MONOTONICITY OF NON-PLURIPOLAR MONGE-AMPERE MASSES

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ABSTRACT. We prove that on a compact Kähler manifold, the non-pluripolar Monge-Ampère mass of a θ-psh function decreases as the singularities increase. This was conjectured by Boucksom-Eyssidieux-Guedj-Zeriahi who proved it under the additional assumption of the functions having small unbounded locus. As a corollary we get a comparison principle for θ-psh functions, analogous to the comparison principle for psh functions due to Bedford-Taylor.

1. INTRODUCTION

Let \((X,\omega)\) be a compact Kähler manifold and \(\theta\) be a smooth real \((1,1)\)-form on \(X\). Recall that a function \(\phi: X \to [-\infty, \infty)\) is said to be \(\theta\)-psh if it is upper semicontinuous, \(L^1_{\text{loc}}\) and \(\theta + dd^c \phi \geq 0\) in the sense of currents. Note that by the \(dd^c\)-lemma any closed positive \((1,1)\)-current \(T\) in the class \([\theta]\) can be written as \(T = \theta + dd^c \phi\) where \(\phi\) is \(\theta\)-psh.

When \(\phi\) is smooth one defines the Monge-Ampère measure \(MA_\theta(\phi) := (\theta + dd^c \phi)^n\), which will be a semipositive \((n,n)\)-form, and clearly
\[
\int_X MA_\theta(\phi) = \int_X \theta^n.
\]

By the fundamental work of Bedford-Taylor [BT82, BT87] one can still define a Monge-Ampère measure \(MA_\theta(\phi)\), which will be a finite positive measure, only assuming \(\phi\) to be locally bounded. When \(\phi\) is \(\theta\)-psh on \(X\) and bounded away from a closed complete pluripolar set \(A \subset X\), the Monge-Ampère measure \((\theta + dd^c \phi)^n\) can be defined in \(X \setminus A\) in the sense of Bedford-Taylor [BT82]. One extends this measure trivially on \(X\), but a priori this could fail to be locally finite. However, it was proved by Boucksom-Eyssidieux-Guedj-Zeriahi in [BEGZ10] that the extended measure is locally finite (for \(\theta\) Kähler this was first shown by Guedj-Zeriahi [GZ07]); in fact, the total mass is bounded by some constant only depending on the cohomology class \([\theta]\) (e.g. the volume \(\text{vol}([\theta])\)). For a general \(\theta\)-psh function \(\phi\) one considers \(\phi_t := \max(\phi, V_{\theta} - t), t > 0\), where \(V_{\theta}\) is a \(\theta\)-psh function with minimal singularities, which is bounded away from a closed complete pluripolar set by a result of Demailly [Dem92]). By the analysis above one can define the Monge-Ampère operator \(MA_\theta(\phi_t)\) as a positive Radon measure on \(X\). Since the Monge-Ampère operator is local in the plurifine topology [BT87] one gets that the measures \(\mathbb{1}_{\{\phi > V_{\theta} - t\}} MA_\theta(\phi_t)\) are nondecreasing in \(t\). The nonpluripolar Monge-Ampère measure of \(\phi\), denoted by \(MA_\theta(\phi)\), is defined as the strong limit of \(\mathbb{1}_{\{\phi > V_{\theta} - t\}} MA_\theta(\phi_t)\) as \(t \to +\infty\). By construction one sees that \(MA_\theta(\phi)\) has finite total mass. But in contrast to the case when \(\phi\) is smooth (or locally bounded) the mass \(\int_X MA_\theta(\phi)\) does not only depend on \([\theta]\). To describe the dependence of the Monge-Ampère mass on \(\phi\) we recall the following definition.

**Definition 1.1.** If \(\phi\) and \(\psi\) are two \(\theta\)-psh functions we say that \(\phi\) is less singular than \(\psi\) if \(\phi \geq \psi + O(1)\).
Also recall that $\phi$ is said to have small unbounded locus if it is locally bounded away from a closed complete pluripolar subset of $X$. In [BEGZ10, Thm 1.16] Boucksom-Eyssidieux-Guedj-Zeriahi proved that if $\phi$ is less singular than $\psi$ and both $\phi$ and $\psi$ have small unbounded locus then

$$\int_X MA_\theta(\phi) \geq \int_X MA_\theta(\psi).$$

They conjectured this to be true even without the assumption of small unbounded locus. In this paper we prove their conjecture, i.e.:

**Theorem 1.2.** Let $\phi$ and $\psi$ be two $\theta$-psh functions. If $\phi$ is less singular than $\psi$ then

$$\int_X MA_\theta(\phi) \geq \int_X MA_\theta(\psi).$$

(1)

**Remark 1.3.** In fact, [BEGZ10, Thm 1.16] contains a more general statement about the pseudoeffectivity of certain differences of cohomology classes of non-pluripolar products (see Theorem 2.2).

