TOTALLY REAL EMBDDINGS WITH PRESCRIBED POLYNOMIAL HULLS

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Abstract. We embed compact $C^\infty$ manifolds into $\mathbb{C}^n$ as totally real manifolds with prescribed polynomial hulls. As a consequence we show that any compact $C^\infty$ manifold of dimension $d$ admits a totally real embedding into $\mathbb{C}^{\left\lfloor \frac{3d^2}{2} \right\rfloor}$ with non-trivial polynomial hull without complex structure.

1. Introduction

We will prove the following result.

Theorem 1.1. Let $X \subset \mathbb{C}^m$ be a totally real $C^\infty$ submanifold of dimension $d > 1$ and let $K \subset X$ be a compact subset. Let $M$ be a compact $C^\infty$ manifold of dimension $d$ (possibly with boundary) and assume that there exists a $C^\infty$ embedding $\phi : X \to M$. Then there exists a totally real $C^\infty$ embedding $\psi : M \to \mathbb{C}^{m+\ell}$, where $\ell = \max\{\left\lfloor \frac{3d^2}{2} \right\rfloor - m, 0\}$, such that

(i) $\psi \circ \phi = \text{id}$ on $K$, and
(ii) $\psi(M) = \psi(M) \cup \hat{K}$.

Notice that by a slight abuse of notation we denote the compact set $K \times \{0\} \subset \mathbb{C}^{m+\ell}$ also by the letter $K$.

Our motivation for proving this result is the following recent result by Izzo and Stout.

Theorem 1.2 (Izzo–Stout [9]). Let $M$ be a compact $C^\infty$ surface (possibly with boundary). Then $M$ embeds into $\mathbb{C}^3$ as a totally real $C^\infty$ surface $\Sigma$ such that $\hat{\Sigma} \setminus \Sigma$ is not empty and does not contain any analytic disc.

In the case where $M$ is a torus, Theorem 1.2 was proved by Izzo, Samuelsson Kalm and the second named author [8]. In a recent paper Gupta [4] gave an explicit example of an isotropic torus in $\mathbb{C}^3$ whose polynomial hull consists only of an annulus attached to the torus. We note that the torus embedded in [8] is also isotropic, the image being the graph of a real valued function on the torus in $\mathbb{C}^2$.

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As a corollary to Theorem 1.1 we prove the following generalization of Theorem 1.2 to every dimension \( d > 1 \).

**Corollary 1.3.** Let \( M \) be a compact \( C^\infty \) manifold (possibly with boundary) of dimension \( d > 1 \). Then \( M \) embeds into \( \mathbb{C}^{\lfloor \frac{3d}{2} \rfloor} \) as a totally real \( C^\infty \) submanifold \( \Sigma \) such that \( \Sigma \setminus \Sigma \) is not empty and does not contain any analytic disc.

The same result was proved in [8] for manifolds of dimension \( d > 2 \), with dimension of the target space \( 2d + 4 \) (and with a better dimension of the target space for certain specific 2-manifolds). Notice that by [7, Theorem 2.1] the dimension \( \lfloor \frac{3d}{2} \rfloor \) is optimal for totally real embeddings of \( d \)-dimensional manifolds, which implies that Corollary 1.3 is optimal.

We emphasize that Theorem 1.1 is a simple consequence of the general fact that any \( C^\infty \) manifold of dimension \( d \) embeds into \( \mathbb{C}^{\lfloor \frac{3d}{2} \rfloor} \) as a totally real submanifold, and the following general genericity result regarding polynomial hulls and totally real embeddings (to be proved in Section 4):

**Theorem 1.4.** Let \( M \) be a compact \( C^\infty \) manifold (possibly with boundary) of dimension \( d < n \) and let \( f : M \to \mathbb{C}^n \) be a totally real \( C^\infty \) embedding. Let \( K \subset \mathbb{C}^n \) be a compact polynomially convex set. Then for any \( k \geq 1 \) and for any \( \varepsilon > 0 \) there exists a totally real \( C^\infty \) embedding \( f_\varepsilon : M \to \mathbb{C}^n \) such that

1. \( \|f_\varepsilon - f\|_{C^k(M)} < \varepsilon \),
2. \( f_\varepsilon - f = 0 \) on \( f^{-1}(K) \),
3. \( K \cup f_\varepsilon(M) = K \cup f_\varepsilon(M) \).

In the case \( K = \emptyset \), Theorem 1.4 was proved by Forstneric and Rosay for \( d = 2 \) and \( n \geq 3 \), by Forstneric [2] for \( d \leq \frac{2n}{3} \), and by Løw and the second named author [10] for \( d < n \).

