Stationary solution to the Navier-Stokes equations in the scaling invariant Besov space and its regularity

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Abstract

We consider the stationary problem of the Navier-Stokes equations in $\mathbb{R}^n$ for $n \geq 3$. We show existence, uniqueness and regularity of solutions in the homogeneous Besov space $B^{-1+\frac{n}{p},q}_p$, which is the scaling invariant one. As a corollary of our results, a self-similar solution is obtained. For the proof, several bilinear estimates are established. The essential tool is based on the paraproduct formula and the imbedding theorem in homogeneous Besov spaces.

Introduction.

Let us consider the stationary Navier-Stokes equation in $\mathbb{R}^n$ for $n \geq 3$:

$$
\begin{aligned}
-\Delta u + u \cdot \nabla u + \nabla \pi &= f, \\
\text{div } u &= 0,
\end{aligned}
$$

(NS)

where $u = u(x) = (u^1(x), \cdots, u^n(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, respectively, while $f = f(x) = (f^1(x), \cdots, f^n(x))$ denotes the given external force. In the literature, there are a number of results on weak and strong solutions to the Navier-Stokes equations, and modern techniques arising in functional and harmonic analysis have been applied to solve fundamental problems on both stationary and non-stationary cases such as existence, uniqueness, regularity and stability. The existence of the solution of the stationary Navier-Stokes equations (NS) had been founded by Leray [19], Ladyzhenskaya [18] and Fujita [9]. Then Finn [8] established a fundamental framework of the stationary problem in exterior domains. It was shown by Heywood [10] that the solution of (NS) can be obtained as a limit of the solutions of the non-stationary equation. Their methods can be also applied to the whole space $\mathbb{R}^n$, and these results have been developed...
up to the present. For instance, Chen [6] improved regularity of the solution of (NS) in $L^n$, and Secchi [20] treated the solution in $L^n \cap L^p$ with $p > n$.

Among these results, it seems important to find more general spaces where the solution of (NS) is obtained. For that purpose, we shall introduce homogeneous Besov spaces. As the first step, it is necessary to seek a suitable function space to handle (NS). To this end, the scaling argument is quite useful. Indeed, it is easy to show that if $\{u, \pi, f\}$ satisfies (NS), then so does the family $\{u_\lambda, \pi_\lambda, f_\lambda\}$ for all $\lambda > 0$, where $u_\lambda(x) = \lambda u(\lambda x)$, $\pi_\lambda(x) = \lambda^2 \pi(\lambda x)$ and $f_\lambda(x) = \lambda^3 f(\lambda x)$.

We call the spaces $X$ and $Y$ with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ scaling invariant for the velocity $u$ and the external force $f$ with respect to (NS) if it holds that $\|u_\lambda\|_X = \|u\|_X$ and $\|f_\lambda\|_Y = \|f\|_Y$ for all $\lambda > 0$, respectively. For instance, in the usual Lebesgue space $L^p$, $L^n(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ are scaling invariant for $u$ and $f$ with respect to (NS) since it holds that $\|u_\lambda\|_{L^n} = \|u\|_{L^n}$ and $\|f_\lambda\|_{L^2} = \|f\|_{L^2}$ for all $\lambda > 0$. Since $L^n \hookrightarrow B_{p,\infty}^{-1+\frac{n}{q}}$ for all $n < p \leq \infty$ and $L^2 \hookrightarrow B_{p,\infty}^{-3+\frac{n}{p}}$ for all $n/3 \leq p \leq \infty$, homogeneous Besov spaces enable us to treat more general spaces than Lebesgue ones. In comparison with the Lebesgue spaces, the advantage of Besov spaces to solve the non-stationary problems in $\mathbb{R}^3$ has been established by Cannone [3] and Cannone-Planchon [5]. Indeed, they proved the existence theorem for the Cauchy problem with the initial data $u_0 \in B_{p,q}^{-1+\frac{n}{q}}$ with $n < p < \infty$ which is larger than $L^n$. More abstract result including [3] and [5] was shown by Cannone-Meyer [4], which made fully use of the Littlewood-Paley decomposition. Kato [13] and Kozono-Yamazaki [16] obtained the corresponding results to the Morrey space.

The purpose of the present paper is to prove existence, uniqueness and regularity of the solutions of (NS) in the homogeneous Besov spaces with the scaling invariant norms. More precisely, if the external force $f$ is sufficiently small in $B_{p,q}^{-1+\frac{n}{p}}$ with $1 \leq p < n$, $1 \leq q \leq \infty$, then there exists a unique solution $u \in B_{p,q}^{-1+\frac{n}{p}}$ of (NS). Since such homogeneous Besov spaces for $f$ and $u$ include homogeneous functions with degrees $-3$ and $-1$, respectively, our existence theorem necessarily yields the self-similar solution of (NS). The advantage of making use of homogeneous Besov spaces stems from the fact that the scaling argument works, while it is not the case in inhomogeneous Besov spaces. On account of such a scaling property as that of usual $L^p$-spaces, we are successful to construct the stationary solution $u \in B_{p,q}^{-1+\frac{n}{p}}$ for small $f \in B_{p,q}^{-3+\frac{n}{p}}$. On the other hand, there is also disadvantage of homogeneous Besov spaces since the function has to be handled modulo polynomials. So, it is not clear whether the solution $u$ satisfies (NS) almost everywhere in $\mathbb{R}^n$. To give a definite answer to such ambiguity, under a certain additional regularity assumptions on $f$, we shall show that our solution $u$ becomes in fact, a strong solution of (NS) which means that $u$ belongs to the usual Sobolev space $H^{2+r}$ for some $1 < r < \infty$ and satisfies (NS) almost everywhere in $\mathbb{R}^n$. Compared with the non-stationary problem, there seems to be less results in the stationary one which are discussed in the framework of the Besov spaces. A similar result to that in the Morrey space was obtained by Kozono-Yamazaki [17].

This paper is organized as follows. In the next section, we shall state our main theorems with some remarks. First, we shall show a fundamental existence and uniqueness theorem in $B_{p,q}^{-1+\frac{n}{p}}$. As a corollary, we prove the existence of self-similar solutions. We shall also emphasize that if the external force has some additional regularity, then our solution may be regarded as the strong solution of (NS) in $H^{2+r}$ with some $1 < r < \infty$. For the proof of our main theorems, the
generalized Leibnitz rule of the Hölder-type inequality in the Besov spaces plays an important role. So, in Section 2 we shall make use of the paraproduct formula due to Bony [2] and establish several bilinear estimates related to the nonlinear structure $u \cdot \nabla u$ for the solenoidal vector field $u$. Finally, we shall prove the main theorems in Section 3.

1 Results.

Before stating our main results, let us recall homogeneous and inhomogeneous Besov spaces. For that purpose, we first introduce the Littlewood-Paley decomposition \( \{ \varphi_j \}_{j=\infty}^{\infty} \). We take \( \phi \in C_0^\infty(\mathbb{R}^n) \) in such a way that \( \text{supp } \phi = \{ \xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2 \} \) satisfying \( \sum_{j=\infty}^{\infty} \phi(2^{-j}\xi) = 1 \) for all \( \xi \neq 0 \). The functions \( \varphi_j \) are defined as \( \mathcal{F}\varphi_j(\xi) = \phi(2^{-j}\xi), j \in \mathbb{Z} \), where \( \mathcal{F} \) denotes the Fourier transform. Let \( \psi \) be as \( \mathcal{F}\psi(\xi) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j}\xi) \). For \( 1 \leq p \leq \infty \) and \( s \in \mathbb{R} \), the homogeneous Besov space \( \dot{B}^s_{p,q} \) is defined by \( \dot{B}^s_{p,q} := \{ f \in \mathcal{S}'/\mathcal{P}; \| f \|_{\dot{B}^s_{p,q}} < \infty \} \) with the seminorm

\[
\| f \|_{\dot{B}^s_{p,q}} = \left\{ \begin{array}{ll}
\sum_{j=\infty}^{\infty} \left( 2^{sj} \| \varphi_j * f \|_{L^p} \right)^q & \text{for } 1 \leq q < \infty,
\sup_{j \in \mathbb{Z}} 2^{sj} \| \varphi_j * f \|_{L^p} & \text{for } q = \infty,
\end{array} \right.
\]

where \( \mathcal{P} \) is the set of polynomials in \( \mathbb{R}^n \). We also define the corresponding inhomogeneous Besov space \( B^s_{p,q} \) on By \( B^s_{p,q} := \{ f \in \mathcal{S}'/\| f \|_{B^s_{p,q}} < \infty \} \) with the norm

\[
\| f \|_{B^s_{p,q}} = \left\{ \begin{array}{ll}
\| \psi * f \|_{L^p} + \left( \sum_{j=\infty}^{\infty} \left( 2^{sj} \| \varphi_j * f \|_{L^p} \right)^q \right)^{\frac{1}{q}} & \text{for } 1 \leq q < \infty,
\| \psi * f \|_{L^p} + \sup_{j \in \mathbb{Z}} 2^{sj} \| \varphi_j * f \|_{L^p} & \text{for } q = \infty.
\end{array} \right.
\]

For more precise, see e.g., Bergh-Löfström [1].

