\textbf{C*-ALGEBRAS OF ALGEBRAIC DYNAMICAL SYSTEMS AND RIGHT LCM SEMIGROUPS}

NATHAN BROWNLOWE, NADIA S. LARSEN, AND NICOLAI STAMMEIER

\textbf{Abstract.} We introduce algebraic dynamical systems, which consist of an action of a right LCM semigroup by injective endomorphisms of a group. To each algebraic dynamical system we associate a C*-algebra and describe it as a semigroup C*-algebra. As part of our analysis of these C*-algebras we prove results for right LCM semigroups. More precisely we discuss functoriality of the full semigroup C*-algebra and compute its K-theory for a large class of semigroups. We introduce the notion of a Nica-Toeplitz algebra of a product system over a right LCM semigroup, and show that it provides a useful alternative to study algebraic dynamical systems.

1. Introduction

Dynamical systems arising from injective endomorphisms of countable, discrete groups have provided an important source of algebraic data from which to build C*-algebras. The case of group automorphisms can by now be considered classical, and is modelled C*-algebraically by the much-studied crossed product. More recent work in the area has focussed on injective endomorphisms which are not surjective. The study of C*-algebras associated to a single injective endomorphism of a group with finite cokernel includes Hirshberg’s work \cite{Hir02} on endomorphisms of amenable groups, and Cuntz and Vershik's work \cite{CV13} on endomorphisms of abelian groups, while the situation with an injective endomorphism with infinite cokernel was considered by Vieira \cite{Vie13}.

Through recent work in \cite{BLS17, Sta15} the authors have sought to build a unifying framework for the above constructions, while at the same time significantly broadening the scope of the theory. In \cite{Sta15} the third-named author generalised the single-endomorphism setting to actions of semigroups of injective endomorphisms via the notion of an \textit{irreversible algebraic dynamical system}, which consists of an action of a countably generated, free abelian monoid \( P \) (with identity) by injective group endomorphisms subject to an independence condition which originated from \cite{CV13}. A construction of very similar flavour was proposed by the first two authors, see \cite[Example 5.10]{HLS12b} for details. The C*-algebra from \cite{Sta15} associated to an irreversible algebraic dynamical system has a presentation by generators and relations that encodes the action, and can be viewed alternatively as a Cuntz-Nica-Pimsner algebra. In \cite{BLS17} the authors looked at similar actions \( \theta : P \curvearrowright G \), and studied them via the semidirect product semigroup \( G \rtimes_\theta P \). The focus of \cite{BLS17} is on the semigroup C*-algebra, in the sense of Li \cite{Li12}, associated to \( G \rtimes_\theta P \).

In many cases, the semidirect products \( G \rtimes_\theta P \) from \cite{BLS17} are examples of \textit{right LCM semigroups}. These are left cancellative semigroups with the property that principal
right ideals intersect to be either empty or another principal right ideal. Their C*-algebras, both the semigroup C*-algebra and a boundary-quotient type C*-algebra, have garnered recent attention [Nor14, BRRW14, Star15]. Right LCM semigroups lead to a tractable and yet rich class of semigroup C*-algebras that includes the C*-algebras associated to quasi-lattice ordered pairs as introduced by Nica [Nic92], and the Toeplitz type C*-algebras associated to a self-similar actions in [LRRW14] as a counterpart to the Cuntz-Pimsner algebras from [Nek04]. In [BLS17], the authors pursued the study of the semigroup C*-algebras of right LCM semigroups by establishing uniqueness results.

This paper serves two major purposes. The first, and our original motivation for this work, is to significantly extend the work of [Sta15] by introducing the notion of an algebraic dynamical system and studying C*-algebras that encode the action. An algebraic dynamical system consists of an action \( \theta : P \curvearrowright G \) of a right LCM semigroup (with identity) \( P \) by injective endomorphisms of a group \( G \) which harmonise with the structure of the principal right ideals in \( P \). This setting is more general than the one considered in [Sta15] and gives a large class of new examples. To every algebraic dynamical system \((G, P, \theta)\) we associate a C*-algebra \( A[\theta] \) that is universal for a unitary representation of \( G \) and a representation of \( P \) by isometries satisfying relations which encode the dynamics. These relations lead to a Wick ordering on \( A[\theta] \). Our first main result says that \( A[\theta] \) is canonically isomorphic to the full semigroup C*-algebra of \( G \rtimes_{\theta} P \). This explains and motivates our renewed interest in the study of the full semigroup C*-algebra of right LCM semigroups. Therefore, a continuation of the work in [BLS17] in order to build a general theory of semigroup C*-algebras associated to right LCM semigroups constitutes the second major purpose of this paper.

Conceivably, \( A[\theta] \) resembles a Toeplitz type crossed product for a generalised Exel-Larsen system naturally arising from \((G, P, \theta)\). To make this idea precise we appeal to Fowler’s product systems of right-Hilbert bimodules over semigroups [Fow02]. Explicitly, we show that his theory, that was originally developed for quasi-lattice ordered pairs, can be taken a considerable step further to the setting of right LCM semigroups. Motivated by our analysis of the C*-algebra associated to algebraic dynamical systems, we prove a number of results for right LCM semigroups:

1. We investigate the functoriality of \( S \mapsto C^*(S) \) and determine the right notion of a morphism for the category with objects right LCM semigroups so that the construction of the full semigroup C*-algebra becomes functorial. We also characterise surjectivity of the induced morphism, and injectivity in the case that both left regular representations are faithful.

2. We use the machinery of Cuntz, Echterhoff and Li [CEL15] to show that, for a large class of right LCM semigroups \( S \), the \( K \)-theory of \( C^*(S) \) is given by the \( K \)-theory of \( C^r(S^*) \), where \( S^* \) denotes the group of units in \( S \).

3. We extend the notion of a compactly aligned product system of right-Hilbert bimodules, and of a Nica covariant representation of such a product system, from the quasi-lattice ordered setting first introduced by Fowler [Fow02] to the setting of right LCM semigroups. This leads to the definition of a Nica-Toeplitz algebra for product systems over right LCM semigroups.

Our description of \( A[\theta] \) as the semigroup C*-algebra \( C^*(G \rtimes_{\theta} P) \) of the right LCM semigroup \( G \rtimes_{\theta} P \) allows us to apply the general theory from (1) and (2). We
thus characterise the morphisms between algebraic dynamical systems which induce morphisms of their $C^*$-algebras, and we compute the $K$-theory of $A[G, P, \theta]$ for a large class of algebraic dynamical systems $(G, P, \theta)$. We also apply the general theory built in (3) to the setting of algebraic dynamical systems, where we construct a compactly aligned product system $M$ from an algebraic dynamical system $(G, P, \theta)$ whose Nica-Toeplitz algebra $\mathcal{NT}(M)$ is canonically isomorphic to $A[G, P, \theta]$. This allows us to conclude that, for right LCM semigroups $S, C^*(S)$ can be modelled as the Nica-Toeplitz algebra of the product system over $S$ with fibres $C$. We would like to stress the point that there is an abundance of compactly aligned product systems $M$, built from algebraic dynamical systems, for which the left action on some (or all) of the fibres of $M$ is not by generalised compacts, as has frequently been the case in the context of quasi-lattice ordered pairs.

We finish with a section on examples, the most basic of which is the system $(\mathbb{Z}, P, \cdot)$ in which $P \subset \mathbb{Z}^\times$ is generated by a family of relatively prime nonzero integers acting on $\mathbb{Z}$ by multiplication. Despite the straightforward nature of these systems, we show that they provide an interesting class of $C^*$-algebras that warrant further study. Other examples of algebraic dynamical systems include the more general $(R, P, \cdot)$ in which $R$ is a ring of integers in a number field and $P$ a right LCM subsemigroup of $R^\times$ acting by multiplication, and shift systems $(\bigoplus_P G_0, P, \theta)$ in which $G_0$ is countable, $P$ right LCM, and $\theta$ the natural shift action on $\bigoplus_P G_0$.

The structure of the paper is as follows: in Section 2 we introduce algebraic dynamical systems $(G, P, \theta)$ and their $C^*$-algebras $A[G, P, \theta]$. In Section 3 we address the question of functoriality of $S \mapsto C^*(S)$, and in Section 4 we prove that $A[G, P, \theta] \cong C^*(G \rtimes_{\theta} P)$, and apply the results of Section 3 to the context of $A[G, P, \theta]$. In Section 5 we prove that for a large class of right LCM $S$ we have $K_i(C^*(S)) \cong K_i(C^*_r(S^*))$, where $C^*_r(S^*)$ is the reduced $C^*$-algebra of the group of units $S^*$ in $S$. In Section 6 we introduce the Nica-Toeplitz algebra for a product system over a right LCM semigroup, and then in Section 7 we describe $A[G, P, \theta]$ as such a Nica-Toeplitz algebra. We finish in Section 8 with a discussion of examples of algebraic dynamical systems and their $C^*$-algebras. The flow of the paper is a steady back and forth between algebraic dynamical systems and general right LCM semigroups. We feel that this arrangement reflects in a proper way the fruitful interplay between the two topics in the context of $C^*$-algebraic constructions.

We would like to point out that algebraic dynamical systems are not the same as algebraic dynamics, a terminology going back to [Sch95]. The objects of interest in this area are actions of countable, discrete groups on compact (abelian) groups, see [LS15] for a recent survey of highly interesting connections to other fields. We remark that algebraic dynamical systems $(G, P, \theta)$ can be thought of as algebraic dynamics provided that $G$ is abelian and $P$ is a group. We hope that our choice does not cause confusion, but rather stimulates an examination of the connections between these two concepts.

While this manuscript was under review, its contents served as fruitful grounds for [BS16], where many of the results from [Sta15] are extended to algebraic dynamical systems, building heavily on [Star15]. This lead to a generalisation of the boundary quotient diagram of [BaHLR12] from $\mathbb{N} \times \mathbb{N}^\times$ to a broad class of right LCM semigroups, see [Sta17]. The insights into the internal structure of right LCM semigroups gained from this construction then allowed the authors (in collaboration with Zahra Afsar)
to determine the structure of equilibrium states on $C^*(S)$ for a natural dynamics for a surprisingly large class of right LCM semigroups $S$, encompassing $\mathbb{N} \rtimes \mathbb{N}^*$, dilation matrices, self-similar group actions, Baumslag-Solitar monoids, and many arising as semidirect products from algebraic dynamical systems, see [ABLS].

**Acknowledgements:** The present work has advanced substantially during multiple visits of the first and the third author to Oslo in 2013 and 2014, and we would like to thank the group in Oslo for their hospitality. This work has also benefited from a joint week spent at the BIRS mini-workshop 13w5152 in November 2013 and we are grateful to the organisers Alan Carey and Marcelo Laca for inviting the three of us. The third author acknowledges support provided by DFG through SFB 878 and by ERC through AdG 267079. We thank S. Echterhoff for directing our attention to [CCH16].

# 2. Algebraic Dynamical Systems and Their $C^*$-Algebras

In [Sta15, Definition 1.5], the third author introduced the notion of irreversible algebraic dynamical systems and studied associated $C^*$-algebras, which under mild assumptions turn out to be unital UCT Kirchberg algebras. Recall that an irreversible algebraic dynamical system $(G,P,\theta)$ is given by a countable, discrete group $G$, a countably generated, free abelian semigroup $P$ with identity, and a $P$-action $\theta$ on $G$ by injective group endomorphisms such that

$$\theta_p(G) \cap \theta_q(G) = \theta_{pq}(G)$$

if and only if $pP \cap qP = pqP$.

Note that due to the above displayed condition and injectivity of $\theta_p$, the group $G$ needs to be infinite whenever $P$ is nontrivial. Moreover, the only group automorphism of $G$ present in such a dynamical system is $\text{id}_G$.

We are now going to provide a vast generalisation of the concept of an irreversible algebraic dynamical system $(G,P,\theta)$, where $P$ is allowed to be any countable right LCM semigroup with identity, for example a group, and relations of the images of $\theta_p$ and $\theta_q$ no longer imply relations for $pP$ and $qP$. Recall that a semigroup $S$ is right LCM if it is left cancellative and for any $p,q$ in $S$, the intersection of principal right ideals $pS \cap qS$ is either empty or of the form $rS$ for some $r \in S$, see [BRRW14].

All identities for semigroups will be denoted by 1. If $S$ is a semigroup with identity, we let $S^*$ denote the group of invertible elements in $S$.

**Definition 2.1.** An algebraic dynamical system is a triple $(G,P,\theta)$ consisting of a countable, discrete group $G$, a countable, right LCM semigroup $P$ with identity, and an action $\theta$ of $P$ by injective group endomorphisms of $G$ such that for all $p,q \in P$:

$$\theta_p(G) \cap \theta_q(G) = \theta_{pq}(G) \text{ whenever } r \in P \text{ satisfies } pP \cap qP = rP.$$ 

Following [BLS17] we say that $\theta$ respects the order on $P$ whenever (2.1) is satisfied. Note that if $r \in P$ satisfies $rP = r'P$, then there is $x \in P^*$ such that $r = rx$. Hence one has $\theta_r(G) = \theta_r(\theta_x(G)) = \theta_x(G)$ as $\theta_x$ is a group automorphism of $G$.

The class of algebraic dynamical systems contains various types of examples, see Section 8. In particular, it includes the irreversible algebraic dynamical systems from [Sta15]. We showed in [BLS17, Proposition 8.2] that $G \rtimes_\theta P$ is a right LCM semigroup.
whenever \((G, P, \theta)\) is a triple as in Definition 2.1. In later sections we shall exploit this connection, but for the moment we focus our attention on the triple \((G, P, \theta)\).

**Definition 2.2.** For an algebraic dynamical system \((G, P, \theta)\) let \(\mathcal{A}[G, P, \theta]\) be the universal \(C^*\)-algebra generated by a unitary representation \(u\) of \(G\) and an isometric representation \(s\) of \(P\) satisfying

\[
\begin{align*}
(A1) \quad s_p u_g &= u_{\theta_p(g)} s_p \quad \text{for all } p \in P, g \in G, \text{ and} \\
(A2) \quad s_p u_g s_q &= \begin{cases} 
  u_k s_p s_{q'}^* u_{\ell} & \text{if } pP \cap qP = rP, pp' = qq' = r \\
  0 & \text{and } g = \theta_p(k) \theta_q(\ell^{-1}) \text{ for some } k, \ell \in G,
\end{cases}
\end{align*}
\]

We need to check that (A2) does not depend on the choice of \(r\) or \((k, \ell)\). If \(r' \in P\) satisfies \(r'P = rP\), then there exists \(x \in P^*\) such that \(r' = rx\). If \(p'' = q'' \in P\) satisfy \(pp'' = qq'' = r'\), then \(p'' = p' x, q'' = q' x\), and hence \(s_{p''} s_{q''} = s_{p'} s_{q'}\).

