REGULARITY AND ASYMPTOTIC APPROACH TO SEMILINEAR ELLIPTIC EQUATIONS WITH SINGULAR POTENTIAL

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Abstract. We study weak solutions of the problem

\[
\begin{aligned}
-\Delta u &= \frac{\lambda}{|x|^2} u + u^p \quad \text{in } \Omega \setminus \{0\} \\
u &\geq 0 \quad \text{in } \Omega \setminus \{0\} \\
u|_{\partial \Omega} &= 0
\end{aligned}
\]

where \( \Omega \subseteq \mathbb{R}^N \) is a smooth bounded domain containing the origin, \( N \geq 3 \), \( 1 < p < \frac{(N + 2)(N - 2)}{4} \) and \( -\infty < \lambda \leq \bar{\lambda} := \frac{(N - 2)^2}{4} \). We present a regularity estimate around the origin, generalizing previous results of other authors for the case \( \lambda \geq 0 \). For the case of radially symmetric solutions on the unit ball \( \Omega = B_1(0) \) we present a very good approximation for the shape of the solution in the limit when \( \lambda \to -\infty \).

MSC:35J25

1. Introduction

In this paper we present results concerning non-trivial solutions of the elliptic problem

\[
\begin{aligned}
-\Delta u &= \frac{\lambda}{|x|^2} u + u^p \quad \text{in } \Omega \setminus \{0\} \\
u &\geq 0 \quad \text{in } \Omega \setminus \{0\} \\
u|_{\partial \Omega} &= 0
\end{aligned}
\]

where \( 0 \in \Omega \subseteq \mathbb{R}^N \) is a smooth domain, \( N \geq 3 \), \( p > 1 \) and \( \lambda \in \mathbb{R} \). This is a case of a non-linear Schrodinger equation with singular potential. The term \( \lambda |x|^{-2} \) is called Calogero potential, where positive and negative sign for \( \lambda \) means attractive and repulsive field, respectively.

A weak solution of the problem (1.1) is supposed to be critical point of the functional

\[
J[u] = \frac{1}{2} \int_{\Omega} \left\{ \nabla u^2 - \frac{\lambda}{|x|^2} u^2 \right\} - \frac{1}{p + 1} \int_{\Omega} u^{p+1},
\]

defined on the Sobolev space \( W^{1,2}_0(\Omega) \). Hardy’s inequality allows to use variational methods to prove existence of non-trivial solutions when \( \lambda \leq \bar{\lambda} \), where \( \bar{\lambda} = (N - 2)^2/4 \) is the best constant. On the other hand \( |x|^{-2} \) is the smallest singularity for which the functional \( J \) can be well-defined in \( W^{1,2}_0(\Omega) \). Furthermore, this singularity is critical in the sense that bootstrapping argument do not improves integrability of \( u \) at 0, and so regularity near zero must be investigated by other tools. Concerning the range of the parameters in (1.1), we observe that if \( \lambda > \bar{\lambda} \) then \( J \) in unbounded from below, and actually there is just the trivial solutions \( u \equiv 0 \). Still, if \( p \geq \frac{(N + 2)(N - 2)}{4} \) or \( \Omega \) is star-shaped, Pohozaev’s identity implies that the problem has only the trivial solution.

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For the case of our interest, i.e., \( \lambda < \bar{\lambda} \) and \( 1 < p < (N + 2)/(N - 2) \), the problem always has a non trivial weak solution in \( W^{1,2}_0(\Omega) \) and questions of existence, multiplicity and regularity of solutions are strongly sensitive on the sign of \( \lambda \). For the case of a spherical domain \( \Omega = B_1(0) \), it was shown in [6] that when \( \lambda > 0 \) the problem (1.1) has a unique solution, and this solution is, by using moving plane method, radially symmetric about 0. In [8] the authors show uniqueness of radial solutions for all \( \lambda < \bar{\lambda} \) and that there exist branches of non-radial solutions bifurcating from the radial branch for \( \lambda < 0 \). When \( \Omega \) is a general bounded domain, a local estimate near 0 was proved in [11] for the case \( \lambda > 0 \). The first result of the present work is a global regularity estimate for solutions of (1.1) for all values of \( \lambda \) (generalizing the results in [11] and [8]). We also note that, using a different method, Cao and Han in Lemma 2.2 of [4] proved the same estimate for eigenfunctions of Hardy-type linear operator.

Theorem 1.1. If \( \lambda < \bar{\lambda} \) and \( 1 < p < (N + 2)/(N - 2) \) then any weak solution \( u \in W^{1,2}_0(\Omega) \) of (1.1) satisfies

\[
\tag{1.3}
 u(x) \leq C_\lambda |x|^{\sqrt{\alpha - \lambda} - \sqrt{\alpha}}
\]

for all \( x \in \Omega \). This estimate is optimal in the sense that the exponent of \( |x| \) cannot be increased.

A remarkable consequence of (1.3) is that if \( \lambda < 0 \) then the solution vanishes at the origin in any bounded domain of \( \mathbb{R}^N \). So the monotonicity propriety of Gidas, Ni and Nirenberg ([9]) never hold if \( \lambda < 0 \) in any bounded symmetric domain.

Note also that as \( \lambda \to -\infty \) the derivatives of all orders vanishes at the origin.

For the case \( \Omega = \mathbb{R}^N \), existence and multiplicity results were proved in [14]. The author shows that if \( p = (N + 2)/(N - 2) \) and \( 0 \leq \lambda < \bar{\lambda} \) then (1.1) has an unique solution in the space \( D^{1,2}(\mathbb{R}^N) = \{ v \in L^2(\mathbb{R}^N) : |
abla v| \in L^2(\mathbb{R}^N) \} \). Terracini also proved that if \( \lambda \) is sufficiently negative then there exist at least two different solutions in the space \( D^{1,2}(\mathbb{R}^N) \), one radial and another non-radial.

The second question developped in this paper concerns radial solutions of (1.1), i.e., solutions of the problem

\[
\begin{aligned}
-\frac{N}{r} \frac{d}{dr} u' - \frac{N - 1}{r^2} u &= \lambda r^{N-2} u + u^p \\
&\text{in } (0, 1) \\
u &\geq 0 \\
u'(0) = u(1) = 0
\end{aligned}
\]  

(1.4)

It is already proved in [8] that (1.4) has an unique non-trivial solution. Here we proved an accurate approximation of shape of this solution when \( \lambda \to -\infty \). After a suitable transformation it will be shown that (1.4) becomes

\[
\begin{aligned}
-\frac{d^2 v}{dr^2} - \frac{M_\lambda - 1}{r} v &= v^p \\
v &> 0 \\
v(1) = 0
\end{aligned}
\]  

(1.5)

where \( M_\lambda \) is defined in (2.8).

Equation (1.5) has some remarkable aspects: the left hand side resembles the radial laplacian with a noninteger dimension \( M_\lambda \); on other hand, by using the formula (2.8), we have that

\[
\frac{M_\lambda + 2}{M_\lambda - 2} - p = \left( \frac{N + 2}{N - 2} - p \right) b_\lambda \to 0 \quad \text{as } \lambda \to -\infty,
\]

where \( b_\lambda \), defined in (2.5), vanishes when \( \lambda \to -\infty \). It means that equation (1.5) is asymptotically critical when \( \lambda \to -\infty \) (i.e., \( b \to 0 \)) with critical exponent. For this
reason it is natural to develop an approach like in [2] and [12].

Our first result is the following “local behavior” near the origin. In the statements we refer to the parameters \(a_\lambda, b_\lambda, M_\lambda\) defined in (2.6), (2.5), (2.8).