Next, we rewrite (NS) to the generalized form by means of the abstract setting of the functional analysis. Let \( P \) be the projection operator from \( L^p \) onto the solenoidal space \( L^p_\delta = \{ u \in L^p; \text{div } u = 0 \} \). It is known that \( P \) has the expression \( P = \{ P_{jk} \}_{1 \leq j,k \leq n} \) with \( P_{jk} = \delta_{jk} + R_j R_k, j,k = 1, \cdots, n, \) where \( \delta_{jk} \) denotes the Kronecker symbol and \( R_k = \frac{\partial}{\partial x_k} (-\Delta)^{\frac{1}{2}} \) denotes the Riesz transform. Since \( R_k, k = 1, 2, \cdots, n \) is a bounded operator in \( L^p \) for \( 1 < p < \infty \), \( P \) is also bounded from \( L^p \) onto \( L^p_\delta \) for \( 1 < p < \infty \). However, \( P \) is unbounded in \( L^p \) for \( p = 1 \) and for \( p = \infty \). On the other hand, \( P \) is bounded in the homogeneous Besov space \( \dot{B}^s_{p,q} \) for all \( 1 \leq p \leq \infty \), \( 1 \leq q \leq \infty \) and \( s \in \mathbb{R} \). It should be noted by the Hausdorff-Young inequality that

\[
\| \varphi_j * R_k f \|_{L^p} = \left\| \sum_{l=j-1}^{j+1} \varphi_l * R_k (\varphi_j * f) \right\|_{L^p} \leq \sum_{l=j-1}^{j+1} \left\| F^{-1} \left( \hat{\varphi_l}(\xi) \frac{i\xi_k}{|\xi|} \right) * \varphi_j * f \right\|_{L^p} \leq 3 \| \Phi_k \|_{L^1} \| \varphi_j * f \|_{L^p}, \quad k = 1, \cdots, n
\]

for all \( 1 \leq p \leq \infty \) and for all \( j \in \mathbb{Z} \) with \( \Phi_k = F^{-1}(\phi(\xi) \frac{i\xi_k}{|\xi|}) \) in \( L^1 \), from which we see that \( R_k, k = 1, \cdots, n \) is bounded in \( \dot{B}^s_{p,q} \) even for \( p = 1 \) and \( p = \infty \). Since we need to find the
solution $u$ of (NS) with $\text{div } u = 0$, let us introduce the space $\dot{B}^s_{p,q} \equiv PP^s_{p,q}$. Since $P u = u$, $P(\nabla u) = 0$ and since $P$ commutes with $-\Delta$, application of $P$ to both sides of (NS) yields that $-\Delta u + P(u \cdot \nabla u) = Pf$. Since $\text{div } u = 0$, it holds that $u \cdot \nabla u = \nabla \cdot u \otimes u$, and hence we see that $u$ can be expressed by

$$u = (-\Delta)^{-1} P(u \cdot \nabla u) + (-\Delta)^{-1} Pf$$

$$= P(-\Delta)^{-1} \nabla \cdot (u \otimes u) + P(-\Delta)^{-1} f$$

$$= K(u \otimes u) + P(-\Delta)^{-1} f,$$

where $K \equiv P(-\Delta)^{-1}\nabla$. may be regarded as the Fourier multiplier with the differential order $-1$. More precisely, $K g = (K g_1, \cdots, K g_n)$ has an expression

$$K g_j(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{k,l=1}^n \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \frac{1}{|\xi|^2} i \xi l F g_{kl}(\xi) d\xi, \quad j = 1, \cdots, n$$

for $n \times n$ tensors $g = (g_{kl})_{1 \leq k, l \leq n}$. Then we have the following proposition.

**Proposition 1.1** Let $1 \leq p \leq p_0$ and $-\infty < s_0 \leq s + 1 < \infty$ satisfy $s_0 - n/p_0 = s + 1 - n/p$. Let $1 \leq q \leq \infty$. $K$ is a bounded operator from $\dot{B}^s_{p,q}$ to $\dot{B}^0_{p,q}$ with the estimate

$$\|K g\|_{\dot{B}^0_{p,q}} \leq C\|g\|_{\dot{B}^s_{p,q}},$$

(1.1)

for all $g \in \dot{B}^s_{p,q}$, where $C = C(n, p, p_0, q, s, s_0)$.

For the proof of Proposition 1.1, see the Appendix.

Here and in what follows, we denote by $C$ various constants which may change from line to line. In particular, we denote by $C = C(\ast, \cdots, \ast)$ the constants depending only on the quantities appearing in the parentheses.

Our main results now read as follows. We first state existence and uniqueness of solutions for small data.

**Theorem 1.1** Let $n \geq 3$. For every $1 \leq p < n$ and $1 \leq q \leq \infty$ there is a constant $\delta = \delta(n, p, q) > 0$ such that if $f \in \dot{B}^{-3+\frac{2}{p}}_{p,q}$ satisfies $\|f\|_{\dot{B}^{-3+\frac{2}{p}}_{p,q}} < \delta$, then there exists a solution $u \in \dot{B}^{-1+\frac{\alpha}{p}}_{p,q}$ of (E). Moreover, there exists a constant $\eta = \eta(n, p, q) > 0$ such that if $u$ and $v$ are two solutions of (E) in the class $\dot{B}^{-1+\frac{\alpha}{p}}_{p,q}$ satisfying $\|u\|_{\dot{B}^{-1+\frac{\alpha}{p}}_{p,q}} \leq \eta$, $\|v\|_{\dot{B}^{-1+\frac{\alpha}{p}}_{p,q}} \leq \eta$, then it holds that $u \equiv v$.

An immediate consequence of the above theorem is the existence of self-similar solutions.

**Corollary 1.1** Let $n \geq 3$. Let $1 \leq p < n$ and $q = \infty$. If $f \in \dot{B}^{-3+\frac{2}{p}}_{p,\infty}$ is a homogeneous function with degree $-3$, i.e., $f(\lambda x) = \lambda^{-3} f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ and if $f$ satisfies $\|f\|_{\dot{B}^{-3+\frac{2}{p}}_{p,\infty}} < \delta$, then the solution $u$ given by Theorem 1.1 is a homogeneous function with degree $-1$, i.e., $u(\lambda x) = \lambda^{-1} u(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda > 0$, which means that $u$ may be regarded as a self-similar solution of (NS).
Remark 1.1 (i) In the above theorems, the space $\dot{B}^{-1+\frac{n}{p}}_{p,q}$ for solutions $u$ and the space $\dot{B}^{-3+\frac{n}{p}}_{p,q}$ for the given external force are both scaling invariant with respect to (NS).

(ii) Notice that the Dirac delta function $\delta \in \dot{B}^{-n+\frac{n}{p}}_{p,\infty}$ for all $n \geq 1$ and all $1 \leq p \leq \infty$. Hence in $\mathbb{R}^3$, if $f$ is written in the form $f = \varepsilon b \delta$ with the unit vector $b \in \mathbb{R}^3$ and the small constant $\varepsilon$ depending on $1 \leq p < 3$, then we have a solution $u \in \dot{B}^{-1+\frac{n}{p}}_{p,q}$ of (E) in $\mathbb{R}^3$. It seems an interesting question whether such a solution $u$ coincides with a Landau solution due to Sverák [21, Theorem 1].

(iii) The restrictions on the dimension $n$ and the integral exponent $p$ such as $n \geq 3$ and $1 \leq p < n$ are due to validity of the paraproduct formula which plays a key role for the proof of the bilinear estimate in Lemma 2.1.

(iv) Recently, Tsurumi [22] proved ill-posedness of (NS) in $\dot{B}^{-1}_{\infty,\infty}$ in the sense that the norm inflation occurs in (E) for the solution map $f \in \dot{B}^{-3}_{\infty,1} \mapsto u \in \dot{B}^{-1}_{\infty,\infty}$. Hence, it seems an interesting question whether Theorem 1.1 does hold for $n \leq p < \infty$.

We next show that under some additional assumption on $f$, our solution $u$ has more regularity.

**Theorem 1.2** Let $n \geq 3$, and let $1 \leq p < n$, $1 \leq q, \tilde{q} \leq \infty$. Suppose that $1 \leq r \leq \infty$ and $s \in \mathbb{R}$ satisfy

$$1 - \frac{n}{p} < s, \quad \frac{n}{r} < 1 - \frac{1-s}{n}. \quad (1.2)$$

Then there exists a constant $\delta' = \delta'(n, p, q, r, s, \tilde{q}) \leq \delta$ such that if $f \in \dot{B}^{-3+\frac{n}{p}}_{p,q} \cap \dot{B}^{-\frac{n}{r}+\frac{1}{q}}_{r,q}$ satisfies that $\|f\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}} < \delta'$, then the solution $u$ given by Theorem 1.1 has an additional regularity such as $u \in \dot{B}^{-1+\frac{n}{p}}_{p,q} \cap \dot{B}^{s}_{r,q}$.

In Theorems 1.1 and 1.2, we show that the solution $u$ satisfies (E) in $\dot{B}^{-1+\frac{n}{p}}_{p,q}$. Although the original system (NS) of equations is reduced to (E), it is necessary to identify (E) with an equivalence class in modulo polynomials. To prove that our solution $u$ given by Theorems 1.1 and 1.2 satisfies (NS) in the usual sense, it should be clarified that $u$ and its derivatives $D^\alpha u$ for $|\alpha| \leq 2$ belong to usual $L^r$-spaces. For that purpose, we consider the homogeneous Sobolev space $\dot{H}^{s,\tilde{r}} = \{v \in S'/\mathbb{P}; \|(\Delta)^{\frac{s}{2}} v\|_{L^{\tilde{r}}} < \infty\}$, and show that $u \in \dot{H}^{\tilde{s},\tilde{r}}$ provided $f \in \dot{H}^{-\tilde{s},\tilde{r}}$ under some restriction on $\tilde{r}$ and $\tilde{s}$.