Now suppose \(k_1, \ell_1 \in G\) satisfy \(\theta_p(k_1) \theta_q(\ell_1^{-1}) = g = \theta_p(k) \theta_q(\ell^{-1})\) and \(pP \cap qP = rP\) for some \(r \in P\). But \(G\) is a group, so we have \(\theta_p(k_1^{-1} k_1) = \theta_q(\ell^{-1} \ell_1)\). As \(\theta\) respects the order on \(P\), this element belongs to \(\theta_r(G)\). Since \(\theta_p\) and \(\theta_q\) are injective, we get \(k_1^{-1} k_1 = \theta_p(k_2)\) and \(\ell^{-1} \ell_1 = \theta_q(\ell_2)\) for some \(k_2, \ell_2 \in G\). Note that \(\theta_p(k_1^{-1} k_1) = \theta_q(\ell^{-1} \ell_1)\) is the same as \(\theta_r(k_1) = \theta_r(\ell_1)\), and injectivity of \(\theta_r\) then implies that \(k_2 = \ell_2\). Using (A1) we conclude

\[
u_{k_1} s_{p'} s_{q'}^* u_{\ell_1} = u_k u_{k^{-1} k_1} s_p s_{q'}^* u_{\ell_1^{-1}} u_{\ell_1} = u_k s_{p'} u_{k_2} u_{\ell_2} s_{q'}^* u_{\ell_1} = u_k s_{p'} s_{q'}^* u_{\ell_1}.
\]

**Remark 2.3.** We chose (A2) as one of our defining relations of \(\mathcal{A}[G, P, \theta]\) because the expressions of the form \(s_p^* u_g s_q\) are precisely the ones which need to be expressible in terms of \(u_{g_1} s_{p_1} s_{q_2}^* u_{g_2}\), \(g_i \in G, p_i \in P\) in order to get a Wick ordering on \(\mathcal{A}[G, P, \theta]\), i.e. monomials in the generators where all starred generators appear on the right of all non-starred generators, see [JSW95].

We like to think of \(\mathcal{A}[G, P, \theta]\) as a Toeplitz type crossed product of \(C^*(G)\) by the \(P\)-action \(\theta\). If one looks into the literature for semigroup crossed products, say by abelian semigroups, one encounters a covariance condition that deals with the analogue of \(s_p^* u_g s_q\) by means of a transfer operator. This was introduced by Ruy Exel in [Exe03] for \(N\) and was extended to abelian semigroups by the second author in [Lar10]. This development gave rise to the notion of Exel-Larsen systems. In fact, for each algebraic dynamical system, one can associate a transfer operator \(L\) to the action \(\alpha : C^*(G) \to C^*(G)\) induced by \(\theta\). Explicitly, \(L\) is the semigroup homomorphism from the opposite semigroup \(P^\text{op}\) of \(P\) to the semigroup of unital, positive, linear maps \(C^*(G) \to C^*(G)\) given by \(L_p(\delta_g) = \chi_{\theta_p(G)}(g) \delta_{\theta_p^{-1}(g)}\), where \(\delta_g\) denote the generating unitaries in \(C^*(G)\). In this way, we can regard \(\mathcal{A}[G, P, \theta]\) as a Toeplitz type crossed product for the generalised Exel-Larsen system \((G, P, \alpha, L)\), see Section 7 for details.

The step of moving from abelian to more general semigroups \(P\) requires taking care of expressions of the form \(s_p^* s_q\) as well. If \(P\) happens to be right LCM, Nica covariance as it appeared first in [Nic92] provides a natural means to replace \(s_p^* s_q\) by \(s_q s_{p'}^*\) if \(pP \cap qP = rP\) with \(r = pp' = qq'\) and declare it to vanish otherwise.

Altogether we see that (A2) combines the idea of a covariance condition coming from a transfer operator for \(\theta\) and Nica covariance for the semigroup \(P\). Note, however, that
terms of the form \( s_p^*q_g s_q \) with \( g \neq 1 \) and \( p, q \in \mathbb{P} \setminus \mathbb{P}^r \) such that \( r \mathbb{P} \supset \mathbb{P}P \cup q \mathbb{P} \) implies \( r \in \mathbb{P}^r \) for all \( r \in \mathbb{P} \).

For each \( g \in \mathcal{G}, p \in \mathbb{P} \) we denote the projection \( u_g s_p s_p^*u_q^* \) by \( e_{(g, p)} \). In the following lemma we show that relation \((A2)\) is equivalent to a more familiar looking relation involving the product of these projections \( e_{(g, p)} \).

**Lemma 2.4.** Let \((\mathcal{G}, \mathbb{P}, \theta)\) be an algebraic dynamical system, \( u \) a unitary representation of \( \mathcal{G} \) and \( s \) an isometric representation of \( \mathbb{P} \) satisfying \((A1)\). Then \( u \) and \( s \) satisfy \((A2)\) if and only if

\[
    e_{(g, p)} e_{(h, q)} = \begin{cases} 
        e_{(g \theta_p(k), r)} & \text{if } p \mathbb{P} \cap q \mathbb{P} = r \mathbb{P} \text{ for some } r \in \mathbb{P} \\
        g \theta_p(k) \in h \theta_q(G) \text{ for some } k \in \mathcal{G},& \text{otherwise.}
    \end{cases}
\]

**Proof.** Suppose \((A2)\) holds. Since \( e_{(g, p)} e_{(h, q)} = u_g s_p (s_p^* u_g^{-1} q_s) s_q^* u_q^* \), we know from \((A2)\) that \( e_{(g, p)} e_{(h, q)} \) is zero unless \( p \mathbb{P} \cap q \mathbb{P} = r \mathbb{P} \) and \( g^{-1} h = \theta_p(k) \theta_q(\ell^{-1}) \) for some \( k, \ell \in \mathcal{G} \). If these conditions hold, and \( p', q' \in \mathbb{P} \) with \( pp' = qq' = r \), then \((A1)\) and \((A2)\) give

\[
    e_{(g, p)} e_{(h, q)} = u_g s_p (s_p^* u_g^{-1} q_s) s_q^* u_q^* = u_g s_p u_k s_p^* s_q^* u_q^* = u_g \theta_p(k) s_r s_q^* u_q^* = e_{(g \theta_p(k), r)}.
\]

Now \((2.2)\) holds because \( g^{-1} h = \theta_p(k) \theta_q(\ell^{-1}) \) for some \( k, \ell \in \mathcal{G} \) if and only if \( g \theta_p(k) \in h \theta_q(G) \) for some \( k \in \mathcal{G} \).

Suppose \((2.2)\) holds. Let \( g \in \mathcal{G} \) and \( p, q \in \mathbb{P} \). Then \( s_p^* u_g s_q = s_p^* e_{(1, p)} e_{(g, q)} u_g s_q \) vanishes according to \((2.2)\) unless \( p \mathbb{P} \cap q \mathbb{P} = r \mathbb{P} \) for some \( r \in \mathbb{P} \) and \( \ell \mathbb{P} \cap q \mathbb{P} = r \mathbb{P} \) for some \( k, \ell \in \mathcal{G} \). Note that this is the same condition as in \((A2)\). If we let \( p', q' \in \mathbb{P} \) satisfy \( pp' = qq' = r \), we arrive at

\[
    s_p^* u_g s_q = s_p^* e_{(1, p)} e_{(g, q)} u_g s_q = s_p^* u_{\theta_p(k)} s_p^* s_q^* u_q^* = s_p^* u_{\theta_p(k)} s_p^* u_q^* = s_p^* u_{\theta_p(k)} s_p^* u_q^*,
\]

which establishes \((A2)\). \( \square \)

**Notation 2.5.** Given \( p \in \mathbb{P} \), we will refer to a complete set of representatives for \( \mathcal{G} \) as a transversal of \( \mathcal{G} \) and denote it by \( T_p \).

**Remark 2.6.** Note that \((2.2)\) implies that for each \( p \in \mathbb{P} \) and each transversal \( T_p \) of \( \mathcal{G} \), the projections \( \{ e_{(g, p)} \mid g \in T_p \} \) form a collection of mutually orthogonal projections.

We would like to discuss the question of functoriality for the mapping \((\mathcal{G}, \mathbb{P}, \theta) \mapsto \mathcal{A}[\mathcal{G}, \mathbb{P}, \theta] \). To this end, we introduce a natural notion of a morphism between algebraic dynamical systems.

**Definition 2.7.** Suppose \((\mathcal{G}_1, \mathbb{P}_1, \theta_1)\) and \((\mathcal{G}_2, \mathbb{P}_2, \theta_2)\) are algebraic dynamical systems. A morphism from \((\mathcal{G}_1, \mathbb{P}_1, \theta_1)\) to \((\mathcal{G}_2, \mathbb{P}_2, \theta_2)\) is a pair \((\phi_G, \phi_P)\), where

(i) \( \phi_G : \mathcal{G}_1 \to \mathcal{G}_2 \) is a group homomorphism,
(ii) \( \phi_P : \mathbb{P}_1 \to \mathbb{P}_2 \) is an identity preserving semigroup homomorphism, and
(iii) \( \phi_G \circ \theta_{1, p} = \theta_{2, \phi_P(p)} \circ \phi_G \) holds for all \( p \in \mathbb{P}_1 \).

The proof of the following lemma is straightforward and therefore omitted.
Lemma 2.8. Suppose \((G_1, P_1, \theta_1)\) and \((G_2, P_2, \theta_2)\) are algebraic dynamical systems. If \((\phi_G, \phi_P)\) is a morphism from \((G_1, P_1, \theta_1)\) to \((G_2, P_2, \theta_2)\), then \((g, p) \mapsto (\phi_G(g), \phi_P(p))\) defines an identity preserving semigroup homomorphism \(\phi_G \times \phi_P : G_1 \times_{\theta_1} P_1 \to G_2 \times_{\theta_2} P_2\).

Remark 2.9. By Lemma 2.8, every morphism between algebraic dynamical systems gives rise to a homomorphism between the corresponding semigroups. The converse is false in general. More precisely, for given algebraic dynamical systems \((G_1, P_1, \theta_1)\) and \((G_2, P_2, \theta_2)\), a homomorphism \(\varphi : G_1 \times_{\theta_1} P_1 \to G_2 \times_{\theta_2} P_2\) is induced by a morphism of the algebraic dynamical systems if and only if \(\varphi(G_1 \times \{1\}) \subseteq G_2 \times \{1\}\) and \(\varphi(\{1\} \times P_1) \subseteq \{1\} \times P_2\).

It is a natural question whether every morphism \((\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)\) induces a \(*\)-homomorphism \(\varphi_G \times \varphi_P : \mathcal{A}[G_1, P_1, \theta_1] \to \mathcal{A}[G_2, P_2, \theta_2]\). We shall postpone an answer to this question until the end of Section 4.

3. \(C^*\)-algebras of right LCM semigroups and functoriality

In [Li12], Li investigated the functoriality of the assignment \(S \mapsto C^*(S)\) in the context of \(ax + b\)-semigroups over integral domains, and applied his findings to the Toeplitz type \(C^*\)-algebras associated to number fields in [CDL13]. He remarked that functoriality is not likely to be easily described for arbitrary \(S\). Here we show that it can be successfully approached in case that \(S\) is a right LCM semigroup with identity.

We first recall Li’s construction of the full and the reduced \(C^*\)-algebra associated to a discrete left cancellative semigroup \(S\). A set \(X \subseteq S\) is a right ideal if it is closed under right multiplication with any element of \(S\). For each right ideal \(X\), the sets
\[ pX = \{px \mid x \in X\} \quad \text{and} \quad p^{-1}X = \{y \in S \mid py \in X\}. \]
are also right ideals. Li [Li12, p.4] defines \(\mathcal{J}(S)\) to be the smallest family of right ideals of \(S\) satisfying
\begin{align*}
\text{(a)} & \quad S, \emptyset \in \mathcal{J}(S); \text{ and} \\
\text{(b)} & \quad X \in \mathcal{J}(S) \text{ and } p \in S \text{ implies } pX \text{ and } p^{-1}X \in \mathcal{J}(S).
\end{align*}
The elements of \(\mathcal{J}(S)\) are called \emph{constructible right ideals}. The general form of a constructible right ideal is given in [Li12, Equation (5)]. We note that \(\mathcal{J}(S)\) is also closed under finite intersections, a fact that can be derived from (a) and (b) using \(pS \cap qS = p(p^{-1}(qS))\).

**Definition 3.1.** The set of constructible right ideals \(\mathcal{J}(S)\) is called \textit{independent} if for every \(X, X_1, \ldots, X_n \in \mathcal{J}(S)\) we have
\[ X_j \varsubsetneq X \quad \text{for all } 1 \leq j \leq n \implies \bigcup_{j=1}^{n} X_j \varsubsetneq X. \]
Alternatively, \(\mathcal{J}(S)\) is independent if \(\bigcup_{j=1}^{n} X_j = X \implies X_j = X\) for some \(j\).

Let us recall Li’s definition of the full semigroup \(C^*\)-algebra for \(S\).

**Definition 3.2.** Let \(S\) be a discrete left cancellative semigroup. The \textit{full semigroup} \(C^*\)-algebra \(C^*(S)\) is the universal \(C^*\)-algebra generated by isometries \((v_p)_{p \in S}\) and projections \((e_X)_{X \in \mathcal{J}(S)}\) satisfying
Theorem 3.3. Suppose $S_1$ and $S_2$ are right LCM semigroups with identity. Let $(\tilde{v}_p)_{p \in S_1}$ and $(v_p)_{p \in S_2}$ be families of generating isometries in $C^*(S_1)$ and $C^*(S_2)$, respectively. An identity preserving semigroup homomorphism $\phi : S_1 \to S_2$ induces a *-homomorphism $\varphi : C^*(S_1) \to C^*(S_2)$ such that $\tilde{v}_p \mapsto v_{\phi(p)}$ if and only if $\phi$ satisfies

$$e_{\phi(p)}S_2 \cap \phi(q)S_2 = \phi(pS_1 \cap qS_1)S_2$$

for all $p,q \in S_1$. 

