symmetry results can be given, based on certain integral identities and inequalities, and the maximum principle. H. F. Weinberger’s proof ([We]) of symmetry, even if it is known to work only for problem (1.1)-(1.2) and other few instances, leaves open the option of considering less regular settings. These ideas can be extended to the Soap Bubble Theorem, using R. C. Reilly’s argument ([Re]), in which the relevant hypersurface is regarded as the zero level surface of the solution of (1.1).

In this paper, we shall further analyze the analogies in Weinberger’s and Reilly’s arguments to obtain quantitative estimates of the desired radial symmetries. Roughly speaking, we shall address the problem of estimating how close to a sphere is a hypersurface Γ, if either its mean curvature $H$ is close to be constant or, alternatively, if the normal derivative on Γ of the solution of (1.1) is close to be constant.

In both problems, the radial symmetry of Γ will follow from that of the solution $u$ of (1.1). In fact, we will show that in Newton’s inequality

$$(\Delta u)^2 \leq N |\nabla^2 u|^2,$$

that holds pointwise in $\Omega$ by Cauchy-Schwarz inequality, the equality sign is identically attained in $\Omega$. Such equality holds if and only if $u$ is a quadratic polynomial $q$ of the form

$$q(x) = \frac{1}{2} (|x - z|^2 - a),$$

for some choice of $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$. The boundary condition in (1.1) will then tell us that Γ must be a sphere centered at $z$.

The starting points of our analysis are the following two integral identities:

$$(\Delta u)^2 \leq N |\nabla^2 u|^2,$$

$$1 \frac{1}{N-1} \int_{\Omega} \left( |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right) dx + \int_{\Gamma} (u_\nu - R)^2 dS_x = \int_{\Gamma} (H_0 - H) (u_\nu - q_\nu) u_\nu dS_x + \int_{\Gamma} (H_0 - H) (u_\nu - R) q_\nu dS_x.$$

The two identities hold regardless of how the point $z$ or the constant $a$ are chosen in (1.5). In (1.6) and (1.7), $R$ and $H_0$ are reference constants given by

$$R = \frac{N |\Omega|}{|\Gamma|}, \quad H_0 = \frac{1}{R} = \frac{|\Gamma|}{N |\Omega|}.$$

If $u_\nu$ is constant on $\Gamma$, that constant must be equal to $R$, by the identity

$$\int_{\Gamma} u_\nu dS_x = N |\Omega|,$$

and hence we obtain symmetry by using (1.10), since the equality sign must hold in (1.4) (in fact, $-u > 0$ in $\Omega$ by the maximum principle). If, on the other hand, $H$ is constant on $\Gamma$, then the Minkowski’s identity,

$$\int_{\Gamma} H q_\nu dS_x = |\Gamma|,$$

implies that $H \equiv H_0$ on $\Gamma$, and hence once again symmetry follows by an inspection of (1.7), instead.

Identity (1.6), which appears without proof in [MP, Remark 2.5], puts together Weinberger’s identities and some remarks of L. E. Payne and P. W. Schaefer [PS]. (1.7) is a slight modification of one that was proved in [MP, Theorem 2.2] and
will turn out to be useful to improve our desired quantitative estimates. For the reader’s convenience, we will present the proofs of (1.6) and (1.7) in Section 2.

Identity (1.6) certainly holds under Serrin’s smoothness assumptions ($\Gamma \in C^2$), but it is clear that it can be easily extended by approximation to the case of $\Gamma \in C^{1,\alpha}$, since in that case $u_\nu$ is continuous on $\Gamma$. Nevertheless, it should be noticed that, to infer the radial symmetry of $\Omega$, it suffices to show that the surface integral on the right-hand side of (1.6) is zero.

The main motivation of this paper is to investigate how the use of (1.6) and (1.7) benefits the study of the stability of the radial configuration in the Soap Bubble Theorem and Serrin’s problem. Technically speaking, one may look for two concentric balls $B_{\rho_i}(z)$ and $B_{\rho_c}(z)$, centered at $z \in \Omega$ with radii $\rho_i$ and $\rho_c$, such that

$$B_{\rho_c}(z) \subset \Omega \subset B_{\rho_i}(z)$$

and

$$\rho_c - \rho_i \leq \psi(\eta),$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function vanishing at 0 and $\eta$ is a suitable measure of the deviation of $u_\nu$ or $H$ from being a constant.

Our main result for Serrin’s problem is the following theorem. Here and in the sequel, $d_\Omega$ denotes the diameter of $\Omega$, $r_i$ and $r_c$ are the relevant radii in the interior and exterior uniform sphere conditions for $\Omega$, and $\delta r(z)$ is the distance of $z$ to $\Gamma$.

**Theorem 1.1 (Stability for Serrin’s problem).** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\Gamma$ of class $C^2$ and $R$ be the constant defined in (1.8). Let $u$ be the solution of problem (1.1) and $z \in \Omega$ be any of its critical points.

Then (1.11) holds with $\rho_i$ and $\rho_c$ such that

$$\rho_c - \rho_i \leq C \|u_\nu - R\|_{2, \Gamma}^{2/(N+2)} \quad \text{if} \quad \|u_\nu - R\|_{2, \Gamma} < \varepsilon.$$

The constants $C$ and $\varepsilon$ depend on $N$, $d_\Omega$, $r_i$, $r_c$, $\delta r(z)$.

The problem of stability for Serrin’s problem was considered for the first time in [ABR]. There, for a $C^{2,\alpha}$-regular domain $\Omega$, it is considered a positive solution $u$ of semilinear equation (1.2), such that $u = 0$ on $\Gamma$, and it is proved that (1.11) and (1.12) hold with $\psi(\eta) = C |\log \eta|^{-1/N}$ and $\eta = \|u_\nu - c\|_{C^1(\Gamma)}$. (Here and in the remaining paragraphs of this introduction, for the dependence of the constants $C$ on the geometric and regularity parameters associated to $\Omega$ and, when applicable, $f$, we refer the reader to the cited references.) The proof in [ABR] is based on a quantitative study of the method of moving planes. In [CMV], in the same general framework and with a similar proof, that stability estimate has been improved. It is in fact shown that (1.11) and (1.12) hold with $\psi(\eta) = C \eta^\tau$ and

$$\eta = \sup_{x,y \in \Gamma, x \neq y} \frac{|u_\nu(x) - u_\nu(y)|}{|x - y|}.$$

The exponent $\tau \in (0, 1)$ can be computed for a general setting and, if $\Omega$ is convex, is arbitrarily close to $1/(N + 1)$.

Theorem 1.1 improves on a different technique, based on Weinberger’s integral identities and first employed in [BNST]. There, a quantitative estimate of Hölder type was obtained for the first time, by using the weaker deviation $\eta = \|u_\nu - c\|_\infty$ of $u_\nu$ from some reference constant. In [BNST], it is also considered the deviation in $L^1$-norm $\eta = \|u_\nu - c\|_{1, \Gamma}$ and, assuming an additional uniform bound on $u_\nu$, it is shown that $\Omega$ can be approximated in measure by a finite number of mutually disjoint balls $B_i$. The error in the approximation is $\psi(\eta) = C \eta^{1/(4N + 9)}$. Recently, the approach of [BNST] has been greatly improved in [FG]. There, rather than by
and (1.12), the closeness of $\Omega$ to a ball is measured by the following slight modification of the so-called Fraenkel asymmetry:

\[
A(\Omega) = \inf \left\{ \frac{|\Omega \Delta B_x|}{|B_x|} : x \text{ center of a ball } B^x \text{ with radius } R \right\}.
\]

Here, $\Omega \Delta B_x$ denotes the symmetric difference of $\Omega$ and $B_x$, and $R$ is the constant defined in (1.8). In [Fe], the deviation of $u_\nu$ from a constant is measured in $L^2$-norm and the estimate obtained is of Lipschitz type: $A(\Omega) \leq C \|u_\nu - R\|_{2, \Gamma}$.