**Theorem 1.2.** Let \(u_\lambda\) be a solution to (1.5) and set \(v_\lambda : (0, 1) \to \mathbb{R},\)

\[ v_\lambda(r) = b_\lambda^\frac{2}{p} r^{\alpha_\lambda} u_\lambda(r^{\beta_\lambda}) \]  

(1.6)

Then the scaled function

\[ w_\lambda(r) = \frac{1}{\|v_\lambda\|_\infty} v_\lambda \left( \frac{\sqrt{M_\lambda(M_\lambda - 2)}}{\|v_\lambda\|_\infty} r \right) \]  

(1.7)

satisfy

\[ w_\lambda(r) \to \frac{1}{(1 + r^2)^{\frac{p-2}{4}}} \text{ in } C_{loc}^1([0, +\infty)) \text{ as } \lambda \to -\infty. \]  

(1.8)

The result of the previous theorem is a little bit surprising. Indeed, although \(p\) is a fixed number in the subcritical range \((1, (N + 2)/(N - 2))\), as \(\lambda \to -\infty\) problem (1.5) behaves like an asymptotically critical problem. We do not know about any analogous phenomenon in the literature. Some interesting results for nonlinearities approaching the critical power and \(\mu > 0\) can be found in [3].

Our final result deals with with the global behavior on the solution in \((0, 1)\). For convenience we define the constant

\[ A_0 = \frac{1}{p - 1} \left\lceil \frac{1}{8(p + 1)} \right\rceil \frac{2p}{\Gamma\left( \frac{p+1}{p-1} \right)} \Gamma\left( \frac{2p+2}{p-1} \right) \frac{(N + 2 - p)}{(N - 2 - p)} \]  

(1.9)

and the parameter

\[ \beta_\lambda^* = \frac{\sqrt{8(p + 1)}}{p - 1} A_0^{\frac{p-1}{2}} b_\lambda^{\frac{p-1}{2}}. \]  

(1.10)

**Theorem 1.3.** Let \(1 < p < (N + 2)/(N - 2)\) and \(u_\lambda(x) = u_\lambda(|x|)\) the unique non-trivial radial solution of problem (1.1) on \(\Omega = B_1\). The following estimates hold:

(i) We have

\[ \lim_{\lambda \to -\infty} \sup_{0 < r \leq 1} \left| b_\lambda^{\frac{p+1}{2}} r^{\alpha_\lambda/b_\lambda} u_\lambda(r) - A_0^{-1/2} \left( 1 + (r^{1/b_\lambda}/\beta_\lambda^*)^2 \right)^{-\frac{2}{p-2}} \right| = 0. \]

In particular, for fixed \(0 \leq r < 1\) we have

\[ \lim_{\lambda \to -\infty} b_\lambda^{\frac{p+1}{2}} r^{\alpha_\lambda/b_\lambda} u_\lambda(r) = A_0^{-1/2}. \]

(ii) Let \(r_\lambda\) a maximum point for \(u_\lambda\). Then

\[ \lim_{\lambda \to -\infty} b_\lambda^{\frac{p+1}{2}} u_\lambda(r_\lambda) = \left[ \frac{2(p + 1)}{(p - 1)^2} \right]^{\frac{1}{p-2}} \quad \text{and} \quad \lim_{\lambda \to -\infty} \frac{r_\lambda^{1/b_\lambda}}{\beta_\lambda^*} = 1. \]

(iii) The total energy of the problem blows up at the limit. Precisely,

\[ \lim_{\lambda \to -\infty} b_\lambda^{\frac{p+1}{2}} \int_{B_1} \left\{ |\nabla u_\lambda|^2 - \lambda \frac{u_\lambda^2}{|x|^2} \right\} = \frac{N\omega_N}{2} \left[ \frac{8(p + 1)}{p - 1} \right]^{\frac{p+1}{2}} \frac{\Gamma\left( \frac{p+1}{p-1} \right)^2}{\Gamma\left( \frac{2p+2}{p-1} \right)}. \]

where \(\omega_N\) is the measure of the unit ball in \(\mathbb{R}^N\).
Then we find that the problem (1.1) is equivalent to
$$\lim_{\lambda \to -\infty}$$
where
$$\omega$$
the map
$$\Delta$$
Consider the expression of $$\Delta u$$ in polar coordinates $$\rho = |x|$$, $$\omega = x/|x|$$:
$$\Delta u \equiv \frac{\partial^2 u}{\partial \rho^2} + \frac{N - 1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \Delta_{S^{N-1}} u,$$
where $$\Delta_{S^{N-1}} u$$ means the Laplace-Beltrami operator on the unit sphere applied to the map $$\omega \in S^1 \to u(\rho \omega)$$. Applying this formula to (2.2) we get
$$\Delta u = b^{-2} r^{-2b-a} \left\{ r^2 \frac{\partial^2 v}{\partial r^2} + (bN + 1 - 2a - 2b) r \frac{\partial v}{\partial r} + (a^2 + 2ab - abN) v + b^2 \Delta_{S^{N-1}} v \right\},$$
where $$r = |y|$$. It allows to rewrite equation (1.1) in terms of $$v$$:
$$0 = \frac{\partial^2 v}{\partial r^2} + (bN + 1 - 2a - 2b) \frac{1}{r} \frac{\partial v}{\partial r} + (a^2 + 2ab + b^2 \lambda - abN) \frac{v}{r^2} + b^2 \Delta_{S^{N-1}} v + b^2 r^{2b+a-ap-2} v^p \}.$$
We choose $$a, b$$ such that
$$a^2 + 2ab + b^2 \lambda - abN = 0, \quad 2b + a - ap = 2, \quad b > 0.$$ (2.4)
Solving for $$b$$ we get
$$b = b_\lambda = \left[ \frac{p - 1}{2} \left( \sqrt{\lambda} - \sqrt{\bar{\lambda}} \right) + 1 \right]^{-1},$$ (2.5)
where $$\bar{\lambda} = (N - 2)^2 / 4$$. Notice that $$b_\lambda$$ is increasing with respect to $$\lambda$$, observing that
$$\lim_{\lambda \to -\infty} b_\lambda = 0, \quad b_0 = 1$$ and $$b_\lambda < \infty$$ since $$1 < p < (N + 2)/(N - 2)$$. $$0 < b_\lambda < 1$$. The value of $$a_\lambda$$ in terms of $$b_\lambda$$ is
$$a_\lambda = \frac{2b_\lambda - 2}{p - 1}. $$ (2.6)
Then we find that the problem (1.1) is equivalent to
$$\begin{cases}
- \frac{\partial^2 v}{\partial r^2} - \frac{M_\lambda - 1}{r} \frac{\partial v}{\partial r} - \frac{b_\lambda^2}{r^2} \Delta_{S^{N-1}} v = b_\lambda^2 v^p \quad & \text{in } \Omega_{b_\lambda} \\
v \geq 0 \quad & \text{in } \Omega_{b_\lambda} \\
v \big|_{\partial \Omega_{b_\lambda}} = 0
\end{cases}$$ (2.7)
Next we see a useful dense subset of $M$ up on the first term in the inner product (2.9) by setting $s = \alpha$ with respect to where $q = 2$. Lemma 2.2. Denote by $D$.

Proof. Let $X$ containing the origin, let $D$ be the completion of $C^\infty_0(U)$, the space of continuously differentiable functions with compact support in $U$, with respect to the topology induced by the inner product

$$\langle \phi, \phi' \rangle_{X_\alpha} = \int_U r^{\alpha - 2} \phi \phi' + \int_U r^\alpha \nabla \phi \cdot \nabla \phi'.$$

We need some basic proprieties about this Hilbert space. The first one is a well known embedding theorem (see [3]):

Lemma 2.1 (Caffarelli-Kohn-Nirenberg). Let $\alpha > 2 - N$ and $0 \leq s \leq 1$, where $N \geq 3$. Then for all $\phi \in C^\infty_c(\mathbb{R}^N)$ we have

$$\left( \int_{\mathbb{R}^N} |y|^{q/(\alpha - 2 - s)} |\phi|^q dy \right)^{2/q} \leq C_{\alpha,s} \int_{\mathbb{R}^N} |y|^\alpha |\nabla \phi|^2 dy,$$

where $q = 2N/(N - 2 + 2s)$. Additionally, the best constant $C_{\alpha,s}$ is increasing with respect to $\alpha$.

By density, this inequality also holds in the space $X_\alpha(U)$, and so we can give up on the first term in the inner product (2.9) by setting $s = 1$ and $q = 2$. In particular, since $M_\lambda > 2$, we have that $\alpha = M_\lambda - N$ is admissible in Lemma 2.1. Next we see a useful dense subset of $X_\alpha$.

