SUFFICIENTLY COLLAPSED IRREDUCIBLE
ALEXANDROV 3-SPACES ARE GEOMETRIC

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Abstract. We prove that sufficiently collapsed, closed and irreducible three-dimensional Alexandrov spaces are modeled on one of the three-dimensional Thurston geometries, excluding the hyperbolic one. This extends a result of Shioya and Yamaguchi, originally formulated for Riemannian manifolds, to the Alexandrov setting.

Contents

1. Introduction 1
2. Preliminaries 3
3. Double branched covers of collapsing Alexandrov spaces 13
4. Outline of the proof of Theorem A 14
5. Two-dimensional limit space 15
6. One-dimensional limit space 20
7. Zero-dimensional limit space 27
References 27

1. Introduction

In Riemannian geometry, collapse imposes strong geometric and topological restrictions on the spaces on which it occurs. Recall that there are eight three-dimensional Thurston geometries: $S^3$, $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\tilde{\text{SL}}_2(\mathbb{R})$, Nil and Sol (see [29]). A closed (i.e. compact and without boundary) 3-manifold is geometric if it admits a geometric structure modeled on...
one of these geometries. In this context, Shioya and Yamaguchi [30] obtained
the following result for sufficiently collapsed Riemannian 3-manifolds.

**Theorem** (Shioya and Yamaguchi [30, Corollary 0.9]). Let $D > 0$ and let
$\mathcal{M}(3, D)$ be the class of closed Riemannian 3-manifolds with diameter at
most $D$ and sectional curvature bounded below by $-1$. For any $D > 0$,
there exists a constant $\varepsilon = \varepsilon(D) > 0$ such that if a closed, prime 3-manifold
with infinite fundamental group admits a Riemannian metric contained in
$\mathcal{M}(3, D)$ with volume $< \varepsilon$, then it admits a geometric structure modeled on
one of the six geometries $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\text{SL}_2(\mathbb{R})$, Nil and Sol.

Hyperbolic geometry is ruled out by the fact that a manifold with such a
google has non-vanishing simplicial volume (see [31, Theorem 6.2]), and
the fact that manifolds with positive simplicial volume do not collapse (see
[12, pp. 67–68]). We point out that, although the geometry $\mathbb{S}^3$ appears in the
original statement of the result in [30], the assumption that the fundamental
group of the manifold is infinite precludes it from admitting spherical geom-
etry. Note, in any case, that by Perelman’s proof of Thurston’s Elliptiza-
tion Conjecture, any closed 3-dimensional manifold with finite fundamental
group has spherical geometry.

Closed Riemannian manifolds are special cases of Alexandrov spaces (with
curvature bounded below). In this article, we extend the preceding theorem
to the Alexandrov setting. In dimension three, the structure of Alexandrov
spaces is fairly well understood (see [9, 18]). Topologically, these spaces
are either 3-manifolds or quotients of 3-manifolds by orientation reversing
involutions with only isolated fixed points (see [9]).

Before stating our main theorem, we need some definitions. Recall that
a non-trivial closed 3-manifold $M$ is *prime* if it cannot be presented as a
connected sum of two non-trivial closed 3-manifolds. A closed 3-manifold
is *irreducible* if every embedded 2-sphere bounds a 3-ball. It is known that,
with the exception of manifolds homeomorphic to $\mathbb{S}^3$, $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \times \mathbb{S}^2$ (the
non-trivial 2-sphere bundle over $\mathbb{S}^1$), a closed 3-manifold is prime if and
only if it is irreducible (see [15, Lemma 3.13]). Since $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^1 \times \mathbb{S}^2$
are geometric, and $\mathbb{S}^3$ has finite fundamental group, one can think of the
theorem above as a statement about irreducible 3-manifolds. Therefore, in
generalizing this theorem to Alexandrov spaces we will focus our attention
on the irreducible case. In view of this, we need to define *irreducibility* for
this more general class of spaces.

**Definition.** Let $X$ be a closed Alexandrov 3-space. We say that $X$ is
*irreducible* if every embedded 2-sphere in $X$ bounds a 3-ball and, in the case
that the set of topologically singular points of $X$ is non-empty, it is further
required that every 2-sided $\mathbb{R}P^2$ bound a $K(\mathbb{R}P^2)$, a cone over $\mathbb{R}P^2$.

With this definition in hand, we state our main result.

**Theorem A.** For any $D > 0$ there exists $\varepsilon = \varepsilon(D) > 0$ such that, if $X$ is
a closed, irreducible Alexandrov 3-space with curv $\geq -1$, diam $X \leq D$, and
$\text{vol } X < \varepsilon$, then $X$ admits a geometric structure modeled on one of the seven geometries $\mathbb{R}^3$, $S^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\text{SL}_2(\mathbb{R})$, Nil and Sol.

In Theorem A, by the volume of an Alexandrov 3-space, we mean its 3-dimensional Hausdorff measure, normalized so that volume of 3-dimensional Riemannian manifolds agrees with the usual Riemannian volume.

**Remark B.** As in the Riemannian case, one can rule out the appearance of hyperbolic geometry by combining the fact that the simplicial volume of a collapsing Alexandrov space is zero (see [19, Corollary 1.7]) with the fact that the simplicial volume of a hyperbolic manifold must be bounded below by the Riemannian volume (see [31, Theorem 6.2]). For more details, see the end of Section 4.

We prove Theorem A by carefully studying the metric and topological structure of collapsed irreducible Alexandrov 3-spaces and their orientable double branched covers (in the case where the space is not a manifold). The extensive work of Mitsuishi and Yamaguchi on collapsed Alexandrov 3-spaces [18], combined with the irreducibility hypothesis, allows us to obtain fairly explicit topological descriptions of closed, collapsed, irreducible Alexandrov 3-spaces, and to exhibit them as geometric 3-manifolds or their quotients by orientation-reversing involutions with only isolated fixed points.

Classification results of Alexandrov spaces with (local) circle actions (see [10, 21]) and classical results on involutions on 3-manifolds (see [16, 17]) also play an important role in the proof.

Our article is divided as follows. In Section 2 we collect results on general Alexandrov 3-spaces, as well as on those spaces that admit collapse or (local) isometric circle actions. In Section 3 we show that a non-manifold Alexandrov 3-space collapses if and only if its canonical orientable double branched Alexandrov cover also collapses. We prove Theorem A in Sections 4–7.

**Acknowledgements.** The authors thank John Harvey, for pointing out reference [19], and the referee, for helpful comments.

### 2. Preliminaries

**Three-dimensional Alexandrov spaces.** In this section we give a brief account of the basic structural properties of closed Alexandrov spaces of dimension 3. We refer the reader to [9] for a more detailed account.

Let $X$ be a closed Alexandrov 3-space. The space of directions $\Sigma_x X$ at each point $x$ in $X$ is a closed Alexandrov 2-space with curvature bounded below by 1. Therefore, by the Bonnet-Myers Theorem [2, Theorem 10.4.1], the fundamental group of $\Sigma_x X$ is finite. This in turn implies that $\Sigma_x X$ is homeomorphic to a 2-sphere $S^2$ or a real projective plane $\mathbb{R}P^2$. A point in $X$ whose space of directions is homeomorphic to $S^2$ is called a topologically regular point. Otherwise, the point is called topologically singular. The set of topologically regular points is open and dense in $X$. 
By Perelman’s Conical Neighborhood Theorem [26], every point \( x \) in \( X \) has a neighborhood pointed-homeomorphic to the cone over the \( \Sigma_x X \). This result implies that the set of topologically singular points of \( X \) is finite and \( X \) is homeomorphic to a compact 3-manifold \( X_o \) with a finite number of \( \mathbb{RP}^2 \)-boundary components to which one glues in cones over \( \mathbb{RP}^2 \). It is an easy consequence of this description that \( X \) must have an even number of topologically singular points. Note first that the number of topologically singular points of \( X \) is the number of \( \mathbb{RP}^2 \)-boundary components of \( X_o \).

Letting \( D(X_o) \) be the double of \( X_o \), we obtain, using the Mayer-Vietoris sequence for the decomposition of \( D(X_o) \) as the union of two copies of \( X_o \) glued along \( \partial X_o \), that

\[
\chi(D(X_o)) = 2\chi(X_o) - \chi(\partial X_o).
\]

Since \( D(X_o) \) is a closed 3-manifold, its Euler characteristic is zero. Hence, the preceding equation implies that \( \chi(\partial X_o) \) is even. Since each connected component of \( \partial X_o \) is a real projective space, and \( \chi(\mathbb{RP}^2) = 1 \), it follows that \( X_o \) has an even number of boundary components (cf. [15, Proof of Theorem 9.5]). Therefore, \( X \) has an even number of topologically singular points.

A closed Alexandrov 3-space \( X \) whose set of topologically singular points is non-empty can also be described as a quotient of a closed, orientable, topological 3-manifold \( \tilde{X} \) by an orientation-reversing involution \( \iota : \tilde{X} \to \tilde{X} \) with only isolated fixed points. The 3-manifold \( \tilde{X} \) is the so-called orientable double branched cover of \( X \) (see, for example, [9, Lemma 1.7]). In addition to this topological description, it is possible to lift the metric on \( X \) to \( \tilde{X} \), so that \( \tilde{X} \) becomes an Alexandrov space with the same lower curvature bound as \( X \) and \( \iota \) is an isometry with respect to the lifted metric. In particular, \( \iota \) is equivalent to a smooth involution on \( \tilde{X} \) regarded as a smooth 3-manifold.

We refer the reader to [9, Lemma 1.8] and [13, Section 5] for more details.

The geometric structure of Alexandrov 3-spaces was further studied in [9]. We recall that \( X \) is geometric (or admits a Thurston geometry) if it is a quotient of one of the eight 3-dimensional Thurston geometries by some cocompact lattice (see [29]). We say that \( X \) admits a geometric decomposition if \( X \) can be cut along a family of spheres, projective planes, tori, and Klein bottles in such a way that the resulting pieces are geometric. The geometrization of closed Alexandrov 3-spaces was obtained in [9], i.e. it was proved that every closed Alexandrov 3-space admits a geometric decomposition into geometric Alexandrov 3-spaces.

**Collapsing Alexandrov 3-spaces.** Let \( \{X_i\}_{i=1}^\infty \) be a sequence of \( n \)-dimensional Alexandrov spaces with diameters uniformly bounded above by \( D > 0 \) and \( \text{curv} \geq k \) for some \( k \in \mathbb{R} \). After passing to a subsequence, Gromov’s Precompactness Theorem implies that there exists an Alexandrov space \( Y \) with diam \( Y \leq D \) and \( \text{curv} Y \geq k \) such that \( X_i \to Y \) in the Gromov-Hausdorff sense. As in the Riemannian case, the sequence \( X_i \) is said to collapse to \( Y \) if \( \text{dim} Y < n \). We will also say that an \( n \)-dimensional Alexandrov space \( X \) collapses (or that
it is a \textit{collapsing Alexandrov space} if there exists a sequence of Alexandrov
metrics \(\{d_i\}_{i=1}^\infty\) on \(X\), such that \(\{(X,d_i)\}_{i=1}^\infty\) is a collapsing sequence. In
[18], Mitsuishi and Yamaguchi obtained a topological classification of closed
collapsing 3-dimensional Alexandrov spaces, describing them as a union of
certain pieces. We now give a brief account of those pieces with topological
or metric singularities.