The proof of Theorem 1.1 simplifies and improves arguments of Feldman [Fe] by the use of our first identity (1.6). The obtained estimate (1.13) makes better than [CMV], even if we replace the Lipschitz semi-norm by an $L^2$-deviation. In addition, Feldman’s inequality is improved with a stronger control of symmetry at the cost of a slight decrease of the continuity exponent. As pointed out in Remarks 3.8 and 3.9, the dependence on the point $z$ of the constants in (1.13) can (so far) be removed when $\Omega$ is convex or by assuming some additional requirement (see Remark 3.9).

We will also prove in passing the inequality

\[
\rho_e - \rho_i \leq C \|H_0 - H\|_{2, \Gamma}^{\frac{1}{N+2}},
\]

which clearly improves that obtained in [BNST] (see Theorem 3.6). The constants $C$ and $\varepsilon$ depend on $N$, $d_\Omega$, $r_i$, $r_e$, $\delta_\Gamma(z)$.

Theorem 1.2 (Stability for the Soap Bubble Theorem). Let $N \geq 2$ and let $\Gamma$ be the connected boundary of class $C^2$ of a bounded domain $\Omega \subset \mathbb{R}^N$. Denote by $H$ the mean curvature at points of $\Gamma$ and let $H_0$ be the constant defined in (1.8).

Then (1.11) holds for some point $z \in \Omega$ and

(i) if $N = 2$ or $N = 3$, then

\[
\rho_e - \rho_i \leq C \|H_0 - H\|_{2, \Gamma};
\]

(ii) if $N \geq 4$, then

\[
\rho_e - \rho_i \leq C \|H_0 - H\|_{2, \Gamma}^{2/(N+2)} \text{ if } \|H_0 - H\|_{2, \Gamma} < \varepsilon.
\]

The constants $C$ and $\varepsilon$ depend on $N$, $d_\Omega$, $r_i$, $r_e$, $\delta_\Gamma(z)$.

Theorem 1.2 enhances in all dimensions estimates obtained in [CM] for strictly mean convex surfaces with the uniform deviation $\|H_0 - H\|_{\infty, \Gamma}$. Also, it improves by a factor of 2 the results derived by the authors in [MP]. More importantly, it gains the (optimal) Lipschitz stability for the cases $N = 2, 3$ and for a general class of hypersurfaces. That kind of stability was already obtained in [CV] and [KM, Theorem 1.8], but with a uniform deviation and for strictly mean convex hypersurfaces.

As a final important achievement, by arguments similar to those of [Fe], we also get in Theorem 4.6 the (optimal) inequality for the asymmetry (1.14):

\[
A(\Omega) \leq C \|H_0 - H\|_{2, \Gamma}.
\]

In the remaining paragraphs of this introduction, we shall pinpoint the main remarks that lead us to the proof of Theorems 1.1 and 1.2.

We start to simplify matters by noticing that, by (1.6), the harmonic function $h = q - u$ satisfies:

\[
\int_\Omega (-u) |\nabla^2 h|^2 \, dx = \frac{1}{2} \int_\Gamma (R^2 - u_\nu^2) h_\nu \, dS_x.
\]
Also, notice that $h = q$ on $\Gamma$ and hence
\[
(1.19) \quad \max_{\Gamma} h - \min_{\Gamma} h = \frac{1}{2} (\rho_2^2 - \rho_1^2) \geq \frac{1}{2} (|\Omega|/|B|)^{1/N} (\rho_2 - \rho_1).
\]

Now, observe that (1.18) holds regardless of the choice of the parameters $z$ and $a$ defining $q$. We will thus complete the first step of our proof by choosing $z \in \Omega$ in a way that the oscillation of $h$ on $\Gamma$ on the left-hand side of (1.19) (which is indeed the oscillation of $h$ on $\Omega$) can be bounded in terms of the left-hand side of (1.18).

To carry out this plan, we use three ingredients. First, we choose $z \in \Omega$ as a minimum (or any critical) point of $u$ and, as done in [MP, Lemma 3.3], we show that
\[
\max_{\Gamma} h - \min_{\Gamma} h \leq C \left( \int_{\Omega} h^2 \, dx \right)^{1/(N+2)}.
\]

Secondly, we observe that, depending on the regularity of $\Gamma$, we can easily obtain the bound
\[
C \delta_{\Gamma}(x)^\alpha \leq -u \quad \text{on} \quad \Omega,
\]
where $\alpha = 1$ or 2. Thirdly, we apply two integral inequalities to $h$ and its first (harmonic) derivatives. One is the Hardy-Poincaré-type inequality
\[
\int_{\Omega} v(x)^2 \, dx \leq C \int_{\Omega} \delta_{\Gamma}(x)^\alpha |\nabla v(x)|^2 \, dx,
\]
that is applied to the first (harmonic) derivatives. It holds for fixed $\alpha \in [0,1]$ and for any harmonic function $v \in W^{1,2}(\Omega)$ that is zero at some given point in $\Omega$ (in our case that point will be $z$, since $\nabla h(z) = 0$). The other one is applied to $h$ and is the Poincaré-type inequality
\[
\int_{\Omega} v^2 \, dx \leq C \int_{\Omega} |\nabla v|^2 \, dx,
\]
that holds for any harmonic function $v \in W^{1,2}(\Omega)$ with zero mean value on $\Omega$.

Thus, summing up the last 5 inequalities gives:
\[
(1.20) \quad \max_{\Gamma} h - \min_{\Gamma} h \leq C \left( \int_{\Omega} (u - R)^2 \, dx \right)^{1/(N+2)}.
\]

Next, we work on the right-hand side of (1.18). The important observation is that, if $u_\nu - R$ tends to 0, also $h_\nu$ does. Quantitatively, this fact can be expressed by the inequality
\[
\|h_\nu\|_{2,\Gamma} \leq C \|u_\nu - R\|_{2,\Gamma},
\]
that can be derived from [Fe]. Thus, (1.18) will follow by using this inequality, after an application of Hölder’s inequality to the right-hand side of (1.18).

In order to prove Theorem 1.2 we use the new identity (1.7). In fact, discarding the first summand at its left-hand side and applying Hölder’s and the last inequality to its right-hand side yield that
\[
\|u_\nu - R\|_{2,\Gamma} \leq C \|H_0 - H\|_{2,\Gamma}.
\]

Theorem 1.2 then follows again from (1.7) and the following estimate already obtained in [MP]:
\[
\max_{\Gamma} h - \min_{\Gamma} h \leq C \left( \int_{\Omega} |\nabla^2 h|^2 \, dx \right)^{\tau_N/2},
\]
where $\tau_N = 1$ for $N = 2, 3$ and $\tau_N = 2/(N + 2)$ for $N \geq 4$.

The paper is organized as follows. In Section 2 we collect the relevant identities on which our results are based. Section 3 is dedicated to the stability of Serrin’s problem. Subsection 3.2 contains the estimates on harmonic functions that are instrumental to derive (1.15) and (1.13). Then, in Subsection 3.3 we prove Theorems
1. and 3.6 by assembling all the obtained relevant identities and inequalities. In Section 4, we prove the new stability for the Soap Bubble Theorem (Theorem 1.2).