Lemma 2.2. Denote by $D(U)$ the subspace of $C^1_0(U)$ of the functions vanishing at some neighbourhood of the origin. Then $D(U)$ is a dense subspace of $X_\alpha(U)$.

Proof. Let $\phi \in C^1_0(U)$ and $\eta$, smooth such that

$$\eta_x = \begin{cases} 0 & \text{if } |y| < \epsilon \\ 1 & \text{if } |y| > 2\epsilon. \end{cases}$$

and we can assume $|\nabla \eta_x| \leq 4/\epsilon$. Then

$$\int_{B_\epsilon} r^\alpha |\nabla (\eta_x \phi - \phi)|^2 \leq C \left( \int_{B_{2\epsilon}} r^\alpha |\nabla \phi|^2 + \int_{B_{2\epsilon}} r^\alpha \frac{4}{\epsilon^2} \phi^2 \right)$$

$$\leq C' \left( \int_{B_{2\epsilon}} r^\alpha |\nabla \phi|^2 + \int_{B_{2\epsilon}} r^\alpha - 2 \phi^2 \right).$$

So $\eta_x \phi$ approaches $\phi$ in $X_\alpha$ norm when $\epsilon \to 0$.

The relation between the energies of the problems for $u$ and $v$ is stated below.

Lemma 2.3. The transformation $L = L[u]$ is invertible as map from $W^{1,2}_0(\Omega)$ onto $X_{M_\lambda - N}(\Omega_b)$. Furthermore we have the energy identity

$$\int_{\Omega} \left\{ |\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \right\} = \frac{1}{b_\lambda} \int_{\Omega_b} \left\{ r^{M_\lambda - N} \left( \frac{\partial v}{\partial r} \right)^2 + b_\lambda^2 \lambda r^{M_\lambda - N - 2} |\nabla_{S^{N-1}} v|^2 \right\}. $$
Remark. Observe that, since $\lambda < \lambda$, Hardy’s inequality gives that the left hand side of the last relation is an equivalent norm of $W^{1,2}_0(\Omega)$. Moreover, the right hand side is an equivalent norm of $X_{M-N}(\Omega_b)$, being enough to remember the Pythagorean identity

$$|\nabla \varphi|^2 = \frac{\partial \varphi}{\partial r}^2 + \frac{1}{r^2} |\nabla_{S^{N-1}} \varphi|^2.$$ (2.10)

It follows that $\mathcal{L}$ is actually an isometric bijection.

Proof. Let $u \in D(\Omega)$ (see Lemma 2.2), then $v = \mathcal{L}[u]$ is in $D(\Omega_b)$. Let us call

$$I = \int_{\Omega} \left( \frac{\partial u}{\partial \rho} \right)^2 - \frac{\lambda}{|x|^2} u^2 \right) dx, \quad J = \int_{\Omega} \frac{1}{|x|^2} |\nabla_{S^{N-1}} u|^2 \, dx.$$

The relation between $u$ and $v = \mathcal{L}[u]$ can be written in polar coordinates as $u(\rho, \omega) = r^{-a} v(r, \omega)$ where $\rho > 0, \omega \in S_1(0), r = \rho^{1/b}$. The first derivative transforms as

$$\frac{\partial u}{\partial \rho} = \frac{1}{b} r^{-a-b+1} \left( \frac{\partial v}{\partial r} - \frac{a}{r} v \right).$$ (2.11)

Consider the change of variables $y = |x|^{1/b-1} x$, whose determinant of jacobian matrix is $|dx/|dy| = b|y|^{-N(1-b)}$. Then

$$b \int_{\Omega_b} \left( \frac{\partial u}{\partial \rho} \right)^2 \, dy = \int_{\Omega_b} r^{-2a-2b+2-N(1-b)} \left( \frac{\partial v}{\partial r} - \frac{a}{r} v \right)^2 \, dy$$

$$= \int_{\Omega_b} r^{M-N} \left( \frac{\partial v}{\partial r} - \frac{a}{r} v \right)^2 \, dy \quad \text{(using (2.8))}$$

$$= \int_{\Omega_b} r^{M-N} \left( \frac{\partial v}{\partial r} \right)^2 \, dy + a^2 \int_{\Omega_b} r^{M-N-2} v^2 \, dy - a \int_{\Omega_b} r^{M-N-1} v \frac{\partial v}{\partial r} \, dy.$$

Concerning the last integral, integrating by parts we get

$$\int_{\Omega_b} r^{M-N-1} v \frac{\partial v}{\partial r} \, dy = \sum \int_{\Omega_b} |y|^{M-N-2} y_i v \frac{\partial v}{\partial y_i} \, dy =$$

$$= - \sum \int_{\Omega_b} \frac{\partial}{\partial y_i} (|y|^{M-N-2} y_i v^2) \, dy = -(M-2) \int_{\Omega_b} |y|^{M-N-2} v^2 \, dy$$

We therefore have

$$b \int_{\Omega_b} \left( \frac{\partial u}{\partial \rho} \right)^2 \, dy = \int_{\Omega_b} r^{M-N} \left( \frac{\partial v}{\partial r} \right)^2 \, dy + a(a + M - 2) \int_{\Omega_b} r^{M-N-2} v^2 \, dy.$$

Changing variables as before we also have that

$$\int_{\Omega_b} \frac{|u|^2}{|x|^2} \, dx = b \int_{\Omega_b} r^{M-N-2} v^2 \, dy.$$

Relations (2.4) give that $a(a + M - 2) = b^2 \lambda$, and so we get

$$I = \frac{1}{b} \int_{\Omega_b} r^{M-N} \left( \frac{\partial v}{\partial r} \right)^2 \, dy.$$

The computation of $J$ is much easier since the change of variables do not acts on tangential directions. We get

$$J = b \int_{\Omega_b} r^{M-N-2} |\nabla_{S^{N-1}} v|^2.$$
And so by Pythagorean identity (2.10) we have

\[ I + J = \int_{\Omega} \left\{ \nabla u^2 - \lambda \frac{u^2}{|x|^2} \right\}, \]

which is the desired identity. It shows that \( \mathcal{L} \) is an isometry between the spaces \( \mathcal{D}(\Omega) \) and \( \mathcal{D}(\Omega_b) \). Since these are, by Lemma 2.2, dense subspaces of \( W^{1,2}_0(\Omega) \) and \( X_{M-N}(\Omega_b) \), respectively, we have that \( \mathcal{L} \) univocally extends to an isometry between the last spaces. \( \square \)

**Lemma 2.4.** The function \( v \) defined in (2.1) verifies

\[ \int_{\Omega_b} \left\{ r^{M-N} \frac{\partial v}{\partial r} \frac{\partial \phi}{\partial r} + b^2 r^{M-N-2} \nabla_{SN-1} v \cdot \nabla_{SN-1} \phi \right\} = b^2 \int_{\Omega_b} r^{M-N} v^p \phi \]  

(2.12)

for all \( \phi \in X_{M-N}(\Omega_b) \).

**Proof.** Let \( u \) a solution for (1.1), i.e., a critical point of the functional \( J \) stated in (1.2). It satisfies

\[ \int_{\Omega} \left\{ \nabla u \cdot \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi \right\} = \int_{\Omega} u^p \varphi \]

for all \( \varphi \in W^{1,2}_0(\Omega) \). Define

\[ Q_1[\varphi, \tilde{\varphi}] = \int_{\Omega} \left\{ \nabla \tilde{\varphi} \cdot \nabla \varphi - \frac{\lambda}{|x|^2} \tilde{\varphi} \varphi \right\}, \]

\[ Q_2[\phi, \tilde{\phi}] = \int_{\Omega_b} \left\{ r^{M-N} \frac{\partial \phi}{\partial r} \frac{\partial \tilde{\phi}}{\partial r} + b^2 r^{M-N-2} \nabla_{SN-1} \phi \cdot \nabla_{SN-1} \tilde{\phi} \right\}. \]

Both are bilinear maps in \( W^{1,2}_0(\Omega) \) and \( X_{M-N}(\Omega_b) \), respectively. Given \( \varphi, \tilde{\varphi} \in W^{1,2}_0(\Omega) \) and \( \phi, \tilde{\phi} \in X_{M-N}(\Omega_b) \), we have

\[ 4Q_1[\varphi, \tilde{\varphi}] = Q_1[\varphi + \tilde{\varphi}, \varphi + \tilde{\varphi}] - Q_1[\varphi - \tilde{\varphi}, \varphi - \tilde{\varphi}] = -Q_2[\phi + \tilde{\phi}, \phi + \tilde{\phi}] - Q_2[\phi - \tilde{\phi}, \phi - \tilde{\phi}] \]

by Lemma 2.3.