Let us recall the fundamental comparison principle for psh functions due to Bedford-Taylor [BT82, Thm. 4.1]:

**Theorem 1.4.** Let $\Omega$ be a bounded open set in $\mathbb{C}^n$ and $u$ and $v$ two locally bounded psh functions on $U$ such that $\liminf z \to \partial \Omega u(z) - v(z) \geq 0$ (i.e. $u \geq v$ on $\partial \Omega$). Then

$$\int_{\{u < v\}} MA(v) \leq \int_{\{u < v\}} MA(u).$$

As was observed in [BEGZ10], Theorem 1.2 implies an analogous comparison principle for $\theta$-psh functions:

**Corollary 1.5.** Let $\phi$ and $\psi$ be $\theta$-psh and assume $\phi$ is less singular than $\psi$. Then

$$\int_{\{\phi < \psi\}} MA_\theta(\psi) \leq \int_{\{\phi < \psi\}} MA_\theta(\phi).$$

In order to prove Theorem 1.2 we will, given $\theta$, $\phi$ and $\psi$, construct a related form $\tilde{\theta}$ together with $\tilde{\theta}$-psh functions $\Phi$ and $\Psi$ on $X \times \mathbb{P}^N$ for large $N$. By construction $\Phi$ will be less singular than $\Psi$ and they will also have small unbounded locus so we know from [BEGZ10, Thm 1.16] that

$$\int_{X \times \mathbb{P}^N} MA_{\tilde{\theta}}(\Phi) \geq \int_{X \times \mathbb{P}^N} MA_{\tilde{\theta}}(\Psi).$$

(2)

We will then establish a formula for the Monge-Ampère masses of $\Phi$ and $\Psi$ involving the Monge-Ampère masses of $\phi$ and $\psi$ (Prop. 3.1), so that invoking (2) for larger and larger $N$ yields the desired inequality (1).

1.1. Related work. Boucksom-Eyssidieux-Guedj-Zeriahi also proved a comparison principle [BEGZ10, Cor. 2.3]:

**Theorem 1.6.** For any two $\theta$-psh functions $\phi$ and $\psi$ we have

$$\int_{\{\phi < \psi\}} MA_\theta(\psi) \leq \int_{\{\phi < \psi\}} MA_\theta(\phi) + \text{vol}([\theta]) - \int_X MA_\theta(\phi).$$

If $\phi$ has minimal singularities then $\int_X MA_\theta(\phi) = \text{vol}([\theta])$ and thus

$$\int_{\{\phi < \psi\}} MA_\theta(\psi) \leq \int_{\{\phi < \psi\}} MA_\theta(\phi).$$
In [BEGZ10, Rem. 2.4] Boucksom-Eyssidieux-Guedj-Zeriahi noted that when \( \phi \) is less singular than \( \psi \) and both have small unbounded locus, Theorem 2.2 can be used to show that

\[
\int_{\{ \phi < \psi \}} MA_{\theta}(\psi) \leq \int_{\{ \phi < \psi \}} MA_{\theta}(\phi).
\]

In [Dar13] Darvas proved that if \( \theta \) is Kähler then a \( \theta \)-psh function \( \psi \) has full Monge-Ampère mass (i.e. \( \int_X MA_{\theta}(\psi) = \int_X \theta^n \)) iff whenever \( \phi \) is \( \theta \)-psh and locally bounded we have that \( P_{[\psi]}(\phi) = \phi \). Here \( P_{[\psi]}(\phi) \) is a certain kind of envelope introduced in [RWN14], defined as the usc regularization of the supremum of all \( \theta \)-psh functions \( \phi' \) such that \( \phi' \leq \phi \) and \( \phi' \leq \psi + O(1) \). Recently Darvas-Di Nezza-Lu generalized this result to the case when \([\theta]\) is just big.

These results have had important applications to the study of the geometry of the space of full mass currents, but are not in an obvious way connected to the monotonicity on the Monge-Ampère masses. However, interestingly their proof uses so called geodesic rays constructed from the \( \theta \)-psh function \( \psi \). A geodesic ray can be thought of as a \( \pi_X^* \theta \)-psh function on \( X \times \mathbb{D} \), where \( \mathbb{D} \) is the unit disc, and one could also extend it to a \( \tilde{\theta} := \pi_X^* \theta + \pi_p^* \omega_{FS} \)-psh function on \( X \times \mathbb{P}^1 \). This is similar to our construction of a \( \theta \)-psh functions \( \Phi \) on \( X \times \mathbb{P}^N \) when \( N = 1 \). Note though that in contrast to our proof the methods in [Dar13, DDNL16] do not rely on the calculation of the Monge-Ampère mass of the geodesic ray, as this is automatically zero. Nevertheless our construction was initially inspired by these papers [Dar13, DDNL16] and it would be very interesting to know if possibly there are more links between the results.