Our result on additional regularity of the solution $u$ of (E) reads as follows:

**Theorem 1.3** Let $n \geq 3$, and let $1 \leq p < n$, $1 \leq q \leq \infty$. Suppose that $1 < \tilde{r} < \infty$ and $\tilde{s} \geq 0$ satisfy

$$q \leq \tilde{r}, \quad \frac{n}{\tilde{r}} - n + 1 < \tilde{s} < \min\left\{\frac{n}{p}, \frac{n}{\tilde{r}}\right\}. \quad (1.3)$$

Then there exists a constant $\delta'' = \delta''(n, p, q, \tilde{r}, \tilde{s}) \leq \delta$ such that if $f \in \dot{B}^{-3+\frac{n}{p}}_{p,q} \cap \dot{H}^{-\tilde{s},\tilde{r}}$ satisfies that $\|f\|_{\dot{B}^{-3+\frac{n}{p}}_{p,q}} < \delta''$, then the solution $u$ given by Theorem 1.1 has an additional regularity such as $u \in \dot{B}^{-1+\frac{n}{p}}_{p,q} \cap \dot{H}^{\tilde{s},\tilde{r}}$. In particular, by taking $\tilde{s} = 0$ and $\tilde{r} = p$, we have that $u \in \dot{B}^{-1+\frac{n}{p}}_{p,q} \cap L^p$, which implies that the solution $u$ belongs to the inhomogeneous Besov space $\dot{B}^{-1+\frac{n}{p}}_{p,q}$.
Remark 1.2 (i) The restrictions on $s$ such that $s > 1 - \frac{n}{p}$ in Theorem 1.2 is necessary for application of such a formula as in Lemma 2.2.

(ii) In Theorems 1.2 and 1.3, we need smallness of $f$ only on the scaling invariant norm, i.e., $\mathring{B}^{s-\frac{n}{2}}_{p,q}$. It should be noted that smallness assumptions for $f$ in $\mathring{B}^{s-2}_{r,q}$ and in $\mathring{H}^{s-2,r}$ are both redundant.

(iii) By taking $\tilde{r}_1 \geq \max\{p, q\}$ for $i = 0, 1, 2$ in such a way that
\[ \frac{n}{n-1} < \tilde{r}_0, \quad \tilde{r}_1 < n, \quad \tilde{r}_2 < \frac{n}{2}, \]
we see from Theorem 1.3 that if $f \in \mathring{H}^{-2,\tilde{r}_0} \cap \mathring{H}^{-1,\tilde{r}_1} \cap L^{\tilde{r}_2}$, then our solution $u$ has the additional regularity that $u \in L^{\tilde{r}_0}$, $\nabla u \in L^{\tilde{r}_1}$ and $\nabla^2 u \in L^{\tilde{r}_2}$. This implies that $u$ satisfies (NS) almost everywhere in $\mathbb{R}^n$.

(iv) Assume that $f$ is written as $f = \text{div} F$ with small $F \in L^{\frac{n}{2}}$. Then it follows from Jawerth [11, Theorem 2.1 (iii)] that $f \in \mathring{H}^{-1,\frac{n}{2}} = \mathring{F}^{-1}_{\frac{n}{2},2} \hookrightarrow \mathring{B}^{s-\frac{n}{2}}_{p,q}$ for $\frac{n}{2} < p < n$. Here $\mathring{F}^{s}_{p,q}$ denotes the homogeneous Triebel-Lizorkin space. Furthermore, for such $p$ and $q = \frac{n}{2}$ we see that $\tilde{r} = \frac{n}{2}$ and $\tilde{s} = 1$ fulfill the condition (1.3). Hence, it follows from Theorem 1.3 that there exists a solution $u \in \mathring{H}^{1,\frac{n}{2}}$ of (E). This implies that our theorem includes the existence result of solutions to (NS) with the external force written in the divergence form in the usual Lebesgue space $L^{\frac{n}{2}}$. See e.g., Chen [6].

2 Bilinear estimates.

Let us first introduce some imbedding theorems in the homogeneous Besov and Triebel-Lizorkin spaces. Recall that the homogeneous Triebel-Lizorkin space $\dot{F}^{s}_{p,q}$ for $1 \leq p < \infty$, $1 \leq q \leq \infty$ is the set of all $f \in S'/\mathcal{P}$ such that $\|f\|_{\dot{F}^{s}_{p,q}} < \infty$, where
\[
\|f\|_{\dot{F}^{s}_{p,q}} = \left\{ \begin{array}{cl}
\left(\sum_{j=-\infty}^{\infty} |2^{nj}\varphi_j * f|^q \right)^{\frac{1}{q}} & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj}|\varphi_j * f|_{L^p} & \text{for } q = \infty.
\end{array} \right.
\]

Proposition 2.1 (i) Let $s_1 \geq s_2$, $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q \leq \infty$. If $s_1 - n/p_1 = s_2 - n/p_2$, then it holds that
\[
\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q}.
\]

(ii) Let $s_0 > s$, $1 \leq r_0 < r \leq \infty$, $1 \leq q \leq r$, $1 < \beta < \infty$. If $s_0 - n/r_0 = s - n/r$, then it holds that
\[
\dot{B}^{s_0}_{r_0,q} \hookrightarrow \dot{F}^{s}_{r,\beta}.
\]

Proof. The proof for (i) is straightforward, so we may omit it. For precise, see, e.g., Bergh-Löfström [1, Theorem 6.5.1].

(ii) Since $1 \leq q \leq r$, we have
\[
\dot{B}^{s_0}_{r_0,q} \hookrightarrow \dot{B}^{s_0}_{r_0,r}.
\]
It follows from Jawerth [11, Theorem 2.1 (iii)] that for $s_1 > s_2$, $1 < p_1 < p_2 < \infty$ with $s_1 - n/p_1 = s_2 - n/p_2$ it holds $\dot{F}^{s_1}_{p_1,\beta} \hookrightarrow \dot{B}^{s_2}_{p_2,p_1}$, where $\beta' \equiv \beta/(\beta - 1)$. Hence by duality, we have that
\[
\dot{B}^{-s_2}_{p_2,p_1'} \hookrightarrow \dot{F}^{-s_1}_{p_1',\beta'}.
\]
We may take \(s_2, p_1, p_2\) in such a way that \(s_2 = -s_0, p_1 = r', p_2 = r_0'\), respectively. Indeed, since \(1 < r_0 < r < \infty\), we have \(1 < p_1 < p_2 < \infty\). Moreover, since \(s_0 - n/r_0 = s - n/r\), it holds that \(s_1 = s_2 + n/p_1 - n/p_2 = -s_0 + n/r_0 - n/r = -s\). Hence it follows from (2.2) that
\[
\hat{B}^0_{r_0,r} \hookrightarrow \hat{F}^s_{r,s}. \tag{2.3}
\]

The desired imbedding is a consequence of (2.1) and (2.3). This completes the proof.

Next, we consider the Leibnitz rule in the homogeneous Besov space.

**Proposition 2.2** (i) Let \(1 \leq p, q \leq \infty, s > 0, \alpha > 0\) and \(\beta > 0\). Assume that \(1 \leq p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty\) satisfy \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}\). If \(f \in \dot{B}^{s+\alpha}_{p_1,q} \cap \dot{B}^{-\beta}_{p_1,\infty}\) and \(g \in \dot{B}^{-\alpha}_{p_2,\infty} \cap \dot{B}^{s+\beta}_{p_2,q}\), then we have \(fg \in \dot{B}^s_{p,q}\) with the estimate
\[
\|fg\|_{\dot{B}^s_{p,q}} \leq C\left(\|f\|_{\dot{B}^{s+\alpha}_{p_1,q}}\|g\|_{\dot{B}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{B}^{-\beta}_{p_1,\infty}}\|g\|_{\dot{B}^{s+\beta}_{p_2,q}}\right) \tag{2.4}
\]
where \(C = C(p, p_1, p_2, \tilde{p}_1, \tilde{p}_2, q, s, \alpha, \beta)\).

(ii) Let \(1 \leq p, q \leq \infty\) and \(s > 0\). Assume that \(1 \leq p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leq \infty\) satisfy \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{p}_2}\). If \(f \in \dot{B}^s_{p_1,q} \cap L^{\tilde{p}_1}\) and \(g \in L^{p_2} \cap \dot{B}^{-\beta}_{p_2,q}\), then we have \(fg \in \dot{B}^s_{p,q}\) with the estimate
\[
\|fg\|_{\dot{B}^s_{p,q}} \leq C\left(\|f\|_{\dot{B}^s_{p_1,q}}\|g\|_{L^{p_2}} + \|f\|_{L^{\tilde{p}_1}}\|g\|_{\dot{B}^{-\beta}_{p_2,q}}\right) \tag{2.5}
\]
where \(C = C(p, p_1, p_2, \tilde{p}_1, \tilde{p}_2, q, s)\).