In particular, \( Q_1[\varphi, u] = Q_2[\phi, v] \), i.e.

\[ \int_{\Omega} \left\{ \nabla u \cdot \nabla \varphi - \frac{\lambda}{|x|^2} u \varphi \right\} = \frac{1}{b} \int_{\Omega_b} \left\{ r^{M-N} \frac{\partial v}{\partial r} \frac{\partial \phi}{\partial r} + b^2 r^{M-N-2} \nabla_{SN-1} v \cdot \nabla_{SN-1} \phi \right\}. \]

where \( \phi = \mathcal{L}[\varphi] \). Moreover using the change of variables \( y = |x|^{1/b-1}x \) (as in the previous lemma) we obtain

\[ \frac{1}{b} \int_{\Omega} u^p v \, dx = \int_{\Omega_b} |y|^{-a \lambda} v(y)^p |y|^{-a} \phi(y) \, dx \, dy \]

\[ = \int_{\Omega_b} |y|^{-a \lambda - a(1-b)} v^p \phi \, dy = \int_{\Omega_b} |y|^{M-N} v^p \phi \, dy, \]

where we used (2.6) and (2.5). It follows that that \( v \) satisfies (2.12). \( \square \)

3. **Proof of Theorem 1**

Recalling that

\[ u(x) = |x|^{-\frac{2 \lambda}{b \lambda}} v(|x|^{\frac{1}{b \lambda}-1} x) \]

and

\[ -\frac{a \lambda}{b \lambda} = \sqrt{\lambda} - \lambda - \sqrt{\lambda} \]

(by (2.5) and (2.6)), we see that it is enough to prove that \( \sup v < \infty \). We stress that we already know that \( v \) is smooth in \( \Omega_b \setminus \{0\} \), smoothness inherited from \( u \).
We will divide the proof in three steps. First we derive an integral (test) version of (2.7) and exhibit an inequality for powers of \(v\). In the second step we show that \(v\) belongs to \(L^{p+1}\). Finally, by using Moser iteration we conclude that \(v\) is bounded.

**Step 1.** We want to show that the estimate

\[
(\beta - 1) \int_{\Omega} r^\alpha v^{\beta - 2} |\nabla v|^2 \leq C \int_{\Omega_b} r^\alpha v^{\beta - 1 + \beta},
\]

where \(C = C(N, p) > 0\), holds for all \(2 - N < \alpha < M_\lambda - N\) if \(\beta > 1\) and for \(\alpha = M_\lambda - N\) if \(\beta = 2\). The last case follows by taking \(\phi = v\) in (2.12) and using the identity (2.4). To show for the case \(\alpha < M_\lambda - N\) we begin to writing equation (2.7) in the following way

\[
- \left( \frac{\partial^2 v}{\partial r^2} + \frac{\alpha + N - 1}{r} \frac{\partial v}{\partial r} \right) - \frac{b_1^2}{r^2} \Delta_{S^{N-1}} v - \frac{M_\lambda - N - \alpha}{r} \frac{\partial v}{\partial r} = b_2^2 v^p.
\]

Let \(\eta\) a smooth radial function such that

\[
\eta = \begin{cases} 
0 & \text{if } |y| \leq \epsilon \\
1 & \text{if } |y| \geq 2\epsilon.
\end{cases}
\]

with the additional assumption \(0 \leq \partial \eta/\partial r \leq 2/\epsilon\). Define \(\phi = \eta v^{\beta - 1}, \beta > 1\). Since \(v\) is smooth away from 0 we have that \(\phi\) is smooth in \(\Omega_b\). By Gauss-Green formula we have

\[
\int_{\Omega_b} r^\alpha \phi \left( - \frac{\partial^2 v}{\partial r^2} - \frac{\alpha + N - 1}{r} \frac{\partial v}{\partial r} \right) dy = \int_{\Omega_b} r^{1-N} \phi \frac{\partial}{\partial r} \left( r^{\alpha + N - 1} \frac{\partial v}{\partial r} \right) dy + \int_{\Omega_b} \sum_i \frac{\partial}{\partial y_i} \eta v^{\beta - 1} \frac{\partial v}{\partial r} dy.
\]

Analogously, using integration by parts in \(S_1(0)\) we find

\[
\int_{\Omega_b} r^\alpha \phi \left( - \frac{1}{r^2} \Delta_{S^{N-1}} v \right) = \int_{\Omega_b} r^{\alpha - 2} \nabla_{S^{N-1}} \phi \cdot \nabla_{S^{N-1}} v.
\]

So we derive the following equation

\[
\int_{\Omega_b} r^\alpha \frac{\partial \phi}{\partial r} \frac{\partial v}{\partial r} + b_2^2 \int_{\Omega_b} r^{\alpha - 2} \nabla_{S^{N-1}} \phi \cdot \nabla_{S^{N-1}} v - (M - N - \alpha) \int_{\Omega_b} r^{\alpha - 1} \phi \frac{\partial v}{\partial r} = b_2^2 \int_{\Omega_b} r^\alpha \phi v^p.
\]

Now let us expand the terms in (3.2) and make \(\epsilon \to 0\). For the first one,

\[
\int_{\Omega_b} r^\alpha \frac{\partial \phi}{\partial r} \frac{\partial v}{\partial r} = \int_{\Omega_b} r^\alpha v^{\beta - 1} \frac{\partial \eta}{\partial r} \frac{\partial v}{\partial r} + (\beta - 1) \int_{\Omega_b} r^\alpha \eta v^{\beta - 2} \left( \frac{\partial v}{\partial r} \right)^2.
\]

Then for any \(\delta > 0\):

\[
\left| \int_{\Omega_b} r^\alpha v^{\beta - 1} \frac{\partial \eta}{\partial r} \frac{\partial v}{\partial r} \right| \leq \frac{1}{2\delta} \int_{\Omega_b} r^\alpha v^{\beta - 2} \frac{\partial \eta}{\partial r} \frac{\partial v}{\partial r}^2 + \frac{\delta}{2} \int_{\Omega_b} r^\alpha v^{\beta - 2} \frac{\partial \eta}{\partial r} \left( \frac{\partial v}{\partial r} \right)^2.
\]

It gives the estimate

\[
\int_{\Omega_b} r^\alpha \frac{\partial \phi}{\partial r} \frac{\partial v}{\partial r} \geq \int_{\Omega_b} r^\alpha v^{\beta - 2} \left( \frac{\partial v}{\partial r} \right)^2 \left[ (\beta - 1) \eta - \frac{\delta}{2} \frac{\partial \eta}{\partial r} \right] - \frac{1}{2\delta} \int_{\Omega_b} r^\alpha v^{\beta - 2} \frac{\partial \eta}{\partial r} \left( \frac{\partial v}{\partial r} \right)^2.
\]
Now let us look for the third integral in (3.2). We have
\[ \int_{\Omega_b} r^{a-1} \phi \frac{\partial v}{\partial r} = \frac{1}{\beta} \int_{\Omega_b} r^{a-1} \eta_v \frac{\partial \phi}{\partial r} = \frac{1}{\beta} \sum_i \int_{\Omega_b} y_i |y|^{a-2} \eta_v \frac{\partial \phi}{\partial y_i} = \]
\[ - N + \alpha - 2 \frac{1}{\beta} \int_{\Omega_b} |y|^{a-2} \eta_v \frac{\partial \phi}{\partial y} - \frac{1}{\beta} \sum_i \int_{\Omega_b} y_i |y|^{a-2} \frac{\partial \eta_v}{\partial y_i} \leq - \frac{1}{\beta} \int_{\Omega_b} |y|^{a-1} \frac{\partial \eta_v}{\partial y}. \]