Proof. Let $\{e_{pS_1}\}_{p \in S_1}$ and $\{e_{pS_2}\}_{p \in S_2}$ be the standard projections in $C^*(S_1)$ and $C^*(S_2)$, so $e_{pS_1} = \tilde{v}_p \tilde{v}_p^*$ for $p \in S_1$ and similarly for $S_2$.

We show first that (3.1) is a necessary condition. Assume therefore that $\phi$ induces a *-homomorphism $\varphi : C^*(S_1) \to C^*(S_2)$. Given $p,q \in S_1$, we observe that $pS_1 \cap qS_1 \neq \emptyset$ forces $\phi(p)S_2 \cap \phi(q)S_2 \supset \phi(pS_1 \cap qS_1)S_2 \neq \emptyset$. Conversely, $\phi(p)S_2 \cap \phi(q)S_2 \neq \emptyset$ together with the equation

$$e_{\phi(p)}S_2 \cap \phi(q)S_2 = e_{\phi(p)}S_2 e_{\phi(q)}S_2 = \varphi(\tilde{e}_{pS_1})\varphi(\tilde{e}_{qS_1}) = \varphi(\tilde{e}_{pS_1} \tilde{e}_{qS_1})$$

implies that $pS_1 \cap qS_1 \neq \emptyset$. Hence we have $\phi(p)S_2 \cap \phi(q)S_2 \neq \emptyset$ if and only if $pS_1 \cap qS_1 \neq \emptyset$. 

Assuming both intersections to be non-empty, the fact that $S_1$ and $S_2$ are right LCM implies that there exist $r_1 \in S_1$ and $r_2 \in S_2$ such that $\phi(p)S_2 \cap \phi(q)S_2 = r_2S_2$ and $pS_1 \cap qS_2 = r_1S_1$. It follows that $r_2S_2 \supset \phi(r_1)S_2$. If $\phi(r_1)S_2$ was a proper subset of $r_2S_2$, then the image of $e_{r_2S_2} - e_{\phi(r_1)S_2}$ under the left regular representation would be non-zero. Since the assumption on $\phi$ implies that $e_{r_2S_2} = \varphi(\tilde{e}_{r_1S_1}) = e_{\phi(r_1)S_2}$, we would obtain a contradiction. Hence, $r_2 \in \phi(r_1)S_2$ and thus (3.1) is satisfied.

Conversely, let us assume (3.1) holds. We claim that $(v_{\phi(p)})_{p \in S_1}$ and $(e_{\phi(q)})_{q \in S_1}$ satisfy (L1)-(L4). Since $S_1$ and $S_2$ are right LCM, it suffices to prove (L4) because (L1)-(L3) are immediate. Let $p,q \in S_1$. If $pS_1 \cap qS_1 = \emptyset$, which by (3.1) is equivalent to $\phi(p)S_2 \cap \phi(q)S_2 = \emptyset$, then

$$e_{\phi(p)}S_2 e_{\phi(q)}S_2 = e_{\phi(pS_2 \cap \phi(q))S_2} = 0,$$

as required for (L4). If $pS_1 \cap qS_1 \neq \emptyset$, then $pS_1 \cap qS_1 = rS_1$ for some $r \in S_1$. Now (3.1) ensures that

$$e_{\phi(p)}S_2 e_{\phi(q)}S_2 = e_{\phi(pS_2 \cap \phi(q))S_2} = e_{\phi(pS_1 \cap qS_1)S_2} = e_{\phi(r)S_2}.$$
Remark 3.4. Note that the set on the right hand side in (3.1) is always contained in the set on the left. As we can see from the proof, (3.1) is equivalent to the requirement that

$$\phi : S_1 \rightarrow S_2$$

is injective, then

$$\phi$$

is a semigroup homomorphism satisfying

$$\phi(S_1) \rightarrow \phi(S_2) \text{ given by } \emptyset \rightarrow \emptyset \text{ and } pS_1 \mapsto \phi(p)S_2$$

is compatible with finite intersections. In fact, as $S_1$ and $S_2$ are right LCM semigroups with identity, the image $\mathcal{J} := \phi(\mathcal{J}(S_1))$ is the smallest family of right ideals in $S_2$ which satisfies

(a) $\emptyset, S_2 \in \mathcal{J}(\phi)$; and
(b) $X \in \mathcal{J}(\phi)$ and $p \in S_1$ implies $\phi(p)X$ and $\phi(p)^{-1}X \in \mathcal{J}(\phi)$.

Therefore we interpret (3.1) as a condition which characterises when $\mathcal{J}(\phi)$ is a subfamily of constructible right ideals of $S_2$ corresponding to $\phi(S_1)$.

Remark 3.5. The condition (3.1) is essentially the same as (b) from [Li12, Lemma 2.18]. The apparent difference is that the similar identity in [Li12, Lemma 2.18] has images under $\phi$ of right ideals in the left-hand side as well as in the right-hand side. Since we deal with right LCM semigroups, we can phrase the condition only using images of right ideals in the right-hand side of (3.1). It is striking that for right LCM semigroups there is no need to require [Li12, Lemma 2.18 (a)], and (b) turns out to be the precise condition which is needed to ensure that $\phi$ induces a *-homomorphism on the level of the full semigroup C*-algebra.

Proposition 3.6. Let $S_1$ and $S_2$ be right LCM semigroups with identity and suppose $\phi : S_1 \rightarrow S_2$ is a semigroup homomorphism satisfying (3.1). Then the induced *-homomorphism $\varphi : C^*(S_1) \rightarrow C^*(S_2)$ is surjective if and only if $\phi$ is surjective. If $\varphi$ is injective, then $\phi$ is injective. The converse holds when the left regular representation implements an isomorphism $C^*(S_i) \cong C^*_\varphi(S_i)$ for each $i = 1, 2$.

Proof. Since $\phi(S_1)$ is a subsemigroup of $S_2$ and $\varphi(C^*(S_1)) \cap \{v_q \mid q \in S_2\} = \{v_{\phi(p)} \mid p \in S_1\}$, the statement about surjectivity of $\varphi$ is immediate. Assume that $\phi$ is not injective. If $p, q \in S_1, p \neq q$ are such that $\phi(p) = \phi(q)$, then $\tilde{v}_p \neq \tilde{v}_q$ but $\varphi(\tilde{v}_p) = v_{\phi(p)} = v_{\phi(q)} = \varphi(\tilde{v}_q)$, showing that $\varphi$ is not injective either.

Finally, suppose that $\phi$ is injective and that the left regular representation implements canonical isomorphisms $C^*(S_i) \cong C^*_\varphi(S_i)$ for $i = 1, 2$. The left regular representation of $S_2$ on $\ell^2(S_2)$ restricts to a representation of $\phi(S_1)$. Compressing this by the projection onto $\phi(S_1)$ gives rise to a representation on $\ell^2(\phi(S_1)) \subset \ell^2(S_2)$. As $\phi$ is injective, this is simply a representation of $S_1$ which is unitarily equivalent to the left regular representation on $\ell^2(S_1)$. Since $C^*(S_1) \cong C^*_\varphi(S_1)$, we deduce that this compression of $\varphi$ is injective. Hence $\varphi$ itself is injective.

We shall examine (3.1) more closely for right LCM semigroups built from algebraic dynamical systems in the next section.


In this section we show that, for an algebraic dynamical system $(G, P, \theta)$, the C*-algebra $A[G, P, \theta]$ is isomorphic to the full semigroup C*-algebra of the semidirect product $G \rtimes_\theta P$, which is a right LCM semigroup by [BLS17 Proposition 8.2]. As an application of this identification we study functoriality of the assignment $(G, P, \theta) \mapsto A[G, P, \theta]$ based on the findings of Section 3. Throughout this section we let $(G, P, \theta)$ be an algebraic
dynamical system. We start with a description of the structure of the constructible right ideals \( \mathcal{J}(G \rtimes_\theta P) \) of \( G \rtimes_\theta P \).

**Notation 4.1.** We denote the principal right ideals \((g, p)\)(\( G \rtimes_\theta P \)) by \( X_{(g,p)} \).

**Proposition 4.2.** Let \( X_{(g,p)} \) and \( X_{(h,q)} \) be principal right ideals of \( G \rtimes_\theta P \), for \( g, h \in G \) and \( p, q \in P \). Then

\[
X_{(g,p)} \cap X_{(h,q)} = \begin{cases} 
X_{(g\theta_p(k),r)} & \text{if } pP \cap qP = rP \text{ for some } r \in P \text{ and } g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Hence the family of constructible right ideals \( \mathcal{J}(G \rtimes_\theta P) \) is independent. If \((T_p)_{p \in P}\) is a family of transversals, then \( \mathcal{J}(G \rtimes_\theta P) = \{\emptyset\} \cup \{X_{(g,p)} \mid p \in P, g \in T_p\} \).

**Proof.** The formula for the intersection of principal ideals can be found in the proof of [BLS17, Proposition 8.2]. The remaining claims follow from [BLS17, Lemma 3.3 and Corollary 3.6].

**Corollary 4.3.** \( C^*(G \rtimes_\theta P) \) is the closed linear span of \( \{v_{(g,p)} v_{(h,q)}^* \mid g, h \in G, p, q \in P\} \).

**Proof.** Since \( G \rtimes_\theta P \) is right LCM, this follows from [BLS17, Lemma 3.11].

We now state the main result of this section.

**Theorem 4.4.** There is an isomorphism \( \varphi : A[G, P, \theta] \to C^*(G \rtimes_\theta P) \) satisfying

\[
\varphi(u_g) = v_{(g,1)} \quad \text{and} \quad \varphi(s_p) = v_{(1,p)},
\]

for all \( g \in G, p \in P \).

**Proof.** Straightforward calculations show that \( g \mapsto v_{(g,1)} \) is a unitary representation of \( G \) and \( p \mapsto v_{(1,p)} \) is an isometric representation of \( P \). We show that these representations satisfy (A1) and (A2). Recall that \( e_{X_{(g,p)}} = v_{(g,p)} v_{(g,p)}^* \). Fix \( g, h \in G \) and \( p, q \in P \). Then

\[
v_{(1,p)} v_{(g,1)} (L1) = v_{(1,p)(g,1)} = v_{(\theta_p(g),p)} (L1) = v_{(\theta_p(g),1)p} v_{(1,p)},
\]

so (A1) holds. To get (A2), we just check that (2.2) holds. We have

\[
v_{(g,1)} v_{(1,p)} v_{(g,1)} v_{(1,p)}^* v_{(g,1)}^* = v_{(g,p)} v_{(g,p)}^* = e_{X_{(g,p)}}.
\]

Proposition 4.2 gives that

\[
e_{X_{(g,p)}} e_{X_{(h,q)}} = \begin{cases} 
e_{(g\theta_p(k),r)} & \text{if } pP \cap qP = rP \text{ and } g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\
\emptyset & \text{otherwise},
\end{cases}
\]

which is (2.2). The universal property of \( A[G, P, \theta] \) gives therefore a homomorphism \( \varphi : A[G, P, \theta] \to C^*(G \rtimes_\theta P) \) satisfying \( \varphi(u_g) = v_{(g,1)} \) and \( \varphi(s_p) = v_{(1,p)} \), for all \( g \in G, p \in P \). To prove the result we use the universal property of \( C^*(G \rtimes_\theta P) \) to find an inverse \( \varphi' \) for \( \varphi \).

Consider the elements \( u_g s_p, e_{(g,h)} \in A[G, P, \theta] \). Each \( u_g s_p \) is an isometry with range projection equal to \( e_{(g,p)} \). The elements \( \{u_g s_p, e_{(h,q)} \mid g, h \in G, p, q \in P\} \cup \{0\} \) are easily seen to satisfy (L1)–(L4); condition (L1) follows from an application of (A1), (L2) and (L3) are immediate, and (L4) follows from (2.2) and the formula for intersection of ideals.
from Proposition 4.2. The universal property of $C^\ast(G \rtimes \theta P)$ now gives a homomorphism $\varphi' : C^\ast(G \rtimes \theta P) \to \mathcal{A}[G, P, \theta]$ which satisfies $v_{(g, 1)} \mapsto u_g$ and $v_{(1, p)} \mapsto s_p$, for all $g \in G$, $p \in P$. Hence $\varphi$ and $\varphi'$ are mutually inverse, so $\varphi$ is an isomorphism. 

Let us return to the question of functoriality for $(G, P, \theta) \mapsto \mathcal{A}[G, P, \theta]$ raised at the end of Section 2. Now that we have realised $\mathcal{A}[G, P, \theta]$ as the semigroup $C^\ast$-algebra of the right LCM semigroup $G \rtimes \theta P$, we can appeal to (3.1) which gives us a precise condition under which a morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ of algebraic dynamical systems induces a $*$-homomorphism $\varphi_G \rtimes \varphi_P : \mathcal{A}[G_1, P_1, \theta_1] \to \mathcal{A}[G_2, P_2, \theta_2]$.

**Definition 4.5.** A morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ of algebraic dynamical systems is called admissible, if, in addition to (i)–(iii) of Definition 2.7, the following two conditions hold for all $p, q \in P_1$:

1. $\phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2$.
2. $\phi_G^{-1}(\phi_G(G_1) \cap \theta_{2, \phi_P(p)}(G_2) \theta_{2, \phi_P(q)}(G_2)) = \theta_{1, p}(G_1) \theta_{1, q}(G_1)$ if $P_1 \cap qP_1 \neq \emptyset$.

**Remark 4.6.** Let us comment briefly on these extra requirements:

(a) For both (iv) and (v), the inclusion $\supseteq$ holds for every morphism $(\phi_G, \phi_P)$, so the statement is about the reverse inclusion.