2. Identities for Serrin’s problem and the Soap Bubble Theorem

We begin by setting some relevant notations. By \( \Omega \subset \mathbb{R}^N, \ N \geq 2, \) we shall denote a bounded domain, that is a connected bounded open set, and call \( \Gamma \) its boundary. By \( |\Omega| \) and \( |\Gamma| \), we will denote indifferently the \( N \)-dimensional Lebesgue measure of \( \Omega \) and the surface measure of \( \Gamma \). When \( \Gamma \) is of class \( C^1 \), \( \nu \) will denote the (exterior) unit normal vector field to \( \Gamma \) and, when \( \Gamma \) is a hypersurface of class \( C^2 \), \( H(x) \) will denote its mean curvature (with respect to \( -\nu(x) \)) at \( x \in \Gamma \).

We will also use the letter \( q \) to denote the quadratic polynomial defined in (1.5), where \( z \) is any point in \( \mathbb{R}^N \) and \( a \) is any real number; furthermore, we will always use the letter \( h \) to denote the harmonic function

\[
    h = q - u.
\]

Finally, as already mentioned in the introduction, for a point \( z \in \Omega \) to be determined, \( \rho_i \) and \( \rho_e \) shall denote the radius of the largest ball contained in \( \Omega \) and that of the smallest ball that contains \( \Omega \), both centered at \( z \); in formulas,

\[
    \rho_i = \min_{x \in \Gamma} |x - z| \quad \text{and} \quad \rho_e = \max_{x \in \Gamma} |x - z|.
\]

To start, we will provide the proof of identity (1.6) appearing in [MP, Remark 2.5]. The proof is as minimal as possible. It follows the tracks of and improves on the work of Weinberger [We] and its modification due to Payne and Schaefer [PS].

Identity (1.6) is a consequence of two formulas involving the solution \( u \) of (1.1).

One is the differential identity

\[
    |\nabla^2 u|^2 - \left( \frac{\Delta u}{N} \right)^2 = \Delta P,
\]

that associates the Cauchy-Schwarz deficit on the left-hand side with the P-function,

\[
    P = \frac{1}{2} |\nabla u|^2 - u.
\]

The other one is the Rellich-Pohozaev identity (see [Po]):

\[
    (N + 2) \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Gamma} (u_{\nu})^2 \, q_{\nu} \, dS_x.
\]

Notice that (2.2) also implies that \( P \) is subharmonic, since the left-hand side is non-negative by Cauchy-Schwarz inequality.

Theorem 2.1 (Fundamental identity for Serrin’s problem). Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with boundary \( \Gamma \) of class \( C^{1,\alpha}, \ 0 < \alpha \leq 1, \) and \( R \) be the positive constant defined in (1.8). Then the solution \( u \) of (1.1) satisfies identity (1.6), that is it holds that

\[
    \int_{\Omega} (-u) \left\{ |\nabla^2 u|^2 - \left( \frac{\Delta u}{N} \right)^2 \right\} \, dx = \frac{1}{2} \int_{\Gamma} (u_{\nu}^2 - R^2) (u_{\nu} - q_{\nu}) \, dS_x.
\]

In particular, if the right-hand side of (1.6) is non-positive, \( \Gamma \) must be a sphere (and hence \( \Omega \) a ball) of radius \( R \). The same conclusion clearly holds if \( u_{\nu} \) is constant on \( \Gamma \).

Proof. First, suppose that \( \Gamma \) is of class \( C^{2,\alpha} \), so that \( u \in C^{2,\alpha}(\Omega) \). Integration by parts then gives:

\[
    \int_{\Omega} (u \Delta P - P \Delta u) \, dx = \int_{\Gamma} (u P_{\nu} - u_{\nu} P) \, dS_x.
\]
Thus, since $u$ satisfies (1.1), we have that

$$\int_\Omega (-u) \Delta P \, dx = -N \int_\Omega P \, dx + \frac{1}{2} \int_\Gamma u_\nu^2 \, dS_x,$$

being $P = |\nabla u|^2/2 = u_\nu^2/2$ on $\Gamma$.

Notice that we can write $u_\nu^2 = (u_\nu^2 - R^2)(u_\nu - q_\nu) + R^2(u_\nu - q_\nu) + u_\nu^2q_\nu$. Next, by the divergence theorem and (2.3) we compute:

$$N \int_\Omega P \, dx = \frac{N}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega u \Delta u \, dx = \left( \frac{N}{2} + 1 \right) \int_\Omega |\nabla u|^2 \, dx = \frac{1}{2} \int_\Gamma u_\nu^2q_\nu \, dS_x.$$

Thus, this identity, (2.4), and (2.2) give (1.6), since

$$\int_\Gamma (u_\nu - q_\nu) \, dS_x = 0,$$

being $u - q$ harmonic in $\Omega$.

If $\Gamma$ is of class $C^{1, \alpha}$, then $u \in C^{1, \alpha}(\Omega) \cap C^2(\Omega)$. Thus, by a standard approximation argument, we conclude that (1.6) holds also in this case.

Now, if the right-hand side of (1.6) is non-positive, then the integrand at the left-hand side must be zero, being non-negative by (1.3) and the maximum principle for $u$. Then (1.4) must hold with the equality sign, since $u < 0$ on $\Omega$, by the strong maximum principle. Being $\Delta u = N$, we infer that $\nabla^2 u$ coincides with the identity matrix $I$. Thus, $u$ must be a quadratic polynomial $q$ of the form (1.5), for some $z \in \mathbb{R}^N$ and $a \in \mathbb{R}$.

Since $u = 0$ on $\Gamma$, then $|x - z|^2 = a$ for $x \in \Gamma$, that is $a$ must be positive and

$$\sqrt{a} |\Gamma| = \int_\Gamma |x - z| \, dS_x = \int_\Gamma (x - z) \cdot \nu(x) \, dS_x = N \, |\Omega|.$$

In conclusion, $\Gamma$ must be a sphere centered at $z$ with radius $R$.

Finally, if $u_\nu \equiv c$ on $\Gamma$ for some constant $c$, then

$$c |\Gamma| = \int_\Gamma u_\nu \, dS_x = N \, |\Omega|,$$

that is $c = R$, and hence we can apply the previous argument.

Corollary 2.2. Let $u$ be the solution of (1.1) and set $h = q - u$. Then, $h$ is harmonic in $\Omega$ and (1.13) holds true, that is

$$\int_\Omega (-u) |\nabla^2 h|^2 \, dx = \frac{1}{2} \int_\Gamma (R^2 - u_\nu^2) h_\nu \, dS_x.$$

Moreover, if the center $z$ of the polynomial $q$ in (1.5) is chosen in $\Omega$, then the oscillation of $h$ on $\Gamma$ can be bounded below as in (1.19).

Proof. Simple computations give that $|\nabla^2 h|^2 = |\nabla^2 u|^2 - (\Delta u)^2/N$ and $h_\nu = q_\nu - u_\nu$, and hence (1.13) easily follows from the fundamental identity (1.6) for Serrin’s problem.

Notice that $h = q$ on $\Gamma$. Thus, the equality in (1.19) follows from (2.1), by choosing $z$ in $\Omega$. The inequality in (1.19) is implied by $\rho_c + \rho_i \geq \rho_c \geq (|\Omega|/|B|)^{1/N}$, since $B_{\rho_c} \supset \Omega$.

Remark 2.3. The assumptions on the regularity of $\Gamma$ can further be weakened. For instance, if $\Omega$ is a (bounded) convex domain, then inequality (3.3) below imply that $u_\nu$ is essentially bounded on $\Gamma$ with respect to the $(N - 1)$-dimensional Hausdorff measure on $\Gamma$. Thus, an approximation argument again gives that (1.6) holds true.