Very recently, after a preprint of this paper was made available on the arXiv, Darvas-Di Nezza-Lu posted the preprint [DDNL17] which, crucially using our Theorem 1.2, proves a more general statement of the conjecture in [BEGZ10, Thm 1.16] allowing mixed Monge-Ampère masses (see [DDNL17]). Interestingly, this monotonicity result is then used in [DDNL17] to solve Monge-Ampère equations in classes of currents with prescribed singularities, thus increasing the scope of the variational method championed by Berman-Boucksom-Guedj-Zeriahi [BBGZ13].

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2. Preliminaries

Let \((X, \omega)\) be a compact Kähler manifold and \( \theta \) be a smooth real \((1, 1)\)-form on \( X \). Recall from the introduction that a function \( \phi : X \to (-\infty, \infty) \) is said to be \( \theta \)-psh if it is upper semicontinuous, \( L^1_{loc} \), and \( \theta + dd^c \phi \geq 0 \) in the sense of currents, and that by the \( dd^c \)-lemma any closed positive \((1, 1)\)-current \( T \) in the class \([\theta]\) can be written as \( T = \theta + dd^c \phi \) where \( \phi \) is \( \theta \)-psh.

The set of \( \theta \)-psh functions is denoted by \( PSH(X, \theta) \). The cohomology class \([\theta]\) is called pseudoeffective if it contains a closed positive current, i.e. if \( PSH(X, \theta) \) is nonempty, while \([\theta]\) is said to be big if for some \( \epsilon > 0 \), \([\theta - \epsilon \omega]\) is pseudoeffective.

A \( \theta \)-psh function is said to have analytic singularities if locally it can be written as \( c \ln(\sum |g_i|^2) + f \) where \( c > 0 \), \( g_i \) is a finite collection of local holomorphic functions and \( f \) is smooth. By a deep regularization result of Demailly [Dem92], if \([\theta]\) is big then there exists a \( \theta \)-psh function with analytic singularities. Since any proper analytic subset is closed and complete pluripolar we note that if \( \phi \) is less singular than some function with analytic singularities then \( \phi \) has small unbounded locus.
A \( \theta \)-psh function is said to have minimal singularities if it is less singular than all other \( \theta \)-psh functions.

We now come to the notion of non-pluripolar positive products of closed positive currents. This theory was first developed in the local setting by Bedford-Taylor [BT82, BT87] and later in the geometric setting of compact Kähler manifolds by Boucksom-Eyssidieux-Guedj-Zeriahi [BEGZ10].

Let \( T_i, i = \{1, \ldots, p\} \) be closed positive \((1, 1)\)-currents on \( X \), and for each \( i \) let \( \theta_i \) be a smooth form in the class of \( T_i \). We assume that each \([\theta_i]\) is big. For each \( i \) we have \( \theta_i \)-psh function \( \phi_i \) such that \( \theta_i + dd^c \phi_i = T_i \). When each \( \phi_i \) is bounded Bedford-Taylor proved in [BT82] that one can define the product \( \langle T_1 \wedge \ldots \wedge T_p \rangle \) recursively by

\[
\langle T_1 \wedge \ldots \wedge T_k \rangle := \langle T_1 \wedge \ldots \wedge T_{k-1} \rangle \wedge \theta_k + dd^c (\phi_k \langle T_1 \wedge \ldots \wedge T_{k-1} \rangle),
\]

and that will be a closed positive \((p, p)\)-current.

This product is symmetric and multilinear [BEGZ10, Prop. 1.4].