(b) Condition (iv) is nothing but (3.1) for $\phi_P : P_1 \to P_2$. As long as we do not have more knowledge on $P_1$ and $P_2$, we cannot expect to get anything better than this, compare Theorem 3.3. But the existence of a semigroup homomorphism $\phi_P : P_1 \to P_2$ satisfying (iv) has structural consequences: If $P_2$ is such that the family of principal right ideals ordered by reverse inclusion is directed, then the same is true of $P_1$. In particular, this is the case if $P_2$ is a group.

(c) Condition (v) holds if $\phi_G$ is injective or the action $\theta_1$ is by automorphisms of $G_1$.

(d) Note that (v) forces $\theta_{1, p} \in \text{Aut}(G_1)$ whenever $\theta_{2, \phi_P(p)} \in \text{Aut}(G_2)$. Hence a morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ with $P_1^* \not\subseteq \phi_G^{-1}(P_2^*)$ is not admissible.

Before we proceed to the general considerations on functoriality we look at admissible morphisms in two particular situations.

**Proposition 4.7.** Let $P_1$ be a group. Then every morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ of algebraic dynamical systems is admissible.

**Proof.** If $P_1$ is a group, then $P_1^* = P_1$ and $\phi_P(P_1) \subseteq P_2^*$. Hence $\phi_P(p)P_2 \cap \phi_P(q)P_2 = P_2 = \phi_P(pP_1 \cap qP_1)P_2$ for all $p, q \in P_1$, which establishes (iv). Moreover, $\theta_1$ and $\theta_{2, \phi_P(\cdot)}$ are $P_1$-actions by group automorphisms of $G_1$ and $G_2$, respectively, so

$$\phi_G^{-1}(\phi_G(G_1) \cap \theta_{2, \phi_P(p)}(G_2) \theta_{2, \phi_P(q)}(G_2)) = G_1 \theta_{1, p}(G_1) \theta_{1, q}(G_1)$$

for all $p, q \in P_1$. This shows (v). 

**Proposition 4.8.** Let $P_1 = \mathbb{F}_n^+$ for some $1 \leq n \leq \infty$. A morphism of algebra dynamical systems $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ is admissible if and only if it satisfies $\phi_G^{-1}(\phi_G(G_1) \cap \theta_{2, \phi_P(p)}(G_2)) = \theta_{1, p}(G_1)$ for all $p \in P_1$.

**Proof.** Fix $p, q \in P_1$. Since $pP_1$ and $qP_1$ are disjoint unless $pP_1 \subset qP_1$ or $qP_1 \subset pP_1$, we can assume $q \in pP_1$, i.e., $q = pr$ for some $r \in P_1$. In this case,

$$\phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(p)P_2 \cap \phi_P(p)\phi_P(r)P_2 = \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2$$
The next two results show that admissible morphisms yield the right class of morphisms with respect to functoriality for \((G, P, \theta) \mapsto \mathcal{A}[G, P, \theta]\).

**Lemma 4.9.** Suppose \((\phi_G, \phi_P) : (G_1, P_1, \theta_1) \rightarrow (G_2, P_2, \theta_2)\) is a morphism of algebraic dynamical systems. Then \(\phi_G \times \phi_P : G_1 \times_{\theta_1} P_1 \rightarrow G_2 \times_{\theta_2} P_2\) satisfies (3.1) if and only if \((\phi_G, \phi_P)\) is admissible.

**Proof.** For convenience, let us denote \(S_i := G_i \times_{\theta_i} P_i\) for \(i = 1, 2\) and \(\phi := \phi_G \times \phi_P\). Pick \(p, q \in P_1\). We need to show that the equality

\[
(4.1) \quad \phi(g, p)S_2 \cap \phi(h, q)S_2 = \phi((g, p)S_1 \cap (h, q)S_1)S_2
\]

holds for all \(g, h \in G_1\) if and only if we have

\[
(iv) \quad \phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2,
\]

and

\[
(v) \quad \phi_G^{-1}(\phi_G(G_1) \cap \theta_2,\phi_P(p)(G_2) \theta_2,\phi_P(q)(G_2)) = \theta_1,\phi(G_1) \theta_1,\phi(G_1) \quad \text{if } pP_1 \cap qP_1 \neq \emptyset.
\]

It was noticed in Remark 3.4 and Remark 4.6 (a) that, for all of these equations, the set on the right is contained in the set on the left. Moreover, Proposition 4.2 implies that \((g, p)S_1 \cap (h, q)S_1\) is non-empty if and only if

\[
pP_1 \cap qP_1 \neq \emptyset \quad \text{and} \quad g^{-1}h \in \theta_1,\phi(G_1) \theta_1,\phi(G_1).
\]

Likewise, \(\phi(g, p)S_2 \cap (h, q)S_2\) is non-empty if and only if

\[
(4.2) \quad \phi_P(p)P_2 \cap \phi_P(q)P_2 \neq \emptyset \quad \text{and} \quad \phi_G(g^{-1}h) \in \theta_2,\phi_P(p)(G_2) \theta_2,\phi_P(q)(G_2).
\]

Now suppose (4.1) holds. For \(g = h = 1\) and \(p, q \in P_1\), (4.1) implies \(\phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2\), which is (iv). Next, assume that \(pP_1 \cap qP_1 \neq \emptyset\) and take \(g = 1, h \in \phi_G^{-1}(\phi_G(G_1) \cap \theta_2,\phi_P(p)(G_2) \theta_2,\phi_P(q)(G_2))\). Then we have \(\phi_1,\phi(p)S_2 \cap \phi(h, q)S_2 \neq \emptyset\). By (4.1), this implies \((1, p)S_1 \cap (h, q)S_1 \neq \emptyset\). Hence \(h \in \theta_1,\phi(G_1) \theta_1,\phi(G_1)\) and (v) holds.

Conversely, suppose (iv) and (v) are satisfied. If \(\phi(g, p)S_2 \cap (h, q)S_2\) is empty, then there is nothing to show, so assume it is non-empty. As noted before, this means that we have (4.2). In this case, (v) implies \(g^{-1}h \in \theta_1,\phi(G_1) \theta_1,\phi(G_1)\). Together with (iv) this yields (4.1) for \((g, p)\) and \((h, q)\).

**Corollary 4.10.** A morphism \((\phi_G, \phi_P) : (G_1, P_1, \theta_1) \rightarrow (G_2, P_2, \theta_2)\) of algebraic dynamical systems induces a \(*\)-homomorphism \(\varphi_G \times \varphi_P : \mathcal{A}[G_1, P_1, \theta_1] \rightarrow \mathcal{A}[G_2, P_2, \theta_2]\) if and only if \((\phi_G, \phi_P)\) is admissible.

**Proof.** This follows from applying Lemma 4.9, Theorem 3.3 and Theorem 4.4.

5. **The K-theory for right LCM semigroups that are left Ore**

The goal of this section is to compute the \(K\)-theory of \(C^*_r(S)\) for a large class of right LCM semigroups. In view of Theorem 4.4, we will thus obtain the \(K\)-theory of \(\mathcal{A}[G, P, \theta]\) whenever \(G \times_{\theta} P\) is left Ore and the left regular representation from \(C^*(G \times_{\theta} P)\) onto \(C^*_r(G \times_{\theta} P)\) is an isomorphism. As we shall see, right LCM left Ore semigroups form a
natural class to which the results in \cite{CEL15} \S7 apply nicely: thus, as an outcome we will obtain a stronger statement than just a computation of the $K$-groups.

Recall that a semigroup $S$ is right reversible if every pair of non-empty left ideals has non-empty intersection. If $S$ is cancellative and right reversible, then it is called a left Ore semigroup. It is well-known and an essential reason for the importance of the class of left Ore semigroups that a semigroup non-empty intersection. If $S$ is cancellative and right reversible, then it is called a left Ore semigroup.

**Proof.** It suffices to note that for all $s, t$ in $S$ we have

\[
s^{-1}t \in G(S) \iff s^{-1}t \cdot (1^{-1} \cdot S) = (1^{-1} \cdot S) \iff 1^{-1} \cdot tS = 1^{-1} \cdot sS \iff tS = sS \iff s^{-1}t \in S^*.
\]

\[\square\]

**Lemma 5.1.** The action of $G(S)$ on $Y$ is transitive. In particular, $\{S\}$ is a complete set of representatives for the quotient $G(S) \setminus Y$.

**Proof.** Let $X_1 = t_1s$ and $X_2 = t_2s$ for $t_1, t_2 \in S$ be two elements of $\mathcal{J}(S) \setminus \{\emptyset\}$, see \cite{BLS17} Lemma 3.3. Let $s_1, s_2 \in S$. We need to find $g \in G(S)$ such that $g \cdot (s_1^{-1} \cdot X_1) = s_2^{-1} \cdot X_2$. Note that $r \cdot (1^{-1} \cdot S) = 1^{-1} \cdot rS$ for all $r \in S$. Hence, the choice $g = s_2^{-1}t_2t_1^{-1}s_1$ works.

\[\square\]

**Lemma 5.2.** The stabiliser group $G(S)^S$ of $S \in Y$ is $S^*$.

**Proof.** It suffices to note that for all $s, t$ in $S$ we have

\[
s^{-1}t \in G(S)^S \iff s^{-1}t \cdot (1^{-1} \cdot S) = (1^{-1} \cdot S) \iff 1^{-1} \cdot tS = 1^{-1} \cdot sS \iff tS = sS \iff s^{-1}t \in S^*.
\]

\[\square\]
It is proved in [CEL15, Lemma 4.2] that the reduced semigroup $C^*$-algebra $C_r^*(S)$ is Morita equivalent to a certain reduced crossed product $D_r^{(\infty)}(S) \rtimes_r \mathcal{G}(S)$. Let $\{V_s\}_{s \in S}$ denote the family of isometries on $l^2(S)$ that generates $C_r^*(S)$ and consider the homomorphism

$$\Psi : C_r^*(S^*) \to C_r^*(S), \lambda_{s^{-1}t} \mapsto V_s^*V_t$$

for $s, t \in S$ from [CEL15] Lemma 7.2.

We are now ready to state the main result of this subsection. With the preparation above, the proof is a direct application of [CEL15, Corollary 7.4]. As in [CEL15], $K^*$ stands for the direct sum of $K_0$ and $K_1$ viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group and $K^*$ is the $K$-homology.

**Theorem 5.3.** If the enveloping group $\mathcal{G}(S)$ of $S$ satisfies the Baum-Connes conjecture with coefficients in the $\mathcal{G}(S)$-algebras $c_0(Y)$ and $D_r^{(\infty)}(S)$, then $K_*(\Psi) : K_*(C_r^*(S^*)) \to K_*(C_r^*(S))$ is an isomorphism. If, moreover, $c_0(Y) \rtimes \mathcal{G}(S)$ and $D_r^{(\infty)}(S) \rtimes \mathcal{G}(S)$ satisfy the UCT or $\mathcal{G}(S)$ satisfies the strong Baum-Connes conjecture with coefficients in $c_0(Y)$ and $D_r^{(\infty)}(S)$, then $K^*(\Psi) : K^*(C_r^*(S^*)) \to K^*(C_r^*(S))$ is an isomorphism.

We now apply Theorem 5.3 to the semigroups arising from algebraic dynamical systems $(G, P, \theta)$. Our main result is a considerable generalisation of [CEL13, Theorem 6.3.4].

**Proposition 5.4.** If $(G, P, \theta)$ is an algebraic dynamical system, then the group of invertible elements $(G \rtimes_\theta P)^*$ is isomorphic to $G \rtimes_\theta P^*$ and $G \rtimes_\theta P$ is left Ore precisely when $P$ is left Ore.

**Proof.** The claim about the group of units follows, for example, from [BLS17, Lemma 2.4]. It is clear that $P$ needs to be right reversible and right cancellative whenever $G \rtimes_\theta P$ has these two properties. In other words, $P$ itself has to be a left Ore semigroup as left cancellation is assumed throughout. So let us suppose that this $P$ is left Ore. If we have $(g, \theta_{p_1}(h), p_1q) = (g\theta_{p_2}(h), p_2q)$, then right cancellation for $P$ and then for $G$ forces $p_1 = p_2$ and $g_1 = g_2$. To see that $G \rtimes_\theta P$ is right reversible, consider the left ideals $(G \rtimes_\theta P)(g, p)$ and $(G \rtimes_\theta P)(h, q)$. By assumption, there exists $r \in P$ with $r = p\prime p = q\prime q$ and hence

$$(1, q')(h, q) = (\theta_q(h), r) = (\theta_q(h)\theta_p(g)^{-1}, p')(g, p) \in (G \rtimes_\theta P)(g, p) \cap (G \rtimes_\theta P)(h, q).$$

So $G \rtimes_\theta P$ is right cancellative, and hence is a left Ore semigroup.

Recall that $\lambda : C^*(G \rtimes_\theta P) \to C^*_r(G \rtimes_\theta P)$ denotes the left regular representation and $\varphi : \mathcal{A}[G, P, \theta] \to C^*(G \rtimes_\theta P)$ is the isomorphism from Theorem 4.4. We have the following consequence of Theorem 5.3.

**Corollary 5.5.** Let $(G, P, \theta)$ be an algebraic dynamical system such that $P$ is a left Ore semigroup and let $\mathcal{A}[G, P, \theta]$ be its associated $C^*$-algebra. Assume that $\lambda$ is an isomorphism. If $G \rtimes_\theta P$ satisfies the Baum-Connes conjecture with coefficients in the $\mathcal{G}(G \rtimes_\theta P)$-algebras $c_0(Y)$ and $D_r^{(\infty)}(G \rtimes_\theta P)$, then

$$K_*((\lambda \circ \varphi)^{-1} \circ \Psi_{G \rtimes_\theta P}) : K_*(C_r^*(G \rtimes_\theta P^*)) \to K_*(\mathcal{A}[G, P, \theta])$$

is an isomorphism.
If, moreover, $c_0(Y) \rtimes_r G \rtimes_{\theta} P$ and $D^1(G \rtimes_{\theta} P) \rtimes_r G \rtimes_{\theta} P$ satisfy the UCT or $G(G \rtimes_{\theta} P)$ satisfies the strong Baum-Connes conjecture with coefficients in $c_0(Y)$ and $D^1(G \rtimes_{\theta} P)$, then

$$K^*(\lambda \circ \varphi)^{-1} \circ \Psi_{G \rtimes_{\theta} P} : K^*(C^*_r(G \rtimes_{\theta} P)) \to K^*(A[G, P, \theta])$$

is an isomorphism.