To obtain Corollary 1.3 it is crucial that Alexander [1] has constructed a suitable compact set \( K \) in the totally real distinguished boundary of the bidisk in \( \mathbb{C}^2 \) (see Section 2).

2. **Proof of Theorem 1.1 and Corollary 1.3**

In this section we prove Theorem 1.1 and Corollary 1.3 assuming Theorem 1.4 which will be proved in the next section.

We recall first the following result [3, Lemma 5.3]. By generic we mean open and dense in the Whitney \( C^\infty \)-topology.

**Lemma 2.1.** Let \( M \) be a compact \( C^\infty \) manifold (possibly with boundary) of dimension \( d \). Then a generic \( C^\infty \) embedding \( \psi : M \to \mathbb{C}^{\lfloor \frac{3d}{2} \rfloor} \) is totally real.

**Proof of Theorem 1.1:** Let \( U' \) be a relatively compact open subset of \( X \) containing \( K \). Set \( U := \phi(U') \), and let \( \psi_1 : \phi(X) \to \mathbb{C}^m \) be the inverse of \( \psi \). Let \( \psi_2 : M \to \mathbb{C}^m \) be a \( C^\infty \) mapping which agrees with \( \psi_1 \) on \( U \). Notice that since \( 2(m + \ell) \geq 2d + 1 \), by Whitney’s theorem (see e.g. [5, Theorem 2.13]) a generic \( C^\infty \) mapping from \( M \) to \( \mathbb{C}^{m+\ell} \) is an embedding, and since \( m + \ell \geq \lfloor \frac{3d}{2} \rfloor \), by Lemma 2.1 a generic \( C^\infty \) embedding of \( M \) into \( \mathbb{C}^{m+\ell} \) is totally real. Let \( \psi_3 : M \to \mathbb{C}^{m+\ell} \) be a small perturbation of \( \psi_2 \) which is a totally real \( C^\infty \) embedding.
Let $\chi \in C^0_0(U)$ with $\chi \equiv 1$ in a neighborhood of $\phi(K)$ and $0 \leq \chi \leq 1$. Then if the perturbation $\psi_3$ was small enough, the mapping from $M$ to $\mathbb{C}^{m+\ell}$ defined as $\psi_4 := \chi \cdot \psi_1 + (1-\chi)\psi_3$ is a totally real $C^\infty$ immersion injective in $U$. There exists a small $C^\infty$ perturbation $\psi : M \to \mathbb{C}^{m+\ell}$ which is a totally real $C^\infty$ embedding and such that $\psi|_{\phi(K)} = \psi_4|_{\phi(K)}$ (see e.g. [5, Theorem 2.4.3]) where the approximation is in $C^0$-norm but the proof can be easily adapted to our case) which implies that $K \subset \psi(M)$. Notice that $\psi_4|_{\phi(K)} = \psi_1|_{\phi(K)}$, and thus $\psi \circ \phi = \text{id}$ on $K$. Since $m + \ell > d$, we can apply Theorem 1.4 with $K$ replaced by $\hat{K}$ and Theorem 1.1 follows.

To prove Corollary 1.3 we need the following result of Alexander:

**Theorem 2.2.** (Alexander, [1]) The standard torus $T^2 := \{(e^{i\theta}, e^{i\psi}) : \vartheta, \psi \in \mathbb{R}\}$ in $\mathbb{C}^2$ contains a compact subset $K$ such that $\hat{K} \setminus K$ is not empty but contains no analytic discs. Such a set can be found in every neighborhood of the diagonal in $T^2$.

**Proof of Corollary 1.3:** Choose a small annular neighbourhood $A$ of the diagonal in $T^2$, and let $K \subset A$ be the compact subset given by Theorem 2.2. Define a totally real $C^\infty$ submanifold $X$ of $\mathbb{C}^d$ by

$$X = \{(z_1, z_2, z_3, \ldots, z_d) : (z_1, z_2) \in A, \text{Im}(z_j) = 0 \text{ and } -\varepsilon < \text{Re}(z_j) < \varepsilon \text{ for } j = 3, \ldots, d\},$$

where $\varepsilon > 0$. Now $X$ embeds into (any small portion of) the interior of $M$. By Theorem 1.1, there exists a totally real $C^\infty$ embedding $\psi : M \to \mathbb{C}^d$ such that $K \subset \psi(M)$ and $\hat{\psi(M)} = \psi(M) \cup \hat{K}$. Recall [12, Theorem 6.3.2] that for a polynomially convex subset $H$ of a totally real $C^\infty$ submanifold the uniform algebra $\mathcal{P}(H)$, consisting of all the functions on $H$ that can be approximated uniformly by polynomials, coincides with the algebra of continuous functions on $H$. Hence the hull $\hat{K}$ cannot be contained in $\psi(M)$, and the result follows from Theorem 2.2.