Recall from [BT87, BEGZ10] that the plurifine topology is the one generalized by sets of the form \( U \cap \{ v > 0 \} \) where \( U \) is open and \( v \) is some psh function on \( U \). The product \( \langle T_1 \wedge \ldots \wedge T_p \rangle \) is local in the plurifine topology [BEGZ10, Prop. 1.4(a)], i.e. if \( \phi_i = \psi_i \) on a plurifine open set \( O \) then

\[
1_O \langle \theta_1 + dd^c \phi_1 \wedge \ldots \wedge \theta_p + dd^c \phi_p \rangle = 1_O \langle \theta_1 + dd^c \psi_1 \wedge \ldots \wedge \theta_p + dd^c \psi_p \rangle.
\]

For general \( \theta_i \)-psh functions \( \phi_i \) (i.e. not necessarily bounded) one does the following. For each \( i \) we let \( V_{\theta_i} \) be a \( \theta_i \)-psh function with minimal singularities. We see from the regularity result of Demailly [Dem92] that each \( V_{\theta_i} \) has small unbounded locus. For \( t > 0 \) one lets

\[
R_t := \langle \theta_1 + dd^c \max(\phi_1, V_{\theta_1} - t) \wedge \ldots \wedge \theta_p + dd^c \max(\phi_p, V_{\theta_p} - t) \rangle.
\]

Now there is some closed complete pluripolar subset \( A \subset X \) away from which each \( V_{\theta_i} \), and hence also \( \max(\phi_i, V_{\theta_i} - t) \) is bounded. Thus \( R_t \) is a welldefined closed positive \((p, p)\)-current on \( X \setminus A \), and by [BEGZ10, Prop. 1.6] the trivial extension of \( R_t \) to \( X \) is welldefined and locally finite. Indeed if \( \omega \) is a Kähler form then \( \int_X \omega_{n-p} \wedge R_t \) is bounded by some constant only depending on the cohomology classes of \( \theta_i \) and \( \omega \).

By plurifine locality the currents \( 1_{U \cap \{ \phi_i > V_{\theta_i} - t \}} R_t \) are nondecreasing in \( t \), and the non-pluripolar positive product \( \langle \theta_1 + dd^c \phi_1 \wedge \ldots \wedge \theta_p + dd^c \phi_p \rangle \) is then defined as the strong limit of these currents as \( t \to +\infty \). By the estimates the limit is a welldefined locally finite positive \((p, p)\)-current. Also by [BEGZ10, Thm. 1.8] it is closed.

This nonpluripolar positive product is again symmetric and multilinear [BEGZ10, Prop. 1.4(c)]). Importantly non-pluripolar products never puts mass on pluripolar sets.

When \( p = n = \dim_{\mathbb{C}} X \), \( \langle T_1 \wedge \ldots \wedge T_n \rangle \) is a positive measure, and when the \( n \) currents are all equal \( T_i = \theta + dd^c \psi \), then \( \langle (\theta + dd^c \psi)^n \rangle \) is known as the (non-pluripolar) Monge-Ampère measure of \( \psi \), which we also denote by \( MA_\theta(\psi) \). From the symmetry and multilinearity of the non-pluripolar product we get that if \( \phi \) and \( \psi \) are \( \theta \)-psh the measure \( MA_\theta((1-t)\phi + t\psi) \) depends continuously on \( t \in [0, 1] \).

A basic fact is that if on some open set \( U \) \( \psi \) is \( C^{1,1} \) (or more generally if \( dd^c \psi \) has coefficients in \( L^\infty \)) then

\[
1_U MA_\theta(\psi) = 1_U (\theta + dd^c u)^n.
\]

Here the right hand side simply denotes the measure one gets by taking the appropriate determinant of the coefficient functions.
The following convergence result for Monge-Ampère measures by Bedford-Taylor [BT82, Thm. 2.1] is absolutely fundamental.

**Theorem 2.1.** Let $U$ be an open set and $u_k$ be a decreasing sequence of $\theta$-psh functions such that $u := \lim_{k \to \infty} u_k$ is locally bounded on $U$ ($u$ will then automatically be $\theta$-psh on $U$). Then on $U$ the measures $MA_\theta(u_k)$ converge weakly to $MA_\theta(u)$.

Let us state the full version of monotonicity result of Boucksom-Eyssidieux-Guedj-Zeriahi from [BEGZ10, Thm. 1.16] mentioned in the Introduction.

**Theorem 2.2.** Assume we have two $p$-tuples of currents $T_i = \theta_i + dd^c \phi_i$ and $T'_i = \theta_i + dd^c \psi_i$ such that for each $i$ $\phi_i$ is less singular than $\psi_i$, and furthermore each $\phi_i$ and $\psi_i$ has small unbounded locus. Then the cohomology class of $(T_1 \wedge ... \wedge T_p) - (T'_1 \wedge ... \wedge T'_p)$ is pseudoeffective, i.e. contains a closed positive $(p, p)$-current. In the particular case that $p = n$, $\theta_1 = \theta$, $\phi_1 = \phi$ and $\psi_1 = \psi$ this means precisely that

$$\int_X MA_\theta(\phi) \geq \int_X MA(\psi).$$

### 3. A Construction on $X \times \mathbb{P}^N$

Assume $\theta$ is a smooth closed real $(1, 1)$-form such that $[\theta]$ is big and fix $\phi \in PSH(X, \theta)$. Pick $N \in \mathbb{N}$ and let $\omega_{FS}$ denote the Fubini-Study form on $\mathbb{P}^N$. If $\pi_X$ and $\pi_{\mathbb{P}^N}$ are the projections from $X \times \mathbb{P}^N$ to $X$ and $\mathbb{P}^N$ respectively we let $\tilde{\theta} := \pi_X^* \theta + \pi_{\mathbb{P}^N}^* \omega_{FS}$. We clearly have that $[\tilde{\theta}]$ also is big.