Corollary 5.5 applies whenever $G \rtimes_{\theta} P$ satisfies the strong Baum-Connes conjecture with commutative coefficients, in particular it applies when $G \rtimes_{\theta} P$ is amenable. To check amenability, one possible strategy is as follows: by [CEL13, Proposition 6.1.3], there is an embedding of $G \rtimes_{\theta} P$ into a semidirect product $G^{(\infty)} \rtimes_{\theta^{(\infty)}} P^{-1}P$, where $G^{(\infty)}$ is the inductive limit of $(G, \theta_\mu)$ over $P$. The universal property of $G(G \rtimes_{\theta} P)$ gives an injective group homomorphism $G(G \rtimes_{\theta} P) \hookrightarrow G^{(\infty)} \rtimes_{\theta^{(\infty)}} P^{-1}P$. Thus amenability of $G(G \rtimes_{\theta} P)$ will follow from that of $G^{(\infty)} \rtimes_{\theta^{(\infty)}} P^{-1}P$. For instance, the latter is amenable whenever $G$ and $P^{-1}P$ are amenable.

6. Nica-Toeplitz algebras of product systems over right LCM semigroups

Our next aim is to describe $A[G, P, \theta]$ as a $C^*$-algebra associated to a product system of right-Hilbert $C^*(G)$-bimodules. As the natural product system to consider is fibred over the right LCM semigroup $P$, we need to extend the existing theory to build Nica-Toeplitz algebras for such product systems. In this section we define the notions of a compactly aligned product system over a right LCM semigroup, and of a Nica covariant representation of such a product system. We use Parseval frames to find a condition under which a product system over a right LCM semigroup is compactly aligned, and we show that the Fock representation is Nica covariant.

Let $A$ be a $C^*$-algebra and $S$ a (countable, discrete) left cancellative semigroup with identity 1. Recall from [Fow02] that a product system of right-Hilbert $A$-bimodules over $S$ is a semigroup $M = \coprod_{p \in S} M_p$, where each $M_p$ is a right-Hilbert $A$-bimodule; $x \otimes_A y \mapsto xy$ determines an isomorphism of $M_p \otimes_A M_q$ onto $M_{pq}$ for all $p, q \in S$ such that $p \neq 1$; the bimodule $M_1$ is $A \rtimes_A A$; and the products from $M_1 \times M_p \to M_p$ and $M_p \times M_1 \to M_p$ are given by the module actions of $A$ on $M_p$.

Recall that for each $p \in S$ and each $\xi, \eta \in M_p$ the operator $\Theta_{\xi, \eta} : M_p \to M_p$ defined by $\Theta_{\xi, \eta}(\mu) := \langle \xi, \langle \eta, \mu \rangle \rangle_p$ is adjointable with $\Theta^*_{\xi, \eta} = \Theta_{\eta, \xi}$. The space $K(M_p) := \overline{\text{span}}\{\Theta_{\xi, \eta} \mid \xi, \eta \in M_p\}$ is a closed two-sided ideal in $\mathcal{L}(M_p)$ called the algebra of generalised compact operators on $M_p$. For $p, q \in S$, there is a homomorphism $i^p_q : \mathcal{L}(M_p) \to \mathcal{L}(M_q)$ characterised by

$$i^p_q(T)(\eta) = (T\xi)\eta \quad \text{for all } \xi \in M_p, \eta \in M_q.$$  

In other words, $i^p_q(T) = T \otimes i^p_q$. For $q \notin pS$ we set $i^p_q = 0$.

A representation $\psi$ of $M$ in a $C^*$-algebra $B$ is a map $M \to B$ such that

1. each $\psi_p := \psi|_{M_p} : M_p \to B$ is linear, $\psi_1 : A \to B$ is a homomorphism,
2. $\psi_p(\xi)\psi_q(\eta) = \psi_{pq}(\xi, \eta)$ for all $p, q \in S$, $\xi \in M_p$, and $\eta \in M_q$; and
3. $\psi_p(\xi)\psi_q(\eta) = \psi_1(\langle \xi, \eta \rangle_p)$ for all $p \in S$, and $\xi, \eta \in M_p$. 


For each $p \in S$ there is a homomorphism $\psi^{(p)} : \mathcal{K}(M_p) \to B$ characterised by

$$\psi^{(p)}(\Theta_{\xi,\eta}) = \psi_p(\xi)\psi_p(\eta)^*.$$  

In [Fow02 Definition 5.7] Fowler introduced the notion of a compactly aligned product system for quasi-lattice ordered pairs. We now extend this notion to product systems over right LCM semigroups.

**Definition 6.1.** A product system $M$ over a right LCM semigroup $S$ is called *compactly aligned* if for all $p, q, r \in S$ such that $pS \cap qS = rS$ and all $T_p \in \mathcal{K}(M_p), T_q \in \mathcal{K}(M_q)$ we have $\iota^r_p(T_p)\iota^r_q(T_q) \in \mathcal{K}(M_r)$.

**Remark 6.2.** Recall that if $r$ is a right least common multiple of $p$ and $q$, then all right least common multiples of $p$ and $q$ are of the form $rx$ for $x \in S^*$. Note that $M_x$ is a Morita equivalence from $A$ to $A$ for every $x \in S^*$ because $M_x \otimes_A M_{x^{-1}} \cong AA_A \cong M_{x^{-1}} \otimes_A M_x$. Hence $T \mapsto T \otimes \text{id}$ defines an adjunction between $M_x$ and $M_{x^{-1}}$ in the sense of [CCH16 Definition 2.17]. Combining this with [CCH16 Lemma 3.7], we deduce that the natural isomorphism $\iota_r^{rx} : \mathcal{L}(M_r) \to \mathcal{L}(M_{rx})$ restricts to an isomorphism from $\mathcal{K}(M_r)$ to $\mathcal{K}(M_{rx})$. Thus, $\iota_p^r(T_p)\iota_q^r(T_q)$ belongs to $\mathcal{K}(M_r)$ if and only if $\iota_p^{rx}(T_p)\iota_q^{rx}(T_q)$ belongs to $\mathcal{K}(M_{rx})$ for all $x \in S^*$.

We will shortly present a class of compactly aligned product systems. Before doing so we recall that a *Parseval frame* for a right-Hilbert module $Y$ over a unital $C^*$-algebra $A$ is a countable family $(\xi_i)_{i \in I} \subset Y$ that satisfies the *reconstruction formula* $\eta = \sum_{i \in I} \xi_i : \langle \xi_i, \eta \rangle$ for all $\eta \in Y$, where the convergence is in norm in $Y$. Parseval frames are also referred to as standard normalised tight frames, which are the main objects of interest in [FL02]. For our purposes the reconstruction formula is more relevant than the equivalent Parseval-type identity in $A$ that defines a standard normalised tight frame.

**Lemma 6.3.** Let $A$ be a unital $C^*$-algebra, $S$ a right LCM semigroup with identity, and $M = \bigsqcup_p M_p$ a product system of right-Hilbert $A$-bimodules such that the following condition is satisfied:

(\text{CA}) \text{ Whenever } p, q, r \in S \text{ satisfy } pS \cap qS = rS, \text{ there are Parseval frames } (\xi_i^{(p)})_{i \in I_p} \text{ for } M_p \text{ and } (\xi_j^{(q)})_{j \in I_q} \text{ for } M_q, \text{ such that } \iota_p^r(\Theta_{\xi_i^{(p)}})\iota_q^r(\Theta_{\xi_j^{(q)}}) \text{ is in } \mathcal{K}(M_r) \text{ for all } i \in I_p, j \in I_q.

Then $M$ is compactly aligned.

**Proof.** Let $p,q,r \in S$ satisfy $pS \cap qS = rS$. By linearity and continuity it suffices to show $\iota_p^r(\Theta_{\eta_1,\eta_2})\iota_q^r(\Theta_{\zeta_1,\zeta_2}) \in \mathcal{K}(M_r)$ for all $\eta_1, \eta_2 \in M_p, \zeta_1, \zeta_2 \in M_q$. We claim that, due to the reconstruction formula, we have

$$\sum_{i \in I_p} \Theta_{\eta_1,\eta_2} \Theta_{\xi_i^{(p)} \zeta_i^{(p)}} \xrightarrow{\text{F \subset I_p finite}} \Theta_{\eta_1,\eta_2}, \text{ and}
\sum_{j \in I_q} \Theta_{\zeta_1 \zeta_2} \Theta_{\xi_j^{(q)} \zeta_j^{(q)}} \xrightarrow{\text{F \subset I_q finite}} \Theta_{\zeta_1,\zeta_2},$$

with convergence in norm in $\mathcal{K}(M_r)$. Firstly, note that it suffices to consider the second case, where the rank one operators corresponding to the elements from the Parseval
frame are to the left, because the involution on $\mathcal{L}(M_p)$ is continuous. We start by observing that

$$\sum_{j \in F'} \Theta_{\xi_j^{(q)}} \Theta_{\xi_1, \xi_2} - \Theta_{\xi_1, \xi_2} = \Theta \left( \sum_{j \in F'} \xi_j^{(q)} \langle \xi_j^{(q)}, \xi_j^{(q)} \rangle \right).$$

It is routine to see that $\|\Theta_{\xi_1, \xi_2}\| \leq \|\xi_1\| \cdot \|\xi_2\|$ holds for all $\xi_1, \xi_2 \in M$, where $M$ is a right-Hilbert module over $A$. Using this observation, we obtain

$$\| \sum_{j \in F'} \Theta_{\xi_j^{(q)}} \Theta_{\xi_1, \xi_2} - \Theta_{\xi_1, \xi_2} \| = \| \Theta \left( \sum_{j \in F'} \xi_j^{(q)} \langle \xi_j^{(q)}, \xi_j^{(q)} \rangle \right) - \Theta_{\xi_1, \xi_2} \| \leq \| \sum_{j \in F'} \xi_j^{(q)} \langle \xi_j^{(q)}, \xi_j^{(q)} \rangle - \Theta_{\xi_1, \xi_2} \|.$$

Therefore $\sum_{j \in F'} \Theta_{\xi_j^{(q)}} \Theta_{\xi_1, \xi_2}$ converges in norm to $\Theta_{\xi_1, \xi_2}$ as $F' \nearrow I_q$ and, likewise, $\sum_{i \in F} \Theta_{\xi_i^{(p)}} \Theta_{\xi_1, \xi_2}$ converges in norm to $\Theta_{\xi_1, \xi_2}$ as $F \nearrow I_p$. From this we deduce

$$\sum_{(i,j) \in F \times F'} t_p^r (\Theta_{\xi_i^{(p)}} \Theta_{\xi_j^{(q)}}) t_q^r (\Theta_{\xi_j^{(q)}} \Theta_{\xi_1, \xi_2}) F \times F' \nearrow I_p \times I_q \text{ finite} \quad \impliedby \quad t_p^r (\Theta_{\xi_i^{(p)}} \Theta_{\xi_1, \xi_2}) = \sum_{(i,j) \in F \times F'} t_p^r (\Theta_{\xi_i^{(p)}} \Theta_{\xi_j^{(q)}}) t_q^r (\Theta_{\xi_j^{(q)}} \Theta_{\xi_1, \xi_2}).$$

where, again, we have convergence with respect to the operator norm on $\mathcal{L}(M_r)$. By condition (CA), the operator

$$\sum_{(i,j) \in F \times F'} t_p^r (\Theta_{\xi_i^{(p)}} \Theta_{\xi_j^{(q)}}) t_q^r (\Theta_{\xi_j^{(q)}} \Theta_{\xi_1, \xi_2})$$

is in $\mathcal{K}(M_r)$ for each finite set $F \times F' \subset I_p \times I_q$. Since $\mathcal{K}(M_r)$ is a norm closed ideal in $\mathcal{L}(M_r)$, the result follows.

To associate a $C^*$-algebra to product systems over right LCM semigroups we now introduce the notion of Nica covariant representations of such product systems, which is an extension of Fowler’s Nica covariance [Fow02, Definition 5.7] for product systems of quasi-lattice ordered pairs. In the following, we let $\varphi_p : A \to \mathcal{L}(M_p)$ denote the $*$-homomorphism that implements the left action in $M_p$.

**Definition 6.4.** A representation $\psi : M \to B$ of a compactly aligned product system $M$ over a right LCM semigroup $S$ in a $C^*$-algebra $B$ is called **Nica covariant** if for all $p, q \in S$ and $T_p \in \mathcal{K}(M_p), T_q \in \mathcal{K}(M_q)$ we have

$$\psi^{(p)}(T_p) \psi^{(q)}(T_q) = \begin{cases} \psi^{(r)}(t_p^r (T_p) t_q^r (T_q)) & \text{if } p S \cap q S = r S \text{ for some } r \in S, \\ 0 & \text{otherwise}. \end{cases}$$

**Remark 6.5.** An explanation is in order about this definition. Suppose we have $p, q \in S$ such that $p S \cap q S = r S$ for some $r \in S$. Let $T_p \in \mathcal{K}(M_p)$ and $T_q \in \mathcal{K}(M_q)$. Let $x \in S^*$, so that $r x$ is another right LCM for $p$ and $q$. Then a necessary condition for a representation $\psi : M \to B$ to be Nica covariant is that

$$\psi^{(r)} \left( t_p^r (T_p) t_q^r (T_q) \right) = \psi^{(r x)} \left( t_p^{r x} (T_p) t_q^{r x} (T_q) \right).$$

Since $t_p^{r x} = t_r^{r x} \circ t_p^r$ and similarly for $t_q^{r x}$, and since $t_p^{r x}$ is a homomorphism on $\mathcal{L}(M_r)$, we have

$$t_p^{r x} (T_p) t_q^{r x} (T_q) = t_r^{r x} (t_p^r (T_p) t_q^r (T_q)).$$

Using this, (6.3) will follow if $t_r^{r x} (t_p^r (T_p) t_q^r (T_q)) \in \mathcal{K}(M_{r x})$ and $\psi^{(r x)} \circ t_r^{r x} = \psi^{(r)}$. The first requirement is always satisfied because by Remark 6.2, the homomorphism $t_r^{r x}$
restricts to an isomorphism from $\mathcal{K}(M_r)$ onto $\mathcal{K}(M_{rx})$ and $\iota_p^r(T_p)i_q^r(T_q) \in \mathcal{K}(M_r)$ by compact alignment. In the next result we present a class of product systems for which the requirement $\psi^{(rx)} \circ \iota_p^x = \psi^{(r)}$ holds for every representation $\psi$ of $M$.