## 3. Proof of the main lemma

For any compact set $K \subset \mathbb{C}^n$ set $h(K) := \overline{K \setminus K}$. We will sometimes denote $\hat{K}$ by $[K]^\circ$.

The proof of the following main lemma is essentially contained in [10, Proposition 6.7] where it is given in the case of $C^1$-regularity. We give a proof for completeness.

**Lemma 3.1.** Let $M$ be a compact $C^\infty$ manifold (possibly with boundary) of dimension $d \leq n$ and let $f : M \to \mathbb{C}^n$ be a totally real $C^\infty$ embedding. Let $K \subset \mathbb{C}^n$ be a compact polynomially convex set, and let $U$ be a neighborhood of $K$. Then for any $k \geq 1$ and for any $\varepsilon > 0$ there exists a totally real $C^\infty$ embedding $f_\varepsilon : M \to \mathbb{C}^n$ such that

1. $\|f_\varepsilon - f\|_{C^k(M)} < \varepsilon$,
2. $f_\varepsilon - f = 0$ in a neighborhood of $f^{-1}(K)$,
3. $K \cup f_\varepsilon(M) \subset U \cup f(M)$.

The following result is proved in [10, Proposition 4]. Recall that a small enough $C^1$-perturbation of an embedding of a compact manifold is still an embedding (see e.g. [5, Theorem 1.4]).

**Proposition 3.2.** Let $M$ be a compact $C^1$ manifold (possibly with boundary) and let $f : M \to \mathbb{C}^n$ be a totally real $C^1$ embedding. Let $U' \subset U \subset \mathbb{C}^n$ be open sets. Then there exists an open neighborhood $\Omega$ of $f(M)$ such that
(1) if $S \subset M$ is closed and $K \subset U'$ is compact, then
$$h(K \cup f(S)) \subset U' \cup \Omega \Rightarrow h(K \cup f(S)) \subset U,$$

(2) if $\tilde{f}$ is a sufficiently small $C^1$-perturbation of $f$, then (1) holds with $\tilde{f}$ in place of $f$.

The following lemma is proved in [10, Corollary 2]

**Lemma 3.3.** Let $K \subset \mathbb{C}^n$ be a compact subset. Let $M$ be a compact $C^1$ manifold (possibly with boundary) and let $f : M \to \mathbb{C}^m$ be a totally real $C^1$ embedding. Let $S \subset M$ be a closed subset. Let $U$ be an open set such that $h(K \cup f(S)) \subset U$.

Then there exists a constant $c > 0$ such that if $\|f - \tilde{f}\|_{C^1(M)} < c$, then $h(K \cup \tilde{f}(S)) \subset U$.

**Proof of Lemma 3.1:** The proof is in two steps. Let $I$ denote the interval $[0, 1] \subset \mathbb{R}$.

**First Step:** The first step is proving the lemma in the case $M = I^d$. The proof is by induction on $d$. Fix a strictly positive $d \in \mathbb{N}$, and assume that the result holds for embeddings of $I^{d-1}$.

Since $K$ is polynomially convex, it admits fundamental system of open neighborhoods which are Runge and Stein open sets. Thus there exists a Runge and Stein open set $U' \subset \mathbb{C}^n$ such that $K \subset U' \subset U$. Since $U'$ is Runge and Stein, it admits a normal exhaustion by polynomially convex subsets. Hence there exists a polynomially convex set $K' \subset U'$ such that $K \subset \text{int}(K')$.

Let us fix some notation.

(1) for $J \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$, $0 \leq \alpha_j \leq J - 1$, we denote by $I^J_\alpha$ the cube
$$I^J_\alpha := \left[\frac{\alpha_1}{J}, \frac{\alpha_1 + 1}{J}\right] \times \cdots \times \left[\frac{\alpha_d}{J}, \frac{\alpha_d + 1}{J}\right],$$

(2) for $1 \leq m \leq d$, $0 \leq j \leq J$, we denote by $G^J_{m,j}$ the $(d-1)$-dimensional cube
$$G^J_{m,j} := I \times \cdots \times I \times \left\{\frac{j}{J}\right\} \times I \cdots \times I,$$

(3) we denote by $G^J$ the grid which is the union of all $(d-1)$-dimensional cubes
$$G^J := \bigcup_{m,j} G^J_{m,j},$$

(4) we denote by $G^J(\delta)$ the closed $\delta$-neighborhood of the grid $G^J$,
$$G^J(\delta) := \{x \in I^d : \text{dist}(x, G^J) \leq \delta\},$$

(5) we denote by $Q^J_\alpha(\delta) \subset I^J_\alpha$ the smaller $d$-dimensional cube
$$Q^J_\alpha(\delta) := I^J_\alpha \setminus G^J(\delta),$$

(6) we denote by $S^J$ an $n$-tuple of distinct cubes $I^J_\alpha$.