**Proof.** (i) We make use of the following paraproduct formula of \(fg\) due to Bony [2]. Our method is related to Christ-Weinstein [7, Proposition 3.3] and Kozono-Shimada [15, Lemma 2.1].
\[
f \cdot g = \sum_{k=-\infty}^{\infty} (\varphi_k \ast f)(P_k g) + \sum_{k=-\infty}^{\infty} (P_k f)(\varphi_k \ast g) + \sum_{k=-\infty}^{\infty} \sum_{|k-j| \leq 2} (\varphi_k \ast f)(\varphi_j \ast g)
\]
\[
= h_1 + h_2 + h_3, \tag{2.6}
\]
where \(P_k g = \sum_{l=-\infty}^{k-3} \varphi_l \ast g\). We first consider the case \(1 \leq q < \infty\). Since
\[
\text{supp } \mathcal{F}((\varphi_k \ast f)(P_k g)) \subset \{\xi \in \mathbb{R}^n; 2^{k-2} \leq |\xi| \leq 2^{k+2}\},
\]
\[
\text{supp } \mathcal{F} \varphi_j = \{\xi \in \mathbb{R}^n; 2^{j-1} \leq |\xi| \leq 2^{j+1}\},
\]
we have that
\[
\|h_1\|_{\dot{B}^s_{p,q}} = \left\{ \sum_{j=-\infty}^{\infty} \left(2^{jq} \|\varphi_j \ast h_1\|_{L^p}\right)^\frac{1}{q} \right\} \frac{1}{q}
\]
\[
= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{jq} \left\| \sum_{k=-\infty}^{\infty} \varphi_j \ast ((\varphi_k \ast f)(P_k g)) \right\|_{L^p} \right)^\frac{1}{q} \right\} \frac{1}{q}
\]
\[
= \left\{ \sum_{j=-\infty}^{\infty} \left(2^{jq} \left\| \sum_{|k-j| \leq 2} \varphi_j \ast ((\varphi_k \ast f)(P_k g)) \right\|_{L^p} \right)^\frac{1}{q} \right\} \frac{1}{q}.
\]
Since \( \varphi_j(x) = 2^j \phi(2^j x) \) for all \( j \in \mathbb{Z} \), it holds by the Hausdorff-Young and the Hölder inequalities that

\[
\| \varphi_j * ((\varphi_k * f)(P_kg)) \|_{L^p} \leq \| \varphi_j \|_{L^1} \| (\varphi_k * f)(P_kg) \|_{L^p} \leq \| \mathcal{F}^{-1} \phi \|_{L^1} \| \varphi_k * f \|_{L^{p_1}} \| P_kg \|_{L^{p_2}}
\]

for all \( j, k \in \mathbb{Z} \). Hence it follows from the Minkowski inequality that

\[
\| h_1 \|_{B^{p,q}_{p_1,q}} \leq C \left\{ \sum_{i=-\infty}^{\infty} \left( 2^i \sum_{k \sim j \leq 2} \| \varphi_k * f \|_{L^{p_1}} \| P_kg \|_{L^{p_2}} \right) \right\}^{\frac{q}{2}}
\]

\[
= C \left\{ \sum_{i=-\infty}^{\infty} \left( 2^i \sum_{k \sim j \leq 2} \| \varphi_j * f \|_{L^{p_1}} \| P_jg \|_{L^{p_2}} \right) \right\}^{\frac{q}{2}}
\]

\[
\leq C \sum_{i=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} \left( 2^j \| (\varphi_j * f) \|_{L^{p_1}} \| P_jg \|_{L^{p_2}} \right) \right) \frac{q}{2}
\]

\[
= C \sum_{i=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} \left( 2^j \sum_{k \sim j \leq 2} \| \varphi_i * f \|_{L^{p_1}} \| P_i g \|_{L^{p_2}} \right) \right) \frac{q}{2}
\]

\[
\leq C \sum_{i=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} \left( 2^j \sum_{k \sim j \leq 2} \| \varphi_i * f \|_{L^{p_1}} \| P_i g \|_{L^{p_2}} \right) \right) \left( \sum_{l=-\infty}^{\infty} \left( 2^l \| \varphi_i \|_{L^{p_1}} \| P_i g \|_{L^{p_2}} \right) \right) \frac{q}{2}
\]

\[
= C \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{i=-\infty}^{\infty} \left( 2^i \| \varphi_k * g \|_{L^{p_2}} \right) \right) \frac{q}{2}
\]

\[
\leq C \sum_{k \in \mathbb{Z}} \sup_{i \in \mathbb{Z}} 2^{-\alpha k} \| \varphi_k * g \|_{L^{p_2}} \left( \sum_{i=-\infty}^{\infty} \left( 2^i \| \varphi_i \|_{L^{p_1}} \right) \right) \frac{q}{2}
\]

\[
= C \left\{ \sum_{k \in \mathbb{Z}} \left( \sum_{i=-\infty}^{\infty} \left( 2^i \| \varphi_i \|_{L^{p_1}} \right) \right) \frac{q}{2}
\]

\[
= C \| g \|_{\dot{B}^{-\alpha}_{p_2,\infty}} \| f \|_{\dot{B}^{\alpha}_{p_1,\infty}},
\]

(2.7)

where \( C = C(n, p, p_1, p_2, q, s, \alpha) \). In the above estimate it should be noted that \( \sum_{l=3}^{\infty} 2^{-\alpha l} < \infty \) since \( \alpha > 0 \). In the case \( q = \infty \), we see similarly to (2.7) that

\[
\| h_1 \|_{B^{p,q}_{p_1,q}} \leq C \sup_{k \in \mathbb{Z}} 2^{-\alpha k} \| \varphi_k * g \|_{L^{p_2}} \sup_{i \in \mathbb{Z}} 2^{(s+\alpha) i} \| \varphi_i \|_{L^{p_1}} \sum_{l=3}^{\infty} 2^{-\alpha l} = C \| g \|_{\dot{B}^{-\alpha}_{p_2,\infty}} \| f \|_{\dot{B}^{\alpha}_{p_1,\infty}},
\]

with \( C = C(n, p, p_1, p_2, s, \alpha) \), from which and (2.7) it follows that

\[
\| h_1 \|_{B^{p,q}_{p_1,q}} \leq C \| g \|_{\dot{B}^{-\alpha}_{p_2,\infty}} \| f \|_{\dot{B}^{\alpha}_{p_1,\infty}} \| f \|_{\dot{B}^{\alpha}_{p_1,\infty}} \| f \|_{\dot{B}^{\alpha}_{p_1,\infty}} \| f \|_{\dot{B}^{\alpha}_{p_1,\infty}}
\]

\[
\text{for all } 1 \leq q \leq \infty,
\]

(2.8)

where \( C = C(n, p, p_1, p_2, q, s, \alpha) \).

Replacing the role of \( f \) by \( g \), we obtain similarly to (2.7) and (2.8) that

\[
\| h_2 \|_{B^{p,q}_{p_1,q}} \leq C \| f \|_{\dot{B}^{-\beta}_{p_1,\infty}} \| g \|_{\dot{B}^{\beta}_{p_2,\infty}} \| g \|_{\dot{B}^{\beta}_{p_2,\infty}} \| g \|_{\dot{B}^{\beta}_{p_2,\infty}}
\]

\[
\text{for all } 1 \leq q \leq \infty,
\]

(2.9)
where \( C = C(n, p, \tilde{p}_1, \tilde{p}_2, g, s, \beta) \).

Next we treat \( h_3 \) in \( B^s_{p, q} \). Let us consider the case \( 1 < q < \infty \). Since
\[
\text{supp } \mathcal{F}((\varphi_k * f)(\varphi_l * g)) \subset \{ \xi \in \mathbb{R}^n; |\xi| \leq 2^{\max \{k, l\} + 2} \},
\]
we have that
\[
\| h_3 \|_{B^s_{p, q}} = \left\{ \sum_{j=-\infty}^{\infty} 2^{sj} \| \varphi_j * h_3 \|_{L^p} \right\}^{\frac{1}{q}}
\]
\[
= \left\{ \sum_{j=-\infty}^{\infty} 2^{sj} \left( \sum_{k=-\infty}^{\infty} \sum_{|l-k| \leq 2} \| \varphi_j * (\varphi_k * f)(\varphi_l * g) \|_{L^p} \right) \right\}^{\frac{1}{q}}
\]
\[
= \left\{ \sum_{j=-\infty}^{\infty} 2^{sj} \left( \sum_{\max \{k, l\} \geq j-2} \sum_{|l-k| \leq 2} \| \varphi_j * (\varphi_k * f)(\varphi_l * g) \|_{L^p} \right) \right\}^{\frac{1}{q}}
\]
\[
\leq \left\{ \sum_{j=-\infty}^{\infty} 2^{sj} \left( \sum_{r \geq -4} \sum_{|t| \leq 2} \| \varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g) \|_{L^p} \right) \right\}^{\frac{1}{q}}
\]
By the Hausdorff-Young and the Hölder inequalities, it holds that
\[
\| \varphi_j * (\varphi_{j+r} * f)(\varphi_{j+r+t} * g) \|_{L^p} \leq \| \varphi_j \|_{L^p} \| (\varphi_{j+r} * f)(\varphi_{j+r+t} * g) \|_{L^p} \leq \| \mathcal{F}^{-1} \varphi \|_{L^q} \| \varphi_{j+r} * f \|_{L^1} \| \varphi_{j+r+t} * g \|_{L^q}
\]
for all \( j, r, t \in \mathbb{Z} \). Hence it follows from the Minkowski inequality that
\[
\| h_3 \|_{B^s_{p, q}} \leq C \left\{ \sum_{j=-\infty}^{\infty} 2^{sj} \sum_{r \geq -4} \sum_{|t| \leq 2} \| \varphi_{j+r} * f \|_{L^p_1} \| \varphi_{j+r+t} * g \|_{L^p_2} \right\}^{\frac{1}{q}}
\]
\[
\leq C \sum_{r \geq -4} \sum_{|t| \leq 2} \left\{ \sum_{j=-\infty}^{\infty} 2^{sj} \| \varphi_{j+r} * f \|_{L^p_1} \| \varphi_{j+r+t} * g \|_{L^p_2} \right\}^{\frac{1}{q}}
\]
\[
= C \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} 2^{sa_t} \left\{ \sum_{j=-\infty}^{\infty} 2^{(s+\alpha)(j+r)} \| \varphi_{j+r} * f \|_{L^p_1} 2^{-\alpha(j+r+t)} \| \varphi_{j+r+t} * g \|_{L^p_2} \right\}^{\frac{1}{q}}
\]
\[
\leq C \sup_{l \in \mathbb{Z}} 2^{-\alpha l} \| \varphi_l * g \|_{L^p_2} \sum_{r \geq -4} 2^{-sr} \sum_{|t| \leq 2} 2^{sa_t} \left\{ \sum_{k=-\infty}^{\infty} 2^{(s+\alpha)k} \| \varphi_k * f \|_{L^p_1} \right\}^{\frac{1}{q}}
\]
\[
= C \| g \|_{B^{-\alpha}_{p, \infty}} \| f \|_{B^{p+\alpha}_{p_1, q}},
\]
(2.10)
where $C = C(n, p, p_1, p_2, q, s, \alpha)$. In the above estimate it should be noted that $\sum_{r \geq -4} 2^{-sr} < \infty$ since $s > 0$. In case $q = \infty$, similarly to (2.10), we have that