Putting the last two estimates together we find
\[ \int_{\Omega_b} r^{a} \frac{\partial \phi}{\partial r} \frac{\partial v}{\partial r} - (M - N - \alpha) \int_{\Omega_b} r^{a-1} \phi \frac{\partial v}{\partial r} \geq \]
\[ \int_{\Omega_b} r^{a} \frac{\partial \phi}{\partial r} \left( \frac{\partial v}{\partial r} \right)^2 \left( \beta - 1 \right) = A, \]
\[ + \int_{\Omega_b} r^{a-1} v^{\beta-2} \left( \frac{\partial \eta_v}{\partial r} \right)^2 \left( M - N - \alpha - \frac{r}{2\beta} \right) \]
\[ := I_\epsilon + J_\epsilon. \]

Now let us choose \( \delta = \delta(\epsilon) = \epsilon \beta/(M - N - \alpha) \). Then we have \( J_\epsilon \geq 0 \) (the support on the integrand in \( J_\epsilon \) is \( B_{2\lambda} \setminus B_r \) since \( \partial \eta_v/\partial r \equiv 0 \) outside of this set). Denote by \( A_\epsilon \) the function inside the brackets in \( I_\epsilon \). Then \( A_\epsilon \) converges to \( \beta - 1 \) in \( \Omega_b \).

Furthermore,
\[ |A_\epsilon(y)| \leq \begin{cases} 
\beta - 1 & \text{if } y \notin B_{2\lambda} \setminus B_r \\
\beta - 1 + \beta/(M - N - \alpha) & \text{if } y \in B_{2\lambda} \setminus B_r 
\end{cases} \]

and so \( A_\epsilon \) is bounded. Therefore
\[ \liminf_{\epsilon \to 0} (I_\epsilon + J_\epsilon) \geq (\beta - 1) \int_{\Omega_b} r^{a} v^{\beta-2} \left( \frac{\partial v}{\partial r} \right)^2. \]

On other hand (3.2) gives
\[ I_\epsilon + J_\epsilon + (\beta - 1) b^2 \int_{\Omega_b} r^{a-2} \eta_v v^{\beta-2} |\nabla S^{N-1} v|^2 + b^2 \int_{\Omega_b} r^{a-2} v^{\beta-2} |\nabla S^{N-1} v|^2 = 0 \text{ because } \eta \text{ is radial} \]
\[ \leq b^2 \int_{\Omega_b} r^{a} \eta_v v^{p+1 + \beta}. \]

We can pass to the limit when \( \epsilon \to 0 \) due to the monotone convergence theorem (since we can assume \( \eta_v \leq \eta \) if \( \epsilon' \leq \epsilon \)). We find
\[ (\beta - 1) \left\{ \int_{\Omega_b} r^{a} v^{\beta-2} \left( \frac{\partial v}{\partial r} \right)^2 + b^2 \int_{\Omega_b} r^{a-2} v^{\beta-2} |\nabla S^{N-1} v|^2 \right\} \leq b^2 \int_{\Omega_b} r^{a} v^{p+1 + \beta}. \]

Using that \( 1, b\lambda < B \) and the Pythagorean identity (2.10) we obtain (3.1).

**Step 2.** The aim of this step is to prove that
\[ \int_{\Omega_b} v^{p+1} < \infty. \]

By simplicity, we consider the constant \( C \) in (3.1) equal 1. Choosing \( \beta = 2 \) in (3.1) we have
\[ \int_{\Omega_b} r^{a} |\nabla v|^2 \leq \int_{\Omega_b} r^{a} v^{p+1}. \tag{3.3} \]

Since \( 2 < p + 1 < 2^* \), we can choose
\[ s = \frac{N - 2}{2(p + 1)} \left( \frac{N + 2}{N - 2} - p \right) \in (0, 1). \]
such that \( q = p + 1 \) in Lemma 2.1. Let \( \alpha = (p + 1)(\alpha/2 - s) \). Then (3.3) becomes
\[
\left( \int_{\Omega_b} r^\alpha v^{p+1} \right)^{2/(p+1)} \leq C_{\alpha,s} \int_{\Omega_b} r^\alpha v^{p+1}.
\] (3.4)

We want to iterate this inequality setting \( \alpha_{k+1} = (p + 1)(\alpha_k/2 - s) \), starting with \( \alpha_0 = M - N \). Indeed, the right-hand side of (3.3) is finite for \( \alpha = M - N \) since, by (3.2) and Lemma 2.3, it is bounded by the energy of the solution \( u \). The solution of the recursive equation is
\[
\alpha_k = \left( \frac{p + 1}{2} \right)^k \left[ \alpha_0 + \frac{2s(p + 1)}{p - 1} \left( \frac{2}{p + 1} \right)^k - 1 \right].
\]

Then \( \alpha_k < 0 \) if and only if
\[
\left( \frac{2}{p + 1} \right)^k < \frac{4 - (M - 2)(p + 1)}{4 - (N - 2)(p + 1)} = b, \quad \text{by (2.8) and (2.6).}
\] (3.5)

So there is a minimum \( k_0 = k_0(\lambda) \) such that \( \alpha_{k_0} < 0 \). Observe that \( k_0 \to \infty \) as \( \lambda \to -\infty \).

Iterating (3.4) we obtain
\[
\left( \int_{\Omega_b} r^\alpha v^{p+1} \right)^{\frac{k}{k-1}} \leq \prod_{j=0}^{k-1} \left( C_{\alpha_j,s} \right) \left( \frac{r^{M-N} v^{p+1}}{\int_{\Omega_b} r^\alpha v^{p+1}} \right)^{\frac{k}{k-1}} \int_{\Omega_b} r^{M-N} v^{p+1}.
\]

Since \( (\alpha_j) \) is decreasing and the best constant \( C_{\alpha,s} \) is increasing with respect to \( \alpha \) (see Lemma 2.1), we have \( C_{\alpha_j,s} \leq C_{\alpha_0,s} \) and therefore
\[
\prod_{j=0}^{k-1} \left( C_{\alpha_j,s} \right) \left( \frac{r^{M-N} v^{p+1}}{\int_{\Omega_b} r^\alpha v^{p+1}} \right)^{\frac{k}{k-1}} \leq \prod_{j=0}^{\infty} \left( C_{\alpha_0,s} \right) \left( \frac{r^{M-N} v^{p+1}}{\int_{\Omega_b} r^\alpha v^{p+1}} \right)^{\frac{k}{k-1}} := C
\]

So for all \( k = 1, 2, ..., k_0 \) we have
\[
\left( \int_{\Omega_b} r^\alpha v^{p+1} \right)^{\frac{k}{k-1}} \leq C \int_{\Omega_b} r^{M-N} v^{p+1} < \infty.
\]

where \( C = C(N; p) \). This last integral is bounded by the norm of \( v \) in \( X_{M-N}(\Omega_b) \), by choosing \( \phi = v \) in (2.12). Since \( \alpha_{k_0-1} \geq 0 \), there is a \( 0 < t_0 \leq 1 \) such that \( 0 = (1 - t_0)\alpha_{k_0} + t_0\alpha_{k_0-1}, \) i.e., \( t_0 = \alpha_{k_0} / (\alpha_{k_0} - \alpha_{k_0-1}) \). By Holder inequality
\[
\int_{\Omega_b} v^{p+1} \leq \left( \int_{\Omega_b} r^{\alpha_{k_0} v^{p+1}} \right)^{1-t_0} \left( \int_{\Omega_b} v^{\alpha_{k_0-1} v^{p+1}} \right)^{t_0} < \infty.
\]

Step 3. In this final step we perform the Moser iteration. This is a standard argument. Indeed, we just need (3.1) with \( \alpha = 0 \) and the estimate \( \int_{\Omega_b} v^{p+1} < \infty \). Then the same proof of Theorem 8.15 in [10] applies. It gives precisely that
\[
\sup v < \infty,
\]
and it ends the proof.