We now present the proof of (1.7), which is a modification of [MP, formula (2.6)].
Theorem 2.4 (Fundamental identity for the Soap Bubble Theorem). Let $\Omega$ be a bounded domain with boundary $\Gamma$ of class $C^2$. Then (1.7) holds true, that is

$$\frac{1}{N-1} \int_{\Omega} \left\{ |\nabla^2 u|^2 - \frac{(\Delta u)^2}{N} \right\} \, dx + \frac{1}{R} \int_{\Gamma} (u_\nu - R)^2 \, dS_x = \int_{\Gamma} (H_0 - H) (u_\nu - q_\nu) \, u_\nu \, dS_x + \int_{\Gamma} (H_0 - H) (u_\nu - R) q_\nu \, dS_x.$$

Proof. We proceed similarly to the proof of Theorem 2.1, but with two main differences: in place of Pohozaev’s identity, we use Minkowski’s identity; we use the following well-known formula for the laplacian of $u$:

$$\Delta u = u_{\nu\nu} + (N-1) H u_\nu.$$

This last identity holds pointwise on any regular level surface of $u$ if we agree to still denote by $\nu$ the vector field $\nabla u / |\nabla u|$ (it is clear that, on $\Gamma$, this coincides with the normal).

We begin with the divergence theorem:

$$\int_{\Omega} \Delta P \, dx = \int_{\Gamma} P_{\nu} \, dS_x.$$

To compute $P_{\nu}$, we observe that $\nabla u$ is parallel to $\nu$ on $\Gamma$, that is $\nabla u = (u_\nu) \nu$ on $\Gamma$. Thus,

$$P_{\nu} = (\nabla^2 u \cdot \nabla u, \nu) - u_{\nu}(\nabla^2 u)_{\nu, \nu} - u_{\nu} = u_{\nu} \Delta u - (N-1) H u_\nu - u_{\nu} = (N-1) (1 - H u_\nu) u_\nu,$$

where we have used (1.1). Therefore,

$$\frac{1}{N-1} \int_{\Omega} \Delta P \, dx = \int_{\Gamma} (1 - H u_\nu) u_\nu \, dS_x.$$

Now, straightforward calculations that use (1.1), (1.10), and (1.9) tell us that

$$\int_{\Gamma} (H_0 - H) (u_\nu - q_\nu) \, u_\nu \, dS_x + \int_{\Gamma} (H_0 - H) (u_\nu - R) q_\nu \, dS_x = H_0 \int_{\Gamma} u_\nu^2 \, dS_x - \int_{\Gamma} H u_\nu^2 \, dS_x,$$

while

$$\frac{1}{R} \int_{\Gamma} (u_\nu - R)^2 \, dS_x = H_0 \int_{\Gamma} u_\nu^2 \, dS_x - \int_{\Gamma} u_\nu \, dS_x.$$

The conclusion then follows by a simple inspection, from these two formulas, (2.2), and (2.5).

Corollary 2.5. Let $\Omega$ be a bounded domain with boundary $\Gamma$ of class $C^2$ and set $h = q - u$. Then it holds that

$$\frac{1}{N-1} \int_{\Omega} |\nabla^2 h|^2 \, dx + \frac{1}{R} \int_{\Gamma} (u_\nu - R)^2 \, dS_x = - \int_{\Gamma} (H_0 - H) h_\nu \, u_\nu \, dS_x + \int_{\Gamma} (H_0 - H) (u_\nu - R) q_\nu \, dS_x.$$
3. Stability for Serrin’s overdetermined problem

3.1. Notations. As already mentioned in the introduction, The diameter of Ω is indicated by \( d_Ω \), while \( δ_Γ(x) \) denotes the distance of a point \( x \) to the boundary \( Γ \).

Even if the fundamental identity for Serrin’s problem, (1.6), holds for less regular domains, in order to consider the stability issue, we shall assume that \( Ω \) is a bounded domain with boundary \( Γ \) of class \( C^2 \). In fact, under this assumption, \( Ω \) has the properties of the uniform interior and exterior sphere condition, whose respective radii we have designated by \( r_i \) and \( r_e \). In other words, there exists \( r_e > 0 \) (resp. \( r_i > 0 \)) such that for each \( p ∈ Γ \) there exists a ball contained in \( \mathbb{R}^N \setminus Ω \) (resp. contained in \( Ω \)) of radius \( r_e \) (resp. \( r_i \)) such that its closure intersects \( Γ \) only at \( p \).

We shall later see that, when \( Ω \) is a convex domain we can remove the assumption on the regularity of \( Γ \).

The assumed regularity of \( Ω \) ensures that the unique solution of (1.1) is of class at least \( C^{1,α}(Ω) \). Thus, we can define

\[
M = \max_{Ω} |∇u| = \max_{Γ} u_ν.
\]

As shown in [MP, Theorem 3.10], the following bound holds for \( M \):

\[
M ≤ c_N \frac{d_Ω(d_Ω + r_e)}{r_e},
\]

where \( c_N = 3/2 \) for \( N = 2 \) and \( c_N = N/2 \) for \( N ≥ 3 \). Notice that, when \( Ω \) is convex, we can choose \( r_e = +∞ \) in (3.2) and obtain

\[
M ≤ c_N d_Ω.
\]

For other similar estimates present in the literature, see [MP, Remark 3.11].

3.2. Some estimates for harmonic functions. As already sketched in the introduction, the desired stability estimate for the spherical symmetry of \( Ω \) will be obtained by linking the oscillation of \( h \) on \( Γ \) to the integral

\[
\int_{Ω} (-u)|∇^2 h|^2 dx,
\]

by means of identity (1.15).

To fulfill this agenda, we start by relating the factor \(-u\) appearing in that quantity to the function \( δ_Γ(x) \); we do it in the following lemma.

Lemma 3.1. Let \( Ω ⊂ \mathbb{R}^N \), \( N ≥ 2 \), be a bounded domain such that \( Γ \) is made of regular points for the Dirichlet problem, and let \( u \) be the solution of (1.1). Then

\[
-u(x) ≥ \frac{1}{2} δ_Γ(x)^2 \quad \text{for every} \quad x ∈ Ω.
\]

Moreover, if \( Γ \) is of class \( C^2 \), then it holds that

\[
-u(x) ≥ \frac{r_i}{2} δ_Γ(x) \quad \text{for every} \quad x ∈ Ω.
\]

Proof. If every point of \( Γ \) is regular, then a unique solution \( u ∈ C^0(Ω) ∩ C^2(Ω) \) exists for (1.1). Now, for \( x ∈ Ω \), let \( r = δ_Γ(x) \) and consider the ball \( B = B_r(x) \).

Let \( w \) be the solution of (1.1) in \( B \), that is \( w(y) = (|y-x|^2 - r^2)/2 \). By comparison we have that \( w ≥ u \) on \( B \) and hence, in particular, \( w(x) ≥ u(x) \). Thus, we infer the first inequality in the lemma.

If \( Γ \) is of class \( C^2 \), (3.3) certainly holds if \( δ_Γ(x) ≥ r_i \). If \( δ_Γ(x) < r_i \), instead, let \( z \) be the closest point in \( Γ \) to \( x \) and call \( B \) the ball of radius \( r_i \) touching \( Γ \) at \( z \) and containing \( x \). Up to a translation, we can always suppose that the center of the ball
$B$ is the origin $0$. If $w$ is the solution of (1.1) in $B$, that is $w(y) = (|y|^2 - r_i^2)/2$, by comparison we have that $w \geq u$ in $B$, and hence

$$-u(x) \geq \frac{1}{2} (|x|^2 - r_i^2) = \frac{1}{2} (r_i + |x|)(r_i - |x|) \geq \frac{1}{2} r_i (r_i - |x|).$$

This implies (3.4), since $r_i - |x| = \delta_r(x)$.

By the last lemma, we can estimate the right-hand side of (1.18) from below in terms of the integral

$$\int_{\Omega} |\nabla h|^2 \delta_{2\alpha} dx,$$

with $\alpha = 1$ or $1/2$. For this kind of integral, useful estimates are present in the literature. We shall briefly report on some of them.