\[
\|h_3\|_{\dot{B}^{s}_{p,q}} \leq C \sup_{l \in \mathbb{Z}} 2^{-sl} \|\varphi_l * g\|_{L^p_{2}} \sum_{r \geq -4} 2^{-sr} \sum_{|l| \leq 2} 2^{at} \sup_{k \in \mathbb{Z}} 2^{(s+\alpha)k} \|\varphi_k * f\|_{L^p_{1}}
\]

and (2.11). It should be noticed that

\[
h_3(x) = 2^{nq} \psi(2^{k}x) = \psi_{2^{-k}}(x), \quad \forall k \in \mathbb{Z},
\]

where $f_{\varepsilon}(x) = \varepsilon^{-n} f(x/\varepsilon)$ for $\varepsilon > 0$. Hence we have $\|\sum_{l=-\infty}^{k} \varphi_l\|_{L^1} = \|\psi\|_{L^1}$ for all $k \in \mathbb{Z}$, and it holds that

\[
\|P_l g\|_{L^p_{2}} = \left\| \sum_{l=-\infty}^{i-3} \varphi_l * g \right\|_{L^p_{2}} = \|\psi_{2^{-i-3}} * g\|_{L^p_{2}} \leq \|\psi\|_{L^1} \|g\|_{L^p_{2}} \quad \text{for all } i \in \mathbb{Z},
\]

from which and (2.12) it follows that

\[
\|h_1\|_{\dot{B}^{s}_{p,q}} \leq C \|g\|_{L^p_{2}} \|f\|_{\dot{B}^{s}_{p,1,q}},
\]

where $C = C(n, p, p_1, p_2, q, s, \alpha)$. In case $q = \infty$, we have that

\[
\|h_1\|_{\dot{B}^{s}_{p,\infty}} \leq C \sup_{l \in \mathbb{Z}} \|P_l g\|_{L^p_{2}} \sup_{l \in \mathbb{Z}} 2^{at} \|\varphi_l * f\|_{L^p_{1}} \leq C \|g\|_{L^p_{2}} \|f\|_{\dot{B}^{s}_{p,1,\infty}},
\]

from which and (2.13) it follows that

\[
\|h_1\|_{\dot{B}^{s}_{p,q}} \leq C \|g\|_{L^p_{2}} \|f\|_{\dot{B}^{s}_{p,1,q}} \quad \text{for all } 1 \leq q \leq \infty,
\]
where $C = C(n, p, p_1, p_2, q, s)$.

Replacing the role of $f$ by $g$, we have similarly to (2.14) that

$$\|h_2\|_{\dot{B}^{s}_{p,q}} \leq C\|f\|_{L^{p_1}} \|g\|_{\dot{B}^{s}_{p_1,q}}$$

for all $1 \leq q \leq \infty$, (2.15)

where $C = C(n, p, \tilde{p}_1, \tilde{p}_2, q, s)$.

Concerning the estimate of $h_3$ in $\dot{B}^{s}_{p,q}$ for $1 \leq q < \infty$, we have similarly to (2.10) that

$$\|h_3\|_{\dot{B}^{s}_{p,q}} \leq C \sum_{r \geq -4} 2^{-sr}\sum_{|s| \leq 2} \left\{ \sum_{j = -\infty}^{\infty} (2^{s(j+r)} \| \varphi_{j+r} * f \|_{L^{p_1}} \| \varphi_{j+r+t} * g \|_{L^{p_2}})^{\frac{1}{q}} \right\}$$

$$\leq C \sup_{i \in \mathbb{Z}} \| \varphi_i * g \|_{L^{p_2}} \sum_{r \geq -4} 2^{-sr}\sum_{i = -\infty}^{\infty} (2^{si} \| \varphi_i * f \|_{L^{p_1}})^{\frac{1}{q}}$$

$$\leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^{s}_{p_1,q}},$$

(2.16)

where $C = C(n, p, p_1, p_2, q, s)$.

In case $q = \infty$, we have similarly to the above that

$$\|h_3\|_{\dot{B}^{s}_{p,\infty}} \leq C \sum_{r \geq -4} 2^{-sr}\sum_{|s| \leq 2} \sup_{j \in \mathbb{Z}} 2^{s(j+r)} \| \varphi_{j+r} * f \|_{L^{p_1}} \| \varphi_{j+r+t} * g \|_{L^{p_2}}$$

$$\leq C \sup_{i \in \mathbb{Z}} \| \varphi_i * g \|_{L^{p_2}} \sup_{l \in \mathbb{Z}} 2^{sl} \| \varphi_l * f \|_{L^{p_1}} \sum_{r \geq -4} 2^{-sr}$$

$$\leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^{s}_{p_1,\infty}},$$

from which and (2.16) it follows that

$$\|h_3\|_{\dot{B}^{s}_{p,q}} \leq C \|g\|_{L^{p_2}} \|f\|_{\dot{B}^{s}_{p_1,q}}$$

for all $1 \leq q \leq \infty$, (2.17)

where $C = C(n, p, p_1, p_2, s)$. Now the desired estimate (2.5) is a consequence of (2.14), (2.15) and (2.17). This proves Proposition 2.2.

Let us recall the operator $K$ given by Proposition 1.1. The following three lemmata of the bilinear estimate play important roles for proofs of our main theorems.

**Lemma 2.1** Let $n \geq 3$ and let $1 \leq p < n$, $1 \leq q \leq \infty$. For $u, v \in \dot{B}^{\frac{n-1+s}{p}}_{p,q}$ we have $K(u \otimes v) \in \dot{B}^{\frac{n-1+s}{p}}_{p,q}$ with the estimate

$$\|K(u \otimes v)\|_{\dot{B}^{\frac{n-1+s}{p}}_{p,q}} \leq C \|u\|_{\dot{B}^{\frac{n-1+s}{p}}_{p,q}} \|v\|_{\dot{B}^{\frac{n-1+s}{p}}_{p,q}},$$

(2.18)

where $C = C(n, p, q)$.

**Proof.** Taking $p = p_0$, $s = -2 + n/p$ in Proposition 1.1, we have that $s_0 = -1 + n/p$, and so it holds that

$$\|K(u \otimes v)\|_{\dot{B}^{\frac{n-1+s}{p}}_{p,q}} \leq C \|u \otimes v\|_{\dot{B}^{\frac{n-1+s}{p}}_{p,q}},$$