\textit{The space \(B(\text{pt})\).} Let \(D^2 \times S^1 \subset \mathbb{R}^2 \times \mathbb{C}\) be equipped with the usual flat
product metric. An isometric involution \(\alpha\) on \(D^2 \times S^1\) is defined by
\[
\alpha((x,y),e^{i\theta}) := ((-x,-y), e^{-i\theta}).
\]
The space \(B(\text{pt}) := D^2 \times S^1 / \alpha\) is an Alexandrov space of curv \(\geq 0\) with two
topologically singular points corresponding to the image in the quotient of
the points \(((0,0),e^{i\theta})\) and \(((0,0),e^{i\pi})\), which are fixed by \(\alpha\) (cf. [18, Exam-
ple 1.2]). There is a projection \(p\) : \(B(\text{pt}) \to K_1(S^1)\) sending an interval join-
ing the topologically singular points to the vertex \(o\) of the cone. To describe
it, observe that the quotient of \(D^2 \subset \mathbb{R}^2\) by the involution \((x,y) \mapsto (-x,-y)\)
is homeomorphic to \(D^2\), and metrically is isometric to \(K_1(S^1)\), where the \(S^1\)
taken has length \(\pi\). The projection \(p\) : \(B(\text{pt}) \to K_1(S^1)\) is then obtained by
mapping
\[
[(x,y),e^{i\theta}] \to [(x,y)].
\]
This projection is a fibration on \(K_1(S^1) \setminus \{o\}\).

An alternate, topological description of \(B(\text{pt})\) appears after [18, Exam-
ple 2.60]: choose two copies of cones over \(\mathbb{R}P^2\), select a disc \(D_i^2\), \(i = 0,1,\)
on each \(\mathbb{R}P^2\)-boundary, and glue both cones using some homeomorphism
\(\varphi : D_i^2 \to D_i^2\). The resulting space does not depend on the chosen \(\varphi\), and is
homeomorphic to \(B(\text{pt})\). It is clear that its boundary is obtained by taking
two Möbius bands glued by their boundaries, i.e. a Klein bottle.

\textit{Spaces with 2-dimensional souls.} We now describe three different closed
Alexandrov 3-spaces as quotients of certain involutions (cf. [20]):

\begin{enumerate}
\item \(B(S_2) := S^2 \times [-1,1]/(\sigma,-\text{id})\), where \(S^2\) is a sphere of non-negative
curvature in the Alexandrov sense with an isometric involution \(\sigma : S^2 \to S^2\) topologically conjugate to the involution on the 2-sphere
given by the suspension of the antipodal map on the circle. The
resulting space is homeomorphic to \(\text{Susp}(\mathbb{R}P^2) \setminus \text{int}(D^3)\), where \(D^3 \subset \text{Susp}(\mathbb{R}P^2)\) is a closed 3-ball consisting of topologically regular points
(cf. [18, Remark 2.62]).
\item \(B(S_4) := T^2 \times [-1,1]/(\sigma,-\text{id})\), where \(T^2\) is a flat torus and the
involution \(\sigma : T^2 \to T^2\) maps \((z_1,z_2)\) to \((z_1,\bar{z}_2)\) (observe that \(T^2/\sigma\)
is homeomorphic to \(S^2\)). This space has four topologically singular
points, corresponding to the four fixed points of the involution; this
can be seen by observing that at each such point, the differential of

\end{enumerate}
the involution acts as the antipodal map on the unit tangent sphere. Its oriented branched cover is $\mathbb{T}^2 \times [-1, 1]$.

(iii) $B(\mathbb{RP}^2) := K^2 \times [-1, 1]/(\sigma, -\text{id})$, where $K^2$ is a flat Klein bottle and $\sigma : K^2 \to K^2$ is an isometric involution topologically conjugate to the unique involution on $K^2$ whose quotient is $\mathbb{RP}^2$.

**Generalized Seifert fiber spaces.** A generalized Seifert fibration of a topological 3-orbifold $M$ over a topological 2-orbifold $B$ (both possibly with boundaries) is a map $f : M \to B$ whose fibers are homeomorphic to circles or bounded closed intervals. It is required that for every $x \in B$, there is a neighborhood $U_x$ homeomorphic to a 2-disk such that

(i) if $f^{-1}(x)$ is homeomorphic to a circle, then there exists a fiber-preserving homeomorphism of $f^{-1}(U_x)$ to a Seifert fibered solid torus in the usual sense, and

(ii) if $f^{-1}(x)$ is homeomorphic to an interval, then there is a fiber-preserving homeomorphism of $f^{-1}(U_x)$ to the space $B(\text{pt})$, with respect to the fibration $(B(\text{pt}), p^{-1}(o)) \to (K_1(S^1), o)$.

Furthermore, for any compact component $C$ of $\partial B$ there is a collar neighborhood $N$ of $C$ in $B$ such that $f|_{f^{-1}(N)}$ is a usual circle bundle over $N$. We say that $M$ is a generalized Seifert fibered space and we use the notation $M = \text{Seif}(B)$.

**Generalized solid tori and Klein bottles.** A generalized solid torus (GST) (respectively, generalized solid Klein Bottle (GSKB)) is a topological 3-orbifold $Y$ with boundary homeomorphic to a torus (respectively, a Klein bottle). It admits a map $Y \to S^1$ such that the fibers are homeomorphic to either a 2-disk or a Möbius band, and the fiber type can only change at a finite number of corner points in $S^1$. We refer the reader to [18, Definition 1.4] for the precise definitions. A GST (respectively, a GSKB) is said to be of type $N$ if it has $2N$ topologically singular points. The GST of type 0 are defined to be $S^1 \times D^2$ and $S^1 \times \text{Mo}$, where Mo denotes the Möbius strip. Similarly, one defines the GSKB of type 0 as $S^1 \times D^2$ and $S^1 \times \text{Mo}$. For convenience, we will denote a generalized solid torus (respectively, generalized solid Klein bottle) of type $N$ by $\text{GST}_N$ (respectively, $\text{GSKB}_N$).

**I-bundles over the Klein bottle.** These are obtained as disk bundles of certain line bundles over the Klein bottle $K^2$. They are easily described as quotients of $\mathbb{R}^3$ under certain isometric actions [34]. Except for the trivial bundle $K^2 \times I$, the rest are as follows:

(i) $K^2 \times I$: this is the disc bundle in the orientable three manifold obtained as quotient of $\mathbb{R}^3$ under the group generated by

$$(x, y, z) \xrightarrow{\tilde{r}} (x + 2, y, z), \quad (x, y, z) \xrightarrow{\tilde{\sigma}} (-x, y + 1, -z).$$

Its boundary is given by a 2-torus.
(ii) $K^2 \tilde{\times} I$: this is the disc bundle in the non-orientable three manifold obtained as quotient of $\mathbb{R}^3$ under the group generated by

$$(x, y, z) \overset{\tau}{\mapsto} (x + 1, y, -z), \quad (x, y, z) \overset{\sigma}{\mapsto} (-x, y + 1, -z).$$

Its boundary is given by a Klein bottle.

The identity map in $\mathbb{R}^3$ induces a twofold Riemannian covering map $\pi : K^2 \tilde{\times} I \to K^2 \hat{\times} I$. At the fundamental group level, $\pi$ is an injective homomorphism that sends $\tilde{\tau} \mapsto \hat{\tau}^2$ and $\tilde{\sigma} \mapsto \hat{\sigma}$. Furthermore, since the fundamental group of $K^2$ is the dihedral group, and this group contains a unique subgroup of index 2, it follows that $K^2 \tilde{\times} I$ is the unique twofold cover of $K^2 \hat{\times} I$.

We summarize the results of [18] in Tables 1, 2 and 4, which contain the classifications of collapsing Alexandrov 3-spaces $X_i$, for sufficiently large $i$, with a 2-, 1- and 0-dimensional limit space $Y$, respectively.

**Table 1. Two-dimensional limit space $Y$**

<table>
<thead>
<tr>
<th>dim $Y$</th>
<th>$\partial Y$</th>
<th>Homeomorphism type of $X_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>= $\emptyset$</td>
<td>$\text{Seif}(Y)$</td>
</tr>
<tr>
<td></td>
<td>$\neq \emptyset$</td>
<td>$\text{Seif}(Y) \cup_{\partial Y} \left( \bigsqcup_i GST_{N_i} \bigsqcup \bigsqcup_j GSKB_{N_j} \right)$</td>
</tr>
</tbody>
</table>

**Table 2. One-dimensional limit space $Y$**

<table>
<thead>
<tr>
<th>dim $Y$</th>
<th>$\partial Y$</th>
<th>Homeomorphism type of $X_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>= $\emptyset$</td>
<td>$F$-fiber bundle over $S^1$ with $F = S^2, \mathbb{R}P^2, T^2$ or $K^2$</td>
</tr>
<tr>
<td></td>
<td>$\neq \emptyset$</td>
<td>$B \cup_{\partial B} B'$ where $\partial B = \partial B'$, and $B$, $B'$ are given in Table 3</td>
</tr>
</tbody>
</table>

**Local circle actions.** In this section we summarize the classification of closed Alexandrov spaces of dimension 3 admitting a local isometric circle action obtained in [10]. This classification will be useful in subsequent sections.

Let $X$ be a closed Alexandrov 3-space. A local circle action on $X$ is a decomposition of $X$ into (possibly degenerate) disjoint, simple, closed curves called fibers, each having a tubular neighborhood which admits an effective circle action whose orbits are the curves of the decomposition. The local circle action is isometric if the circle actions on each tubular neighborhood of the fibers are isometric with respect to the restricted metric.

The different fiber types of a local circle action on $X$ depend on the corresponding isotropy group, where one considers these fibers as orbits of the isometric circle action on a small tubular neighborhood around them.
Table 3. Homeomorphism type of $B$ and $B'$ in Table 2 ($B$ and $B'$ may differ)

<table>
<thead>
<tr>
<th>Homeomorphism type of $\partial B = \partial B'$</th>
<th>Homeomorphism type of $B$, $B'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^2$</td>
<td>$D^3$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{R}P^3 \setminus D^3$</td>
</tr>
<tr>
<td></td>
<td>$B(S_2)$</td>
</tr>
<tr>
<td>$\mathbb{R}P^2$</td>
<td>$K_1(\mathbb{R}P^2)$</td>
</tr>
<tr>
<td>$T^2$</td>
<td>$D^2 \times S^1$</td>
</tr>
<tr>
<td></td>
<td>$\text{Mo} \times S^1$</td>
</tr>
<tr>
<td></td>
<td>$K^2 \times I$</td>
</tr>
<tr>
<td></td>
<td>$B(S_4)$</td>
</tr>
<tr>
<td>$K^2$</td>
<td>$S^1 \times D^2$</td>
</tr>
<tr>
<td></td>
<td>$K^2 \times I$</td>
</tr>
<tr>
<td></td>
<td>$B(\text{pt})$</td>
</tr>
<tr>
<td></td>
<td>$B(\mathbb{R}P^2)$</td>
</tr>
</tbody>
</table>

Table 4. Zero-dimensional limit space $Y$

<table>
<thead>
<tr>
<th>dim $Y$</th>
<th>Homeomorphism type of $X_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>That of a space in Table 1 with curv $Y \geq 0$</td>
</tr>
<tr>
<td></td>
<td>That of a space in Table 2</td>
</tr>
<tr>
<td></td>
<td>That of a closed non-negatively curved Alexandrov space with finite fundamental group</td>
</tr>
</tbody>
</table>

The possible fiber types are the following. An $F$-fiber is a topologically regular point of $X$ which is a fixed point of the local circle action. An $SF$-fiber is a topologically singular point of $X$ which is a fixed point of the local circle action. Observe that $SF$-fibers are isolated. The fibers of type $E$ are those that correspond to $\mathbb{Z}_k$ isotropy, acting in such a way that local orientation is preserved. The fibers of type $SE$ correspond to $\mathbb{Z}_2$ isotropy, reversing the local orientation. Fibers that are not $F$, $SF$, $E$- or $SE$-fibers are called $R$-fibers. The strata of $F$-, $SF$-, $E$-, $SE$- and $R$-fibers are denoted, respectively, by $F$, $SF$, $E$, $SE$ and $R$. The fiber space, denoted by $X^*$, is a 2-dimensional Alexandrov domain, and therefore a topological 2-manifold, possibly with boundary. When present, the boundary is composed of the
SUFFICIENTLY COLLAPSED ALEXANDROV 3-SPACES

A closed Alexandrov 3-space with an isometric local circle action can be decomposed into the following pieces:

(a) building blocks which arise by considering small tubular neighborhoods of connected components of fibers of type \(F\), \(SF\), \(E\) and \(SE\);

(b) an \(S^1\)-fiber bundle (composed only of \(R\)-fibers) with structure group \(O(2)\) over a compact 2-manifold with boundary, which corresponds to the complement in \(X\) of the union of the building blocks in part (a).