We will prove the following statement. There exists a big enough $J$, a small enough $\delta$ and a totally real $C^\infty$ embedding $f_\varepsilon : I^d \to \mathbb{C}^n$ such that

(i) $\|f_\varepsilon - f\|_{C^k(I^d)} < \varepsilon$,

(ii) $f_\varepsilon - f = 0$ in a neighborhood of $f^{-1}(K)$,
(iii) \(f_\varepsilon(I^d_{\alpha}) \cap K \neq \emptyset \Rightarrow f_\varepsilon(I^d_{\alpha}) \subset K',\)
(iv) \(h(K' \cup f_\varepsilon(S^J) \cup f_\varepsilon(G^J(\delta))) \subset U\) for any choice of n-tuple \(S^J\),
(v) for all \(\alpha\) such that \(f_\varepsilon(Q^J_{\alpha}(\delta)) \subset \mathbb{C}^n \setminus K\), there exists an entire function \(g_\alpha \in \mathcal{O}(\mathbb{C}^n)\) such that \(f_\varepsilon(Q^J_{\alpha}(\delta)) \subset \{g_\alpha = 0\} =: Z(g_\alpha),\)
(vi) for all choices of \(n + 1\) different multi-indices \(\alpha^1, \ldots, \alpha^{n+1}\) such that \(f_\varepsilon(Q^J_{\alpha}(\delta)) \subset \mathbb{C}^n \setminus K\) we have \(\bigcap_{1 \leq j \leq n+1} Z(g_{\alpha^j}) = \emptyset\).

Once this is proved, the first step of the proof goes as follows. Assume that
\[q \in [K \cup f_\varepsilon(I^d)].\]

Let \(\mu\) be a representative Jensen measure for the linear functional on \(\mathcal{P}(K \cup f_\varepsilon(I^d))\) defined as the evaluation at \(q\), such that for all entire functions \(g \in \mathcal{O}(\mathbb{C}^n)\) we have
\[\log |g(q)| \leq \int \log |g| d\mu. \tag{3.1}\]

Assume that there exists \(\alpha\) such that \(\mu(f_\varepsilon(Q^J_{\alpha}(\delta))) > 0\) and \(f_\varepsilon(Q^J_{\alpha}(\delta)) \cap K = \emptyset\). By (v) we get that \(\int \log |g_\alpha| d\mu = -\infty\), and by (3.1) we have that \(q \in Z(g_\alpha)\). By (vi), there exists an n-tuple of cubes \(S^J\) such that the measure \(\mu\) is concentrated on the subset
\[K' \cup f_\varepsilon(S^J) \cup f_\varepsilon(G^J(\delta)).\]

It follows that \(q \in [K' \cup f_\varepsilon(S^J) \cup f_\varepsilon(G^J(\delta))]\), and by (iv) we obtain that \(q \in U \cup f_\varepsilon(I^d)\).

We proceed to prove the statement above.

**Claim 1:** If \(J\) is big enough then
\[h(K' \cup f(S^J)) \subset U' \tag{3.2}\]
for any choice of n-tuple \(S^J\). Moreover, for a fixed big enough \(J\), if \(f_0\) is a sufficiently small \(C^1\)-perturbation of \(f\), then (3.2) holds with \(f_0\) in place of \(f\).

**Proof of Claim 1:** Let \(U''\) be an open neighborhood of \(K'\) with \(U'' \subset U',\) Let \(\Omega'\) be given by Proposition 3.2 with data \((f, U', U'')\). Assume by contradiction that there is a sequence \((J_t)_{t \in \mathbb{N}}, J_t \rightarrow \infty\) and a sequence \(S^J_t\) of n-tuples such that
\[h(K' \cup f(S^J_t)) \not\subset U'.\]

Let \(I^{J_t}_{\alpha_1(t)}, \ldots, I^{J_t}_{\alpha_n(t)}\) be the cubes in \(S^J_t\). Up to passing to a subsequence we may assume that for any \(1 \leq j \leq n\), the image \(f(I^{J_t}_{\alpha_j(t)})\) converges to a point \(q_j \in M\) as \(t \rightarrow \infty\). Since \(K' \cup \{q_j\}_{1 \leq j \leq n}\) is polynomially convex, there exists a Runge and Stein neighborhood \(V\) of \(K' \cup \{q_j\}_{1 \leq j \leq n}\) such that \(V \subset U'' \cup \Omega'\). For large enough \(t\) we have that \(K' \cup f(S^J_t) \subset V\), which implies \(h(K' \cup f(S^J_t)) \subset U'' \cup \Omega',\) and thus by Proposition 3.2 we have a contradiction.