**Lemma 3.2** (Hardy-Poincaré-type inequalities). Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\Gamma$ of class $C^{0,\alpha}$, and let $z$ be a point in $\Omega$. Then,

(i) there exists a positive constant $\mu_{\alpha}(\Omega, z)$, such that

$$\left(\int_{\Omega} v^2 dx \right)^{1/2} \leq \mu_{\alpha}(\Omega, z)^{-1} \left(\int_{\Omega} |\nabla v|^2 \delta_{2\alpha} dx \right)^{1/2},$$

for every function $v$ which is harmonic in $\Omega$ and such that $v(z) = 0$;

(ii) there exists a positive constant, $\overline{\mu}_{\alpha}(\Omega)$ such that

$$\left(\int_{\Omega} v^2 dx \right)^{1/2} \leq \overline{\mu}_{\alpha}(\Omega)^{-1} \left(\int_{\Omega} |\nabla v|^2 \delta_{2\alpha} dx \right)^{1/2},$$

for every function $v$ which is harmonic in $\Omega$ and has mean value zero on $\Omega$.

In particular, if $\Gamma$ has a Lipschitz boundary, the number $\alpha$ can be replaced by any exponent in $(0, 1]$.

**Proof.** The assertions (i) and (ii) are easy consequences of a general result of Boas and Straube (see [BS]), that improves a work of Ziemer’s ([Zi]). In case (i), we apply [BS, Example 2.5]). In case (ii), [BS Example 2.1] is appropriate.

The variational problems

(iii) $\mu_{\alpha}(\Omega, z) = \min \left\{ \int_{\Omega} |\nabla v|^2 \delta_{2\alpha} dx : \int_{\Omega} v^2 dx = 1, \Delta v = 0 \text{ in } \Omega, v(z) = 0 \right\}$

and

(iv) $\overline{\mu}_{\alpha}(\Omega) = \min \left\{ \int_{\Omega} |\nabla v|^2 \delta_{2\alpha} dx : \int_{\Omega} v^2 dx = 1, \Delta v = 0 \text{ in } \Omega, \int_{\Omega} v dx = 0 \right\}$

then characterize the two constants.

**Remark 3.3.** (i) Notice that in the special case that $\alpha = 0$ from (3.5) and (3.6) we recover the Poincaré-type inequalities that we proved and used in [MP]. Also, in the case $\alpha = 0$ (3.5) directly follows from the result in [Zi]. Of course, in place of (3.8), the constant in the classical Poincaré inequality would work as well. The addition of the harmonicity of $v$ in (3.8) clearly gives a better constant.

(ii) We have that $\mu_{\alpha}(\Omega, z) \leq \overline{\mu}_{\alpha}(\Omega)$, as one can verify by using the function

$$v_0 = \frac{v - v(z)}{1 + |\Omega| v(z)^2},$$

where $v$ is a minimizer for (3.8).

(iii) In the sequel, we will choose $\alpha = 1/2$ in (3.7) and $\alpha = 0$ in (3.8) and use the simplified notations:

$$\mu(\Omega, z) = \mu_{1/2}(\Omega, z) \quad \text{and} \quad \overline{\mu}(\Omega) = \overline{\mu}_{0}(\Omega).$$
Theorem 3.5. Let \( \alpha \) be the values of the constants chosen such that \( h \in C^2(\Omega) \) and the constants \( \rho \) are given by (3.11):

\[
\int_{\Gamma} |\nabla h|^2 d\Gamma \leq 2 \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \int_{\Omega} (-u)|\nabla^2 h|^2 dx.
\]

Proof. It is clear that \( h \in C^1(\Omega) \cap C^2(\Omega) \). We begin with the following differential identity:

\[
\text{div} \{ v^2 \nabla u - u \nabla (v^2) \} = v^2 \Delta u - u \Delta (v^2) = N v^2 - 2u |\nabla v|^2,
\]

that holds for any \( v \) harmonic function in \( \Omega \), if \( u \) satisfies (1.1). Next, we integrate on \( \Omega \) and, by the divergence theorem, we get:

\[
\int_{\Gamma} v^2 u \nu d\Gamma = N \int_{\Omega} v^2 dx + 2 \int_{\Omega} (-u)|\nabla v|^2 dx.
\]

We use this identity with \( v = h + \epsilon \), and hence we sum up over \( i = 1, \ldots, N \) to obtain:

\[
\int_{\Gamma} |\nabla h|^2 u_i \nu d\Gamma = N \int_{\Omega} |\nabla h|^2 dx + 2 \int_{\Omega} (-u)|\nabla^2 h|^2 dx.
\]

This formula, together with (3.11) and (3.4), gives us:

\[
\int_{\Gamma} |\nabla h|^2 u_i \nu d\Gamma \leq 2 \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \int_{\Omega} (-u)|\nabla^2 h|^2 dx.
\]

The term \( u_i \) at the left-hand side of this last inequality can be bounded from below by \( r_i \), by an adaptation of Hopf’s lemma (see also [MP, Theorem 3.10]). Therefore, (3.9) follows at once. \( \square \)

The crucial step in our analysis is Theorem 3.5 below, in which we associate the oscillation of \( h \), and hence \( \rho_e - \rho_i \), with a weighted \( L^2 \)-norm of its Hessian matrix.

Before stating that, we recall from [MP] the following estimate, that links \( \rho_e - \rho_i \) with the \( L^2 \)-norm of \( h \). In fact, if \( \Omega \) is a bounded domain with boundary of class \( C^2 \), we have that

\[
(3.10) \quad \rho_e - \rho_i \leq a_\Omega M \frac{N}{N+2} |\Omega|^{-\frac{1}{N+2}} \|h\|_{L^2(\Omega)}^{2/(N+2)},
\]

for

\[
(3.11) \quad \|h\|_{L^2(\Omega)} \leq a_\Omega M r_i^{N+2}. \]

The values of the constants \( a_\Omega \) and \( \alpha_\Omega \) can be found in [MP, Lemma 3.3].

Theorem 3.5. Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), be a bounded domain with boundary \( \Gamma \) of class \( C^2 \) and \( z \) be any critical point in \( \Omega \) of the solution \( u \) of (1.1). Consider the function \( h = q - u \), with \( q \) being chosen such that \( h \) has mean value zero on \( \Omega \). Then we have that

\[
(3.12) \quad \rho_e - \rho_i \leq C \left\{ \int_{\Omega} |\nabla^2 h|^2 dx \right\}^{1/(N+2)},
\]

if

\[
(3.13) \quad \int_{\Omega} |\nabla^2 h|^2 dx < \varepsilon^2.
\]

Here,

\[
C = \frac{a_\Omega M \frac{N}{N+2}}{|\Omega|^{\frac{1}{N+2}} \mu(\Omega, \mu(\Omega, z))^{\frac{N}{N+2}}} \quad \text{and} \quad \varepsilon = \alpha_\Omega M r_i^{\frac{N+2}{N}} \sqrt{\mu(\Omega, \mu(\Omega, z))},
\]

and the constants \( a_\Omega \) and \( \alpha_\Omega \) are those in (3.10) and (3.11).
Proof. We apply (3.5) with \( \alpha = 1/2 \) to each first derivative of \( h \) (since \( \nabla h(z) = 0 \)) and obtain that
\[
\int_{\Omega} |\nabla h|^2 \, dx \leq \mu(\Omega, z)^{-1} \int_{\Omega} |\nabla^2 h|^2 \delta_{\Gamma} \, dx.
\]
Hence, we apply (3.6) with \( \alpha = 0 \) to \( h \) and get
\[
\int_{\Omega} h^2 \, dx \leq \mu(\Omega)^{-1} \int_{\Omega} |\nabla^2 h|^2 \delta_{\Gamma} \, dx.
\]
Thus,
\[
\int_{\Omega} h^2 \, dx \leq \mu(\Omega, z)^{-1} \int_{\Omega} |\nabla^2 h|^2 \delta_{\Gamma} \, dx,
\]
and the conclusion follows from (3.10) and (3.11).