Let $Z_i$ be the homogeneous coordinates on $\mathbb{P}^N$ and denote

$$\ln |Z_i|^2_{FS} := \ln \left( \frac{|Z_i|^2}{\sum_{j=0}^N |Z_j|^2} \right).$$

Then $\ln |Z_i|^2_{FS}$ is $\omega_{FS}$-psh with $\omega_{FS} + dd^c \ln |Z_i|^2_{FS}$ being the current of integration along the hyperplane $\{Z_i = 0\}$.

Let $\Sigma_N$ denote the $N$-dimensional unit simplex $\Sigma_N := \{x \in \mathbb{R}^N : x_i \geq 0, \sum x_i \leq 1\}$. To ease notation we will write $|x| := \sum x_i$.

Since $[\tilde{\theta}]$ is big we can pick a $\phi_0 \in PSH(X, \theta)$ with analytic singularities [Dem92] and we now define

$$\Phi := \sup_{x \in \Sigma_N} \{(1 - |x|)(\phi_0 + \ln |Z_0|^2_{FS}) + |x| \phi + \sum_{i=1}^N x_i \ln |Z_i|^2_{FS} - \sum_{i=1}^N x_i^2 \}.$$  

Here $*$ means that we take the usc regularization of the supremum. We see that $\Phi$ is a $\tilde{\theta}$-psh function and since $\phi_0 + \ln |Z_0|^2_{FS}$ has analytic singularities and $\Phi \geq \phi_0 + \ln |Z_0|^2_{FS}$ it follows that $\Phi$ has small unbounded locus.

Key to the proof of Theorem 1.2 will be the following proposition.

**Proposition 3.1.** We have that

$$\int_{X \times \mathbb{P}^N} MA_{\tilde{\theta}}(\Phi) = N \int_0^1 \left( \int_X MA_\theta((1 - t)\phi_0 + t\phi) \right) t^{N-1} dt.$$

To prove Proposition 3.1 we will use the following lemma.
Lemma 3.2. Let $U$ be an open set in $\mathbb{C}^n$, $u$ and $v$ two psh functions on $U$, and we assume that $u$ is smooth while $v$ is locally bounded. Let $\Phi$ be the psh function on $U \times \mathbb{C}^N$ defined as

$$
\Phi := \sup_{x \in \Sigma_N} \left\{ (1 - |x|)u + |x|v + \sum_{i=1}^N x_i \ln |z_i|^2 - \sum_{i=1}^N x_i^2 \right\}.
$$

Then we have that

$$(\pi_U)_*MA(\Phi) = N \int_0^1 MA((1 - |x|)u + |x|v)t^{N-1} \, dt.$$ 

Proof. First we assume that $v$ is smooth.

For $(p, z) \in U \times (\mathbb{C}^*)^N$ we write

$$y_i(p) := \frac{1}{2}(\ln |z_i|^2 + v(p) - u(p))$$

and

$$y(p) := (y_1(p), \ldots, y_N(p)).$$

Then we note that

$$(1 - |x|)u(p) + |x|v(p) + \sum_{i=1}^N x_i \ln |z_i|^2 - \sum_{i=1}^N x_i^2 = u(p) + x \cdot (2y(p) - x)$$

and it follows easily that on $U \times (\mathbb{C}^*)^N :$

$$\Phi(p, z) = u(p) + ||y(p)||^2 - dist^2(y(p), \Sigma_N).$$

Since $y(p)$ depends smoothly on $p$ this shows that $\Phi$ is locally $C^{1,1}$ on $U \times (\mathbb{C}^*)^N$, and hence $MA(\Phi)$ is absolutely continuous with respect to the Lebesgue measure.

Note that the function $f_p(y) := u(p) + ||y||^2 - dist^2(y, \Sigma_N)$ is convex and locally $C^{1,1}$ and that $\nabla f_p(y)$ is the point in $\Sigma_N$ closest to $y$. In particular $f_p$ is not strictly convex on the complement of $\Sigma_N$.