Since there are only a finite number of n-tuples \(S^J\) for a fixed \(J\), the perturbation claim follows from Lemma 3.3. **End proof of Claim 1.**

Choose \(J\) big enough such that Claim 1 holds and that for all \(\alpha\),
\[f(I^d_{\alpha}) \cap K \neq \emptyset \Rightarrow f(I^d_{\alpha}) \subset \text{int}(K'),\]
and such that for all \(\alpha\) the image \(f(I^d_{\alpha})\) is polynomially convex (this can be obtained by [11, Proposition 4.2]).
Proposition 3.2 with data $(f_k)$ such that $\|f_k\|_{C^k(I_{\alpha}^d)} < \theta$ we have that $g(I_{\alpha}^d)$ stays polynomially convex.

Claim 2: There exists a totally real $C^\infty$ embedding $f_2^k : I^d \to \mathbb{C}^n$ such that

1. $\|f_2^k - f\|_{C^k(I^d)} < \max(\theta, \frac{\eta}{2})$,
2. $f_2^k = f$ in a neighborhood of $f^{-1}(K')$,
3. $h(K' \cup f_2^k(S^J) \cup f_2^k(G^J)) \subset U'$ for any choice of $n$-tuple $S^J$.

Moreover, if $f_0$ is a sufficiently small $C^1$-perturbation of $f_2^k$, then (3) holds with $f_0$ in place of $f_2^k$.

Proof of Claim 2: Fix an $n$-tuple $S^J$. Choose an ordering of the $(d - 1)$-dimensional cubes $(G_{m,j}^i)_{m,j}$ and denote them by $G_1, \ldots, G_{\ell}$.

Fix $0 < \eta < 1$ (to be determined later). We will construct inductively a family of totally real $C^\infty$ embeddings $(f_j : I^d \to \mathbb{C}^n)_{0 \leq j \leq \ell}$ such that

1. $\|f_j - f_{j-1}\|_{C^k(I^d)} \leq \eta$,
2. $f_j = f_{j-1}$ in a neighborhood of $f^{-1}(K') \cup S^J \cup \bigcup_{1 \leq i \leq j} G_i$,
3. $h \left(K' \cup f_j(S^J) \cup \bigcup_{1 \leq i \leq j} f_j(G_i)\right) \subset U'$.

If we set $f_0 := f$, then (c) is true by Claim 1. Assume we constructed $f_j$, where $0 \leq j \leq \ell - 1$. Then by (c) the set $h(K' \cup f_j(S^J) \cup \bigcup_{0 \leq i \leq j} f_j(G_i))$ is compact in $U'$, so there exists a set $U''_1$ such that $K' \subset U''_1 \subset U'$ satisfying $h(K' \cup f_j(S^J) \cup \bigcup_{1 \leq i \leq j} f_j(G_i)) \subset U''_1$. Let $\Omega_1$ be given by Proposition 3.2 with data $(f_j, U', U''_1)$. Consider the totally real $C^\infty$ embedding $f_j|_{G_{j+1}} : G_{j+1} \to \mathbb{C}^n$. Since

$$[K' \cup f_j(S^J) \cup \bigcup_{0 \leq i \leq j} f_j(G_i)] \subset U''_1 \cup \Omega_1,$$

by the inductive assumption (the result holds for embeddings of $I^{d-1}$) there exists a totally real $C^\infty$ embedding $\tilde{f}_j|_{G_{j+1}} : G_{j+1} \to \mathbb{C}^n$ such that

1. $\|\tilde{f}_j - f_j\|_{C^k(G_{j+1})}$ is small,
2. $\tilde{f}_j = f_j$ in a neighborhood of $f_j^{-1}(K') \cup S^J \cup \bigcup_{1 \leq i \leq j} G_i$,
3. $h \left(K' \cup \tilde{f}_j(S^J) \cup \bigcup_{1 \leq i \leq j} \tilde{f}_j(G_i) \cup \tilde{f}_j(G_{j+1})\right) \subset U''_1 \cup \Omega_1$.

If $\|\tilde{f}_j - f_j\|_{C^k(G_{j+1})}$ is small enough, then we may extend $\tilde{f}_j$ to a totally real $C^\infty$ embedding $f_{j+1} : I^d \to \mathbb{C}^n$ such that $\|f_{j+1} - f_j\|_{C^k(I^d)} \leq \eta$, which coincides with $f_j$ on $f^{-1}(K') \cup S^J \cup \bigcup_{1 \leq i \leq j} G_i$. By the choice of $\Omega_1$ it follows that

$$h(K' \cup f_{j+1}(S^J) \cup \bigcup_{1 \leq i \leq j+1} f_{j+1}(G_i)) \subset U'.$$

The mapping $f_\ell$ satisfies Claim 2 for the chosen $n$-tuple $S^J$.