(2.19)
where \( C = C(n, p, q) \). Let us first consider the case \( 1 \leq p < n/2 \). Take \( p_1 \) and \( p_2 \) in such a way
\[
p_1 = p, \quad n < p_2, \quad p' = \frac{p}{p - 1} < p_2.
\]
We define \( p_0 \) and \( s_0 \) by
\[
\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_0 = \frac{n}{p_0} - 2.
\]
(2.20)
Since \( 1 \leq p < n/2 \), we have that
\[
1 \leq p_0 \leq p, \quad 0 < s_0, \quad \left( -2 + \frac{n}{p} \right) - \frac{n}{p_0} = s_0 - \frac{n}{p_0}.
\]
(2.21)
It should be noted that the above (2.21) yields \( 1 \leq p_0 \leq p < n/2 \), which necessarily implies that \( n \geq 3 \). Hence it follows from Proposition 2.1(i) that
\[
\|u \otimes v\|_{B^{s_0}_{p_0,q}} \leq C\|u \otimes v\|_{B^{s_0}_{p_0,q}}.
\]
(2.22)
Since \( n < p_2 \), we have \( \alpha \equiv 1 - n/p_2 > 0 \), and we have by Proposition 2.2 (i) that
\[
\|u \otimes v\|_{B^{s_0}_{p_0,q}} \leq C\|u\|_{B^{s_0 + \alpha}_{p_1,q}}\|v\|_{B^{-\alpha}_{p_2,q}} + \|u\|_{B^{-\alpha}_{p_2,q}}\|v\|_{B^{s_0 + \alpha}_{p_1,q}}).
\]
(2.23)
Since \( p_1 = p, p < p_2 \) and since
\[
s_0 + \alpha - \frac{n}{p_1} = -1 = \left( -1 + \frac{n}{p} \right) - \frac{n}{p_2}, \quad -\alpha - \frac{n}{p_2} = -1 = \left( -1 + \frac{n}{p} \right) - \frac{n}{p_2},
\]
it follows from Proposition 2.1(i) that
\[
B^{-\alpha}_{p_2,q} \hookrightarrow B^{s_0 + \alpha}_{p_1,q}, \quad B^{s_0 + \alpha}_{p_1,q} \hookrightarrow B^{-\alpha}_{p_2,q}.
\]
Hence we obtain from (2.23) that
\[
\|u \otimes v\|_{B^{s_0}_{p_0,q}} \leq C\|u\|_{B^{s_0}_{p_0,q}}\|v\|_{B^{s_0}_{p_0,q}}\|v\|_{B^{s_0}_{p_0,q}}.
\]
(2.24)
Now, the desired estimate (2.18) is a consequence of (2.19), (2.22) and (2.24).
We next consider the case \( n/2 \leq p < n \). In such a case, we take \( p_1 \) and \( p_2 \) so that
\[
p_1 = p, \quad n < p_2 < \frac{np}{2p - n}.
\]
Define \( p_0 \) and \( s_0 \) by (2.20). Since
\[
\frac{1}{p} < \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} < \frac{2}{n} + \frac{1}{n} \leq 1,
\]
\[
s_0 = \frac{n}{p_0} - 2 = n\left( \frac{1}{p_2} - \frac{2}{n} \left( \frac{1}{p} - \frac{1}{n} \right) \right) > 0,
\]
we have (2.21), so it holds (2.22). Since \( \alpha \equiv 1 - n/p_2 > 0 \), implied by \( n < p_2 \), in the same way as in the above case, we obtain (2.24), which yields the desired estimate (2.18). This proves Lemma 2.1.
Lemma 2.2 Let \( n \geq 2 \), and let \( 1 \leq p < n, 1 \leq q, \tilde{q} \leq \infty \). Suppose that \( 1 \leq r \leq \infty \) and \( s \in \mathbb{R} \) satisfy

\[
1 - \frac{n}{p} < s, \quad \frac{1}{r} < 1 - \frac{1 - s}{n}.
\]

For \( u, v \in B_{p,q}^{-1 + \frac{\tilde{p}}{p}} \cap B_{r,\tilde{q}}^s \), it holds that \( K(u \otimes v) \in B_{p,q}^{-1 + \frac{\tilde{p}}{p}} \cap B_{r,\tilde{q}}^s \) with the estimate

\[
\|K(u \otimes v)\|_{B_{r,\tilde{q}}^s} \leq C(\|u\|_{B_{p,q}^{-1 + \frac{\tilde{p}}{p}}} \|v\|_{B_{r,\tilde{q}}^s} + \|u\|_{B_{r,\tilde{q}}^s} \|v\|_{B_{r,\tilde{q}}^{-1 + \frac{\tilde{p}}{p}}}),
\]

(2.25)

where \( C = C(n, p, q, \tilde{q}, r, s) \).

Proof. We first consider the case \( 1 - n/p < s \leq 0 \). Let us take \( 1 \leq r_1, r_2 \leq \infty \) in such a way that

\[
r_1 = p, \quad \frac{1}{r} + \frac{1 - s}{n} - \frac{1}{p} < \frac{1}{r_2} < \frac{1}{r}, \quad \frac{1}{r_2} \leq 1 - \frac{1}{p}.
\]

(2.26)

Define \( r_0, s_0 \) and \( \alpha \) by

\[
\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}, \quad s_0 = \frac{n}{r_0} - \frac{n}{r} + s - 1, \quad \alpha = \frac{n}{r} - \frac{n}{r_2} - s.
\]

Since \( s \leq 0 \), we have by (2.26) that \( 1 \leq r_0 \leq \infty \) and \( 1/r_0 = 1/p + 1/r_2 > 1/r + (1-s)/n \geq 1/r \), i.e., \( r_0 < r \). Since \( s - 1 - n/r = s_0 - n/r_0 \), it follows from Proposition 2.1(i) that

\[
\|K(u \otimes v)\|_{B_{r,\tilde{q}}^s} \leq C\|u \otimes v\|_{B_{r_0,\tilde{q}}^{-1}} \leq C\|u \otimes v\|_{B_{r_0,\tilde{q}}^s}
\]

(2.27)

Since \( s \leq 0 \), we see easily by (2.26) that

\[
s_0 = n \left( \frac{1}{r_2} - \left( \frac{1}{r} + \frac{1 - s}{n} - \frac{1}{p} \right) \right) > 0, \quad \alpha = n \left( \frac{1}{r} - \frac{1}{r_2} - \frac{s}{n} \right) \geq n \left( \frac{1}{r} - \frac{1}{r_2} \right) > 0.
\]

Hence, it follows from Proposition 2.2 (i) that

\[
\|u \otimes v\|_{B_{r_0,\tilde{q}}^s} \leq C(\|u\|_{B_{r_1,\tilde{q}}^{s_0}} \|v\|_{B_{r,\tilde{q}}^{-\alpha}} + \|u\|_{B_{r,\tilde{q}}^{-\alpha}} \|v\|_{B_{r_1,\tilde{q}}^{s_0}}).
\]

(2.28)

Moreover, since

\[
p = r_1, \quad \left( -1 + \frac{n}{p} \right) - \frac{n}{p} = s_0 + \alpha - \frac{n}{r_1},
\]

\[
r \leq r_2, \quad s - \frac{n}{r} = - \alpha - \frac{n}{r_2},
\]

we have by Proposition 2.1(i) that

\[
B_{p,q}^{-1 + \frac{\tilde{p}}{p}} \hookrightarrow B_{r_1,\tilde{q}}^{s_0 + \alpha}, \quad B_{r,\tilde{q}}^s \hookrightarrow B_{r_2,\tilde{q}}^{-\alpha},
\]

which yields by (2.28) that

\[
\|u \otimes v\|_{B_{r_0,\tilde{q}}^s} \leq C(\|u\|_{B_{p,q}^{-1 + \frac{\tilde{p}}{p}}} \|v\|_{B_{r,\tilde{q}}^s} + \|u\|_{B_{r,\tilde{q}}^s} \|v\|_{B_{p,q}^{-1 + \frac{\tilde{p}}{p}}}).
\]

(2.29)
From (2.27) and (2.28) we obtain the desired estimate (2.25).

Next, we consider the case $s > 0$. Let us take $1 \leq r_1, r_2 \leq \infty$ in such a way that

$$r_1 = r, \quad \frac{1 - s}{n} < \frac{1}{r_2} < \frac{1}{n}, \quad \frac{1}{r_2} \leq 1 - \frac{1}{r}. \quad (2.30)$$

Define $r_0, s_0$ and $\alpha$ by

$$\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}, \quad s_0 = s - 1 + \frac{n}{r_2}, \quad \alpha = 1 - \frac{n}{r_2}.$$ 

Then we have that $1 \leq r_0 \leq r$, $s - 1 - n/r = s_0 - n/r_0$, and hence a similar argument to (2.27) yields that

$$\|K(u \otimes v)\|_{B^{s_0}_{r_0, q}} \leq C\|u \otimes v\|_{\dot{B}^{s_0}_{0, q}}.$$ 

By (2.30) we see that

$$s_0 = n \left(\frac{1}{r_2} - \frac{1 - s}{n}\right) > 0, \quad \alpha = n \left(\frac{1}{n} - \frac{1}{r_2}\right) > 0,$$

and hence it follows from Proposition 2.2 (i) that

$$\|u \otimes v\|_{B^{s_0}_{r_0, q}} \leq C(\|u\|_{B^{s_0+\alpha}_{r_1, q}} \|v\|_{B^{-\alpha}_{r_2, q}} + \|u\|_{B^{-\alpha}_{r_2, q}} \|v\|_{B^{s_0+\alpha}_{r_1, q}}). \quad (2.32)$$

Moreover, since

$$r = r_1, \quad s_0 - \alpha - \frac{n}{r_1} = s - \frac{n}{r}, \quad p < n < r_2, \quad -\alpha - \frac{n}{r_2} = \left(-1 + \frac{n}{p}\right) - \frac{n}{p},$$

we have by Proposition 2.1(i) that

$$\dot{B}^{s}_{r_0, q} \hookrightarrow \dot{B}^{s_0+\alpha}_{r_1, q}, \quad \dot{B}^{-1+\frac{n}{p}}_{r_0, q} \hookrightarrow \dot{B}^{-\alpha}_{r_2, q},$$

which yields by (2.32) that

$$\|u \otimes v\|_{B^{s_0}_{r_0, q}} \leq C(\|u\|_{B^{s_0}_{r_1, q}} \|v\|_{\dot{B}^{-1+\frac{n}{p}}_{r_0, q}} + \|u\|_{\dot{B}^{-1+\frac{n}{p}}_{r_0, q}} \|v\|_{B^{s_0}_{r_1, q}}). \quad (2.33)$$

From (2.31) and (2.33) we obtain the desired estimate (2.25). This proves Lemma 2.2.