3.3. Stability for Serrin's problem. We collect here our results on the stability of the spherical configuration by putting together the identities of Section 2 and the estimates in the previous subsection.

Theorem 3.5 above gives an estimate from below of the left-hand side of (1.18). In this subsection, we will take care of its right-hand side and prove our main result for Serrin's problem.

Proof of Theorem 1.1. We have that
\[
\int_{\Gamma} (R^2 - u_{\nu}^2) h_{\nu} \, dS_x \leq (M + R) \| u_{\nu} - R \|_{2, \Gamma} \| h_{\nu} \|_{2, \Gamma},
\]
and after an application of Hölder’s inequality. Thus, by Lemma 3.4 (1.18), and this inequality, we infer that
\[
\| h_{\nu} \|^2_{2, \Gamma} \leq \frac{2}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \int_{\Omega} (-u) |\nabla^2 h|^2 \, dx \leq \frac{M + R}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \| u_{\nu} - R \|_{2, \Gamma} \| h_{\nu} \|^2_{2, \Gamma},
\]
and hence
\[
(3.12) \quad \| h_{\nu} \|^2_{2, \Gamma} \leq \frac{M + R}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \| u_{\nu} - R \|_{2, \Gamma}.
\]
Therefore,
\[
\int_{\Omega} |\nabla^2 h|^2 \delta_{\Gamma}(x) \, dx \leq \frac{2}{r_i} \int_{\Omega} (-u) |\nabla^2 h|^2 \, dx \leq \left( \frac{M + R}{r_i} \right)^2 \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right)^2 \| u_{\nu} - R \|^2_{2, \Gamma},
\]
by Lemma 3.1. These inequalities and Theorem 3.5 then give (1.13).

If we want to measure the deviation of \( u_{\nu} \) from \( R \) in \( L^1 \)-norm, we get a smaller stability exponent.

Theorem 3.6 (Stability in \( L^1 \)-norm). Under the same assumptions of Theorem 1.1, we have that
\[
\rho_{\text{e}} - \rho_i \leq C \| u_{\nu} - R \|_{1, \Gamma}^{1/(N+2)} \quad \text{if} \quad \| u_{\nu} - R \|_{1, \Gamma} < \varepsilon,
\]
for some positive constants \( C \) and \( \varepsilon \).
Proof. Instead of applying Hölder’s inequality to the right-hand side of \( 1.18 \), we just use the rough bound:

\[
\int_{\Omega} (-u) |\nabla^2 h|^2 \, dx \leq \frac{1}{2} (M + R) (M + d_\Omega) \int_{\Gamma} |u_\nu - R| \, dS_x,
\]

since \((u_\nu + R) |h_\nu| \leq (M + R) (M + d_\Omega)\) on \(\Gamma\). The conclusion then follows from similar arguments. \(\square\)

Remark 3.7 (On the constants \(C\) and \(\varepsilon\)). For the sake of clarity, we did not display the values of the constants \(C\) and \(\varepsilon\) in Theorems 1.1, 3.6, and the relevant theorems in the sequel. However, their computation will be clear by following the steps of the proofs.

For instance, an inspection of the proofs of Theorem 1.1 informs us that the constants in \(1.13\) are:

\[
C = a_N \frac{M^{\frac{2N}{N + 2}} (M + R)^{\frac{2N}{N + 2}}}{|\Omega|^{\frac{2N}{N + 2}}} \left\{ N + r_i \mu(\Omega, z) \right\}^{\frac{N + 2}{N}} \frac{\sqrt{\mu(\Omega)}}{\mu(\Omega, z)}^{\frac{N + 2}{r_i}}.
\]

and

\[
\varepsilon = \alpha_N \frac{M}{M + R} \frac{\sqrt{\mu(\Omega, z)}}{\sqrt{N + r_i \mu(\Omega, z)}}^{\frac{N + 2}{r_i}}.
\]

Here, the constants \(a_N\) and \(\alpha_N\) are those appearing in \(3.10\) and \(3.11\).

The constants \(C\) and \(\varepsilon\) can be shown to depend only on some geometric parameters of \(\Omega\). In fact, we can use \(3.2\) to bound \(M\) in terms of \(d_\Omega\) and \(r_x\). To estimate the ratio \(R\), we can use the isoperimetric inequality, to bound \(|\Gamma|\) from below in terms of \(|\Omega|^{1-1/N}\), and then a trivial inequality to bound \(|\Omega|^{1/N}\) in terms of \(d_\Omega\).

Remark 3.8 (Estimating \(\mu(\Omega)\) and \(\mu(\Omega, z)\)). For simplicity, in what follows, \(k_N\) denotes a positive number, that only depends on \(N\), whose value may change from line to line.

(i) A lower bound of \(\mu(\Omega)\) can be obtained as follows. In [HS, Theorem 1.3] a more general form of inequality \(3.6\) (without the assumption of harmonicity) is proved for the class of the \(b_0\)-John domains (see [HS] for the definition) and the constants are explicitly computed; in particular, with the aid of [HS], we can easily deduce that

\[
\mu(\Omega)^{-1} \leq k_N |\Omega|^\frac{N}{2} k_0^{2N},
\]

for some constant \(k_N\) only depending on \(N\). A domain of class \(C^2\) is obviously a \(b_0\)-John domain and it is not difficult to show that

\[
b_0 \leq d_\Omega/r_i,
\]

and hence obtain that

\[
(3.14) \mu(\Omega)^{-1} \leq k_N |\Omega|^\frac{N}{2} \left(\frac{d_\Omega}{r_i}\right)^{2N}.
\]

An alternative way to estimate \(\mu(\Omega)\) can be found in [MP, Remark 3.8 (ii)].

A lower bound for \(\mu(\Omega, z)\) can also be obtained, but it may depend on the choice of the particular critical point of \(u\). In fact, by following the argument of [FG Lemma 8], one can adapt the result contained in [HS Theorem 1.3] to the case of harmonic functions vanishing at a given point \(z\) (i.e the case of \((3.5)\)). To explicitly carry out the computations, the definition of bounded \(L_0\)-John domain with given base point is appropriate (see [FG]). In fact, for that class of domains, by means of [FG, Lemma 8], we can deduce that

\[
(3.15) \mu(\Omega, z)^{-1} \leq k_N |\Omega|^\frac{N}{2} L_0^{2N}.
\]
and, by the definition, it is not difficult to prove the following bound:

\[(3.16)\]

\[L_0 \leq \frac{d_\Omega}{\min[r_i, \delta_\Gamma(z)]}\]

Thus, we obtain:

\[(3.17)\]

\[\mu(\Omega, z)^{-1} \leq k_N |\Omega|^{-\frac{d_\Omega}{\min[r_i, \delta_\Gamma(z)]}}^{2N}.\]

Finally, it is clear that we can eliminate the dependence on \(|\Omega|\) in \((3.14)\) and \((3.17)\) in favour of \(d_\Omega\) again.

(ii) The use of \(\delta_\Gamma(z)\) gives an estimate of the number \(L_0\) in terms of an explicit geometrical quantity.

Moreover, in case \(\Omega\) is convex, by using \(\delta_\Gamma(z)\) we are able to completely remove the dependence of the constants on \(z\) and we can affirm that \(C\) and \(\varepsilon\) are controlled by \(r_i\) and \(d_\Omega\), only.