**Lemma 6.6.** Suppose $A$ is a unital C*-algebra, $S$ is a right LCM semigroup, and $M$ is a product system of right-Hilbert $A$-bimodules over $S$. If for each $x \in S^*$ there is an element $1_x \in M_x$ satisfying $\Theta_{1_x,1_x} = \text{id}_{M_x}$, then $\psi^{(rx)} \circ \iota_p^x = \psi^{(r)}$ holds for every representation $\psi$ of $M$ and all $r \in S$.

**Proof.** We claim that for each $r \in S$, $x \in S^*$ and $\mu, \nu \in M_r$ we have $\iota_p^x(\Theta_{\mu,\nu}) = \Theta_{\mu_{1,r},\nu_{1,r}} \in \mathcal{K}(M_{rx})$. To see this, let $\xi \eta \in M_{rx}$ with $\xi \in M_r$ and $\eta \in M_x$. We have

$$
\Theta_{\mu_{1,r},\nu_{1,r}}(\xi \eta) = (\mu_{1,r}) \cdot (\nu_{1,r}, \xi \eta)_{rx} = (\mu_{1,r}) \cdot (1_x, \varphi_x((\nu, \xi), r) \eta)_x
$$

$$
= \mu(\Theta_{1_x,1_x}(\varphi_x((\nu, \xi), r) \eta))
$$

$$
= (\mu \cdot (\nu, \xi), r) \eta = (\Theta_{\mu,\nu}(\xi)) \eta
$$

$$
= \iota_p^x(\Theta_{\mu,\nu})(\xi \eta).
$$

Then the desired conclusion follows by linearity and continuity from

$$
\psi^{(rx)} \circ \iota_p^x(\Theta_{\mu,\nu}) = \psi^{(rx)}(\Theta_{\mu_{1,r},\nu_{1,r}}) = \psi_r(\mu)(\psi_x(1_x)\psi_x(1_x)^* \psi_r(\nu)^*)
$$

$$
= \psi_r(\mu)(\psi_x(\text{id}_{M_x} \psi_r(\nu)^*)
$$

$$
= \psi_r(\mu)(\psi_x(\text{id}_{M_x} \psi_r(\nu)^*)
$$

$$
= \psi^{(r)}(\Theta_{\mu,\nu}).
$$

\[\square\]

**Example 6.7.** Suppose that $M$ is a product system of right-Hilbert $A$-bimodules over a right LCM semigroup $S$. Suppose that we have a collection of automorphisms $\{\alpha_x : x \in S^*\}$ such that each $M_x$ is given by $AA_{\alpha_x}$, with inner product $\langle a, b \rangle_x = \alpha_x^{-1}(a^* b)$. Then the elements $1_x := 1 \in A$ satisfy the hypothesis of Lemma 6.6. The product systems we form in the next section fall in this class of examples.

It was observed [Fow02] that a natural representation for product systems $M$ is the Fock representation. The details of the construction are as follows: The vector space

$$
F(M) = \{(\xi_p)_{p \in S} \mid \xi_p \in M_p, \sum_{p \in S} ||\xi_p||_p^2 < \infty\}
$$

becomes a right-Hilbert $A$-bimodule when equipped with the inner product

$$
\langle (\xi_p)_{p \in S}, (\eta_p)_{p \in S} \rangle = \sum_{p \in S} \langle \xi_p, \eta_p \rangle_p,
$$

together with left and right actions

$$
\varphi_{F(M)}(a)(\xi_p)_{p \in S} = (\varphi_a(\xi_p))_{p \in S} \text{ and } (\xi_p)_{p \in S} \cdot a = (\xi_p \cdot a)_{p \in S}.
$$

It is routine to check that the map $\psi_F : M \to \mathcal{L}(F(M))$ given by

$$
\psi_{F,p}(\xi)(\eta_q)_{q \in S} = (\chi_{pS}(q) \cdot \xi \eta_r)_{q \in S} \text{ for } \xi \in M_p
$$

defines a representation of $M$, where $\chi_{pS}$ denotes the characteristic function on $pS$ and $q = pr$. Note that $M_p \otimes_{\text{A}} M_q \cong M_{pq}$ allows us to write $\eta_{pq} = \sum_{i \in I_{pq}} \eta_{pq,i} \otimes \eta_{pq,i}''$ for suitable
\[ \eta'_{pq,i} \in M_p, \eta''_{pq,i} \in M_q \] (interpreted as a norm limit of finite sums if \( I_{pq} \) is infinite) and we have
\[ \psi_{F,p}(\xi)^* (\eta_{pq}) = \sum_{i \in I_{pq}} \varphi_q((\xi, \eta'_{pq,i})_p) \eta''_{pq,i}. \]

**Proposition 6.8.** If \( M \) is a compactly aligned product system over a right LCM semigroup, then the Fock representation \( \psi_F \) is Nica covariant.

**Proof.** The following is a straightforward adaptation of the proof for the quasi-lattice ordered case from [HLS12], Subsection 2.3. Let \( p, q \in S \) and \( \xi, \eta \in M_p, \xi', \eta' \in M_q \). If \( pS \cap qS = \emptyset \), then \( \psi_{F,p}^{(p)}(\Theta_{\xi,\eta}) \psi_{F,q}^{(q)}(\Theta_{\xi',\eta'})(\zeta)_{r \in S} = 0 \) for all \( (\zeta)_{r \in S} \in F(M) \), so \( \psi_{F,p}^{(p)}(\Theta_{\xi,\eta}) \psi_{F,q}^{(q)}(\Theta_{\xi',\eta'}) = 0 \). So let \( r \in S \) satisfy \( pS \cap qS = rS \). Since \( M \) is compactly aligned we know that \( \iota_r^p(\Theta_{\xi,\eta}) \iota_r^q(\Theta_{\xi',\eta'}) \in K(M_r) \). If \( \iota_r^p(\Theta_{\xi,\eta}) \iota_r^q(\Theta_{\xi',\eta'}) = \Theta_{\xi,\eta} \) for some \( \xi, \eta \in M_r \), then it is routine to check
\[ \psi_{F,p}^{(p)}(\Theta_{\xi,\eta}) \psi_{F,q}^{(q)}(\Theta_{\xi',\eta'}) = \psi_{F}^{(r)}(\Theta_{\xi,\eta}). \]
By linearity and continuity, the required calculations extend to the general case of \( \iota_r^p(\Theta_{\xi,\eta}) \iota_q^r(\Theta_{\xi',\eta'}) \in K(M_r) \). In other words, we have
\[ \psi_{F,p}^{(p)}(\Theta_{\xi,\eta}) \psi_{F,q}^{(q)}(\Theta_{\xi',\eta'}) = \psi_{F}^{(r)}(\iota_r^p(\Theta_{\xi,\eta}) \iota_q^r(\Theta_{\xi',\eta'})), \]
so \( \psi_F \) is Nica covariant. \( \square \)

**Definition 6.9.** For a compactly aligned product system \( M \) over a right LCM semigroup \( S \), the Nica-Toeplitz algebra \( \mathcal{NT}(M) \) is the universal \( C^* \)-algebra generated by a Nica covariant representation of \( M \). We denote the Nica covariant representation generating \( \mathcal{NT}(M) \) as \( j_M \).

Note that \( \mathcal{NT}(M) \) is always nonzero due to Proposition 6.8.

### 7. Nica-Toeplitz Algebras for Algebraic Dynamical Systems

In this section we describe the \( C^* \)-algebra \( A[G, P, \theta] \) as the Nica-Toeplitz algebra of a product system of right-Hilbert \( C^* \)-algebras over the right LCM semigroup \( P \). Throughout this section, let \( (G, P, \theta) \) be an algebraic dynamical system. We denote the canonical generating unitaries in \( C^* \)-algebras \( \gamma_g \), \( g \in G \). The action \( \theta \) induces an action \( \alpha : P \to \text{End} C^* \), i.e. \( \alpha_p(\delta_g) = \delta_{\theta_p(g)} \). Note that injectivity passes from \( \theta \) to \( \alpha \), so \( \alpha \) is an action by unitar injective \( * \)-homomorphisms. We also have an action of the opposite semigroup \( P^{op} \) by unitar, positive, linear maps \( L_p : C^*(G) \to C^*(G) \) given by \( L_p(\delta_g) = \chi_{\theta_p(G)}(g) \delta_{\theta^{-1}_p(g)} \). Each \( L_p \) is a transfer operator for \( (C^*(G), \alpha_p) \), in the sense that \( L_p(\alpha_p(a)b) = aL_p(b) \) for all \( a, b \in C^*(G) \), and so \( (C^*(G), P, \alpha, L) \) is a dynamical system in the style of the Exel-Larsen systems studied in [BR13, Lar10].

**Remark 7.1.** We use the by now standard construction for associating a right-Hilbert \( C^*(G) \)-module to each \( p \in P \): we take \( C^*(G) \) as a vector space and give it a right action \( a \cdot b = a\alpha_p(b) \), and a pre-inner product \( (a, b)_p = L_p(a^*b) \). We mod out by \( N_p := \{ a \in C^*(G) \mid L_p(a^*a) = 0 \} \) and complete to get a right-Hilbert \( C^*(G) \)-module \( M_p \). Left multiplication by elements of \( C^*(G) \) extends to a left action \( \varphi_p \) on \( M_p \) by adjointable operators. We denote the quotient map by \( \pi_p : C^*(G) \to C^*(G)/N_p \) and note that the image of \( \pi_p \) is dense in \( M_p \).
Notation 7.2. For the remainder of this section we will write $E_{g,p}$ for the generalised rank one projection $\Theta_{\pi_p(\delta_g),\pi_p(\delta_h)} \in \mathcal{K}(M_p)$, where $p \in P$ and $g \in G$.

Proposition 7.3. Suppose $\{M_p \mid p \in P\}$ is the collection of right-Hilbert $C^*(G)$-bimodules from Remark 7.1. For each $p \in P$ and each transversal $T_p$ of $G/\theta_p(G)$ the family $\{E_{g,p} \mid g \in T_p\}$ consists of pairwise orthogonal projections in $\mathcal{K}(M_p)$. In particular, the set $\{\pi_p(\delta_g) \mid g \in T_p\}$ is an orthonormal basis for $M_p$. Furthermore, $\varphi_p(C^*(G)) \subset \mathcal{K}(M_p)$ holds if and only if $G/\theta_p(G)$ is finite.

Proof. For each $g, h \in T_p$ we have
\[
\langle \pi_p(\delta_g), \pi_p(\delta_h) \rangle_p = L_p(\delta_{g^{-1}h}) = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{otherwise}. \end{cases}
\]
Thus $E_{g,p}E_{h,p} = 0$ whenever $g, h \in T_p$ are distinct. For each $g \in T_p$ with $g \in g'\theta_p(G)$, and hence
\[
\sum_{g \in T_p} E_{g,p}(\pi_p(\delta_g)) = \sum_{g \in T_p} \chi_{g\theta_p(G)}(h) \pi_p(\delta_g) = \chi_{g'\theta_p(G)}(h) \pi_p(\delta_h) = \pi_p(\delta_h).
\]
By linearity and continuity, this means $\sum_{g \in T_p} E_{g,p}(m) = m$ for all $m \in M_p$, and hence $\{\pi_p(\delta_g) \mid g \in T\}$ is an orthonormal basis for $M_p$.

For the third assertion note that $\varphi_p(1) = \sum_{g \in T_p} E_{g,p}$, which, in case $T_p$ is finite, means that $\varphi_p(1) \in \mathcal{K}(M_p)$, and hence $\varphi_p(C^*(G)) \subset \mathcal{K}(M_p)$. If $T_p$ is infinite, then $\varphi_p(1)$ is an infinite sum (in the strict topology) of mutually orthogonal, equivalent, non-zero projections $E_{g,p}$. Thus, $\varphi_p(1)$ is not compact. \(\square\)

We can use the bimodules $M_p$ to construct a product system of right-Hilbert $C^*(G)$-bimodules over $P$.

Proposition 7.4. Suppose $\{M_p \mid p \in P\}$ is the collection of right-Hilbert $C^*(G)$-bimodules from Remark 7.1. Then the semigroup $M := \bigsqcup_{p \in P} M_p$ with multiplication characterised by
\[
\pi_p(\delta_g)\pi_q(\delta_h) = \pi_{pq}(\delta_{g\theta_p(h)}) \quad \text{for } p, q \in P, \ g, h \in G,
\]
is a product system of right-Hilbert $C^*(G)$-bimodules over $P$.

Proof. The map $(\pi_p(\delta_g), \pi_q(\delta_h)) \mapsto \pi_{pq}(\delta_{g\theta_p(h)})$ is bilinear, and extends to a surjective map from the algebraic balanced tensor product $M_p \otimes_{C^*(G)} M_q$ to $M_{pq}$. This map is equivariant with respect to the left and right actions of $C^*(G)$. It also follows from the calculation
\[
\langle \pi_p(\delta_g_1) \otimes \pi_q(\delta_{h_1}), \pi_p(\delta_{g_2}) \otimes \pi_q(\delta_{h_2}) \rangle = \langle \pi_q(\delta_{h_1}), \langle \pi_p(\delta_g), \pi_p(\delta_g) \rangle_p \cdot \pi_q(\delta_{h_2}) \rangle_q
\]
\[
= L_q(\delta_{h_1}^{-1} L_p(\delta_{g_1}^{-1} g_2) \delta_{h_2})
\]
\[
= L_q(L_p(\alpha_p(\delta_{h_1}^{-1} g_1^{-1} g_2) \alpha_p(\delta_{h_2})))
\]
\[
= L_{pq}(\delta_{g_1}(h_1)^{-1} \delta_{g_2} \delta_{\theta_p(h_2)})
\]
\[
= \langle \pi_{pq}(\delta_{g_1} \theta_p(h_1)), \pi_{pq}(\delta_{g_2} \theta_p(h_2)) \rangle
\]
that $M_p \otimes_{C^*(G)} M_q \to M_{pq}$ preserves the inner product, and hence extends to an isomorphism $M_p \otimes_{C^*(G)} M_q \to M_{pq}$. The remaining conditions for $M := \bigsqcup_{p \in P} M_p$ to be a product system are straightforward to check. \(\square\)
Proposition 7.5. Given $p, q \in P$ and transversals $T_p, T_q$ of $G/\theta_p(G)$, respectively, the map $m_{p,q} : T_p \times T_q \rightarrow G$ given by $(g, h) \mapsto g\theta_p(h)$ is injective and its image is a transversal for $G/\theta_{pq}(G)$. In particular, \{$\pi_{pq}(\delta_{g\theta_p(i)}) \mid g \in T_p, h \in T_q$\} is an orthonormal basis for $M_{pq}$.