A building block is called simple if its boundary is homeomorphic to a torus, and twisted if its boundary is homeomorphic to a Klein bottle (see [10, Section 3]).

Note that we reserved the term building block for neighborhoods of sets of non-regular fibers. Hence, building blocks always contain \(R\)-fibers, since these correspond to principal orbits of the local action.

The set of \(R\)-fibers corresponds to the circle bundle in (b) above only when there are no fibers of any other type (\(F\), \(SF\), \(E\) or \(SE\)). For example, in \(S^1 \times S^2\) with the free action given by letting \(S^1\) act on the circle factor, the set of \(R\)-fibers comprises the whole manifold. When there exist fibers of other types, the set of \(R\)-fibers splits into the \(S^1\) bundle in part (b) and pieces contained the different building blocks of the decomposition. For example, in the isometric action of \(S^1\) on the unit round 3-sphere given by

\[ S^1 \times S^3 \rightarrow S^3, \]

\[ \theta(z_1, z_2) \mapsto (e^{ik\theta} z_1, e^{il\theta} z_2), \]

with \(k\) and \(l\) relatively prime, we have two building blocks of type \(E\), corresponding to small tubular neighborhoods of the exceptional orbits with isotropy \(\mathbb{Z}_k\) and \(\mathbb{Z}_l\). Thus, the two building blocks are solid tori. The \(S^1\) bundle in part (b) corresponds to the complement of these two tubular neighborhoods and is composed of principal orbits of the action. It is a trivial \(S^1\) bundle over \(S^1 \times I\), where \(I\) is a closed interval. The set of \(R\)-fibers is the set of principal orbits and is the complement of the two exceptional orbits in \(S^3\).

The equivariant and topological classification of local circle actions, even in the case of 3-manifolds, is a strict generalization of the corresponding classification of global circle actions. For example Seifert manifolds admit local circle actions but not all of them admit a global action of \(S^1\) (see the Introduction of [24]).

Other examples of closed 3-manifolds with a local circle action that is not induced by a global action are provided by the unit tangent bundles of closed non-orientable surfaces. The structure group of such bundles cannot be reduced to \(SO(2)\) since the bases are non-orientable 2-manifolds. To construct a Riemannian metric such that the local action is isometric one
may proceed as follows. Let Σ be a closed non-orientable surface and let \( P^\Sigma \) be its unit tangent bundle. We let \( \pi : S \to \Sigma \) be the orientable double cover of \( \Sigma \). Equip \( \Sigma \) with an arbitrary Riemannian metric \( g \) and let \( h = \pi^* g_0 \) be the pullback metric on \( S \). Choose a connection \( \theta \) on the pullback circle bundle \( \pi^* P^\Sigma \) and, if necessary, average it with the involution \( \iota : \pi^* P^\Sigma \to \pi^* P^\Sigma \) corresponding to the twofold covering map \( \tilde{\pi} : \pi^* P^\Sigma \to P^\Sigma \), so that \( \theta \) is invariant under \( \iota \). We equip \( \pi^* P^\Sigma \) with any connection metric \( h \) that makes the circle action on \( \pi^* P^\Sigma \) isometric. We now have a Riemannian manifold \((\pi^* P^\Sigma, \tilde{\pi}, h)\) with an isometric involution \( \iota : \pi^* P^\Sigma \to \pi^* P^\Sigma \) which induces a Riemannian quotient metric \( \tilde{g} \) on \( P^\Sigma = \pi^* P^\Sigma / \iota \). In summary, we have the diagram

\[ \begin{array}{ccc}
(\pi^* P^\Sigma, \tilde{h}) & \xrightarrow{\tilde{\pi}} & (P^\Sigma, \tilde{g}) \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_1 \\
(S, h) & \xrightarrow{\pi} & (\Sigma, g)
\end{array} \]

Orlik, Raymond [24], and Fintushel [6] obtained a classification of effective, local circle actions on closed topological 3-manifolds. Their results, which we summarize in the theorem below, generalize the classification of closed topological 3-manifolds with effective circle actions in [28] and [23]:

**Theorem 2.1** (Orlik and Raymond [24], Fintushel [6]). A closed topological 3-manifold \( M \) with an effective local \( S^1 \)-action is determined up to equivariant equivalence by a set of fiber invariants

\[ \{b; \varepsilon, g, (f, k_1), (t, k_2); \{(\alpha_i, \beta_i)\}_{i=1}^n\}. \]

The topological classification of the manifolds in Theorem 2.1 is given in [24, Theorem in p. 143] and [6, Section 3]. Before, recalling the definition of the invariants above, let us state the equivariant classification of Alexandrov 3-spaces with local circle actions:

**Theorem 2.2** ([10, Theorem B]). Let \( X \) be a closed Alexandrov 3-space with a local isometric \( S^1 \)-action. If \( X \) has \( 2r \geq 0 \) topologically singular points, then the following hold:

1. Isometric local circle actions (up to equivariant equivalence) are in one-to-one correspondence with unordered tuples

\[ \{b; \varepsilon, g, (f, k_1), (t, k_2), (s, k_3); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \ldots, r_{s-k_3}); (q_1, q_2, \ldots, q_{k_3})\}. \]

where the admissible values for \( b, \varepsilon, g, (f, k_1), (t, k_2) \) and \( (\alpha_i, \beta_i) \) are given by Theorem 2.1, and \((r_1, r_2, \ldots, r_{s-k_3})\) and \((q_1, q_2, \ldots, q_{k_3})\) are unordered \((s - k_3)\) - and \(k_3\)-tuples of even non-negative integers \( r_i, q_j \), respectively, such that \( r_1 + \ldots + r_{s-k_3} + q_1 + \ldots q_{k_3} = 2r \).

2. There is an equivariant equivalence of \( X \) with

\[ M \# \text{Susp}(\mathbb{R} P^2) \# \cdots \# \text{Susp}(\mathbb{R} P^2), \]

\( r \) summands.
where $M$ is the closed 3-manifold determined by the set of invariants
\[
\{b; \varepsilon, g, (f + s, k_1 + k_3), (t, k_2); \{(\alpha_i, \beta_i)\}_{i=1}^{n}\}
\]
in Theorem 2.1.

We will now define the invariants appearing in Theorems 2.1 and 2.2. To do so, we first recall the classification up to weak bundle equivalence of circle bundles with structure group $O(2)$ over a compact 2-manifold with boundary (see [6, Section 1], also [22, 24, 25]). Recall that these bundles appear as pieces in the decomposition of an Alexandrov 3-space with a local circle action and consist of $R$-fibers.

**Definition 2.3.** Let $\pi_i : E_i \to B_i$, $i = 1, 2$, be circle bundles with structure group $O(2)$. A homeomorphism $\varphi : E_1 \to E_2$ is a weak equivalence if it covers a homeomorphism $f : B_1 \to B_2$, i.e. the diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\varphi} & E_2 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
B_1 & \xrightarrow{f} & B_2
\end{array}
\]

commutes. Whenever $E_1, E_2$ are orientable, we choose an orientation for $E_1$ and $E_2$ and assume that $\varphi$ preserves orientation. If either of the structure groups reduces to $SO(2)$ we demand that $\varphi$ be a bundle map with respect to $SO(2)$. If there exists a weak equivalence between two circle bundles with structure group $O(2)$, we say that the bundles are weakly equivalent.

It is well-known that the fundamental group of a compact, connected 2-manifold $B$ with genus $g$ and $m > 0$ boundary components has the presentation

\[
(a_j, b_j, s_i | s_1 \cdots s_m[a_1, b_1] \cdots [a_g, b_g])
\]
if $B$ is orientable, and

\[
(v_j, s_i | s_1 \cdots s_m v_1^2 \cdots v_g^2)
\]
if $B$ is nonorientable.

**Theorem 2.4 ([7, Theorem 1]).** Let $B$ be a compact, connected 2-manifold of genus $g$ with $m > 0$ boundary components. Then the set of weak equivalence classes of circle bundles over $B$ with structure group $O(2)$ is in one-to-one correspondence with pairs $(\varepsilon, k)$, where $k \geq 0$ is an even integer corresponding to the number of generators $s_i$ in the presentation of $\pi_1(B)$ that reverse orientation along fibers. The symbol $\varepsilon$ can take the letter values $o_1, o_2, n_1, n_2, n_3, n_4$ corresponding to the following classes:

- $o_1$, if $B$ is orientable and all $a_j, b_j$ preserve orientation.
- $o_2$, if $B$ is orientable, all $a_j, b_j$ reverse orientation and $g \geq 1$.
- $n_1$, if $B$ is non-orientable, all $v_j$ preserve orientation and $g \geq 1$.
- $n_2$, if $B$ is non-orientable, all $v_j$ reverse orientation and $g \geq 1$. 
Let $(\varepsilon, k)$ be the pair associated to the bundle of $R$-fibers with possible values as in Theorem 2.4 above. We denote the genus of $X^*$ by $g \geq 0$. We let $f, t, k_1, k_2$ be non-negative integers such that $k_1 \leq f$ and $k_2 \leq t$, where $k_1$ is the number of twisted $F$-blocks and $k_2$ is the number of twisted $SE$-blocks. The number $f - k_1$ is the number of simple $F$-blocks and $t - k_2$ is the number of simple $SE$-blocks. A non-negative integer $n$ will denote the number of $E$-fibers and we let $\{(\alpha_i, \beta_i)\}_{i=1}^n$ be the corresponding Seifert invariants (see [22, Section 1.7] for more details on how Seifert invariants are defined). We also let $b$ denote an integer or an integer mod 2 with the following conditions:

- $b = 0$ if $f + t > 0$ or if $\varepsilon \in \{o_2, n_1, n_3, n_4\}$ and some $\alpha_i = 2$;
- $b \in \{0, 1\}$ if $f + t = 0$ and $\varepsilon \in \{o_2, n_1, n_3, n_4\}$ and all $\alpha_i \neq 2$;
- in the remaining cases $b$ is an arbitrary integer.