In fact, we notice that the dependence of \(M\) on \(r_c\) can be removed, thanks to \((3.3)\). Moreover, we can further eliminate the dependence on \(\delta_\Gamma(z)\) and \(|\Omega|\) appearing in \((3.17)\). Indeed, \(u\) has a unique minimum point \(z \in \Omega\), since \(u\) is analytic and the level sets of \(u\) are convex by a result in \([Ko]\) (see \([MS]\), for a similar argument), and hence we have only one choice for the point \(z\). Thus, an estimate of \(\delta_\Gamma(z)\) from below can be obtained, first, by putting together arguments in \([BMS]\, Theorem 2.7] and \([BMS]\, Remark 2.5], to obtain that

\[\delta_\Gamma(z) \geq \frac{k_N r_i}{|\Omega|^{\frac{2}{d_\Omega}}}.\]

Secondly, by a simple comparison argument, the maximum can be bounded from below by \(r_i^2/2\), and hence we have that

\[\delta_\Gamma(z) \geq k_N \frac{r_i^{2N} |\Omega|^{\frac{2}{d_\Omega}}}{d_\Omega^{2N}}.\]

Again, \(|\Omega|\) can be easily bounded in terms of \(d_\Omega\). By similar arguments, we can take care of \(\varepsilon\).

**Remark 3.9.** The dependence on \(\delta_\Gamma(z)\) of the constants in the relevant estimates can also be removed by choosing the point \(z\) appearing in \((1.5)\) differently.

(i) As already done in \([MP]\, Theorem 3.6 and item (ii) of Remark 4.2], we can choose \(z\) as the center of mass \(z\) \((\Omega)\). In fact, we obtain that \(\int_\Omega \nabla h(x) \ dx = 0\) and we can use \((3.6)\), instead of \((3.5)\), in Lemma 3.4 and Theorems 3.5, 1.1, and 3.6. In this way, we avoid the use of \(\mu(\Omega, z)\) (and hence of \(\delta_\Gamma(z)\)).

However, in this case the extra assumption that \(z \in \Omega\) is needed, since we want that the ball \(B_{r_i}(z)\) (considered in Theorems 3.1, 3.2, and 3.6) be contained in \(\Omega\). (ii) As pointed out by the anonymous referee and as done in \([Fe]\), another possible way to choose \(z\) is \(z = x_0 - \nabla u(x_0)\), where \(x_0\) is the base point in the definition of \(L_0\)-John domain mentioned in Remark 3.8. In fact, we obtain that \(\nabla h(x_0) = 0\) and we can thus use \((3.5)\), with \(\mu(\Omega, z)\) replaced by \(\mu(\Omega, x_0)\), in Lemma 3.4 and Theorems 3.5, 1.1, and 3.6. Moreover, it is easy to check that every \(x_0 \in \Omega\) such that \(\delta_\Gamma(x_0) \geq r_i\) is an appropriate base point for the \(L_0\)-John condition for which it holds that

\[L_0 \leq \frac{d_\Omega}{r_i},\]

since one can use \((3.16)\) with \(z\) replaced by \(x_0\). Hence, from \((3.15)\) we have that

\[\mu(\Omega, x_0)^{-1} \leq k_N |\Omega|^{-\frac{d_\Omega}{r_i}}^{2N}.\]
As in item (i), we should additionally require that \( z \in \Omega \).

Since the estimates in Theorems 1.1 and 3.6 do not depend on the particular critical point chosen, as a corollary, we obtain results of closeness to a union of balls, similar to [MP, Corollary 4.3]: here, we just illustrate the instance of Theorem 1.1.

**Corollary 3.10** (Closeness to an aggregate of balls). Let \( \Gamma, R \) and \( u \) be as in Theorem 1.1. Then, there exist points \( z_1, \ldots, z_n \) in \( \Omega \), \( n \geq 1 \), and corresponding numbers \( \rho_i = \min_{x \in \Gamma} |x - z_j| \) and \( \rho_e = \min_{x \in \Gamma} |x - z_j| \), \( j = 1, \ldots, n \), such that

\[
\bigcup_{j=1}^n B_{\rho_i}(z_j) \subset \Omega \subset \bigcap_{j=1}^n B_{\rho_e}(z_j)
\]

and

\[
\max_{1 \leq j \leq n} (\rho_e - \rho_i) \leq C \left( \int_{\Omega} |\nabla^2 h|^2 \, dx \right)^{1/2} \quad \text{if} \quad \|u - R\|_{2, \Gamma}^2 < \varepsilon,
\]

for some positive constants \( C \) and \( \varepsilon \).

**Proof.** We pick one point \( z_j \) from each connected component of the set of local minimum points of \( u \). By applying Theorem 1.1 to each \( z_j \), the conclusion is then evident.

---

### 4. Enhanced stability for Alexandrov’s Soap Bubble Theorem

In this section, we collect those benefits of the new estimate (1.13) that affect the stability issue for the Soap Bubble Theorem.

We begin by recalling a couple of inequalities concerning the harmonic function \( h \), that we obtained in [MP, Theorem 3.4] and, in the sequel, will play the role of those of Theorem 3.5. It holds that

\[
\rho_e - \rho_i \leq C_0 \left( \int_{\Omega} |\nabla^2 h|^2 \, dx \right)^{1/2},
\]

if \( N = 2, 3 \), and

\[
\rho_e - \rho_i \leq C_0 \left( \int_{\Omega} |\nabla^2 h|^2 \, dx \right)^{1/(N+2)} \quad \text{for} \quad \int_{\Omega} |\nabla^2 h|^2 \, dx < \varepsilon_0^2,
\]

for \( N \geq 4 \). The point \( z \) chosen in the definition (2.1) of \( \rho_i \) and \( \rho_e \) is any critical point of \( u \), as usual.

The positive constants \( C_0 \) and \( \varepsilon_0 \) depend on \( N, |\Omega|, d, r, r_e \), and \( \mu_0(\Omega, z) \) given by (3.7) with \( \alpha = 0 \). For their values, we refer to [MP, Theorem 3.4].

**Remark 4.1.** The parameter \( \mu_0(\Omega, z) \) can be estimated by following the arguments used in item (i) of Remark 3.8 to estimate \( \mu(\Omega, z) \). In fact, we deduce that

\[
\mu_0(\Omega, z)^{-1} \leq k_N |\Omega| \widetilde{L}_N^2.
\]

Another way to estimate \( \mu_0(\Omega, z) \) can be found in [MP, Remark 3.8 (iii)].

Next, we derive the following lemma, that parallels and is a useful consequence of Lemma 3.4.
Lemma 4.2. Let \( N \geq 2 \) and let \( \Gamma \) be the connected boundary of class \( C^2 \) of a bounded domain \( \Omega \subset \mathbb{R}^N \). Denote by \( H \) the mean curvature function for \( \Gamma \) and let \( H_0 \) be the constant defined in (1.3).

Then, the following inequality holds:

\[ (4.3) \quad \|u_\nu - R\|_{2,\Gamma} \leq R \left\{ d_\Omega + \frac{M(M + R)}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \right\} \|H_0 - H\|_{2,\Gamma}. \]

Proof. Discarding the first summand on the left-hand side of (2.6) and applying Hölder’s inequality on its right-hand side gives that

\[ \frac{1}{R} \|u_\nu - R\|_{2,\Gamma}^2 \leq \|H_0 - H\|_{2,\Gamma} (M\|h_\nu\|_{2,\Gamma} + d_\Omega \|u_\nu - R\|_{2,\Gamma}), \]

since \( u_\nu \leq M \) and \( |\nu_0| \leq d_\Omega \) on \( \Gamma \). Thus, inequality (3.12) implies that

\[ \|u_\nu - R\|_{2,\Gamma} \leq R \left\{ d_\Omega + \frac{M(M + R)}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \right\} \|H_0 - H\|_{2,\Gamma} \|u_\nu - R\|_{2,\Gamma}, \]

from which (4.3) follows at once. \( \square \)

Proof of Theorem 1.2. Discarding the second summand on the left-hand side of (2.6) and applying Hölder’s inequality on its right-hand side, as in the previous proof, gives that

\[ \frac{1}{N - 1} \int_\Omega |\nabla^2 h|^2 dx \leq \]

\[ R \left\{ d_\Omega + \frac{M(M + R)}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \right\} \|H_0 - H\|_{2,\Gamma} \|u_\nu - R\|_{2,\Gamma} \leq \]

\[ R^2 \left\{ d_\Omega + \frac{M(M + R)}{r_i} \left( 1 + \frac{N}{r_i \mu(\Omega, z)} \right) \right\}^2 \|H_0 - H\|_{2,\Gamma} \]

where the second inequality follows from Lemma 4.2.