**Lemma 2.3** Let $n \geq 2$, and let $1 \leq p < n$, $1 \leq q < \infty$. Suppose that $1 < \tilde{r} < \infty$ and $\tilde{s} \geq 0$ satisfy

$$q \leq \tilde{r}, \quad \frac{n}{\tilde{r}} - n + 1 < \tilde{s} < \min\left\{\frac{n}{p}, \frac{n}{\tilde{r}}\right\}. \quad (2.34)$$

For $u, v \in B^{-1+\frac{n}{p}}_{p, q} \cap \dot{H}^{\tilde{s}, \tilde{r}}$, we have $K(u \otimes v) \in B^{-1+\frac{n}{p}}_{p, q} \cap \dot{H}^{\tilde{s}, \tilde{r}}$ with the estimate

$$\|K(u \otimes v)\|_{\dot{H}^{\tilde{s}, \tilde{r}}} \leq C(\|u\|_{B^{-1+\frac{n}{p}}_{p, q}} \|v\|_{\dot{H}^{\tilde{s}, \tilde{r}}} + \|u\|_{\dot{H}^{\tilde{s}, \tilde{r}}} \|v\|_{B^{-1+\frac{n}{p}}_{p, q}}), \quad (2.35)$$

where $C = C(n, p, q, \tilde{r}, \tilde{s})$. 

14
Proof. Let us take $1 \leq r_1, r_2 \leq \infty$ in such a way that

$$\max \left\{ \frac{1}{n}, \frac{\hat{s}}{n} \right\} < \frac{1}{r_1} \leq \min \left\{ \frac{1}{p}, 1 - \frac{1}{\overline{r}} + \frac{\hat{s}}{n} \right\}, \quad \frac{1}{r_2} = \frac{1}{\overline{r}} - \frac{\hat{s}}{n}. \hspace{1cm} (2.36)$$

Define $r_0$ and $s_0$ by

$$\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}, \quad s_0 = \frac{n}{r_1}.$$  

Then we have by (2.36) that $1 \leq r_0 \leq \overline{r}$. Since $\hat{H}^{\overline{s}, \overline{r}} = \hat{H}^{\overline{s}, \overline{r}}$, and since $s_0 - n/r_0 = \hat{s} - n/\overline{r}$, $q \leq \overline{r}$, $s_0 - 1 > 0$, it follows from Proposition 2.1(ii) and Proposition 2.2(ii) that

$$\|K(u \otimes v)\|_{\hat{H}^{\overline{s}, \overline{r}}} \leq C\|K(u \otimes v)\|_{\hat{H}^{\overline{s}, \overline{r}}} + C\|u \otimes v\|_{\hat{B}^{0,-1}_{1+\frac{n}{p}}},$$

Moreover, since

$$s_0 - 1 - \frac{n}{r_1} = \left( -1 + \frac{n}{p} \right) - \frac{n}{p}, \quad p \leq r_1,$$

we have by Proposition 2.1(i) and the usual Sobolev imbedding that

$$\hat{B}^{1+\frac{n}{p}}_{1+q} \hookrightarrow \hat{B}^{0,-1}_{1+q}, \quad \hat{H}^{\overline{s}, \overline{r}} \hookrightarrow L^{\overline{r}}_2,$$

which yields by (2.37) that

$$\|K(u \otimes v)\|_{\hat{H}^{\overline{s}, \overline{r}}} \leq C\|u\|_{\hat{B}^{0,-1}_{1+\frac{n}{p}}} \|v\|_{\hat{H}^{\overline{s}, \overline{r}}} + \|u\|_{\hat{H}^{\overline{s}, \overline{r}}} \|v\|_{\hat{B}^{0,-1}_{1+\frac{n}{p}}}.$$  

This completes the proof of Lemma 2.3.

3 Proof of main theorems

3.1 Existence and uniqueness in $\hat{B}^{1+\frac{n}{p}}_{1+q}$; proof of Theorem 1.1

We first prove the existence of the solution to (E). We solve (E) by the successive approximation. For that purpose, let us define the approximating solutions $\{u_j\}$ of (E) by

$$\begin{cases}
  u_0 = P(-\Delta)^{-1}f, \\
  u_{j+1} = K(u_{j} \otimes u_{j}) + u_{0}, \quad j = 0, 1, \cdots
\end{cases} \hspace{1cm} (3.1)$$

Since $f \in \hat{B}^{-3+\frac{n}{p}}_{1+q}$, we see that $u_0 \in \hat{B}^{-1+\frac{n}{p}}_{1+q}$. Assume that $u_j \in \hat{B}^{-1+\frac{n}{p}}_{1+q}$. By Lemma 2.1, we have that $u_{j+1} \in \hat{B}^{-1+\frac{n}{p}}_{1+q}$ with the estimate

$$\|u_{j+1}\|_{\hat{B}^{-1+\frac{n}{p}}_{1+q}} \leq C\|u_j\|_{\hat{B}^{-1+\frac{n}{p}}_{1+q}}^2 + \|u_0\|_{\hat{B}^{-1+\frac{n}{p}}_{1+q}}, \hspace{1cm} (3.2)$$
where \( C = C(n, p, q) \) is independent of \( j \). By induction, it holds that \( u_j \in B_{p,q}^{-1+\frac{n}{p}} \) for all \( j = 0, 1, \cdots \). Taking \( M_j = \| u_j \|_{B_{p,q}^{-1+\frac{n}{p}}} \), we have by (3.2) that

\[
M_{j+1} \leq CM_j^2 + M_0, \quad j = 0, 1, \cdots.
\]

(3.3)

By the standard argument we see from (3.3) that under the condition

\[
M_0 < \frac{1}{4C},
\]

(3.4)

the sequence \( \{M_j\}_{j=0}^\infty \) is subject to the estimate

\[
M_j \leq \alpha = \frac{1 - \sqrt{1 - 4CM_0}}{2C}, \quad j = 0, 1, \cdots.
\]

(3.5)

Take \( w_j = u_{j+1} - u_j \), and we have

\[
w_j = K(u_j \otimes u_j) - K(u_{j-1} \otimes u_{j-1}) = K(u_j \otimes w_{j-1}) + K(w_{j-1} \otimes u_{j-1}).
\]

Letting \( L_j = \| w_j \|_{B_{p,q}^{-1+\frac{n}{p}}} \), we have similarly to (3.3) that

\[
L_j \leq C(M_j + M_{j-1})L_{j-1} \leq 2C\alpha L_{j-1}.
\]

Therefore, it holds that

\[
L_j \leq (2C\alpha)^j L_0, \quad j = 1, 2, \cdots.
\]

By the definition of \( \alpha \) in (3.5), we see that

\[
2C\alpha = 1 - \sqrt{1 - 4CM_0} < 1,
\]

and hence it holds that

\[
\sum_{j=0}^\infty L_j < \infty,
\]

(3.6)

which implies that \( u_j \) converges to some \( u \) in \( B_{p,q}^{-1+\frac{n}{p}} \). Since

\[
M_0 = \| A^{-1} Pf \|_{B_{p,q}^{-1+\frac{n}{p}}} \leq C\| f \|_{B_{p,q}^{-3+\frac{n}{p}}} < C\delta,
\]

(3.7)

with \( C = C(n, p) \), by taking \( \delta = \delta(n, p, q) \) sufficiently small, we see from the above estimate that the condition (3.4) is fulfilled provided \( \| f \|_{B_{p,q}^{-3+\frac{n}{p}}} < \delta \). Now, letting \( j \to \infty \) in (3.1), we see from Lemma 2.1 that the limit \( u \in B_{p,q}^{-1+\frac{n}{p}} \) is a solution of (E).
We next consider the uniqueness. Let $u \in \dot{B}_{p,q}^{1+\frac{n}{p}}$ and $v \in \dot{B}_{p,q}^{1+\frac{n}{p}}$ be the solutions of (E) such that \( \|u\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}} \leq \eta, \|v\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}} \leq \eta \). It follows from Lemma 2.1 that
\[
\|u - v\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}} = \|K(u \otimes (u - v)) + K((u - v) \otimes v)\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}}
\leq C\|u\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}} + \|v\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}}\|v - v\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}}
\leq 2C\eta\|u - v\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}}.
\]
By taking $\eta > 0$ sufficiently small to satisfy $2C\eta < 1$, we obtain $u - v = 0$. This completes the proof of Theorem 1.1.

**Proof of Corollary 1.1.** Since we have a representation $P(-\Delta)^{-1}f(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy$ with the homogeneous kernel function $\Gamma$ with degree $2 - n$, it is easy to see that $u_0 = P(-\Delta)^{-1}f$ is a homogeneous function with degree $-1$. Suppose that $u_j$ is a homogeneous function with degree $-1$. Since
\[
K(u_j \otimes u_j)(x) = \int_{\mathbb{R}^n} K(x-y)u_j(x+y)dy,
\]
with the homogeneous kernel function $K(x)$ with degree $1 - n$, we see that $K(u_j \otimes u_j)$ is also a homogeneous function with degree $-1$. Hence by (3.1) it holds that $u_{j+1}$ is a homogeneous function with degree $-1$. By induction we have that $\{u_j\}_{j=0}^\infty$ is a sequence of homogeneous functions with degree $-1$ in $\dot{B}_{p,\infty}^{1+\frac{n}{p}}$. Since such a homogeneous property of functions is conserved under the convergence in $\dot{B}_{p,\infty}^{1+\frac{n}{p}}$, we conclude that the solution $u$ given in the above proof of Theorem 1.1 is a homogeneous function with degree $-1$ in $\dot{B}_{p,\infty}^{1+\frac{n}{p}}$. This proves the proof of Corollary 1.1.