We let $s, k_3$ be non-negative integers, where $k_3 \leq s$ is the number of twisted $SF$-blocks. Hence $s - k_3$ is the number of simple $SF$-blocks, and we let $(r_1, r_2, \ldots, r_{s-k_3})$ and $(q_1, q_2, \ldots, q_{k_3})$ be $(s - k_3)$- and $k_3$-tuples of non-negative even integers corresponding to the number of topologically singular points in each simple and twisted $SF$-block, respectively. The numbers $k_1, k_2,$ and $k_3$ satisfy $k_1 + k_2 + k_3 = k$. In sum, we associate to any fiber space $X^*$ the set of invariants

$$\{b; \varepsilon, g, (f, k_1), (t, k_2), (s, k_3); \{\alpha_i, \beta_i\}_{i=1}^n; (r_1, r_2, \ldots, r_{s-k_3}); (q_1, q_2, \ldots, q_{k_3})\},$$

determined by the local $S^1$ action.

We conclude this section by recalling the following result which relates isometric local circle actions with collapse. We will use it in Section 5.

**Theorem 2.5** ([10, Corollary 6.2]). Let $\{X_i\}$ be a sequence of closed Alexandrov 3-spaces with curv $\geq -1$ and diam $\leq D$ converging to an Alexandrov surface $X^*$ (possibly with boundary). Further assume that for $i$ large enough, $X_i$ does not have any singular fibers of type $B(\text{pt})$. Then, for $i$ large enough, $X_i$ is homeomorphic to an Alexandrov space with an effective isometric local circle action and the collapse $X_i \to X^*$ occurs along the fibers of the local action. In particular, $X_i$ is homeomorphic to one of the spaces in Theorem 2.2.
3. Double branched covers of collapsing Alexandrov spaces

Let $X$ be a closed, non-orientable Alexandrov 3-space, $\mu$ its 3-dimensional Hausdorff measure, and $S_X$ its set of topologically singular points. We let $\hat{X}$ be the orientable double branched cover of $X$ equipped with $\hat{\mu}$, its 3-dimensional Hausdorff measure. Let $\pi: \hat{X} \to X$ be the canonical projection and $\pi_*\hat{\mu}$, the push-forward measure of $\mu$ with respect to $\pi$.

**Proposition 3.1.** Let $X$ be a closed non-orientable Alexandrov 3-space. Then $\pi_*\hat{\mu} = 2\mu$.

**Proof.** Let $X' = X \setminus S_X$ and observe that, since $S_X$ is a discrete subset of $X$, its $\mu$-measure is zero. Therefore, it suffices to show that $\pi_*\hat{\mu}(B) = 2\mu(B)$ for sufficiently small balls centered at points in $X'$. Observe that we may assume, without loss of generality, that each one of these balls is evenly covered. We have that $\pi^{-1}(B) \subset \hat{X}'$ is the disjoint union of two balls $B_1, B_2$ in $\hat{X}'$ such that $\pi|_{B_i}: B_i \to B$ is an isometry. Therefore, $\hat{\mu}(B_i) = \mu(B)$. The proposition now follows from the definition of the push-forward measure. □

**Theorem 3.2.** Let $\{X_i\}$ be a sequence of closed non-manifold Alexandrov 3-spaces. Then the following hold:

(i) The sequence $\{X_i\}$ converges (in the Gromov-Hausdorff metric) to an Alexandrov space $X_\infty$ if and only if the sequence of orientable branched double covers $\{\hat{X}_i\}$, equipped with their canonical involutions $\sigma_i$, converges in the equivariant Gromov-Hausdorff topology to an Alexandrov space $\hat{X}_\infty$ equipped with a limit involution $\sigma_\infty$ so that $X_\infty = \hat{X}_\infty/\sigma_\infty$.

(ii) The sequence $\{X_i\}$ collapses if and only if the sequence of orientable branched double covers $\{\hat{X}_i\}$ collapses.

**Proof.** We first prove part (i). Let $\{(X_i, d_i)\}$ be a convergent sequence of compact non-manifold Alexandrov 3-spaces. For simplicity, we let $X_i = (X_i, d_i)$ and we let $X_\infty$ be the limit of the sequence. We have an induced sequence of metrics $\hat{d}_i$ on $\hat{X}_i$. Let $\alpha_i: \hat{X}_i \to \hat{X}_i$ be the isometric involution so that $\hat{X}_i/\alpha_i$ is isometric to $X_i$. Observe that the lift of any $\varepsilon$-net in $X_i$ is an $\varepsilon$-net in $\hat{X}_i$ that is invariant under the involution $\sigma_i$. This implies that the sequence $\{\hat{X}_i\}$ has a limit, which we denote by $\hat{X}_\infty$. We define $\alpha_\infty: \hat{X}_\infty \to \hat{X}_\infty$ as the limit of the $\alpha_i$. Now, by [7, Theorem 2-1], for every $i$, we have

$$d_{GH}(\hat{X}_i/\alpha_i, \hat{X}_\infty/\alpha_\infty) \leq C \left( d_{eGH}((\hat{X}_i, \mathbb{Z}_2), (\hat{X}_\infty, \mathbb{Z}_2)) \right)^{1/3},$$

where $d_{eGH}$ stands for the equivariant Gromov-Hausdorff distance. By [8, Proposition 3-6], $\hat{X}_i \overset{eGH}{\to} \hat{X}_\infty$ implies that there is a subsequence $\{\hat{X}_{i_k}\}$ converging in the equivariant Gromov-Hausdorff topology. Furthermore, by the proof of the same proposition, the group acting on the limit is the inverse...
limit of the groups acting on the members of the subsequence, which in this

case are all $\mathbb{Z}_2$. Therefore, $(\tilde{X}_i, \mathbb{Z}_2) \xrightarrow{\text{GH}} (\tilde{X}_\infty, \mathbb{Z}_2)$.

Now, inequality (3.1) implies that $\tilde{X}/\alpha_i \xrightarrow{\text{GH}} \tilde{X}_\infty/\alpha_\infty$. However, since

$\tilde{X}/\alpha_i$ is isometric to $X_i$, we have that $X_i \xrightarrow{\text{GH}} \tilde{X}_\infty/\alpha_\infty$. Since we already

had that $X_i \xrightarrow{\text{GH}} X_\infty$, we conclude that $X_\infty$ is isometric to $\tilde{X}_\infty/\alpha_\infty$. Observe

that the action of $\alpha_\infty$ on $\tilde{X}_\infty$ might be trivial. This shows that, if $\{X_i\}$ GH-

converges to $X_\infty$, then $\{\tilde{X}_i\}$ converges in the equivariant Gromov-Hausdorff
distance to $\tilde{X}_\infty$ with an involution $\sigma_\infty$ so that $\tilde{X}_\infty/\sigma_\infty$ is isometric to $X_\infty$.

On the other hand, equation (3.1) implies that if $\{\tilde{X}_i\}$, equipped with

their canonical involutions $\sigma_i$ converges in the equivariant Gromov-Hausdorff
distance to $\tilde{X}_\infty$ with a limit involution $\sigma_\infty$, then $\{X_i\}$ converges to $X_\infty$ in

the Gromov-Hausdorff distance. This shows part (i) of the theorem. Part

(ii) follows from part (i) and Proposition 3.1.

\begin{flushright}
\Box
\end{flushright}

\textbf{Corollary 3.3.} Let $M$ be a closed Riemannian manifold with positive min-

imal volume admitting an isometric involution $\iota : M \rightarrow M$ with a discrete

set of fixed points. Then the Alexandrov space $M/\iota$ has positive minimal

volume.

4. OUTLINE OF THE PROOF OF THEOREM A

We proceed by contradiction. Suppose the result does not hold. Then

we have a sequence of closed, irreducible Alexandrov 3-spaces $\{X_i\}_{i=1}^\infty$ with

curv $X_i \geq -1$ and diam $X_i \leq D$ such that vol $X_i \rightarrow 0$ and $X_i$ is not geo-

metric. Therefore, passing to a subsequence if necessary, we can assume

that $\{X_i\}$ collapses to a compact Alexandrov space $Y$. We will separate our

analysis in three cases according to the dimension of $Y$. In Sections 5, 6

and 7 we will address the cases in which $Y$ has dimension 2, 1 and 0 respec-

tively. In every case, we conclude that $X_i$ is geometric for a sufficiently big

$i$, obtaining a contradiction.

The following Lemma will be useful in our analysis. We refer the reader

to [32] for the definitions and basic properties of orbifold-covering spaces.

\textbf{Lemma 4.1.} Let $X$ be an Alexandrov 3-space which is not a manifold.

Then $X$ is geometric if and only if its orientable branched double cover $\tilde{X}$
is geometric.

\textbf{Proof.} We assume first that $X$ is geometric. Then, there exists a Thurston

group $M^3$ such that $X$ is isometric to $M^3/G$, where $G$ is a co-compact

group $M^3$ is an orbifold-covering space of $X$ and $X$ is a
good orbifold. Therefore, $X$ admits a Riemannian orbifold metric and we

will henceforth assume that it has such a metric. Moreover, $M^3$ is the

universal orbifold-covering space of $X$.

Now we note that the canonical projection $\pi : \tilde{X} \rightarrow X$ is an orbifold-

covering map. Then, by the universality of $M^3$, there exists an orbifold-

covering map $M^3 \rightarrow \tilde{X}$. Therefore, there exists a subgroup $\tilde{G}$ of the group

of deck transformations such that $\tilde{X}$ is isometric to $M^3/\tilde{G}$. Furthermore,
the metric on $M^3$ is the lift of the metric on $X$ and it coincides with the lift of the metric on $\tilde{X}$. Therefore, $\tilde{X}$ is geometric.

Assume now that $\tilde{X}$ is geometric, i.e. it is isometric to $M^3/\tilde{G}$, where $M^3$ is a Thurston geometry and $\tilde{G}$ a co-compact lattice. Then, as in the previous case, $M^3$ is the universal orbifold-covering space of $\tilde{X}$. Moreover, the canonical projection $\tilde{X} \to X$ is an orbifold-covering map. Therefore, $M^3$ covers $X$ with deck transformation group $G$ and it follows that we can put on $X$ the quotient metric on $M^3/\tilde{G}$. Thus we conclude that $X$ is geometric. □

**Proof of Remark B.** To conclude this section, we explain why a closed collapsing Alexandrov 3-space cannot admit hyperbolic geometry. Let $X$ be a closed, collapsing Alexandrov 3-space and suppose that $X$ admits hyperbolic geometry. We have two cases to consider: either $X$ is a 3-manifold or $X$ has topological singularities.

Suppose first that $X$ is a 3-manifold. Then, by [19, Corollary 1.7] (after passing via the orientable double cover of $X$ if necessary), the simplicial volume $\|X\|$ of $X$ is zero. On the other hand, since $X$ admits a hyperbolic Riemannian metric, by [31, Theorem 6.2] (after passing via the orientable double cover of $X$ if necessary), the simplicial volume of $X$ is non-zero, which is a contradiction.

Suppose now that $X$ is not a 3-manifold. Then its orientable double branched cover $\tilde{X}$ is an orientable topological 3-manifold. By Theorem 3.2, $\tilde{X}$ admits a sequence of collapsing Alexandrov metrics. Hence, by [19, Corollary 1.7], the simplicial volume of $\tilde{X}$ is zero. Now, since $X$ admits hyperbolic geometry, $\tilde{X}$ is also hyperbolic, so its simplicial volume cannot be zero, which is a contradiction. □

5. **Two-dimensional limit space**

In this section we deal with the case that $\{X_i\}_{i=1}^\infty$ collapses to $Y$ with $\dim Y = 2$. Our main result is a proof of Theorem A for this case.