Let $V := \{(p, z) \in U \times (\mathbb{C}^*)^N : y(p) \in \Sigma_N\}$. Note that $\partial V = \{(p, z) \in U \times (\mathbb{C}^*)^N : y(p) \in \partial \Sigma_N\}$, hence $\partial V$ has zero measure with respect to Lebesgue and therefore also $MA(\Phi)$. We also have that $MA(\Phi)$ is zero on $(V^c)^o$ since as we saw $f_p$ fails to be strictly convex there.

It thus follows that

$$MA(\Phi) = 1_V MA(\Phi).$$

On $V$ we have that

$$\Phi(p, z) = u(p) + ||y(p)||^2$$

and thus on $V$

$$dd^c \Phi(p, z) = dd^c u(p) + 2 \sum_i y_i(p) dd^c y_i(p) + \sum_i dy_i(p) \wedge d^c y_i(p) = (1 - |y(p)|)dd^c u(p) + |y(p)| dd^c v(p) + \sum_i dy_i(p) \wedge d^c y_i(p).$$

Here $|y(p)|$ denotes $\sum_i y_i(p)$, and we used that $2dd^c y_i(p) = dd^c v - dd^c u$. Since $\left( \sum_i dy_i(p) \wedge d^c y_i(p) \right)^k = 0$ for any $k > N$ it follows that

$$MA(\Phi) = 1_V ((1 - |y(p)|)dd^c u + |y(p)|dd^c v)^N \wedge \left( \sum_i dy_i(p) \wedge d^c y_i(p) \right)^N.$$
Let \( V_p := V \cap \{(p) \times (\mathbb{C}^*)^N\} \). Since \( dd^c u \) and \( dd^c v \) do not contain any terms with \( dz_i \) or \( d\bar{z}_i \), only the derivatives of \( y_i(p) \) in the \( V_p \)-directions will enter into the expression of \( MA(\Phi) \). It follows that

\[
MA(\Phi) = \mathbb{I}_V ((1 - |y(p)|)dd^c u + |y(p)|dd^c v)^n \wedge (\sum_i d\ln |z_i| \wedge d\bar{c} \ln |z_i|)^N. \tag{7}
\]

Let \( \mu(p) \) denote the map \( z \mapsto y(p) \). We let \( \nu := (\sum_i d\ln |z_i| \wedge d\bar{c} \ln |z_i|)^N \) a simple calculation shows that

\[
\mu(p)_* \nu = N! dx,
\]

where \( dx \) denotes the Lebesgue measure on \( \mathbb{R}^N \).

Now let \( \mu \) denote the map from \( V \) to \( U \times \Sigma N \) given by \( \mu(p, z) := (p, y(p)) \). We then get from (7) and (8) that

\[
\mu_*(MA(\Phi)) = MA_U((1 - |x|)u + |x|v)^N! dx|_{\Sigma N}. \tag{9}
\]

If we let \( \pi' \) denote the projection from \( U \times \Sigma N \) to \( U \) we can write \( \pi_U = \pi' \circ \mu \) and hence by (9)

\[
(\pi_U)_* MA(\Phi) = \pi'_* MA((1 - |x|)u + |x|v)^N! dx|_{\Sigma N} =
\]

\[
= N! \int_{\Sigma N} MA((1 - |x|)u + |x|v) dx. \tag{10}
\]

By homogeneity the volume of \( \Sigma N \cap \{|x| \leq t\} \) is \( t^N / N! \) and it follows that

\[
N! \int_{\Sigma N} MA((1 - |x|)u + |x|v) dx = N! \int_{t=0}^1 MA((1 - t)u + tv)t^{N-1} dt,
\]

with (10) proving the Lemma in the case when \( v \) is smooth.

Now let \( v \) be just locally bounded. Without loss of generality we can assume that \( v \) in fact bounded, and that say \( v > u - C \) for some constant \( C \).

Let \( v_j \) be a sequence of smooth psh functions on \( U \) decreasing to \( v \) and write

\[
\Phi_j := \sup_{x \in \Sigma N} (1 - |x|)u + |x|v_j + \sum_{i=1}^N x_j \ln |z_j|^2 - \sum_{i=1}^N x_i^2.\]

By what we have established

\[
(\pi_U)_* MA(\Phi_j) = N \int_{t=0}^1 MA((1 - t)u + tv_j)t^{N-1} dt. \tag{11}
\]