Proof. Suppose we have $m_{p,q}(g_1, h_1) = m_{p,q}(g_2, h_2)$ for some $g_i \in T_p, h_i \in T_q$. This amounts to $g_1 = g_2\theta_p(h_2h_1^{-1})$, so $g_1 = g_2$ as $T_p$ is a transversal. As $\theta_p$ is injective, we then immediately get $h_1 = h_2$. Thus $m_{p,q}$ is injective. Now given $g \in G$, let $g_1 \in T_p$ be the element satisfying $g \in g_1\theta_p(G)$. Next, choose $h_1 \in T_q$ such that $\theta_p^{-1}(g_1^{-1}g) \in h_1\theta_q(G)$. Then
\[
(g_1\theta_p(h_1))^{-1}g = \theta_p(h_1^{-1}\theta_p^{-1}(g_1^{-1}g)) \in \theta_p(\theta_q(G)) = \theta_{pq}(G),
\]
so $g \in m_{p,q}(g_1, h_1)\theta_{pq}(G)$. Thus $m_{p,q}(T_p \times T_q)$ is a transversal for $G/\theta_{pq}(G)$. $\square$

Proposition 7.5 can also be proven using the fact that orthonormal bases for $M_p$ and $M_q$ yield an orthonormal basis for $M_p \otimes_{C^*(G)} M_q \cong M_{pq}$, see [LR07] Lemma 4.3.

In order to prove that the product system $M$ from Proposition 7.3 is compactly aligned, we need the following lemma. Within its proof, we will make multiple applications of the identity
\[
\Theta_{\pi_p(\delta_{p_1}), \pi_p(\delta_{p_2})}(\pi_p(\delta_{h})) = \chi_{g_2}(\theta_p(G))\pi_p(\delta_{g_1g_2^{-1}h}) \quad \text{for all } p \in P, g_1, g_2, h \in G.
\]

Lemma 7.6. For each $p, q, r \in P$ with $pP \cap qP = rP$ and $g_1, g_2, h_1, h_2 \in G$ we have
\[
t_p(\Theta_{\pi_p(\delta_{p_1}), \pi_p(\delta_{p_2})}) = \sum_{i \in T_{p'}} \Theta_{\pi_r(\delta_{g_1g_2^{-1}h}), \pi_r(\delta_{g_2g_2^{-1}h})}
\]
and
\[
t_q(\Theta_{\pi_q(\delta_{h_1}), \pi_q(\delta_{h_2})}) = \sum_{j \in T_{q'}} \Theta_{\pi_r(\delta_{g_1g_2^{-1}h}), \pi_r(\delta_{g_2g_2^{-1}h})}.
\]

To see that (7.1) holds, fix $s \in G$. Using (6.1) we get
\[
t_p(\Theta_{\pi_p(\delta_{p_1}), \pi_p(\delta_{p_2})})(\pi_r(\delta_{s})) = \Theta_{\pi_r(\delta_{g_1}), \pi_p(\delta_{g_2})}(\pi_p(\delta_{s}))\pi_r(\delta_{g_1g_2^{-1}s}) = \chi_{g_2}(\theta_p(G))(s)\pi_p(\delta_{g_1g_2^{-1}s}) = \chi_{g_2}(\theta_p(G))(s)\pi_r(\delta_{g_1g_2^{-1}s}).
\]

For $i \in T_{p'}$ we have
\[
\Theta_{\pi_r(\delta_{g_1g_2^{-1}h}), \pi_r(\delta_{g_2g_2^{-1}h})}(\pi_r(\delta_{s})) = \chi_{g_2}(\theta_p(i))\theta_r(G)(s)\pi_r(\delta_{g_1g_2^{-1}s}).
\]

Since $g_2\theta_p(G) = \bigsqcup_{i \in T_{p'}} g_2\theta_p(i)\theta_r(G)$, we know that $s$ belongs to $g_2\theta_p(G)$ if and only if there is an $i \in T_{p'}$ with $s \in g_2\theta_p(i)\theta_r(G)$. Note that $i$ is uniquely determined in this case.
It follows that \( \alpha^r_i(\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)})) \) and \( \sum_{i \in T_{y'}} \Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)}) \) agree on the image of \( \pi_r \), and hence on \( M_r \). So (7.1) holds. A similar argument gives (7.2).

Now observe that, for all \( i \in T_{y'}, j \in T_{y'} \), we have
\[
\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)})) \Theta_{\pi_1}(\delta_{y(j)}),\pi_2(\delta_{y(j)})) = \chi_{g_{y(i)}(i)}(G)(h_1\theta_i(j))\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(j)}))
\]

It then follows from (7.1) and (7.2) that \( \alpha^r_i(\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)})) \) \( \alpha^r_j(\Theta_{\pi_1}(\delta_{y(j)}),\pi_2(\delta_{y(j)})) \) \( \neq 0 \) if and only if
\[
h_1\theta_i(j) \in g_{y(i)}(i)\theta_i(G) \text{ for some } i \in T_{y'}, \quad j \in T_{y'}.
\]

As \( r \in pP \), we have \( \theta_r(G)\theta_r(G') = \theta_r(G) \), so the equation from above is equivalent to \( g_{y(i)}^{-1}h_1 \in \theta_i(G)\theta_i(G) \). This in turn is equivalent to the existence of \( k, \ell \in G \) satisfying \( g_{y(i)}(k) = h_1\theta_i(\ell) \). In this case, the sum of \( \alpha^r_i(\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)})) \alpha^r_j(\Theta_{\pi_1}(\delta_{y(j)}),\pi_2(\delta_{y(j)})) \) corresponding to the unique pair \( (i, j) \) with \( k \in i\theta_r(G) \) and \( \ell \in j\theta_r(G) \) is non-zero.

Now suppose \( k_1, k_2, \ell_1, \ell_2 \in G \) satisfy \( g_{y(i)}(k_n) = h_1\theta_i(\ell_n), n = 1, 2 \). We need to show that \( k_1^{-1}k_2 \in \theta_r(G) \) and \( \ell_1^{-1}\ell_2 \in \theta_r(G) \). Rewriting the two equations as
\[
\theta_r(k_1)\theta_i(\ell_1) = g_{y(i)}^{-1}h_1 = \theta_r(k_2)\theta_i(\ell_2)
\]
gives \( \theta_r(k_1^{-1}k_2) = \theta_i(\ell_1^{-1}\ell_2) \). Since \( \theta \) respects the order, we conclude that \( \theta_r(k_1^{-1}k_2) \in \theta_r(G) \cap \theta_i(\ell_1^{-1}\ell_2) \) \( \theta_r(G) \) and, similarly, \( \theta_i(\ell_1^{-1}\ell_2) \in \theta_r(G) \). Using that \( \theta_r \) and \( \theta_i \) are injective and \( pp' = qq' = r \), we conclude that \( k_1^{-1}k_2 \in \theta_r(G) \) and \( \ell_1^{-1}\ell_2 \in \theta_r(G) \).

**Remark 7.7.** Note that if we apply Lemma 7.6 to \( p, q, r \in P \) with \( pP \cap qP = rP \), \( g = g_1 = g_2 \in G \) and \( h = h_1 = h_2 \in G \), we get the identity
\[
\alpha^r_i(\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)})) = \left\{ \begin{array}{ll}
\Theta_{\pi_1}(\delta_{y(i)}),\pi_2(\delta_{y(i)}) & \text{if } g_{y(i)}(k) \in h\theta_i(G) \text{ for some } k \in G, \\
0 & \text{otherwise}.
\end{array} \right.
\]

**Proposition 7.8.** The product system \( M \) from Proposition 7.4 is compactly aligned.

**Proof.** This follows immediately from Lemma 6.3 and (7.3). \( \square \)

We can now state the main result of this section. Recall from Section 6 that \( j_M \) denotes to the universal Nica covariant representation of a compactly aligned product system \( M \); for each \( p \in P \) we denote by \( j_p \) the restriction \( j_M|_{M_p} \).

**Theorem 7.9.** Let \( M \) be the product system given in Proposition 7.4. There is an isomorphism \( \varphi : \mathcal{A}(G, P, \theta) \to \mathcal{N} \mathcal{T}(M) \) satisfying
\[
\varphi(u_g) = j_1(\delta_g) \text{ and } \varphi(s_p) = j_p(\pi_p(1)),
\]
for all \( g \in G, \quad p \in P \).

We will use the following result which provides a characterisation of Nica covariance in terms of the generalised rank one projections \( \mathcal{E}_{g,p} \).

**Proposition 7.10.** Let \( M \) be the product system given in Proposition 7.4. A representation \( \psi \) of \( M \) is Nica covariant if and only if for all \( g, h \in G \) and \( p, q \in P \) we
have

\[(7.4) \quad \psi^{(p)}(\mathcal{E}_{g,p})\psi^{(q)}(\mathcal{E}_{h,q}) = \begin{cases} \psi^{(r)}(\mathcal{E}_{g\theta_p(k),r}) & \text{if } P \cap qP = rP \text{ for some } r \in P \text{ and } g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\ 0 & \text{otherwise.} \end{cases} \]

\[
\]

**Proof.** If \( \psi \) is Nica covariant, then \((7.4)\) follows immediately from \((7.3)\). For the converse direction, we note that, by linearity and continuity, it suffices to check \((6.2)\) for \(\Theta_{\pi_p(\delta_1)},\pi_p(\delta_2)\) and \(\Theta_{\pi_q(\delta_1)},\pi_q(\delta_2)\) for all \(p, q \in P\) and \(g_i, h_i \in G\). We have

\[
\psi^{(p)}(\Theta_{\pi_p(\delta_1)},\pi_p(\delta_2))\psi^{(q)}(\Theta_{\pi_q(\delta_1)},\pi_q(\delta_2)) = \psi^{(p)}(\Theta_{\pi_p(\delta_1)},\pi_p(\delta_2))\psi^{(q)}(\Theta_{\pi_q(\delta_1)},\pi_q(\delta_2)),
\]

which we know by \((7.4)\) is equal to

\[(7.5) \quad \psi^{(p)}(\Theta_{\pi_p(\delta_1)},\pi_p(\delta_2))\psi^{(q)}(\Theta_{\pi_q(\delta_1)},\pi_q(\delta_2)) = \psi^{(r)}(\Theta_{\pi_p(\delta_1)},\pi_p(\delta_2))\psi^{(q)}(\Theta_{\pi_q(\delta_1)},\pi_q(\delta_2)) \quad \text{if } P \cap qP = rP \text{ for some } r \in P \text{ and } g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \quad \text{and is zero, otherwise.}
\]

We immediately see from \((7.4)\) and Lemma \(7.6\) that \((6.2)\) is satisfied when \(P \cap qP = \emptyset\) or \(g_2^{-1}h_1 \notin \theta_q(G)\).

So suppose \(pP \cap qP = rP\) for some \(r \in P\) and \(g\theta_p(k) \in h\theta_q(G)\) for some \(k \in G\). Let \(p', q' \in P\) with \(pp' = qq' = r\), and \(\ell \in G\) with \(g\theta_p(k) = h\theta_q(\ell)\). The expression in \((7.5)\) can be written as

\[
\psi^{(p)}(\pi_p(\delta_1))\psi^{(q)}(\pi_p(\delta_2))^*\psi^{(r)}(\mathcal{E}_{g\theta_p(k),r})\psi^{(q)}(\pi_q(\delta_1))\psi^{(q)}(\pi_q(\delta_2))^*. \quad \text{The expression in } (7.5) \quad \text{can be written as}
\]

\[
\psi^{(p)}(\pi_p(\delta_1))\psi^{(q)}(\pi_p(\delta_2))^*\psi^{(r)}(\mathcal{E}_{g\theta_p(k),r})\psi^{(q)}(\pi_q(\delta_1))\psi^{(q)}(\pi_q(\delta_2))^*. \quad \text{The expression in } (7.5) \quad \text{can be written as}
\]

\[
\psi^{(r)}(\mathcal{E}_{g\theta_p(k),r}) = \psi^r(\pi_r(\delta_1))\psi^r(\pi_r(\delta_2))^* = \psi^r(\pi_r(\delta_1))\psi^r(\pi_r(\delta_2))^*,
\]

Using \(\psi^r(\pi_r(\delta_1))\psi^r(\pi_r(\delta_2))^* = 1 = \psi^r(\pi_r(\delta_1),\pi_r(\delta_2))\), we turn the expression in \((7.5)\) into

\[
\psi^{(p)}(\pi_p(\delta_1))\psi^{(q)}(\pi_p(\delta_2))^*\psi^{(q)}(\pi_q(\delta_1))\psi^{(q)}(\pi_q(\delta_2))^* = \psi^{(r)}(\pi_r(\delta_1),\pi_r(\delta_2)),
\]

But we know from Lemma \(7.6\) that

\[
\psi^{(r)}(\pi_r(\delta_1),\pi_r(\delta_2)) = \psi^{(r)}(\Theta_{\pi_p(\delta_1),\pi_p(\delta_2)}),
\]

So we get

\[
\psi^{(p)}(\Theta_{\pi_p(\delta_1),\pi_p(\delta_2)})\psi^{(q)}(\Theta_{\pi_q(\delta_1),\pi_q(\delta_2)}) = \psi^{(r)}(\pi_r(\delta_1),\pi_r(\delta_2)),
\]

and hence \((6.2)\) is satisfied. \(\square\)

The conclusion we draw from Proposition \(7.10\) is that, for the product system \(M\), it suffices to check Nica covariance on rank one projections coming from suitable orthonormal bases.
Proof of Theorem 7.9. The proof follows closely that of Theorem 4.4. The map \( g \mapsto j_1(\delta_g) \) is a unitary representation of \( G \) in \( \mathcal{N} \mathcal{T}(M) \), and \( p \mapsto j_p(\pi_p(1)) \) is a representation of \( P \) by isometries in \( \mathcal{N} \mathcal{T}(M) \). We have

\[
j_p(\pi_p(1))j_1(\delta_g) = j_p(\pi_p(\delta_{(g,p)})) = j_1(\delta_{(g,p)})j_p(\pi_p(1)),
\]

which is (A1). Instead of (A2), we show (2.2), see Lemma 2.4. First note that \( p \in A \) satisfies (2.2), see Lemma 2.4. First note that \( j_1(\delta_g)j_p(\pi_p(1))j_p(\pi_p(1))^* = j^{(p)}(\mathcal{E}_{g,p}) \). Now (2.2) follows from applying (7.4) to the representation \( j \). So the universal property of \( \mathcal{A}[G,P,\theta] \) gives a homomorphism \( \varphi : \mathcal{A}[G,P,\theta] \to \mathcal{N} \mathcal{T}(M) \) satisfying \( \varphi(u_g) = j_1(\delta_g) \) and \( \varphi(s_p) = j_p(\pi_p(1)) \), for all \( g \in G \), \( p \in P \). This homomorphism is surjective because each \( j_p(\pi_p(\delta_g)) = \varphi(u_g s_p) \) is in the range of \( \varphi \).