4. Asymptotic behaviour of the radial solution

Now we shall focus our attention on the radial solution \( u(x) = u(|x|) = u(r) \) to (1.4) and we study its properties as \( \lambda \to -\infty \). As in the previous section let us set
\[
\bar{v}_\lambda(r) = \mathcal{L}[u_\lambda] = r^{\lambda/2} u(r^{\lambda/2}).
\] (4.1)
Here $a_\lambda$, $b_\lambda$, and $M_\lambda$ are as in (2.5), (2.6), (2.8). Note also that as $\lambda \to -\infty$,
\begin{align*}
a_\lambda &\to \frac{2}{p-1} \\
b_\lambda &\to 0 \\
M_\lambda &\to M_\infty = \frac{2p+2}{p-1} \tag{4.2}
\end{align*}

Then normalize $\hat{v}_\lambda$ as follows,
\begin{equation}
\hat{v}_\lambda(r) = b_{\lambda}^{2/(p-1)} \hat{v}_\lambda(r). \tag{4.3}
\end{equation}

The redefined $v_\lambda$ satisfies the problem,
\begin{equation}
\begin{cases}
-v''_\lambda - \frac{M_\lambda - 1}{r} v_\lambda = v_\lambda^p & \text{in } (0,1) \\
v_\lambda > 0 & \text{in } (0,1) \\
v_\lambda(1) = 0
\end{cases} \tag{4.4}
\end{equation}

We point out that right now is not clear if $v'_\lambda(0) = 0$. However $v_\lambda \in L^\infty$, as shown in the previous section (for $\bar{v}_\lambda$). Moreover (2.12) becomes
\begin{equation}
\int_0^1 r^{M-1} v'_\lambda \phi' = \int_0^1 r^{M-1} v^p \phi \tag{4.5}
\end{equation}

for all $\phi \in C^1_0(0,1)$, or more generally, for any $\phi \in A$ where
\[ A = \{ \phi : [0,1] \to \mathbb{R} \text{ absolutely continuous : } \phi(1) = 0 \} \]

First we prove important some proprieties of (4.4).

**Lemma 4.1.** For every $\lambda \leq \bar{\lambda}$, the solution $v_\lambda$ to (4.4) verifies:

(i) $v_\lambda \in C^1[0,1]$, $v''_\lambda \in L^\infty(0,1)$;
(ii) $|v'_\lambda(r)| \leq C \|v_\lambda\|^p r$, $\|v''_\lambda\|_{\infty} \leq C \|v_\lambda\|^p\infty$ with $C$ independent of $\lambda$. In particular, $v'_\lambda(0) = 0$;
(iii) $\|v_\lambda\|^p_{p-1} \geq C > 0$, where $C$ is independent of $\lambda$;
(iv) The following Pohozaev identity holds
\begin{equation}
\left( \frac{M_\lambda}{p+1} - \frac{M_\lambda - 2}{2} \right) \int_0^1 r^{M-1} v^{p+1}_\lambda = \frac{1}{2} v'_\lambda(1)^2. \tag{4.6}
\end{equation}

**Proof.** For $0 < r < 1$ fixed and $\epsilon > 0$ small enough let us define $\eta_\epsilon : [0,1] \to \mathbb{R}$ by
\[ \eta_\epsilon(s) = \begin{cases} 
1 & \text{if } s \in [0, \epsilon] \\
1 - \frac{(s-r)}{\epsilon} & \text{if } s \in [\epsilon, r+\epsilon] \\
0 & \text{if } s \in [r+\epsilon, 1]
\end{cases} \]

Then $\eta_\epsilon$ is absolutely continuous so and can be insert as test function in (4.5). We get
\[ -\frac{1}{\epsilon} \int_r^{r+\epsilon} s^{M-1} v' = \int_0^{r+\epsilon} s^{M-1} v^p \eta_\epsilon. \]

Letting $\epsilon \to 0$ we get
\[ -r^{M-1} v'(r) = \int_0^r s^{M-1} v^p. \]

Since $v \in L^\infty$ we derive that $|v'(r)| \leq \|v\|_\infty^p \frac{r^M}{M^p} \leq \frac{1}{M^p} \|v\|_\infty^p r$. Using (4.4) we get $\|v''\|_{\infty} \leq C \|v\|_\infty^p$, with $C$ independent of $\lambda$. This shows i) and ii).
In order to prove iii) let \( \ell \geq 2 \) and call \( \mu_{1,\ell} \) be the first eigenvalue of the operator

\[
A_\ell[\phi](r) = -\phi''(r) - \frac{\ell - 1}{r} \phi'(r)
\]

over the space \( \mathcal{A} \) and \( \phi_{1,\ell} \in \mathcal{A} \) be the first positive eigenfunction verifying

\[
\int_0^1 r^{\ell-1} \phi_{1,\ell} \phi = \mu_{1,\ell} \int_0^1 r^{\ell-1} \phi_{1,\ell} \phi
\]

for all \( \phi \in \mathcal{A} \). Clearly we have that \( \mu_{1,\ell} > 0 \) for all \( \ell \geq 2 \). Using \( \phi_{1,M} \) as test function in (4.5) we get

\[
\mu_{1,M} \int_0^1 r^{M-1} v \phi_{1,M} = \int_0^1 r^{M-1} v \phi_{1,M} \leq \|v\|^{p-1}_\infty \int_0^1 r^{M-1} v \phi_{1,M}
\]

and so \( \|v\|^{-1}_\infty \geq \mu_{1,M} \geq C > 0 \), where \( C = \min_{\ell \in [2,M]} \mu_{1,\ell} > 0 \).

The proof of the Pohozaev identity is standard (multiply (4.4) by \( r \) and integrate).

Now we are in position to prove our “local” estimate.

Proof of Theorem 1.2. First we show that \( \|v_\lambda\|_\infty \to \infty \) when \( \lambda \to -\infty \). Suppose by contradiction that there is a subsequence of \( \{v_\lambda\} \) bounded in \( L^\infty \). Therefore also does \( \{v_\lambda^\prime\} \), by Lemma 4.1, and then by Ascoli-Arzela theorem, there is thus a convergence to some \( v_0 \) in \( C^1[0,1] \). Passing the limit in (4.5) we get

\[
\int_0^1 r^{L-1} v_0 \phi' = \int_0^1 r^{L-1} v_0^p \phi
\]

and arguing as in the previous lemma we get that \( v_0 \) satisfies the problem

\[
\begin{align*}
-v_0'' - \frac{M_\infty - 1}{r} v_0' &= v_0^p & \text{in } (0,1) \\
v_0(0) &= 0 \\
v_0(1) &= 0
\end{align*}
\]

Nevertheless, this is a critical problem (since \( p = \frac{M_\infty + 2}{M_\infty - 2} \)) and so admits only the trivial solution (for details, we refer to [7]). Then we would have \( v_0 = 0 \), contradicting iii) of Lemma 4.1 and showing that \( \|v_\lambda\|_\infty \to \infty \).

Consider the scaling

\[
w_\lambda(r) = \alpha_\lambda v_\lambda(\beta_\lambda r)
\]

for

\[
\alpha_\lambda = \|v_\lambda\|_\infty^{-1} \quad \text{and} \quad \beta_\lambda = [M_\lambda(M_\lambda - 2)]^{1/2} \|v_\lambda\|_\infty^{-\frac{M_\lambda - 1}{2}}.
\]

Of course \( \alpha_\lambda, \beta_\lambda \to 0 \) and \( w_\lambda \) satisfies

\[
\begin{align*}
-w_\lambda'' - \frac{M_\lambda - 1}{r} w_\lambda' &= M_\lambda(M_\lambda - 2) w_\lambda^p & \text{in } (0,1/\beta) \\
w_\lambda(0) &= 0 \\
w_\lambda(1/\beta) &= 1 \\
w_\lambda'(0) &= w_\lambda(1/\beta) = 0
\end{align*}
\]

A first integration shows that \( w_\lambda'(0) \leq 0 \), and so \( \|w_\lambda\|_\infty = w_\lambda(0) = 1 \). From this we can deduce, as before for \( v_\lambda \), that \( \{w_\lambda\} \) has a subsequence converging to some \( W_\lambda \).
in $C^1$ norm of any compact subset of $[0, +\infty)$, and this function $W$ therefore satisfies
\[
\begin{cases}
-W'' - \frac{M_\infty - 1}{r} W' = M_\infty (M_\infty - 2) W^{\frac{M_\infty + 2}{M_\infty}} & \text{in } (0, \infty) \\
W > 0 & \text{in } (0, \infty) \\
W(0) = 1 \\
W'(0) = 0
\end{cases}
\]

In [7, Lemma 5.2], it is shown that the previous problem has only the solution
\[
W(r) = \frac{1}{(1 + r^2)^{\frac{M_\infty - 2}{2}}}
\]
and furthermore
\[
w_\lambda(r) \leq W(r) \quad \forall r \in [0, 1/\beta_\lambda].
\]

Thinking $w_\lambda$ as its canonical extension to $[0, \infty)$ we can also deduce, since $W_\lambda$ vanishes at infinity, that $w_\lambda \to W$ uniformly in $[0, \infty)$. This ends the proof.