**Theorem 5.1.** Let $\{X_i\}_i$ be a sequence of closed, irreducible Alexandrov 3-spaces with $\text{curv} \geq -1$, $\text{diam} X_i \leq D$, such that $X_i \to Y$ in the Gromov-Hausdorff distance. If $\dim Y = 2$, then, for sufficiently big $i$, $X_i$ is geometric, and its homeomorphism type is described in Table 1; namely, for sufficiently large $i$, the following hold:

- If $Y$ has empty boundary, then $X_i$ is homeomorphic to $\text{Seif}(Y)$.

- If $Y$ has nonempty boundary, then $X_i$ is homeomorphic to

$$\text{Seif}(Y) \cup \left( \bigcup_{j} GST_{N_i} \right) \cup \left( \bigcup_{j} GSKB_{N_j} \right).$$

In this subsection, we prove Theorem 5.1 by dividing the argument into cases. We begin by stating a technical lemma which we will need throughout
this section. Let us denote the connected sum of $m \geq 0$ copies of a space $X$ by $\#_m X$.

**Lemma 5.2.** Let $M$ be a closed 3-manifold and let $X$ be an Alexandrov 3-space homeomorphic to $M \# (\#_r \text{Susp}(\mathbb{RP}^2))$, for $r \geq 0$. If every embedded 2-sphere bounds a 3-ball, then one of the following holds:

(i) $M$ is prime and $X$ is homeomorphic to $M$,

(ii) $X$ is homeomorphic to $\text{Susp}(\mathbb{RP}^2)$.

**Proof.** If $X$ is a topological manifold then $r = 0$ and the result is the classical fact that an irreducible 3-manifold is prime. Therefore we assume that the set of topologically singular points of $X$ is non-empty. Let us denote $Y = \#_r \text{Susp}(\mathbb{RP}^2)$. Now, let $S \subset M\#Y$ be the 2-sphere which resulted from the connected sum operation. By hypothesis $S$ must bound a 3-ball. Since $Y$ contains topologically singular points, this implies that $M$ is homeomorphic to $S^3$. Hence, $X$ is homeomorphic to $Y$. Now, let us denote $Y_{r-1} = \#_{r-1} \text{Susp}(\mathbb{RP}^2)$ so that $X \cong \text{Susp}(\mathbb{RP}^2)\# Y_{r-1}$. Let $\bar{S} \subset \text{Susp}(\mathbb{RP}^2)\# Y_{r-1}$ be the 2-sphere arising from the connected sum operation. Then, $\bar{S}$ must bound a 3-ball. As in the previous case, the presence of topologically singular points implies that this is only possible if $r - 1 = 0$, i.e. if $X \cong \text{Susp}(\mathbb{RP}^2)$. \hfill \Box

We divide the following analysis in two cases depending whether $X_i$ has fibers of type $B(\text{pt})$ or not.

**Case 5.1** ($k = 0$, $f > 0$). The topological decomposition of Theorem 2.2 applied to this case yields that, depending on the values of the invariants, $X_i$ is homeomorphic to either

\[ S^3 \# \left( \# \mathbb{S}^2 \times \mathbb{S}^1 \right) \# \left( \# \mathbb{R}P^2 \times \mathbb{S}^1 \right) \# \left( \#_{i=1}^n L(\alpha_i, \beta_i) \right) \# \left( \#_r \text{Susp}(\mathbb{RP}^2) \right), \]

or

\[ S^2 \times \mathbb{S}^1 \# \left( \# \mathbb{S}^2 \times \mathbb{S}^1 \right) \# \left( \#_{i=1}^n L(\alpha_i, \beta_i) \right) \# \left( \#_r \text{Susp}(\mathbb{RP}^2) \right), \]
where the value of $\varphi$ depends on $f$, $g$, $\varepsilon$ and $t$. The exact value of $\varphi$ does not play a role in our arguments (see [10], Raymond, Fintushel for the precise definition). Therefore, since $X_i$ is irreducible, Lemma 5.2 implies that only one connected summand of the previous equivariant connected sum decompositions can appear. We observe that the possible connected summands are all geometric. More explicitly, $S^3$, lens spaces and $\text{Susp}(\mathbb{R}P^2)$ admit $S^3$-geometry while $S^2 \times S^1$, $S^2 \times S^1$, $\mathbb{R}P^2 \times S^1$ admit the $(S^2 \times \mathbb{R})$-geometry.

**Case 5.2** ($k = 0$, $f = 0$, $t = 0$). In this case, the condition $f = 0$ implies that the Alexandrov spaces $X_i$ considered are homeomorphic to topological manifolds since any action by isometries on a non-manifold Alexandrov space has fixed points (the SF-points). Moreover, the conditions $f = t = 0$ imply that $X_i$ is a Seifert manifold (see for example [24, Page 150, Lines 16–20]). This in turn implies that $X_i$ is geometric by [29, Theorem 5.3, (ii)]. The possible geometries appearing in this case are $S^3$, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\text{SL}_2(\mathbb{R})$, or Nil.

**Case 5.3** ($k = 0$, $f = 0$, $t > 0$). As in the previous case, the $X_i$ are homeomorphic to topological manifolds since $f = 0$. By the definition of the local circle action invariants, in this case we have that $b = 0$. More explicitly, $X_i$ is determined by the set of invariants

$$\{0; (\varepsilon, g, 0, t); \{(\alpha_i, \beta_i)\}_{i=1}^n\}.$$

Furthermore, since $t > 0$, the $X_i$ are non-orientable. Therefore, by the proof of [24, Theorem 3], the orientable double cover $\tilde{X}_i$ of $X_i$ is determined by the set of invariants

$$\{-n; (\tilde{\varepsilon}, \tilde{g}, 0, 0); (\alpha_1, \beta_1), \{(\alpha_i, \alpha_i - \beta_i)\}_{i=1}^n\}.$$

Here, $\tilde{\varepsilon} = o_1$ if $\varepsilon \in \{o_1, o_2, n_1\}$ and $\tilde{\varepsilon} = n_2$ if $\varepsilon \in \{n_2, n_3, n_4\}$. The value of $\tilde{g}$ is $2g + t - 1$ if $\varepsilon \in \{o_1, o_2, n_2\}$, it is $g + t - 1$ if $\varepsilon = n_1$ and it is $g + t - 2$ if $\varepsilon \in \{n_3, n_4\}$.

We observe then that $\tilde{X}_i$ falls under the considerations of the previous case and therefore is a Seifert manifold. Therefore $X_i$ is doubly covered by a geometric manifold and hence it itself is geometric, the possible geometries being the same as in the last case.

**Case 5.4** ($k \neq 0$, $f > 0$). In this case the classes $o_1$, $o_2$ merge into a single class $o$ while $n_1$, $n_2$, $n_3$ and $n_4$ merge into one class $n$. The decomposition of Theorem 2.2 in this case is the following. If $t > 0$, then $X_i$ is homeomorphic to

$$\left(\# (S^2 \times S^1)\right) \# \left(\# \mathbb{R}P^2 \times S^1\right) \# \left(\sum_{j=1}^n L(\alpha_j, \beta_j)\right) \# \left(\# \text{Susp}(\mathbb{R}P^2)\right).$$
If \( t = 0 \), then \( X_i \) is homeomorphic to
\[
(S^2 \times S^1) \# \left( \#_{\varphi} S^2 \times S^1 \right) \# \left( \#_{t} \mathbb{R}P^2 \times S^1 \right) \# \left( \#_{j=1}^{n} L(\alpha_j, \beta_j) \right) \# \left( \#_r \text{Susp}(\mathbb{R}P^2) \right).
\]
Here, the value of \( \varphi \) is determined by \( f \), \( g \) and \( \varepsilon \), but the precise value is not necessary for our analysis.

As in the case \( k = 0, f > 0 \), since \( X_i \) is irreducible, Lemma 5.2 implies that only one connected summand can appear, and in turn this yields that \( X_i \) is geometric.

**Case 5.5** \((k \neq 0, f = 0)\). As in the previous cases in which \( f = 0 \), the \( X_i \) are homeomorphic to topological manifolds. If \( t = 0 \), then \( X_i \) is a Seifert manifold, whereas if \( t > 0 \) its orientable double cover is a Seifert manifold.

In both cases the space is geometric.

**\( X_i \) contains fibers of type \( B(pt) \).** In this case, for each \( X_i \) we consider its branched orientable double cover \( \tilde{X}_i \). We observe that \( \tilde{X}_i \) does not contain fibers of type \( B(pt) \) since its subset of topologically singular points is empty.

Therefore, by [10, Corollary 6.2] \( \tilde{X}_i \) is homeomorphic to a closed Alexandrov 3-manifold admitting an isometric local circle action determined by a set of invariants
\[
\{ \hat{b}; \hat{\varepsilon}, \hat{g}, (\hat{f}, \hat{k}_1), (\hat{\ell}, \hat{k}_2); \{(\hat{\alpha}_i, \hat{\beta}_i)\}_{i=1}^{n} \}.
\]
However, since \( \tilde{X}_i \) is orientable, we have that \( \hat{f} = 0 \). The cases having \( \tilde{f} = 0 \) are settled as in the case in which \( X_i \) does not contain fibers of type \( B(pt) \).

Hence, in the following we assume that \( \tilde{f} > 0 \). Under this assumption, Theorem 2.2 yields that \( \tilde{X}_i \) is homeomorphic to
\[
(5.1) \quad S^3 \# \left( \#_{\varphi} S^2 \times S^1 \right) \# \left( \#_{j=1}^{n} L(\alpha_j, \beta_j) \right).
\]

We point out that \( S^2 \times S^1 \) does not appear in the decomposition as a consequence of the orientability of \( \tilde{X}_i \). Let us denote by \( \iota_i : X_i \to \tilde{X}_i \) the involution such that \( \iota_i \) is homeomorphic to \( X_i \). Then, \( \iota_i \) induces an involution on the connected sum \( 5.1 \) and we can equip the space with a Riemannian metric which is invariant with respect to \( \iota_i \).

We now observe that, by [4], there is an equivariant (with respect to \( \iota_i \)) Ricci flow with surgery on \( \tilde{X}_i \). Furthermore, by [27, Theorem 1.1] the Ricci flow goes extinct in finite time. Now, [3, Corollary 4.5] implies that the action of \( \iota_i \) is an equivariant connected sum of standard actions on components diffeomorphic to spherical space forms, \( S^2 \times S^1 \) or \( \mathbb{R}P^3 \# \mathbb{R}P^3 \). In particular, the connected summands \( L(\alpha_1, \beta_1) \# \cdots \# L(\alpha_n, \beta_n) \) and \( \#_{\varphi} S^2 \times S^1 \) in (5.1) are invariant under the action of \( \iota_i \).

We now observe the following. If the connected sum decomposition (5.1) of \( \tilde{X}_i \) contains both lens space \( L(\alpha_j, \beta_j) \) and \( (S^2 \times S^1) \)-summands, then the 2-sphere \( S \) dividing \( \tilde{X}_i \) into the summands \( \#_{j=1}^{n} L(\alpha_j, \beta_j) \) and \( \#_{\varphi} S^2 \times S^1 \) can be taken to be invariant under the the action of \( \iota_i \). By the classification...
of the involutions of the 2-sphere, \( S/\iota_i \subset X_i \) is either homeomorphic to \( \mathbb{R}P^2 \) or to a 2-sphere. If \( S/\iota_i \cong \mathbb{R}P^2 \), then we observe that \( S/\iota_i \) does not bound a cone over \( \mathbb{R}P^2 \). Similarly, if \( S/\iota_i \) is a 2-sphere then it does not bound a 3-ball, contradicting the irreducibility of \( X_i \). Therefore, we can assume that \( \tilde{X}_i \) is either a connected sum of lens spaces or a connected sum of copies of \( S^2 \times \mathbb{S}^1 \).