We then iterate this argument for every $n$-tuple $S^J$ (choosing $\eta$ small enough), and we obtain $f_2^k$. The perturbation claim follows from Lemma 3.3.
Let $\Omega$ be an open neighborhood of $f_{\frac{2}{\varepsilon}}(I^{d})$ given by Proposition 3.2 with data $(f_{\frac{2}{\varepsilon}}, U, U')$.

**Claim 3:** There exists a $\delta > 0$ such that

$$h(K' \cup f_{\frac{2}{\varepsilon}}(S^J) \cup f_{\frac{2}{\varepsilon}}(G^{J}(\delta))) \subset U$$

(3.3)

for any choice of $n$-tuple $S^J$. Moreover, if $f_0$ is a sufficiently small $C^1$-perturbation of $f_\frac{2}{\varepsilon}$, then (3.3) holds with $f_0$ in place of $f_\frac{2}{\varepsilon}$.

**Proof of Claim 3:** Fix an $n$-tuple $S^J$. Since $h(K' \cup f_{\frac{2}{\varepsilon}}(S^J) \cup f_{\frac{2}{\varepsilon}}(G^{J})) \subset U'$, there exists a Runge and Stein neighborhood $V \subset \subset \Omega \cup U'$ of $K' \cup f_{\frac{2}{\varepsilon}}(S^J) \cup f_{\frac{2}{\varepsilon}}(G^{J})$. If $\delta > 0$ is small enough we have that $K' \cup f_{\frac{2}{\varepsilon}}(S^J) \cup f_{\frac{2}{\varepsilon}}(G^{J}(\delta)) \subset V$, and thus $h(K' \cup f_{\frac{2}{\varepsilon}}(S^J) \cup f_{\frac{2}{\varepsilon}}(G^{J}(\delta))) \subset U$. Since the number of $n$-tuples is finite, we can choose a $\delta$ which works for all of them. The perturbation claim follows from Lemma 3.3. *End proof of Claim 3.*

We now claim that we may approximate $f_{\frac{2}{\varepsilon}}$ arbitrarily well in $C^k$-norm on $I^{J}_{\alpha}$ by holomorphic automorphisms $G_{\alpha}$ of $\mathbb{C}^{n}$. Granted this for a moment, we proceed as follows. Let $\chi$ be a cutoff function with compact support in $\text{int}(I^{J}_{\alpha})$ such that $\chi \equiv 1$ in a neighborhood of $Q^{J}_{\alpha}(\delta)$. Define

$$f_{\alpha}(x) := f_{\frac{2}{\varepsilon}}(x) + \chi(x)(G_{\alpha}(x) - f_{\frac{2}{\varepsilon}}(x)).$$

(3.4)

Since $d < n$, the $n$-th coordinate of every point in $I^{J}_{\alpha}$ is 0. Hence, if we define $g_{\alpha}$ as the $n$-th coordinate of $G^{-1}_{\alpha}$, then for all $x \in Q^{J}_{\alpha}(\delta)$,

$$g_{\alpha}(f_{\alpha}(x)) = 0.$$  

(Notice that it is here crucial that $d < n$. If $d = n$, there is no way to “push” cubes into analytic hypersurfaces of $\mathbb{C}^{n}$.)

Since $f_{\alpha} = f_{\frac{2}{\varepsilon}}$ on $I^{d} \setminus \text{int}(I^{J}_{\alpha})$, the maps $f_{\alpha}$ patch up as $\alpha$ varies among all multi-indices $\alpha$ such that $f_{\frac{2}{\varepsilon}}(Q^{J}_{\alpha}(\delta)) \cap K = \emptyset$, and we obtain an embedding $f_{\varepsilon} : I^{d} \to \mathbb{C}^{n}$ satisfying the properties (i)-(v).

Finally, by the following sublemma, we may assume, possibly having to perturb the automorphisms $G_{\alpha}$ slightly, that the intersection $\bigcap_{1 \leq j \leq n+1} Z(g_{\alpha j})$ of any collection of $n+1$ zero sets is empty. Thus property (vi) is satisfied.