Now we go further by proving some good asymptotic of $\alpha_\lambda, \beta_\lambda$. This will be done by taking the limit in the Pohozaev identity (4.6). Taking $v$ as test function in equation (2.12) we have
\[
\int_0^1 (v_\lambda')^2 r^{M_\lambda - 1} = \int_0^1 r^{M_\lambda - 1} v_\lambda^{p+1}.
\]

We refer to the classical paper [13] for the inequality
\[
S_M \left( \int_0^1 r^{M-1} v^q \right)^{1/q^*} \leq \left( \int_0^1 r^{M-1} (v')^2 \right)^{1/2}, \quad M > 2,
\]
where $q^* = 2M/(M - 2)$ and
\[
S_M = [M(M - 2)]^{1/2} \left[ \frac{\Gamma(M/2)^2}{2\Gamma(M)} \right]^{1/M}
\]
is the optimal constant. We have the following

**Lemma 4.2.**
\[
\liminf_{\lambda \to -\infty} \int_0^1 r^{M_\lambda - 1} v_\lambda^{p+1} \geq S_\infty^{\frac{p+1}{2}} = S_\infty^{M_\infty}
\]
where $S_\infty = [M_\infty(M_\infty - 2)]^{1/2} \left[ \frac{\Gamma(M_\infty/2)^2}{2\Gamma(M_\infty)} \right]^{1/M_\infty}$.

**Proof.** If $s + t = M_\lambda - 1$ and $t$ such that $t_\lambda q^*/(p + 1) = M_\lambda - 1$ we get by H"older inequality,
\[
\int_0^1 r^{M_\lambda - 1} v_\lambda^{p+1} = \int_0^1 r^s t^{\frac{p+1}{s}} v_\lambda^{\frac{p+1}{s}} \leq \left\{ \int_0^1 (t^{p+1})^{\frac{q^*}{p+1}} \right\}^{\frac{s}{p+1}} \left\{ \int_0^1 (r^t)^\frac{q^*}{s} \right\}^{\frac{s}{p+1}} v_\lambda^{\frac{p+1}{s}}
\]
\[
= \left( \int_0^1 r^{M_\lambda - 1} v_\lambda^{\frac{p+1}{s}} \right)^{\frac{s}{p+1}} M_\lambda^{\frac{p+1}{s}}
\]
Using this inequality with (4.11) in (4.10) we find
\[
S_M M_\lambda^{\frac{p+1}{s}} \frac{1}{\alpha_\lambda} \leq \left( \int_0^1 r^{M_\lambda - 1} v_\lambda^{p+1} \right)^{\frac{1}{s}} v_\lambda^{\frac{p+1}{s}}.
\]
Let
\[ l = \liminf_{\lambda \to -\infty} \int_0^1 r^{M_\lambda - 1} v^{p+1}_\lambda. \]

Observing that \( q^*_\lambda = \frac{2M_\lambda}{M_\lambda - 2} \to \frac{2M_\lambda}{M_\lambda - 2} = p + 1 \) we deduce from (4.13)
\[ S_\infty = \lim_{\lambda \to -\infty} S_{M_\lambda} \leq l^2 - \frac{p+1}{2}. \]
which gives the claim. \( \square \)

**Lemma 4.3.** We have that
\[ \alpha^\lambda \to 1 \quad \text{as} \ \lambda \to -\infty. \quad (4.14) \]

**Proof.** Using the Pohozaev identity (4.6) and Lemma 4.2 we get
\[ |v^\lambda(1)|^2 \geq 2S_\infty^{p+2} \left( \frac{M_\lambda}{p+1} - \frac{M_\lambda - 2}{2} \right) = b_\lambda S_\infty^{p+2} \left( \frac{N + 2}{N - 2} - p \right) \frac{M_\lambda - 2}{p+1} \geq C b_\lambda. \]
Hereafter \( C \) will be a positive constant independent of \( \lambda \). On the other hand, integrating equation (4.8) we find
\[ -\alpha^\lambda \beta^\lambda v^\lambda(1) = -w^\lambda \left( \frac{1}{\beta^\lambda} \right) = \beta^\lambda M_\lambda^{-1} M_\lambda(M_\lambda - 2) \int_0^\infty r^{M_\lambda - 1} w^{p}_\lambda \]
\[ \geq \beta^\lambda M_\lambda^{-1} \left( M_\infty(M_\lambda - 2) \int_0^\infty r^{M_\lambda - 1} (M_\lambda - 2) w^{p}_\lambda + o(1) \right) \leq C \beta^\lambda M_\lambda^{-1}. \]
So we have \((\beta^\lambda M_\lambda^{-1}/\alpha^\lambda)^2 \geq C b_\lambda \) and the definition of \( \beta^\lambda \) gives \( \alpha^\lambda (M_\lambda - 2)(p+1) - C b_\lambda. \) Still, since \( 4 - (M_\lambda - 2)(p+1) = [4 - (N - 2)(p+1)]b_\lambda \) we find
\[ \alpha^\lambda 2^{-(p-1)(p+1)} b_\lambda \geq C b_\lambda. \quad (4.16) \]
By (4.2) and since \( \|v^\lambda\|_\infty \to 0 \) we can assume that \( \alpha^\lambda, b_\lambda < 1 \). By the mean value theorem we have that \( |\alpha^\lambda - 1| \leq |\log \alpha^\lambda| b_\lambda \), and so using (4.16)
\[ |\alpha^\lambda - 1| \leq C |\log \alpha^\lambda| \alpha^\lambda 2^{-(p-1)(p+1)} b_\lambda \to 0 \quad \text{as} \ \lambda \to -\infty \]
since \( \alpha^\lambda, b_\lambda \to 0 \) as \( \lambda \to -\infty \). This gives the claim. \( \square \)

**Proof of Theorem 1.3.** The previous lemmas and the dominated converge theorem allow to pass to the limit in Pohozaev identity. First let us show that
\[ \frac{1}{\alpha^\lambda} \int_0^1 r^{M_\lambda - 1} v^{p+1}_\lambda = \frac{\beta^\lambda}{\alpha^\lambda} \int_0^\infty r^{M_\lambda - 1} w^{p+1}_\lambda \]
\[ = [M_\lambda(M_\lambda - 2)]^{M_\lambda/2} \alpha^\lambda \frac{1}{\alpha^\lambda} \int_0^\infty r^{M_\lambda - 1} w^{p+1}_\lambda \]
\[ \left( \text{by Lemma 4.3 and (4.9)} \right) \to [M_\infty(M_\infty - 2)]^{M_\infty} \frac{M_\infty}{2} \int_0^\infty r^{M_\infty - 1} W^{p+1}_\infty \]
\[ = [M_\infty(M_\infty - 2)]^{M_\infty} \frac{M_\infty}{2} \Gamma \left( \frac{M_\infty}{2} \right)^2 = S_{M_\infty}. \]