**Case 5.6** (\( \tilde{X}_i \) is homeomorphic to \( \# L(\alpha_1, \beta_1) \# \ldots \# L(\alpha_n, \beta_n) \)). By [16, Corollary 3], \( \iota_i = h_{i_1} \# \ldots \# h_{i_m} \) and \( \tilde{X}_i = M_{i_1} \# \ldots \# M_{i_m} \), where each \( h_{i_j} \) is an involution on \( M_{i_j} = A_{i_j} \# Q_{i_j} \# A_{i_j}^c \) arising from operation I-1 (see [16, Page 260]). Here, \( A_{i_j} \) is a connected sum of lens spaces, \( A_{i_j}^c \) is either \( A_{i_j} \) with the orientation reversed, and \( Q_{i_j} \) is either \( \mathbb{R}P^3 \) or \( S^3 \). Moreover, since the involution \( \iota_i : \tilde{X}_i \to \tilde{X}_i \) has fixed points, at least one of the \( h_{i_j} \) must have fixed points. This is the involution we will consider in what follows and fix the index \( i_j \).

If \( Q_{i_j} \) is \( S^3 \), then the involution \( h_{i_j} \) interchanges the summands \( A_{i_j} \) and \( A_{i_j}^c \). In particular \( Q_{i_j} = S^3 \) is invariant under \( h_{i_j} \) and, since the involution has isolated fixed points, its restriction to \( Q_{i_j} = S^3 \) corresponds to the suspension of the antipodal map on \( S^2 \). The quotient of \( A_{i_j} \# Q_{i_j} \# A_{i_j}^c \) by \( h_{i_j} \) is homeomorphic to \( A_{i_j} \# \text{Susp}(\mathbb{R}P^2) \), where the 2-sphere \( S \) dividing the connected sum is the projection of the two 2-spheres \( S_1, S_2 \) that divide the connected sum \( A_{i_j} \# Q_{i_j} \# A_{i_j}^c \) into three summands. Therefore, the sphere \( S \) does not bound a 3-ball, contradicting the irreducibility of \( X_i \).

If \( Q_{i_j} \) is \( \mathbb{R}P^3 \), then we recall that by [17] there is exactly one orientation reversing involution on \( \mathbb{R}P^3 \). However, this involution has a 2-dimensional fixed point set. Since \( \iota_i \) has only isolated fixed points this is a contradiction.

**Case 5.7** (\( \tilde{X}_i \) is homeomorphic to \( \# \varphi S^2 \times S^1 \)). Let us recall that by [16] every involution on a connected sum of closed 3-manifolds is constructed by successive application of the so-called four \( I \)-operations (see [16, Page 260]). In particular \( \iota_i \) can be described in such fashion. We will split our analysis in four cases depending on the type of the last \( I \)-operation used to construct \( \iota_i \). In each case we will conclude that if \( X_i \) is homeomorphic to \( \# \varphi S^2 \times S^1 \), then \( \varphi = 1 \). In the remainder of this case, we follow the notation of [16, Page 260].

**Case 5.7.1** (I-1 operation). Here, \( M_1 \) and \( M_2 \) are connected sums of copies of \( S^2 \times S^1 \). Let \( C_1 \subset M_1 \) be the cell used to perform the I-1 operation. We let \( \pi : \tilde{X}_i \to X_i \) be the canonical projection. Then \( \pi(\partial C_0) \) is a 2-sphere in \( X_i \). Then, \( \pi(\partial C_0) \) bounds a 3-ball in \( X_i \) if and only if either \( M_1/\iota_i \) or \( M_2/\iota_i \) are homeomorphic to a closed 3-ball. Therefore, the question reduces to see if the quotient of a connected sum of copies of \( S^2 \times S^1 \) can be homeomorphic to a 3-ball. To address this question, we observe that if the set of fixed points of \( \iota_i \) is empty, then \( \partial(M_1/\iota_i) = \emptyset \). Then, \( M_1/\iota_i \) cannot be homeomorphic to a 3-ball. Therefore we assume that \( \iota_i \) has fixed points on \( M_1 \). Since the fixed
points of $\iota_i$ correspond to topologically singular points of $X_i$ under $\pi$, the quotient $M_1/\iota_i$ cannot be homeomorphic to a 3-ball. Hence, we conclude that $\pi(\partial C_1)$ does not bound a 3-ball in $X_i$. This is a contradiction since $X_i$ is irreducible.

Case 5.7.2 (I-2 operation). Let $S$ be the sphere used for the connected sum. Then $\pi(S)$ is homeomorphic to an $\mathbb{RP}^2$ in $X_i$ which does not bound a cone over $\mathbb{RP}^2$. Otherwise, if we assume without loss of generality that $M_1/h_1$ is homeomorphic to $K(\mathbb{RP}^2)$ then $M_1$ would be a 3-ball, implying that the connected sum $M_1\#M_2$ is trivial. This is a contradiction.

Case 5.7.3 (I-3 operation). In this case, the quotient of the involution has the form $(M_1/h_1)\#(M_2/h_2)$. If both $h_1$ and $h_2$ have fixed points, then it is clear that $M_i/h_i$ is not homeomorphic to a 3-ball. Therefore, we now assume, without loss of generality, that the fixed point set of $h_1$ is empty. Then, if $M_1/h_1$ is homeomorphic to a 3-ball, it follows that $M_1$ is homeomorphic to a disjoint union of two 3-balls. In particular $M_1$ is disconnected, which is a contradiction.

Case 5.7.4 (I-4 operation). The quotient of the involution in this case is constructed as follows. First, we take out small neighborhoods of two topologically singular points in $X_i$. Then, we glue $\mathbb{RP}^2\times I$ to the resulting space along the $\mathbb{RP}^2$ boundaries. Therefore, the space obtained is not irreducible since it contains a 2-sided $\mathbb{RP}^2$ which does not bound a $K(\mathbb{RP}^2)$. This contradicts the irreducibility of $X_i$.

6. One-dimensional limit space

In this section we analyze the case in which the sequence $\{X_i\}_{i=1}^\infty$ collapses to a one-dimensional space $Y$. Namely, we prove the following case of Theorem A:

Theorem 6.1. Let $\{X_i\}_i$ be a sequence of closed, irreducible Alexandrov 3-spaces with $\text{curv} \geq -1$ and $\text{diam} X_i \leq D$, such that $X_i \to Y$ in the Gromov-Hausdorff distance. If $\text{dim} Y = 1$, then, for sufficiently big $i$, $X_i$ is geometric.

Strategy of the proof: We distinguish two cases, depending on whether $\partial Y = \emptyset$ or $\partial Y \neq \emptyset$. The homeomorphism type of $X_i$ is described in Table 2. There are two cases:

(a) If $Y$ has empty boundary, then $X_i$ is homeomorphic to an $F$-fiber bundle over $\mathbb{S}^1$, where $F = K^2, T^2, S^2$ or $\mathbb{RP}^2$.

(b) If $Y$ has nonempty boundary, then $X_i$ is homeomorphic to a space obtained by gluing two pieces $B$ and $B'$ by a homeomorphism of their boundaries $\phi: \partial B \to \partial B'$, and $B, B'$ are the spaces appearing in Table 3. More specifically, $X_i$ is obtained in one of the four following ways:
We will go over every possibility for $X_i$ and show that in each one of them we can put a geometric structure.

We also point out that, in this section, the assumption of irreducibility is not needed to assert that $X_i$ is geometric. Under such an assumption, at least Cases 6.3.4, 6.3.5, and 6.3.6 are ruled out. However, in those cases the same techniques we use throughout the section yield that $X_i$ is geometric and we include them as well. We also observe that in most cases we do not indicate all the possible geometries that the space in question admits.

**Y does not have boundary.** Under this assumption, $X_i$ is homeomorphic to an $F$-fiber bundle $F \hookrightarrow E \xrightarrow{\xi} S^1$. The possible fibers $F$ are $K^2$, $T^2$, $S^2$ or $\mathbb{R}P^2$. The universal cover of $E$ is calculated in the following way. We let $\xi : \mathbb{R} \to S^1$ be the universal cover projection. Now, we take the pullback $\tilde{\rho} : \xi^*E \to \mathbb{R}$ of $\rho : E \to S^1$ with respect to $\xi$ and observe that $\xi^*E$ is a trivial bundle since the base $\mathbb{R}$ is contractible. Furthermore, the fiber of $\xi^*E$ is homeomorphic to $F$ and therefore $\xi^*E \cong F \times \mathbb{R}$. Moreover, the induced map $\tilde{\xi} : \xi^*E \to E$ is a covering map, and we conclude that $E$ is covered by $F \times \mathbb{R}$.

If $F = K^2$ or $T^2$ then the universal cover of $\xi^*E$ (and consequently of $E$) is $\mathbb{R}^2 \times \mathbb{R} \cong \mathbb{R}^3$. We further observe that using the usual double covers $T^2 \to K^2$ and $S^2 \to \mathbb{R}P^2$ it follows that $T^2 \times \mathbb{R}$ covers any torus and Klein-bottle bundles over $S^1$ and that $S^2 \times \mathbb{R}$ covers any $S^2$ and $\mathbb{R}P^2$ bundles over $S^1$. On the other hand, if $F = S^2$ or $\mathbb{R}P^2$, then the universal cover is $S^2 \times \mathbb{R}$. Hence, $E$ is geometric since its universal cover is geometric for any $F$. We now point out the precise geometries that $E$ admits depending on $F$.

**Case 6.1 ($F = S^2$).** There are two such bundles: $S^2 \times S^1$ and $S^2 \times S^1$. Both of them are covered by $S^2 \times S^1$. Therefore both admit $S^2 \times S^1$-geometry.

**Case 6.2 ($F = T^2$).** We describe $E$ as a mapping torus over $F$ in the following way. Let $S^1 \setminus \{pt\}$ be identified with $I = [a, b]$. Then, the bundle $\rho|_{\rho^{-1}(I)} : \rho^{-1}(I) \to I$ is trivial since $I$ is contractible. Therefore, $\rho^{-1}(I) \cong F \times I$. Hence, the space $E$ is obtained as $F \times I/h$ where $h : \rho^{-1}(a) \to \rho^{-1}(b)$ is a homeomorphism.

Recall that homotopic gluing maps $h$ give rise to isomorphic mapping tori. Then, [5, Theorem 13.2, Theorem 13.5] yield that $E$ may admit the $\mathbb{R}^3$-, Nil- or Sol-geometries depending on the gluing map $h$. 

(1) gluing two spaces (that could be the same) among $D^3$, $\mathbb{R}P^3 \setminus \text{int } D^3$, $B(S_2)$;
(2) gluing two copies of $K_1(\mathbb{R}P^2)$;
(3) gluing two spaces (that could be the same) among $D^2 \times S^1$, $\text{Mo} \times S^1$, $K^2 \times I$ and $B(S_4)$;
(4) gluing two spaces (that could be the same) among $S^1 \times D^2$, $K^2 \times I$, $B(\text{pt})$ and $B(\mathbb{R}P^2)$. 