### 3.2 Additional regularity in $\dot{B}_{r,\tilde{q}}^s$: proof of Theorem 1.2

Let us return the approximating solutions $\{u_j\}_{j=0}^\infty$ defined by (3.1). By (3.2), we have already obtained that $u_j \in \dot{B}_{p,q}^{1+\frac{n}{p}}$ with the estimate $\|u_j\|_{\dot{B}_{p,q}^{1+\frac{n}{p}}} = M_j \leq \alpha$ for all $j = 0, 1, \ldots$. Since $f \in \dot{B}_{r,\tilde{q}}^{-1}$, it is obvious that $u_0 = P(-\Delta)^{-1}f \in \dot{B}_{r,\tilde{q}}^s$. Assume that $u_j \in \dot{B}_{r,\tilde{q}}^s$ with $N_j := \|u_j\|_{\dot{B}_{r,\tilde{q}}^s}$. Then it follows from (3.1) and Lemma 2.2 that
\[
\|u_{j+1}\|_{\dot{B}_{r,\tilde{q}}^s} \leq \|K(u_j \otimes u_j)\|_{\dot{B}_{r,\tilde{q}}^s} + \|u_0\|_{\dot{B}_{r,\tilde{q}}^s}
\leq C'\|u_j\|_{\dot{B}_{r,\tilde{q}}^{1+\frac{n}{p}}} + \|u_0\|_{\dot{B}_{r,\tilde{q}}^s}
\]
with $C' = C'(n,p,q,r,\tilde{q},s)$, which implies that $u_{j+1} \in \dot{B}_{r,\tilde{q}}^s$. By induction, we have that $u_j \in \dot{B}_{r,\tilde{q}}^s$ for all $j = 0, 1, \ldots$. Furthermore, we see from (3.8) that
\[
N_{j+1} \leq C'\alpha N_j + N_0.
\]
Hence under the condition that
\[
\alpha < \frac{1}{C'},
\]

it holds that

\[ N_j \leq \beta \equiv \frac{N_0}{1 - C'\alpha}, \quad j = 0, 1, \ldots. \tag{3.10} \]

Since \( \alpha = \frac{1 - \sqrt{1 - 4C'N_0}}{2C'} \), it is easy to see that the condition (3.9) can be achieved provided \( M_0 \) is sufficiently small. From (3.7), we see that \( M_0 \) can be taken arbitrarily small in accordance with the size of \( \|f\|_{B_{p,q}^{s+\frac{n}{p}}} \). Therefore, there exists a \( \delta' = \delta'(n, p, q, r, s, \tilde{q}) \leq \delta \) such that if

\[ \|f\|_{B_{p,q}^{s+\frac{n}{p}}} \leq \delta', \]

then the condition (3.9) is fulfilled.

Recall \( w_j = u_{j+1} - u_j \). It follows from (3.5), (3.10) and Lemma 2.2 that

\[
\|w_j\|_{\dot{B}_{r,q}^s} = \|K(u_j \otimes w_{j-1}) + K(u_j \otimes u_{j-1})\|_{\dot{B}_{r,q}^s} \\
\leq C' \left( \|u_j\|_{B_{p,q}^{n+\frac{r}{p}}} \|w_{j-1}\|_{\dot{B}_{r,q}^s} + \|u_j\|_{B_{p,q}^{n+\frac{r}{p}}} \|u_{j-1}\|_{\dot{B}_{r,q}^s} \right) \\
+ C' \left( \|u_{j-1}\|_{B_{p,q}^{n+\frac{r}{p}}} \|w_{j-1}\|_{\dot{B}_{r,q}^s} + \|u_{j-1}\|_{B_{p,q}^{n+\frac{r}{p}}} \|u_{j-1}\|_{\dot{B}_{r,q}^s} \right) \\
\leq C' \alpha \|w_{j-1}\|_{\dot{B}_{r,q}^s} + C' \beta L_{j-1}.
\]

Since \( C'\alpha < 1 \), implied by (3.9), from (3.6) and the above estimate we obtain

\[ \sum_{j=0}^{\infty} \|w_j\|_{\dot{B}_{r,q}^s} < \infty, \]

which yields that \( u_j \to u \) in \( \dot{B}_{r,q}^s \). This completes the proof of Theorem 1.2.

### 3.3 Additional regularity in \( \dot{H}^{\tilde{s},\tilde{p}} \); proof of Theorem 1.3

Based on Lemma 2.3, we see that the proof of Theorem 1.3 is quite similar to that of Theorem 1.2. So, it suffices to give the sketch of the proof.

Since \( f \in \dot{H}^{\tilde{s}-2,\tilde{p}} \), we have that \( u_0 = P(-\Delta)^{-1}f \in H^{\tilde{s},\tilde{p}} \). Assume that \( u_j \in H^{\tilde{s},\tilde{p}} \). Then it follows from (3.1), (3.5) and Lemma 2.3 that

\[
\|u_{j+1}\|_{H^{\tilde{s},\tilde{p}}} \leq \|K(u_j \otimes u_j)\|_{H^{\tilde{s},\tilde{p}}} + \|u_0\|_{H^{\tilde{s},\tilde{p}}} \\
\leq C'' \left( \|u_j\|_{B_{p,q}^{\tilde{s}+\frac{\tilde{p}}{p}}} \|u_j\|_{H^{\tilde{s},\tilde{p}}} + \|u_0\|_{H^{\tilde{s},\tilde{p}}} \right) \\
\leq C'' \alpha \|u_j\|_{H^{\tilde{s},\tilde{p}}} + \|u_0\|_{H^{\tilde{s},\tilde{p}}} \tag{3.11}
\]

with \( C'' = C''(n, p, q, \tilde{r}, \tilde{s}) \), which implies that \( u_{j+1} \in H^{\tilde{s},\tilde{p}} \). By induction, we have that \( u_j \in H^{\tilde{s},\tilde{p}} \) for all \( j = 0, 1, \ldots \). Furthermore, we see from (3.11) that under the condition

\[ \alpha < \frac{1}{C''}, \tag{3.12} \]

it holds that

\[ \|u_j\|_{H^{\tilde{s},\tilde{p}}} \leq \frac{\|u_0\|_{H^{\tilde{s},\tilde{p}}}}{1 - C'' \alpha}, \quad j = 1, 2, \ldots. \tag{3.13} \]
Since $\alpha$ defined by (3.5) can be taken arbitrarily small in accordance with the size of $\|f\|_{\dot{B}^{s-\frac{3}{2}}_{p,q}}$, by taking $\delta''$ sufficiently small, we see that under the hypothesis $\|f\|_{\dot{B}^{s-\frac{3}{2}}_{p,q}} \leq \delta''$, the condition (3.12) is achieved. Now, in the same way as in the argument of the last subsection, we conclude from such a bound as in (3.13) that

$$u_j \to u \text{ in } H^{\delta, \tilde{\nu}}.$$

This proves Theorem 1.3.

**Appendix.** Proof of Proposition 1.1. Since the projection $P$ is bounded from $\dot{B}_c^{s_0}$ onto $\dot{B}_p^{s_0}$, it suffices to show that $K' \equiv (-\Delta)^{-1} \nabla$ with the expression

$$K'_g(x) = \frac{1}{(2\pi)^\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \frac{1}{|\xi|^\frac{n}{2}} \sum_{l=1}^n i\xi_l F_{g_l} (\xi) d\xi, \quad k = 1, \ldots, n$$

is a bounded operator from $\dot{B}_p^{s_0}$ to $\dot{B}_p^{s_0}$ with such an estimate as (1.1).

Let us first consider the case $1 \leq q < \infty$. We define $1 \leq r \leq \infty$ by $1/r = 1 - (1/p - 1/p_0)$. By the Hausdorff-Young inequality, we have that

$$\|K'_g\|_{\dot{B}_p^{s_0}} = \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\varphi_j \ast K'_g\|_{L^p} \right)^q \right\}^{\frac{1}{q}} \leq \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\varphi_j \ast K'_g\|_{L^p} \right)^q \right\}^{\frac{1}{q}},$$

where $\varphi_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. It is easy to see that

$$K'_j(x) = 2^{-j} 2^{j^2} \Psi(2^j x) \quad \text{with} \quad \Psi \equiv \sum_{l=1}^n F^{-1} \left( \frac{i\xi_l}{|\xi|^2} \sum_{k=-1}^1 \phi(2^{-k}\xi) \right),$$

which yields that

$$\|K'_j\|_{L^r} \leq 2^{-j} 2^{n(1-\frac{1}{r})j} \|\Psi\|_{L^r} \leq C 2^{-j+n(\frac{1}{2}-\frac{1}{p_0})j} = C 2^{s-\delta_0} j,$$

where $C = C(n, p, p_0)$ is independent of $j \in \mathbb{Z}$. Notice that $\Psi \in S$ because $\text{supp} \sum_{k=-1}^1 \phi(2^{-k}\xi) \subset \{\xi \in \mathbb{R}^n; 2^{-2} \leq |\xi| \leq 2^2\}$. Hence it follows from (3.14) that

$$\|K'_g\|_{\dot{B}_p^{s_0}} \leq C \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\varphi_j \ast g\|_{L^p} \right)^q \right\}^{\frac{1}{q}} = C \|g\|_{\dot{B}_p^{s_0}},$$

19
where \( C = C(n, p, p_0, q, s, s_0) \). In case \( q = \infty \), the proof is quite similar to the above, so we may omit it. This proves Proposition 1.1.

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