**Sublemma 3.4.** Let $\{g_{j}\}_{1 \leq j \leq N} \subset \mathcal{O}(\mathbb{C}^{n})$ be a finite collection of non-constant holomorphic functions, and for each $a_{j} \in \mathbb{C}$ set $g^{a_{j}}_{j} := g_{j} - a_{j}$. Then there exists a dense $G_{\delta}$ set $A \subset \mathbb{C}^{N}$ such that for each $a \in A$ the following holds: for any collection $\{g^{a_{i_{1}}}_{i_{1}}, \ldots, g^{a_{i_{k}+1}}_{i_{k+1}}\}$ with $i_{j} \neq i_{k}$ if $j \neq k$, \ldots
we have that
\[
\bigcap_{l=1}^{n+1} Z(g_{il}^{a_{il}}) = \emptyset. \tag{3.5}
\]

Proof. Denote by \(\Delta_\delta(p) \subset \mathbb{C}\) the disc of center \(p\) and radius \(\delta\), and by \(\mathbb{B}_R^n \subset \mathbb{C}^n\) the ball of radius \(R\) centered at the origin. Fix \(R > 0\) and fix \(I := (i_1, \ldots, i_{n+1})\) where \(i_j \in \{1, \ldots, N\}\) and \(i_j \neq i_k\) if \(j \neq k\). The result immediately follows by the Baire lemma if we prove that the set
\[
A_{I,R} := \{a \in \mathbb{C}^N : \bigcap_{l=1}^{n+1} Z(g_{il}^{a_{il}}) \cap \mathbb{B}_R^n = \emptyset\}
\]
is dense (it is obviously open). Let thus \(a = (a_1, \ldots, a_N) \in \mathbb{C}^N\) be an arbitrary point. The set \(Z(g_{i_1}^{a_{i_1}}) \cap \mathbb{B}_R^n\) consists of a finite number of irreducible components, each of dimension \(n-1\).

Choosing \(a_{i_2} \in \Delta_\delta(a_{i_2})\) such that \(g_{i_2}^{a_{i_2}}\) is not identically zero on the regular part of any of these components, we get that
\[
Z(g_{i_1}^{a_{i_1}}) \cap Z(g_{i_2}^{a_{i_2}}) \cap \mathbb{B}_R^n \tag{3.6}
\]
has a finite number of irreducible components, each of dimension at most \(n-2\), where we have set \(a_{i_1} = a_{i_1}\). Choosing \(a_{i_3} \in \Delta_\delta(a_{i_3})\) such that \(g_{i_3}^{a_{i_3}}\) is not identically zero on any of these components, and continuing in this fashion, we obtain a collection of points \(\{a_{i_l}\}, l = 1, \ldots, n+1\), such that
\[
\bigcap_{l=1}^{n+1} Z(g_{i_l}^{a_{i_l}}) \cap \mathbb{B}_R^n = \emptyset, \tag{3.7}
\]
and such that \(a_{i_l} \subset \Delta_\delta(a_{i_l})\).

We are thus left to show that we may approximate \(f^x_\alpha\) arbitrarily well in \(C^k\)-norm on \(I^J_\alpha\) by holomorphic automorphisms \(G_{\alpha}\) of \(\mathbb{C}^n\). We say that two smooth totally real polynomially convex embeddings \(f, g: I^J_\alpha \rightarrow \mathbb{C}^n\) are connected if there exists a smooth isotopy of embeddings \((G_t: I^J_\alpha \rightarrow \mathbb{C}^n)_{t \in [0,1]}\) such that \(G_0 = f, G_1 = g\) and such that \(G_t(I^J_\alpha)\) is totally real and polynomially convex for all \(t\). If \((h_k: I^J_\alpha \rightarrow \mathbb{C}^n)_{0 \leq k \leq M}\) is a finite family of smooth totally real polynomially convex embeddings such that \(h_0\) is the identity map \(\text{id}_{I^J_\alpha}\) and \(h_k\) is connected to \(h_{k+1}\) for all \(0 \leq k \leq M - 1\), then by the proof of [2, Main Theorem] it follows that \(h_M\) is approximable by automorphisms.

There exists \(\eta > 0\) small enough such that the embedding \(x \mapsto f^x_\alpha(\eta x)\) defined on \(I^J_\alpha\) can be written as
\[
f^x_\alpha(\eta x) = \rho(\beta(x)),
\]
where \(\beta: I^J_\alpha \rightarrow \mathbb{R}^d\) is the orientation preserving diffeomorphism \(\beta(x) := \pi_{\mathbb{R}^d}(f^x_\alpha(\eta x))\) and where \(\rho\) is a graph map \(\rho(x) := (x, \gamma(x))\), where \(D\gamma(0) = 0\) and \(\gamma\) is \(c\)-Lipschitz with \(c > 0\) small enough to obtain that for all \(t \in [0,1]\) the image of \(\beta(I^J_\alpha)\) via the graph map \(\rho_t(x) := (x, t\gamma(x))\) is totally real and polynomially convex (we use [11, Proposition 4.2]).