Inequalities (1.16) and (1.17) then result from (4.1) and (4.2). \( \square \)

Remark 4.3 (On the constants \( C \) and \( \varepsilon \)). Needless to say, the proof of Theorem 1.2 tells us that, by proceeding as in Remarks 3.7, 3.8, and 4.1, we can reduce the dependence of \( C \) and \( \varepsilon \) to the parameters \( N, r_i, r_e, d_\Omega, \) and \( \delta_\Gamma(z) \), if \( \Gamma \) is of class \( C^2 \), and to \( N, r_i, \) and \( d_\Omega \), if \( \Omega \) is also convex or chosen as described in Remark 3.9.

Remark 4.4. (i) Assertion (ii) of Theorem 1.2 can also be proved as a direct corollary of Theorem 1.1 by noting that (4.3) together with (1.13) gives (1.17).

(ii) Analogs of Theorem 3.6 and Corollary 3.10 can be easily derived by following the steps of their proofs.

Remark 4.5. The assumption of smallness of the relevant deviation \( \eta \) required in Theorems 1.1, 3.6, Corollary 3.10, and (ii) of Theorem 1.2 is only apparent, because if \( \eta \geq \varepsilon \), then it is a trivial matter to obtain an upper bound for \( \rho_e - \rho_i \) in terms of \( \eta \). Thus, all the stability estimates that we presented are global.

Another consequence of Lemma 4.2 is the following inequality that shows an optimal stability exponent for any \( N \geq 2 \). The number \( A(\Omega) \), defined in (1.14), is some sort of asymmetry similar to the so-called Fraenkel asymmetry (see [Fr]).

**Theorem 4.6 (Stability by asymmetry).** Let \( N \geq 2 \) and let \( \Gamma \) be the connected boundary of class \( C^2 \) of a bounded domain \( \Omega \subset \mathbb{R}^N \). Denote by \( H \) the mean curvature function for \( \Gamma \) and let \( H_0 \) be the constant defined in (1.8).

Then it holds that

\[ (4.4) \quad A(\Omega) \leq C \|H_0 - H\|_{2,\Gamma}, \]

for some positive constant $C$.

Proof. We use [Fe] inequality (2.14): we have that

$$\frac{|\Omega \Delta B_{R}^{x}|}{|B_{R}^{x}|} \leq C \|u_{\nu} - R\|_{2, \Gamma},$$

where $B_{R}^{x}$ is a ball of radius $R$ (as defined in (1.8)) and centered at the point $z$ described in item (ii) of Remark 3.9. Hence, we obtain that

$$\mathcal{A}(\Omega) \leq C \|u_{\nu} - R\|_{2, \Gamma},$$

by the definition (1.14). Thus, thanks to (4.3), we obtain (4.4).

Here, if we estimate $L_0$ as done in item (ii) of Remark 3.9 we can see that $C$ depends on $N, d_{0}, r_{i}$, and $|\Gamma|/|\Omega|$. \hfill $\Box$

Remark 4.7 (On the asymmetry $\mathcal{A}(\Omega)$). Notice that, for any $x \in \Omega$, we have that

$$\frac{|\Omega \Delta B_{R}^{x}|}{|B_{R}^{x}|} \leq \frac{|B_{R}^{x} \setminus B_{\rho}^{x}|}{|B_{R}^{x}|} = \frac{\rho_{e}^{N} - \rho_{i}^{N}}{R^{N}} \leq \frac{N \rho_{e}^{N-1}}{R^{N}} (\rho_{e} - \rho_{i}),$$

and $\rho_{e} \leq d_{0}$. Thus, if $d_{0}/R$ remains bounded and $(\rho_{e} - \rho_{i})/R$ tends to 0, then the ratio $|\Omega \Delta B_{R}^{x}|/|B_{R}^{x}|$ does it too.

The converse is not true in general. For example, consider a lollipop made by a ball and a stick with fixed length $L$ and vanishing width; as that width vanishes, the ratio $d_{0}/R$ remains bounded, while $|\Omega \Delta B_{R}^{x}|/|B_{R}^{x}|$ tends to zero and $\rho_{e} - \rho_{i} \geq L > 0$.

If we fix $r_{i}$ and $r_{e}$, we have the following result.

Theorem 4.8. Let $\Omega \subset \mathbb{R}^{N}, N \geq 2$, be a bounded domain satisfying the uniform interior and exterior sphere conditions with radii $r_{i}$ and $r_{e}$.

Then, we have that

$$\rho_{e} - \rho_{i} \leq 4 RA(\Omega)^{\frac{1}{N}} \text{ if } \mathcal{A}(\Omega) \leq \left( \frac{r_{i}}{R} \right)^{N}.$$  

Proof. Let $x$ be any point in $\Omega$. It is clear that

$$\text{(4.5) } \max(\rho_{e} - R, R - \rho_{i}) \geq \frac{\rho_{e} - \rho_{i}}{2}.$$  

If that maximum is $\rho_{e} - R$, at a point $y$ where the ball centered at $x$ with radius $\rho_{e}$ touches $\Gamma$, we consider the interior touching ball $B_{r_{i}}$. If $2r_{i} < (\rho_{e} - \rho_{i})/2$ then $B_{r_{i}} \subset \Omega \setminus B_{R}^{x}$ and hence

$$\frac{|\Omega \Delta B_{r_{i}}^{x}|}{|B_{r_{i}}^{x}|} \geq \left( \frac{r_{i}}{R} \right)^{N}.$$  

If, else, $2r_{i} \geq (\rho_{e} - \rho_{i})/2$, $B_{r_{i}}$ contains a ball of radius $(\rho_{e} - \rho_{i})/4$ still touching $\Gamma$ at $y$. Such a ball is contained in $\Omega \setminus B_{r_{i}}^{x}$, and hence

$$\frac{|\Omega \Delta B_{r_{i}}^{x}|}{|B_{r_{i}}^{x}|} \geq \left( \frac{\rho_{e} - \rho_{i}}{4R} \right)^{N}.$$  

Thus, we proved that

$$\frac{|\Omega \Delta B_{r_{i}}^{x}|}{|B_{r_{i}}^{x}|} \geq \min \left\{ \left( \frac{r_{i}}{R} \right)^{N}, \left( \frac{\rho_{e} - \rho_{i}}{4R} \right)^{N} \right\},$$

and the conclusion easily follows, since $x$ was arbitrarily chosen in $\Omega$.

If, else, the maximum in (4.3) is $R - \rho_{i}$, we proceed similarly, by reasoning on the exterior ball $B_{r_{i}}$ and $\mathbb{R}^{N} \setminus \Omega$, instead. \hfill $\Box$
Remark 4.9. Theorem 4.8 and (4.4) give the inequality
\[ \rho_e - \rho_i \leq C \left| \| H_0 - H \|_{2,1}^{1/N} \right. \]
that, for any \( N \geq 2 \), is poorer than that obtained in 1.2.

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References