Hence observing that
\[ \frac{\alpha^\lambda}{\beta^\lambda M_\lambda^{-2}} = [M_\lambda(M_\lambda - 2)]^{-M_\lambda/2} \alpha^\lambda \frac{4-(M_\lambda - 2)(p+1)}{\alpha^\lambda}^{-M_\lambda/2} \]
\[ = [M_\lambda(M_\lambda - 2)]^{-M_\lambda/2} \alpha^\lambda \frac{4-(N - 2)(p+1)}{\alpha^\lambda}, \]
and taking into account Lemma 4.3 we can improve estimate (4.15) as
\[ -\frac{1}{\alpha^\lambda} v^\lambda(1) \to (M_\infty(M_\infty - 2)^{1+M_\lambda/2} \int_0^\infty r^{M_\lambda - 1} W^{p+1}_\infty. \quad (4.18) \]
Direct computation shows that
\[
\int_0^\infty r^{M-1}W^p = \frac{1}{M_\infty}.
\]
Inserting (4.17) and (4.18) in the Pohozaev identity we find
\[
\frac{\alpha^2_\lambda}{b_\lambda} \to \frac{1}{4M_\infty} [M_\infty(M_\infty - 2)]^{-\frac{M_\infty}{2}} \Gamma\left(\frac{M_\infty}{2}\right)^2 \Gamma\left(M_\infty\right)^2 \left(\frac{N + 2}{N - 2} - p\right) := A_0.
\]
In particular, it gives that
\[
\alpha_\lambda \sim b_\lambda^{1/2} \text{ and } \beta_\lambda \sim b_\lambda^{(p-1)/4}.
\]
Then we are motivated to define
\[
\beta^*_\lambda = [M_\infty(M_\infty - 1)]^{1/2}A_0^{-1/2}b_\lambda^{-1/4},
\]
which verifies \(\beta^*_\lambda/\beta_\lambda \to 1\) as \(\lambda \to -\infty\) (both \(A_0\) and \(\beta^*_\lambda\) are the same defined in (1.9) and (1.10)). These estimates are crucial in the final step of the proof. First let us recall how \(u\) is written in terms of \(w\),
\[
u_\lambda(r) = r^{-\frac{b_\lambda}{\alpha_\lambda}}u_\lambda(r^{\frac{1}{\alpha_\lambda}}) = b_\lambda^{-2/(p-1)}r^{-\frac{a_\lambda}{\alpha_\lambda}}v_\lambda(r^{\frac{1}{\alpha_\lambda}}) = b_\lambda^{-2/(p-1)}\alpha_\lambda^{-1}r^{-\frac{a_\lambda}{\alpha_\lambda}}w(r^{\frac{1}{\alpha_\lambda}}/\beta_\lambda).
\]
To prove part (i) we start checking that \(|w_\lambda(r^{\frac{1}{\alpha_\lambda}}/\beta_\lambda) - W(r^{\frac{1}{\alpha_\lambda}}/\beta^*_\lambda)|\) vanishes uniformly in \([0, \infty)\) as \(\lambda \to -\infty\). Since \(w_\lambda \to W\) uniformly in \([0, \infty)\) we have
\[
\sup_{r \geq 0} |w_\lambda(r^{\frac{1}{\alpha_\lambda}}/\beta_\lambda) - W(r^{\frac{1}{\alpha_\lambda}}/\beta^*_\lambda)| \to 0,
\]
and so it remains to prove that \(\sup_{r \geq 0} |W(r^{\frac{1}{\alpha_\lambda}}/\beta_\lambda) - W(r^{\frac{1}{\alpha_\lambda}}/\beta^*_\lambda)| \to 0\). In fact, by the mean value theorem we have that
\[
|W(r_1) - W(r_2)| \leq \sup_{r \geq 0} |rW'(r)| \log r_1 - \log r_2 | \leq C \log \frac{r_1}{r_2},
\]
for all \(r_1, r_2 \in (0, \infty)\). Hence
\[
\sup_{r \geq 0} |W(r^{\frac{1}{\alpha_\lambda}}/\beta_\lambda) - W(r^{\frac{1}{\alpha_\lambda}}/\beta^*_\lambda)| \leq C |\log \frac{\beta^*_\lambda}{\beta_\lambda}| \to 0 \text{ as } \lambda \to -\infty.
\]
By (4.10) we have
\[
b_\lambda^{s_\lambda + a_\lambda/\beta^*_\lambda}u_\lambda(r) = \left(A_0^{-1/2} + o(1)\right) w_\lambda(r^{\frac{1}{\alpha_\lambda}}/\beta_\lambda) \text{ as } \lambda \to -\infty,
\]
which proves part (i).
For part (ii), let \(f_\lambda : [0, \infty) \to \mathbb{R}, f_\lambda(s) = s^{-a_\lambda}w_\lambda(s)\). From the relation between \(u\) and \(w\) we have
\[
f_\lambda(s) = b_\lambda^{2/(p-1)}\alpha_\lambda^{-1}u_\lambda(\beta^*_\lambda s^{b_\lambda}).
\]
We see that \(s_\lambda\) is a maximum point for \(f_\lambda\) if and only if \(r_\lambda = \beta^*_\lambda s^{b_\lambda}\) is a maximum point for \(u_\lambda\). One have that \(f_\lambda(s)\) converges to \(s^{2/(p-1)}W(s)\) uniformly in \([0, \infty)\). For it is enough observe for example that \(f_\lambda(s) \leq (1 + s)^{2/(p-1)}W(s)\), which vanishes at infinity. So there is a compact subset \(K \subseteq [0, \infty)\) such that \(f_\lambda\) has no maximum point outside \(K\) for all \(\lambda\). Therefore if \(s = s_\lambda > 0\) is a maximum point for \(f_\lambda\) we must have \(s_\lambda \in K\) and \(s_\lambda w'(s_\lambda) = a_\lambda w_\lambda(s_\lambda)\), which gives \(-a_\lambda W(s_\lambda) + sW'(s_\lambda) = o(1)(a_\lambda - s_\lambda)\) as \(\lambda \to -\infty\) since \(w_\lambda \to W\) in \(C^2(K)\). We obtain
\[
\frac{-a_\lambda + s_\lambda^2(-a_\lambda - M_\infty + 2)}{(a_\lambda - s)(1 + s_\lambda^2M_\infty/2)} = o(1) \text{ as } \lambda \to -\infty.
\]
The denominator of the last expression is bounded since $s_\lambda \in K$, and so we must have $-a_\lambda + s_\lambda^2 (-a_\lambda - M_\infty + 2) \to 0$, which gives $s_\lambda \to 1$. It shows the second statement of (ii). The first part of (ii) follows using $s = s_\lambda$ in (4.21) and taking the limit with aid of (4.19).

Finally, we prove part (iii) of the theorem. Combining Lemmas 2.3 and 2.4 we have

$$\int_{B_1} \left\{ |\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \right\} = b_\lambda \int_{B_1} r^{M_\lambda - N - p + 1} = b_\lambda N \omega_N \int_0^1 r^{M_\lambda - 1 - p + 1},$$

where $\bar{v} = \mathcal{L}[u]$ and $\omega_N$ is the measure of the unit ball in $\mathbb{R}^N$. Finally by (4.3) we find

$$\int_{B_1} \left\{ |\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \right\} = b_\lambda \frac{r^{p - 1}}{r^{M_\lambda - 1 - p + 1}},$$

and so (4.17) gives the result.

**Remark.** After we got the estimates on $\alpha_\lambda$ and $\beta_\lambda$ a quite standard procedure gives that the function $v_\lambda$ satisfies

$$\lim_{\lambda \to -\infty} \frac{v_\lambda(r)}{M_\lambda^{1 - p/2} - p} = \left( \frac{S_\infty^{M_\lambda}}{2M_\infty} \right)^{1/2} \left( \frac{1}{r^{M_\lambda - 2}} - 1 \right), \quad r > 0 \quad (4.22)$$

and

$$\lim_{\lambda \to -\infty} v_\lambda(0) = \left( \frac{2M_\infty}{S_\infty} \right)^{1/2} \left[ M_\infty(M_\infty - 2)^{(M_\infty - 2)/2}. \right]$$

See for example [12]. However these estimates are not very useful to get good asymptotic for the solution $u_\lambda$ in $(0, 1]$. This is due to the “bad” behavior of the map $r \mapsto r^p$ which collapses the interval $(0, 1)$ in 0.

**References**


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