(1) gluing two spaces (that could be the same) among $D^3$, $\mathbb{R}P^3 \setminus \text{int } D^3$, $B(S_2)$;
(2) gluing two copies of $K_1(\mathbb{R}P^2)$;
(3) gluing two spaces (that could be the same) among $D^2 \times S^1$, $\text{Mo} \times S^1$, $K^2 \times I$ and $B(S_4)$;
(4) gluing two spaces (that could be the same) among $S^1 \times D^2$, $K^2 \times I$, $B(\text{pt})$ and $B(\mathbb{R}P^2)$. 

(2) gluing two copies of $K_1(\mathbb{R}P^2)$;
has boundary. In this case $X$ is homeomorphic to a space obtained
by gluing two pieces $B$ and $B'$ having as possible boundaries $S^2$, $\mathbb{R}P^2$, $T^2$
or $K^2$. We analyze each possible gluing $B \cup B'$. Since sometimes we will
need to compute the double branched cover of such gluing, we will use the
following obvious lemma.

**Lemma 6.2.** Let $X$ be a non-manifold Alexandrov 3-space obtained by glu-
ing two pieces $B$ and $B'$ having as possible boundaries $S^2$, $\mathbb{R}P^2$, $T^2$ or $K^2$
by a homeomorphism $\varphi$. Let $\pi : \tilde{X} \to X$ be the orientable double branched
cover of $X$. Then the following hold:

(i) If only one of the pieces, say, $B$, has topological singularities, then
$\pi^{-1}(B)$ has an isometric involution with only isolated fixed points,
$\partial(\pi^{-1}(B))$ is a 2-fold cover of $\partial B$, and $\pi^{-1}(B')$ is a twofold cover
of $B'$.

(ii) If both pieces $B$ and $B'$ have topological singularities, then $\pi^{-1}(B)$
and $\pi^{-1}(B')$ have isometric involutions with only isolated fixed points
and $\partial(\pi^{-1}(B))$ is a 2-fold cover of $\partial B$.

(iii) If a piece $B$ is an orientable manifold, then $\pi^{-1}(B)$ is given by two
copies of $B$, and the involution sends one copy to the other; each
copy of $B$ is glued to $\pi^{-1}(B')$ by a copy of $\varphi$.

(iv) If a piece $B$ is a non-orientable manifold, then $\pi^{-1}(B)$ is given by
the orientable double cover of $B$, and it is glued to $\pi^{-1}(B')$ by a
twofold cover of $\varphi$.

**Case 6.3** ($\partial B = S^2$). The possible pieces satisfying this assumption are
$D^3$, $\mathbb{R}P^3 \setminus \text{int}D^3$ and $B(S_2)$. We now examine each possible combination of
these pieces.

**Case 6.3.1** ($D^3 \cup D^3$). In this case, $X$ is homeomorphic to $S^3$, which admits
spherical geometry.

**Case 6.3.2** ($D^3 \cup \left( \mathbb{R}P^3 \setminus \text{int}(D^3) \right)$). We have that $\mathbb{R}P^3 \setminus \text{int}(D^3) \cup \partial D^3 \cong
\mathbb{R}P^3$, which admits spherical geometry.

**Case 6.3.3** ($D^3 \cup B(S_2)$). By [18, Remark 2.62], $B(S_2)$ is homeomorphic
to $K_1(\mathbb{R}P^2) \cup_{\text{Mo}} K_1(\mathbb{R}P^2)$. Using a similar argument to [18, Lemma 2.61],
it can be proved that the topology of $B(S_2)$ does not depend on the gluing
map $\text{Mo} \to \text{Mo}$ used. Therefore, using the identity map $\text{id}: \text{Mo} \to \text{Mo}$
we obtain that $B(S_2) \cong \text{Susp}(\mathbb{R}P^2) \setminus D^3$. Hence, $D^3 \cup B(S_2) \cong \text{Susp}(\mathbb{R}P^2)$
which admits spherical geometry. Alternatively, we can use Lemma 6.2. The
description of $B(S_2)$ implies that $\pi^{-1}(B(S_2))$ is homeomorphic to $S^2 \times [-1, 1]$
and the involution sends the point $(x, t)$ to $(\sigma(x), -t)$, where $\sigma : S^2 \to S^2$
is the suspension of the antipodal map on $S^1$. On the other hand, $\pi^{-1}(B^3)$
consists of two copies of $B^3$. Hence $\tilde{X}$ is $S^3$ where the involution sends
$(x, y, z, t)$ to $(x, -y, -z, -t)$. This is the suspension of the antipodal map in
the hyperplane $(y, z, t)$. Hence the quotient is $\text{Susp}(\mathbb{R}P^2)$. 
Case 6.3.4 \( ((\mathbb{R}P^3 \setminus \text{int}(D^3)) \cup B(S_2)) \). By [18, Remark 2.62], this space is homeomorphic to \( \mathbb{R}P^3 \# \text{Susp}(\mathbb{R}P^2) \). Since this space, as the ones in the next two cases, is not irreducible, it does not fall into the family of spaces we are considering. For completeness, however, we will show that it is geometric. We now obtain the orientable branched double cover. We take out small cones centered at each of the two topologically singular points of \( \mathbb{R}P^3 \# \text{Susp}(\mathbb{R}P^2) \). The resulting space is homeomorphic to \( \mathbb{R}P^3 \# (\mathbb{R}P^2 \times [-1, 1]) \). Now, since \( \mathbb{R}P^3 \) is orientable, the orientable double cover of \( \mathbb{R}P^3 \setminus D^3 \) is \( (\mathbb{R}P^3 \setminus D^3) \times \{-1, 1\} \). On the other hand, the orientable double cover of \( (\mathbb{R}P^2 \times [-1, 1]) \setminus D^3 \) is \( (\mathbb{S}^2 \times [-1, 1]) \setminus D^3_1 \cup D^3_2 \), where \( D^3_1 \) and \( D^3_2 \) are two disjoint 3-balls. Therefore, the orientable double cover of \( \mathbb{R}P^3 \# (\mathbb{R}P^2 \times [-1, 1]) \) is
\[
((\mathbb{R}P^3 \setminus D^3) \times \{-1\}) \cup_{\partial D^3_1} ((\mathbb{S}^2 \times [-1, 1]) \setminus D^3_1 \cup D^3_2) \cup_{\partial D^3_2} ((\mathbb{R}P^3 \setminus D^3) \times \{1\}) ,
\]
which by definition is \( \mathbb{R}P^3 \# (\mathbb{S}^2 \times [-1, 1]) \# \mathbb{R}P^3 \). Gluing two 3-balls along the two boundary components of this space, we obtain the branched orientable double cover \( \mathbb{R}P^3 \# (\mathbb{S}^2 \times \mathbb{R}) \# \mathbb{R}P^3 \) \( \cong \mathbb{R}P^3 \# \mathbb{R}P^3 \). The involution can be made isometric with respect to a non-negatively curved metric on \( \mathbb{R}P^3 \# \mathbb{R}P^3 \) with \( (\mathbb{S}^2 \times \mathbb{R}) \)-geometry (see [3, Section 5.2]). Therefore, \( \mathbb{R}P^3 \# \text{Susp}(\mathbb{R}P^2) \) is a closed Alexandrov space admitting a metric with non-negative curvature. We point out that this space should be added to the list of closed, non-negatively curved Alexandrov 3-spaces in Theorem 1.3 in the published version of [9].

Case 6.3.5 \( ((\mathbb{R}P^3 \setminus \text{int}(D^3)) \cup (\mathbb{R}P^3 \setminus \text{int}(D^3)))\). This case was considered in Case (1) of the proof of [30, Theorem 0.6]. By definition of the connected sum, \( \mathbb{R}P^3 \setminus \text{int}(D^3) \cup_{\partial} \mathbb{R}P^3 \setminus \text{int}(D^3) = \mathbb{R}P^3 \# \mathbb{R}P^3 \). Therefore, this space admits \( (\mathbb{S}^2 \times \mathbb{R}) \)-geometry. Observe that this space is not irreducible.

Case 6.3.6 \( (B(S_2) \cup B(S_2)) \). It follows from [18, Remark 2.62], that this space is homeomorphic to \( \text{Susp}(\mathbb{R}P^2) \# \text{Susp}(\mathbb{R}P^2) \). Therefore, this space admits the \( (\mathbb{S}^2 \times \mathbb{R}) \)-geometry (see the proof of [9, Theorem 1.3], Case 2), part (a) of the case on which \( X \) is not a topological manifold). As in the preceding two cases, this space is not irreducible.

Case 6.4 \( (\partial B = \mathbb{R}P^2) \). The only piece satisfying this requirement is \( K_1(\mathbb{R}P^2) \). Therefore, by [18, Lemma 2.61], the only space obtained here is \( \text{Susp}(\mathbb{R}P^2) \) which has spherical geometry.

Case 6.5 \( (\partial B = T^2) \). The collection of pieces that meet this condition is \( D^2 \times S^1, \text{Mo} \times S^1, K^4 \times I, B(S_2) \).

Case 6.5.1 \( (D^2 \times S^1) \cup (D^2 \times S^1)) \). This case was considered in Case (2-i) of the proof of [30, Theorem 0.6]. Gluing two copies of a solid torus \( D^2 \times S^1 \) by a homeomorphism of their boundaries we obtain, by definition, all possible lens spaces. Therefore this space admits either \( S^3 \)- or \( (\mathbb{S}^2 \times \mathbb{R}) \)-geometry (in the case of \( \mathbb{S}^2 \times S^1 \)).
Case 6.5.2 \(((D^2 \times S^1) \cup_\varphi (Mo \times S^1))\). Let us consider the double cover
\[
\rho : S^1 \times I \times S^1 \to Mo \times S^1
\]
given by the involution \(\iota : S^1 \times I \times S^1 \to S^1 \times I \times S^1\) defined by \((x, t, y) \mapsto (-x, -t, y)\). Note that \(\rho\) is the orientable double cover of \(Mo \times S^1\) since it
is a double cover and \(\iota\) is orientation-reversing. We now observe that since
\(D^2 \times S^1\) is orientable, its orientable double cover consists of two disjoint
copies of \(D^2 \times S^1\).

By Lemma 6.2, the orientable double cover of \((D^2 \times S^1) \cup_\varphi (Mo \times S^1)\) is
obtained by gluing each copy of \(D^2 \times S^1\) to a connected component of the
boundary of \(S^1 \times I \times S^1\) via \(\varphi\), thus resulting on \(S^2 \times S^1\); therefore this space
admits \((S^2 \times \mathbb{R})\)-geometry.

Case 6.5.3 \(((D^2 \times S^1) \cup (K^2 \times I))\). This space was considered in Case (2-ii) of the proof of [30, Theorem 0.6]. The space \(K^2 \times I\) is homeomorphic
to \(Mo \times S^1\). This implies that \((D^2 \times S^1) \cup (K^2 \times S^1)\) is homeomorphic to
\(D^2 \times S^1 \cup Mo \times S^1\) which, depending on the gluing homeomorphism, is either
a prism manifold, \(S^1 \times S^2\), or \(\mathbb{R}P^2 \# \mathbb{R}P^3\). Hence, the possible geometries for
this case are the \(S^1\)- and \((S^2 \times \mathbb{R})\)-geometries.