We have that \( \Phi_j \) decreases to \( \Phi \) and so by Theorem 2.1 \( MA(\Phi_j) \) will converge weakly to \( MA(\Phi) \). Also note that \( v_j \geq v > u - C \). Recall that \( MA(\Phi_j) \) is supported on the set \( \tilde{V}_j \). Note that on \( \tilde{V}_j \) for each \( i \) we have that

\[
\frac{1}{2}(\ln |z_i| + v_j - u) \leq 1
\]

and so by assumption

\[
\ln |z_i|^2 \leq 2 - v_j + u \leq 2 + C
\]

on \( \tilde{V}_j \), or in other words

\[
\tilde{V}_j \subseteq U \times e^{C/2+1} \mathbb{D}^N,
\]

where \( e^{C/2+1} \mathbb{D}^N \) denotes the polydisc \( \{z : \forall i, |z_i| < e^{C/2+1}\} \). We thus get that \( MA(\Phi_j) \) is supported on \( U \times e^{C/2+1} \mathbb{D}^N \). Since \( e^{C/2+1} \mathbb{D}^N \) is compact in \( \mathbb{C}^N \) the weak convergence of \( MA(\Phi_j) \) on \( U \times \mathbb{C}^N \) implies that \( (\pi_U)_* MA(\Phi_j) \) converge weakly to \( (\pi_U)_* MA(\Phi) \).
Since again by Theorem 2.1 each $MA((1-t)u+tv_j)$ converge weakly to $MA((1-t)u+tv)$, this proves the Lemma.

We now prove Proposition 3.1.

**Proof.** We wish to show that

$$\left(\pi_X\right)_*MA(\Phi + h) = N \int_{t=0}^1 MA((1-t)u+tv) t^{N-1} dt. \tag{12}$$

The Monge-Ampère measure $MA(\Phi)$ does not charge the analytic set $\{\phi_0 = -\infty\}$ so let us pick a coordinate chart $U \subseteq X \setminus \{\phi_0 = -\infty\}$ where $\theta = dd^c h$ for some smooth function $h$. Then $u := \phi_0 + h$ is a smooth psh function on $U$ while $v := \phi + h$ is simply psh on $U$. We thus need to show that

$$\left(\pi_U\right)_*MA(\Phi + h) = N \int_{t=0}^1 MA((1-t)u+tv) t^{N-1} dt \tag{13}$$

where

$$\Phi + h = \sup_{x \in \Sigma_N} \{(1-|x|)u + |x|v + \sum_{i=1}^N x_i \ln |z_i|^2 - \sum_{i=1}^N x_i^2\}.$$

Pick a constant $C$ and let $\phi_C := \max(\phi, u - C)$, $v_C := \phi_C + h$ and

$$\Phi_C + h = \sup_{x \in \Sigma_N} \{(1-|x|)u + |x|v_C + \sum_{i=1}^N x_i \ln |z_i|^2 - \sum_{i=1}^N x_i^2\}.$$

Since $u$ is smooth and $v_C$ locally bounded on $U$ Lemma 3.2 says that

$$\left(\pi_U\right)_*MA(\Phi_C + h) = N \int_{t=0}^1 MA((1-t)u+tv_C) t^{N-1} dt.$$

By the fact that the Monge-Ampère measures are local in the plurifine topology we thus get that

$$\mathbbm{1}_{\{(\phi < u-C)\}}(\pi_U)_*MA(\Phi + h) = \mathbbm{1}_{\{(\phi < u-C)\}} N \int_{t=0}^1 MA((1-t)u+tv) t^{N-1} dt.$$

Observing that neither $(\pi_U)_*(MA(\Phi + h))$ nor $N \int_{t=0}^1 MA((1-t)u+tv) t^{N-1} dt$ puts any mass on the pluripolar set $\{v = -\infty\}$ we get (13) by letting $C \to \infty$. This establishes (12) while integrating it over $X$ finally yields the Proposition.

**4. PROOFS OF MAIN RESULTS**

We can now prove Theorem 1.2.

**Proof.** Let $\phi$ and $\psi$ be as in the statement of Theorem 1.2. By [BEGZ10, Prop. 1.22], if $[\theta]$ is not big then both Monge-Ampère masses are zero, thus we can assume that $[\theta]$ is big.

Pick a large $N$ and let $\Phi$ and $\Psi$ be defined as above. From the construction it is clear that $\Phi$ is less singular than $\Psi$. Since they also have small unbounded locus we know from [BEGZ10, Thm. 1.16] that

$$\int_{X \times \mathbb{P}^N} MA(\Phi) \geq \int_{X \times \mathbb{P}^N} MA(\Psi).$$
Combined with Proposition 3.1 we get that
\[
N \int_{t=0}^{1} \left( \int_X MA_\theta((1 - t)\phi_0 + t\phi) \right) t^{N-1} dt \geq \nonumber
\]
\[
\geq N \int_{t=0}^{1} \left( \int_X MA_\theta((1 - t)\phi_0 + t\psi) \right) t^{N-1} dt. \tag{14}
\]
Recall from Section 2 that the function
\[
g(t) := \int_X MA_\theta((1 - t)u + t\phi)
\]
is continuous in \( t \in [0, 1] \). It follows that
\[
\lim_{N \to \infty} N \int_{t=0}^{1} \left( \int_X MA_\theta((1 - t)\phi_0 + t\phi) \right) t^{N-1} dt = \int_X MA_\theta(\phi), \tag{15}
\]
and similarly for \( \psi \). The theorem follows from combining (14) and (15). \( \square \)

Let us recall and prove the corollary stated in the Introduction.