We have that
(1) since $\beta$ is orientation preserving, the identity map $\text{id}_{I_d}$ is connected to $\beta$ via an isotopy 
with values in $\mathbb{R}^d$,
(2) the isotopy $\rho_t(\beta(x))$ connects $\beta$ to the embedding $x \mapsto f_{\frac{t}{2}}(\eta x)$,
(3) the embedding $f_{\frac{t}{2}}(\eta x)$ is connected to $f_{\frac{t}{2}}(x)$ via the isotopy $f_{\frac{t}{2}}((\eta + t(1-\eta))x)$.

**Second Step:**

For simplicity we assume that $M$ is without boundary. Let $I_1, \ldots, I_N$ be a cover of $M$ by 
closed cubes, such that there is a collection of subcubes $I_0^i \subset \text{int}(I_i)$ which also cover $M$. 
For each $i$ let $\chi_i$ be a cutoff function compactly supported in $\text{int}(I_i)$ with $\chi_i \equiv 1$ in a neighborhood 
of $I_0^i$. Let $U_j \subset U_{j+1} \subset U$ be open sets for $j = 1, \ldots, N$, and let $\Omega$ be given by Proposition 3.2 
for all the data $(f(M), U_{j+1}, U_j)$ with $j = 1, \ldots, N$. We will proceed to perturb the image $f(M)$ 
cube by cube.

Assume that we have constructed a small $C^k$-perturbation $f_j : M \to \mathbb{C}^n$ of $f$ such that

$$h(K \cup f_j(\cup_{i=1}^j I_0^i)) \subset U_j$$

(it will be clear from the construction how to construct $f_1$).

By Step 1 we let $g_{j+1} : I_{j+1} \to \mathbb{C}^n$ be a small $C^k$-perturbation of $f_j$ such that

$$h(K \cup f_j(\cup_{i=1}^j I_0^i) \cup g_{j+1}(I_{j+1})) \subset \Omega \cup U_j,$$

and $g_{j+1}$ coincides with $f_j$ in a neighbourhood of $\cup_{i=1}^j I_0^i \cup f_j^{-1}(K)$. Defining $f_{j+1}$ by

$$f_{j+1} := f_j + \chi_{j+1}(g_{j+1} - f_j),$$

we obtain

$$h(K \cup f_{j+1}(\cup_{i=1}^{j+1} I_0^i)) \subset U_{j+1}.$$ 

If all the perturbations were small enough, defining $f_\varepsilon := f_N$ yields the result.  \qed

4. **Proof of Theorem 1.4**

Let $\varepsilon > 0$. Let $(U_j)_{j \geq 1}$ be a sequence of open neighborhoods of $K$ such that $U_{j+1} \subset U_j$ for 
all $j \geq 1$ and $K = \bigcap_{j \geq 1} U_j$. Set $f_0 := f$. We perturb $f_0$ inductively using Lemma 3.1, obtaining 
a family of totally real $C^\infty$ embeddings $(f_j : M \to \mathbb{C}^n)_{j \in \mathbb{N}}$ such that,

1. for all $j \geq 1$ we have $K \cup f_j(M) \subset U_j \cup f_j(M),$
2. for all $j \in \mathbb{N}$ we have

$$\|f_{j+1} - f_j\|_{C^{k+j}(M)} < \eta_j,$$

where the sequence $(\eta_j)$ satisfies for all $\ell \in \mathbb{N},$

$$\sum_{j=\ell}^{\infty} \eta_j < \delta_\ell,$$

where $0 < \delta_0 \leq \varepsilon$ is to be chosen and for all $\ell \geq 1$, $\delta_\ell$ is the constant $c(f_\ell, K, U_\ell)$ given 
by Lemma 3.3,
3. for all $j \in \mathbb{N}$ we have $f_{j+1} \equiv f_j$ on $f^{-1}(K)$.
The sequence \((f_j)\) clearly converges to a \(C^\infty\) map \(f_\varepsilon : M \to \mathbb{C}^n\) which coincides with \(f\) on \(f^{-1}(K)\), and such that \(\|f - f_\varepsilon\|_{C^k(M)} < \delta_0\). If \(\delta_0\) is small enough, then \(f_\varepsilon\) is a totally real \(C^\infty\) embedding. We claim that for all \(\ell \geq 1\),

\[
K \cup f_\varepsilon(M) \subset U_\ell \cup f_\varepsilon(M).
\]

This follows from Lemma 3.3, since

\[
\|f_\varepsilon - f_\ell\|_{C^1(M)} \leq \sum_{j=\ell}^{\infty} \eta_j < \delta_\ell.
\]

\[\square\]

References


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