Case 6.5.4 \(((D^2 \times S^1) \cup B(S_4))\). By the description of \(B(S_4)\) in [18, Corollary
2.56], as a quotient of \(T^2 \times [-1, 1]\) with respect to an orientation-reversing
involution with only fixed points, the branched orientable double cover of
\(B(S_4)\) is \(T^2 \times [-1, 1]\). On the other hand, the orientable double cover of
\(D^2 \times S^1\) is a disjoint union \(D^2 \times S^1 \sqcup D^2 \times S^1\). By Lemma 6.2, in the
branched cover the gluing homeomorphisms \(\partial(D^2 \times S^1) \to T^2 \times \{1\}\) and
\(\partial(D^2 \times S^1) \to T^2 \times \{-1\}\) coincide with each other. Therefore,
\[
(D^2 \times S^1) \cup (T^2 \times [-1, 1]) \cup (D^2 \times S^1) \cong (D^2 \times S^1) \cup_{id} (D^2 \times S^1) \cong S^2 \times S^1.
\]
Thus \((D^2 \times S^1) \cup B(S_4)\) admits \((S^2 \times \mathbb{R})\)-geometry.

Case 6.5.5 \(((Mo \times S^1) \cup (Mo \times S^1))\). We consider \(\rho : T^2 \times I \to Mo \times S^1\), the
double cover of Case 6.5.2. Once again, by Lemma 6.2, the orientable cover
of our space is then obtained by gluing pairwise the boundary components
of two copies of \(T^2 \times I\); this results in a fiber bundle over \(S^1\) with fiber an
torus, and will admit an \(\mathbb{R}^3\)-, Nil- or Sol- geometry depending on the gluing
map \(\varphi\).

Case 6.5.6 \(((Mo \times S^1) \cup (K^2 \times I))\). Let us note that the canonical projection
\(\xi : S^1 \times I \times S^1 \to Mo \times S^1\) is the orientable double cover of this piece. On the
other hand, \(K^2 \times I\) being orientable, its orientable double cover is given by
two copies of itself. Thus the orientable double cover of \((Mo \times S^1) \cup (K^2 \times I)\)
is obtained by gluing together both copies of \(K^2 \times I\) along their boundary
where the gluing map respects the \(I\)-bundle structure of each \(K^2 \times I\).

Such space can be obtained as the quotient of \(T^2 \times S^1\) by an isometric
involution of the flat metric: choose some involution \(\sigma : T^2 \to T^2\) with
Therefore, \((\text{Mo} \times S^1) \cup (K^2 \times I)\) admits \(\mathbb{R}^3\)-geometry.

**Case 6.5.7** \(((\text{Mo} \times S^1) \cup B(S_4))\). As in case 6.5.4, \(B(S_4)\) is isometric to a quotient of \(T^2 \times [-1,1]\) by an orientation reversing involution with only fixed points (cf. [18, Corollary 2.56]). Therefore, in similar fashion to the previous two Cases, the space \((\text{Mo} \times S^1) \cup B(S_4)\) is doubly covered by a \(T^2\)-bundle over \(S^1\) and therefore accepts \(\mathbb{R}^3\), Nil or Sol-geometry depending on the gluing.

**Case 6.5.8** \(((K^2 \times I) \cup (K^2 \times I))\). This case was considered in Case (3) of the proof of [30, Theorem 0.6, page 31], where it was observed that the space in question is doubly covered by a \(T^2\)-bundle over \(S^1\). Therefore, the space is geometric, with possible geometries \(\mathbb{R}^3\), Nil or Sol. The precise analysis is analogous to that of Case 6.5.5.

**Case 6.5.9** \(((K^2 \times I) \cup B(S_4))\). The analysis of this case is analogous to that of Case 6.5.6.

**Case 6.5.10** \((B(S_4) \cup B(S_4))\). Since each \(B(S_4)\) lifts to \(T^2 \times I\), the orientable double cover of this space corresponds to \(T^3\), and thus admits either flat, Sol or Nil metric.

**Case 6.6** \((\partial B = K^2)\). In the remaining cases, the pieces we deal with are \(S^1 \times D^2\), \(K^2 \times I\), \(B(pt)\) and \(B(\mathbb{R}P^2)\).

**Case 6.6.1** \(((S^1 \times D^2) \cup (S^1 \times D^2))\). The space \(S^1 \times D^2\) is a solid Klein bottle, and as such, its orientable double cover is \(S^1 \times D^2\) with the involution \(\rho(x,y) = (-x, \overline{y})\). Let \(\varphi : S^1 \times S^1 \to S^1 \times S^1\) be the homeomorphism used to produce the gluing.

Using standard theory for covering spaces, there is a lift of \(\varphi\) that we denote \(\bar{\varphi} : S^1 \times S^1 \to S^1 \times S^1\); thus the orientable cover of our space is obtained by gluing two copies of \(S^1 \times D^2\) by a homeomorphism of the torus, resulting on spaces admitting \(S^3\) or \(S^2 \times \mathbb{R}\)-geometries.

**Case 6.6.2** \(((S^1 \times D^2) \cup (K^2 \times I))\). As in the previous case, \(S^1 \times D^2\) lifts to \(S^1 \times D^2\) in the orientable cover. On the other hand, by the arguments outlined in the final part of Section 2, the orientable twofold cover of \(K^2 \times I\) is \(K^2 \times I\). Thus by Lemma 6.2, the double cover of the space under examination is obtained gluing \(S^1 \times D^2\) to \(K^2 \times I\) along their boundary. This case was examined in case 6.5.3, where it was shown that the allowed geometries were \(S^3\) and \(S^2 \times \mathbb{R}\).

**Case 6.6.3** \(((S^1 \times D^2) \cup B(pt))\). Both of the pieces have \(D^2 \times S^1\) as orientable double cover. Then, by lifting the gluing homeomorphism of \(K^2\) to a homeomorphism of \(T^2\), we get that the space \(S^1 \times D^2 \cup B(pt)\) is doubly covered by \(D^2 \times S^1 \cup \partial D^2 \times S^1\), that is, a lens space or \(S^2 \times S^1\); the geometries can be \(S^3\) or \(S^2 \times \mathbb{R}\).
**Case 6.6.4** $((S^1 \times D^2) \cup B(\mathbb{R}P^2))$. As in the previous cases, $S^1 \times D^2$ has $\mathbb{S}^1 \times D^2$ as orientable double cover. On the other hand, the orientable double branched cover of $B(\mathbb{R}P^2)$ is $K^2 \tilde{\times} I$. One sees that as follows. Recall that $K^2 \tilde{\times} I$ is obtained from $T^2 \times [-1, 1]$ by the $\mathbb{Z}_2$-action induced by the involution

$$T^2 \times [-1, 1] \to T^2 \times [-1, 1]$$

$$((z_1, z_2), t) \mapsto ((z_1, -z_2, -t)).$$

Observe that the quotient of $T^2 \times \{0\}$ is the core Klein bottle in $K^2 \tilde{\times} I$. Denote the points in $K^2 \tilde{\times} I$ by $[z_1, z_2, t]$ and consider the map

$$\varphi : K^2 \tilde{\times} I \to K^2 \tilde{\times} I$$

$$[z_1, z_2, t] \mapsto [z_1, -z_2, t].$$

This map has two fixed points, namely $[i, i, 0]$ and $[i, -i, 0]$. Moreover, the map induces an involution on the core Klein bottle $K^2 \tilde{\times} I \times \{0\}$ of $K^2 \tilde{\times} I$ with the preceding two points as fixed points. By the classification of involutions on $K^2$ (see [20]), $K^2 \times \{0\} / \varphi$ is isometric to an $\mathbb{R}P^2$ with a flat metric with two metric singularities. Comparing this construction with the construction in [18, Corollary 2.56], one sees that the double branched cover of $B(\mathbb{R}P^2)$ is $K^2 \tilde{\times} I$.

It follows from standard covering spaces theory, as in Case 6.6.1, that there is a lift of the gluing homeomorphism. Therefore, the orientable double cover of $(S^1 \times D^2) \cup B(\mathbb{R}P^2)$ is the gluing of a solid torus $S^1 \times D^2$ and $K^2 \tilde{\times} I$ via a homeomorphism of the boundary. Hence, by Case 6.5.3, it admits either $\mathbb{S}^3$- or $(S^2 \times \mathbb{R})$-geometry.

**Case 6.6.5** $((K^2 \tilde{\times} I) \cup (K^2 \tilde{\times} I))$. This is a Klein bottle semibundle; these were classified in [11]. To check that the space is geometric, consider the oriented double cover, obtained by gluing two copies of $K^2 \tilde{\times} I$ along their boundaries. This corresponds to case 6.5.8.

**Case 6.6.6** $((K^2 \tilde{\times} I) \cup B(pt))$. Recall that $K^2 \tilde{\times} I$ is doubly covered by $K^2 \tilde{\times} I$ while the oriented branched cover of $B(pt)$ is $S^1 \times D^2$. This situation was considered in case 6.5.3.

**Case 6.6.7** $((K^2 \tilde{\times} I) \cup B(\mathbb{R}P^2))$. Both pieces have $K^2 \tilde{\times} I$ as orientable double cover and therefore, the orientable double cover of $(K^2 \tilde{\times} I) \cup B(\mathbb{R}P^2)$ is a gluing of two copies of $K^2 \tilde{\times} I$ by a homeomorphism between the boundaries. Then, it follows from Case 6.5.8 that the space in question admits the $\mathbb{R}^3$, Nil- or Sol-geometry depending of the gluing homeomorphism.

**Case 6.6.8** $(B(pt) \cup B(pt))$. The orientable double cover of $B(pt)$ is a solid torus. Therefore, $B(pt) \cup B(pt)$ is doubly covered by a union of two solid tori along the boundary, i.e. a lens space.

**Case 6.6.9** $(B(pt) \cup B(\mathbb{R}P^2))$. The orientable double cover of $B(pt)$ is a solid torus $S^1 \times D^2$, while the orientable double cover of $B(\mathbb{R}P^2)$ is $K^2 \tilde{\times} I$. Therefore, as in Case 6.6.4, it follows from Case 6.6.2 that $B(pt) \cup B(\mathbb{R}P^2)$ admits the $\mathbb{S}^3$- or $(S^2 \times \mathbb{R})$-geometry.
Case 6.6.10 \((B(\mathbb{R}P^2) \cup B(\mathbb{R}P^2))\). The space \(B(\mathbb{R}P^2)\) has \(K^2 \times I\) as orientable double cover and therefore, the branched orientable double cover of \(B(\mathbb{R}P^2) \cup B(\mathbb{R}P^2)\) consists of a gluing of two copies of \(K^2 \times I\) by a homeomorphism of the boundary. The space obtained by such construction was considered in Case 6.5.8, from which it follows that \(B(\mathbb{R}P^2) \cup B(\mathbb{R}P^2)\) admits the \(\mathbb{R}^3\), Nil- or Sol-geometry depending of the gluing.

7. Zero-dimensional limit space

We now address the case in which the limit space \(Y\) is zero-dimensional. From the classification of collapsing closed Alexandrov 3-spaces [18], summarized in Table 4, we see that there are three possibilities for the homeomorphism type of \(X_i\).

In the case that \(X_i\) is a generalized Seifert fiber space \(\text{Seif}(Z)\) (with \(\operatorname{curv} Z \geq 0\)) possibly with attached generalized Solid Tori and Klein Bottles, it follows from our analysis in Section 5 that \(X_i\) is geometric.

The next possibility is that \(X_i\) is homeomorphic to one of the spaces appearing in Section 6. Therefore, we can conclude from our previous analysis that \(X_i\) is geometric.

Finally, if \(X_i\) is a closed, non-negatively curved Alexandrov space, then it follows from [9, Theorem 1.3] that \(X_i\) is geometric. This concludes the proof of Theorem A. \(\square\)

References


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