**Corollary 4.1.** Let \( \phi \) and \( \psi \) be \( \theta \)-psh and assume \( \phi \) is less singular than \( \psi \). Then
\[
\int_{\{\phi < \psi \}} MA_\theta(\psi) \leq \int_{\{\phi < \psi \}} MA_\theta(\phi).
\]

**Proof.** Here we precisely follow [BEGZ10, Cor. 2.3, Rmk. 2.4].

Let \( \epsilon > 0 \) and \( \phi_\epsilon := \max(\phi, \psi - \epsilon) \). By Theorem 1.2 we have that \( \int_X MA_\theta(\phi_\epsilon) = \int_X MA_\theta(\phi) \) and so using the plurifine locality of Monge-Ampère measures we get
\[
\int_X MA_\theta(\phi) = \int_X MA_\theta(\phi_\epsilon) \geq \int_{\{\phi < \psi - \epsilon \}} MA_\theta(\phi_\epsilon) + \int_{\{\phi > \psi - \epsilon \}} MA_\theta(\phi_\epsilon) = \nonumber
\]
\[
= \int_{\{\phi < \psi - \epsilon \}} MA_\theta(\psi) + \int_{\{\phi > \psi - \epsilon \}} MA_\theta(\phi) = \nonumber
\]
\[
= \int_{\{\phi < \psi - \epsilon \}} MA_\theta(\psi) + \int_X MA_\theta(\phi) - \int_{\{\phi \leq \psi - \epsilon \}} MA_\theta(\phi), \nonumber
\]
and hence
\[
\int_{\{\phi < \psi - \epsilon \}} MA_\theta(\psi) \leq \int_{\{\phi \leq \psi - \epsilon \}} MA_\theta(\phi) \leq \int_{\{\phi < \psi \}} MA_\theta(\phi). \nonumber
\]
The result now follows from letting \( \epsilon \) tend to zero. \( \square \)

We will also mention a second corollary, namely the following domination principle:

**Corollary 4.2.** Let \( \phi, \psi \) and \( \rho \) be \( \theta \)-psh, and assume that \( \phi \) is less singular than \( \psi \) and \( \rho \). Then if \( \phi \geq \psi \) a.e. with respect to \( MA(\phi) \) it follows that \( \phi \geq \psi \) a.e. also with respect to \( MA(\rho) \).

**Proof.** Here we precisely follow [BEGZ10, Cor. 2.5].

We can assume that \( \rho \leq \phi \). Let \( \epsilon > 0 \). By Corollary 4.1 we get that
\[
\epsilon^n \int_{\{\phi < (1 - \epsilon)\psi + \epsilon\rho \}} MA_\theta(\rho) \leq \int_{\{\phi < (1 - \epsilon)\psi + \epsilon\rho \}} MA_\theta((1 - \epsilon)\psi + \epsilon\rho) \leq \nonumber
\]
\[
\leq \int_{\{\phi < (1 - \epsilon)\psi + \epsilon\rho \}} MA_\theta(\phi). \tag{16}
\]
But since $\{ \phi < (1 - \epsilon)\psi + \epsilon \rho \} \subseteq \{ \phi < \psi \}$ it follows that
\[
\int_{\{ \phi < (1 - \epsilon)\psi + \epsilon \rho \}} MA_{\theta}(\phi) = 0
\]
and hence by (16)
\[
\int_{\{ \phi < (1 - \epsilon)\psi + \epsilon \rho \}} MA_{\theta}(\rho) = 0.
\]
Letting $\epsilon$ tend to zero yields the result. \qed

This should be compared with the domination principle of Boucksom-Eyssidieux-Guedj-Zeriahi [BEGZ10, Cor. 2.5]:

**Theorem 4.3.** Let $\phi$ and $\psi$ be $\theta$-psh. If $\phi$ has minimal singularities and $\psi \leq \phi$ a.e. with respect to $MA_{\theta}(\phi)$ then $\psi \leq \phi$ everywhere.

**REFERENCES**