It is routine to show that \( \pi_p(\delta_g) \mapsto u_g s_p \) defines a representation of the product system \( M \), and (2.2) gives (7.4), which is Nica covariance. The induced homomorphism \( \psi : \mathcal{N} \mathcal{T}(M) \to \mathcal{A}[G,P,\theta] \) is inverse to \( \varphi \), and hence \( \varphi \) is an isomorphism.

We note the following immediate consequence:

Corollary 7.11. If \( S \) is a right LCM semigroup, then \( C^*(S) \) is isomorphic to the Nica-Toeplitz algebra \( \mathcal{N} \mathcal{T}(M) \) of the product system of right-Hilbert bimodules \( M \) over \( S \) with \( M_p = \mathbb{C} \otimes M_1 \) for all \( p \in S \).

Proof. The triple \( \{1\}, S, \text{id} \) defines an algebraic dynamical system and \( \{1\} \rtimes \text{id} S \cong S \), so Theorem 4.4 and Theorem 7.9 imply

\[
C^*(S) \cong C^*(\{1\} \rtimes \text{id} S) \cong \mathcal{A}[\{1\}, S, \text{id}] \cong \mathcal{N} \mathcal{T}(M)
\]

for the product system \( M \) from Proposition 7.4 associated to \( \{1\}, S, \text{id} \).

8. Examples

Observing that algebraic dynamical systems are a natural generalisation of irreversible algebraic dynamical systems as introduced in [Sta15, Definition 1.5], we already have various examples at our disposal, namely [Sta15, Examples 1.8–1.11,1.14].

Remark 8.1. For each irreversible algebraic dynamical system \( (G,P,\theta) \), the \( C^* \)-algebra \( \mathcal{A}[G,P,\theta] \) can be viewed as the natural Toeplitz extension of the \( C^* \)-algebra \( \mathcal{O}[G,P,\theta] \) from [Sta15]. In fact, \( \mathcal{A}[G,P,\theta] \) is the Nica-Toeplitz algebra of the product system \( M \) associated to \( (G,P,\theta) \) according to Theorem 7.9 and \( \mathcal{O}[G,P,\theta] \) is the Cuntz-Nica-Pimsner algebra of \( M \) if \( (G,P,\theta) \) is of finite type, see [Sta15, Theorem 5.9].

But the class of algebraic dynamical systems is much larger:

Example 8.2. Suppose \( P \) is a discrete group acting on another discrete group \( G \) by automorphisms \( \theta_p, p \in P \). Then \( (G,P,\theta) \) is an algebraic dynamical system and \( \mathcal{A}[G,P,\theta] = C^*(G \rtimes \theta P) = C^*(G) \rtimes P \). In particular, we can take \( P = \{1\} \) and obtain the full group \( C^* \)-algebra for \( G \).

Example 8.3. Similar to the last part in Example 8.2, if we let \( G = \{1\} \), then any right LCM semigroup \( P \) acts on \( G \) by the trivial action and \( \mathcal{A}[G,P,\theta] \) is nothing but the full semigroup \( C^* \)-algebra of \( P \).
Remark 8.4. Since we know that, conversely, $G \rtimes_\theta P$ is a right LCM semigroup for every algebraic dynamical system $(G, P, \theta)$, we see that studying algebraic dynamical systems is a priori more sensible than studying right LCM semigroups. The difference is that an algebraic dynamical system comes equipped with a decomposition of the right LCM semigroup as a semidirect product of a group by a right LCM semigroup. A way of extracting a specific semidirect product description for certain cancellative semigroups out of their semigroup structure is presented in [BLS17, Proposition 2.11].

Example 8.5. The semigroup $\mathbb{N}^\times$ is a right LCM semigroup which acts upon $\mathbb{Z}$ by multiplication in an order preserving way. Hence we get an algebraic dynamical system $(\mathbb{Z}, \mathbb{N}^\times, \cdot)$. By Theorem 4.4, we have $A[\mathbb{Z}, \mathbb{N}^\times, \cdot] \cong C^*(\mathbb{Z} \rtimes \mathbb{N}^\times)$, and by Theorem 7.9 they both are isomorphic to a Nica-Toeplitz algebra for a product system of right-Hilbert $C^*(\mathbb{Z})$-modules over $\mathbb{N}^\times$. It was observed in [BaHLR12, Example 3.9] and in [HLS12a] that the latter $C^*$-algebra is isomorphic to the additive boundary quotient $T_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of $T(\mathbb{N} \rtimes \mathbb{N}^\times)$, the Toeplitz algebra of the affine semigroup over the natural numbers. We note that $(\mathbb{Z}, \mathbb{N}^\times, \cdot)$ is in fact an irreversible algebraic dynamical system and $O[\mathbb{Z}, \mathbb{N}^\times, \cdot] \cong Q_N$.

Example 8.5 yields an alternative perspective on the boundary quotient diagram of [BaHLR12] which seems well-suited for generalisations to other cases that are similar to $N \subset \mathbb{Z}$. The next example provides a reasonable framework for such a task.

Example 8.6. Suppose $R$ is the ring of integers in a number field $K$ and let $(r)$ denote the principal ideal generated by $r \in R$. If we take $G = R$ (equipped with addition) and $P \subset R^\times$ is a subsemigroup satisfying $\{p, q \mid p, q \in P\}$ for all $p, q \in P$, then $(G, P, \cdot)$ is an algebraic dynamical system. Note that the intersection condition is automatically satisfied if $R$ is a principal ideal domain, i.e. the class number of $K$ is one. In this case, we recover the natural extension $C^*(R \rtimes R^\times)$ of the ring $C^*$-algebra $\mathfrak{A}[R]$ from [CL10].

Remark 8.7. Even though there are examples of semigroups $P$ as in Example 8.6 for number fields $K$ with class number bigger than one, it would be convenient to have a systematic treatment which is available for arbitrary subsemigroups of $R^\times$ for arbitrary number fields $K$. This asks for a relaxation of the condition

$$\theta_p(G) \cap \theta_q(G) = \theta_r(G)$$

whenever $pP \cap qP = rP$

to the constraint

$$[\theta_p(G) \cap \theta_q(G) : \theta_r(G)] < \infty$$

whenever $pP \cap qP \supset rP$ and $r$ is minimal in the sense that $pP \cap qP \supset rP$ and $rP \supset r'P$ imply $r'P = rP$. We would like to point out that this condition is somewhat reminiscent of Spielberg’s notion of finite alignment for categories of paths, see [Spi14, Section 3].

Let us end with two basic constructions for algebraic dynamical systems, which have been described in [Sta15, Examples 1.12 and 1.13] for the irreversible case. The first one is built by shifting some suitable group $G_0$ along a right LCM semigroup $P$.

Proposition 8.8. Let $P$ be a countable, right LCM semigroup with identity and $G_0$ a countable group. Then $P$ admits a shift action $\theta$ on $G := \bigoplus P G_0$ and $(G, P, \theta)$ is an algebraic dynamical system.
Proof. The action $\theta$ is given by $(\theta_p((g_q)_{q\in P}))_r = \chi_{g_p}(r) g_{p^{-1}r}$ for all $p, r \in P$. It is apparent that $\theta_p$ is an injective group endomorphism for all $p \in P$. Observing that $\theta_p(G) = 0 \oplus (\bigoplus_{q \in P} G_q)$, we see that $\theta$ also respects the order on $P$. Hence we get an algebraic dynamical system in the sense of Definition 2.1.

Example 8.9. If we apply the construction of Proposition 8.8 to $G_0 = \mathbb{Z}/n\mathbb{Z}$ and $P = \mathbb{N}$, the resulting algebra $A[G, P, \theta]$ is canonically isomorphic to the Toeplitz extension $\mathcal{T}_n$ of the Cuntz algebra $\mathcal{O}_n$, compare [Sta15, Example 3.30].

Example 8.10. Suppose we are given a family $(G^{(i)}, P, \theta^{(i)})_{i \in I}$ of algebraic dynamical systems. Then we can consider $G := \bigoplus_{i \in I} G^{(i)}$ and let $P$ act on $G$ component-wise, i.e. $\theta_p(g^{(i)})_{i \in I} := (\theta^{(i)}_p(g^{(i)}))_{i \in I}$. In this way we get a new algebraic dynamical system $(G, P, \theta)$.

Remark 8.11. It is also possible to decompose an algebraic dynamical system $(G, P, \theta)$. Let $G = \bigoplus_{i \in I} G_i$ be a decomposition of $G$ into $\theta$-invariant subgroups, i.e. $\theta_p(G_i) \subset G_i$ for all $i \in I$. Then $(G_i, P, \theta|_{G_i})$ defines an algebraic dynamical system for each $i \in I$. Indeed, it is easy to see that the subsystem $(G_i, P, \theta|_{G_i})$ inherits the required properties from $(G, P, \theta)$. Besides, we note that the procedure from Example 8.10 is inverse to this method of decomposing.

Example 8.12. For $G = \mathbb{Z}$, the injective group endomorphisms are given by multiplication with elements from $\mathbb{Z}^\times$. For $P \subset \mathbb{Z}^\times$, let $|P|$ denote the multiplicative subgroup of $\mathbb{Z}^\times$ with identity generated by $P$. Let us consider subsemigroups $P_1 = \langle 2, 3 \rangle$, $P_2 = \langle -2, 3 \rangle$, $P_3 = \langle 2, -3 \rangle$ and $P_4 = \langle -2, -3 \rangle$. Each of them gives rise to an irreducible algebraic dynamical system $(\mathbb{Z}, P_i, \cdot)$. It is an instructive exercise to show that the corresponding semidirect products $\mathbb{Z} \rtimes P_i$ are mutually non-isomorphic. However, if we add $-1$ as a generator to each of $P_i$, we always arrive at $P_5 = \langle -1, 2, 3 \rangle$. Moreover, the corresponding embedding $\mathbb{Z} \rtimes P_i \subset \mathbb{Z} \rtimes P_5$ yields injective, admissible morphisms from $(\mathbb{Z}, P_i, \cdot)$ into $(\mathbb{Z}, P_5, \cdot)$. Hence, by Corollary 4.10 there are canonical $\ast$-homomorphisms from $A[\mathbb{Z}, P_i, \cdot]$ to $A[\mathbb{Z}, P_5, \cdot]$ for $i = 1, \ldots, 4$.

In this example the left regular representation implements an isomorphism between the full and the reduced semigroup $C^\ast$-algebras for $\mathbb{Z} \rtimes P_i$ for each $i = 1, \ldots, 5$, see [BLS17, Example 6.3] or [Li12]. Thus by Proposition 3.6 the $C^\ast$-algebras $A[\mathbb{Z}, P_i, \cdot]$, $i = 1, \ldots, 4$ are proper subalgebras of $A[\mathbb{Z}, P_5, \cdot]$ in a natural way. Note that the families of constructible right ideals $J(\mathbb{Z} \rtimes P_i)$ are isomorphic for all five semidirect products. Thus, their diagonal subalgebras are canonically isomorphic, see [Li12]. Since this identification is compatible with translation by $\mathbb{Z}$, we also have an isomorphism on the level of the core subalgebras $\mathcal{F} \cong \mathcal{D} \rtimes \mathbb{Z}$, a fact that follows from [BLS17, Definition 3.12] and [Sta15, Theorem A.5].

It is therefore natural to ask if, and in how far the $C^\ast$-algebras $A[\mathbb{Z}, \langle 2, 3 \rangle, \cdot], A[\mathbb{Z}, \langle -2, 3 \rangle, \cdot], A[\mathbb{Z}, \langle 2, -3 \rangle, \cdot], A[\mathbb{Z}, \langle -2, -3 \rangle, \cdot]$ and $A[\mathbb{Z}, \langle -1, 2, 3 \rangle, \cdot]$ are different. Corollary 5.5 shows that $K_i(A[\mathbb{Z}, P_i, \cdot])$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}$ for $i = 1, \ldots, 4$, and that

$$K_i(A[\mathbb{Z}, P_5, \cdot]) = K_i(C^\ast(\mathbb{Z} \rtimes \mathbb{Z} / 2\mathbb{Z})).$$

Thus, $K$-theory distinguishes $A[\mathbb{Z}, P_i, \cdot]$ from any of the $C^\ast$-algebras $A[\mathbb{Z}, P_j, \cdot]$ for $1, \ldots, 4$. However, finer invariants are needed in order to determine the exact relationship between those four $C^\ast$-algebras.
References


School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia

E-mail address: nathan.brownlowe@sydney.edu.au

Department of Mathematics, University of Oslo, PO BOX 1053 Blindern, 0316 Oslo, Norway

E-mail address: nadiasl@math.uio.no

Department of Mathematics, University of Oslo, PO BOX 1053 Blindern, 0316 Oslo, Norway

E-mail address: nicolsta@math.uio.no