OPTIMAL REGULARITY FOR NONLINEAR ELLIPTIC EQUATIONS WITH RIGHT-HAND SIDE MEASURE IN VARIABLE EXPONENT SPACES

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Abstract. We study a nonlinear elliptic equation with measurable nonlinearity in a nonsmooth domain when the right-hand side is a measure. A global Calderón-Zygmund type estimate in variable exponent spaces is established under an optimal regularity assumption on the nonlinearity and the Reifenberg flatness of the boundary.

1. Introduction

In this paper, we establish global gradient estimates for solutions of the divergence structure nonlinear elliptic equations with measure data in the setting of variable exponent spaces. Many interesting phenomena in the area of applied mathematics naturally involve measure data problems, for instance, the flow pattern of blood in the heart [40, 45], and state-constrained optimal control problems [16, 17, 33].

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with nonsmooth boundary $\partial \Omega$, and $\mu$ be a signed Radon measure on $\Omega$ with finite total variation $|\mu|(\Omega) < \infty$. Consider the Dirichlet problem with measure data

\begin{equation}
\begin{cases}
-\text{div} \ a(Du, x) = \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Here we assume that $\mu$ is defined in $\mathbb{R}^n$ by considering the zero extension to $\mathbb{R}^n \setminus \Omega$, and the vector field $a = a(\xi, x): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in $\xi$ and measurable in $x$, and it satisfies the following conditions:

\begin{equation}
|\xi| |D_\xi a(\xi, x)| + |a(\xi, x)| \leq \Lambda |\xi|,
\end{equation}

\begin{equation}
\lambda |\eta|^2 \leq \langle D_\xi a(\xi, x) \eta, \eta \rangle,
\end{equation}

for every $x, \eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and some constants $\lambda$, $\Lambda$. Note that (1.2) implies that $a(0, x) = 0$ for $x \in \mathbb{R}^n$, and (1.3) yields the following monotonicity condition:

\[ \langle a(\xi_1, x) - a(\xi_2, x), \xi_1 - \xi_2 \rangle \geq \lambda |\xi_1 - \xi_2|^2 \]

for all $x, \xi_1, \xi_2 \in \mathbb{R}^n$ and some constant $\lambda = \lambda(n, \lambda, \gamma_1, \gamma_2) > 0$.

A solution $u$ of (1.1) will be treated in the sense of distribution which does not generally belong to a weak solution in $W^{1,2}_0(\Omega)$ (consider Laplace’s equation with
the Dirac measure). For this reason, it is necessary to generalize a class of solutions below the natural exponent.

**Definition 1.1.** \( u \in W^{1,1}_0(\Omega) \) is a SOLA (Solution Obtained by Limits of Approximations) to the problem (1.1) under the assumptions (1.2) and (1.3) if the vector field \( a(Du, x) \in L^1(\Omega, \mathbb{R}^n) \),

\[
\int_{\Omega} \langle a(Du, x), D\varphi \rangle \, dx = \int_{\Omega} \varphi \, d\mu
\]

holds for all \( \varphi \in C^\infty_c(\Omega) \), and moreover there exists a sequence of weak solutions \( \{u_h\}_{h \geq 1} \in W^{1,2}_0(\Omega) \) of the Dirichlet problems

\[
\begin{cases}
-div a(Du_h, x) = \mu_h & \text{in } \Omega, \\
 u_h = 0 & \text{on } \partial\Omega
\end{cases}
\]  

such that

\[ u_h \to u \text{ in } W^{1,1}_0(\Omega) \text{ as } h \to \infty, \]

where \( \{\mu_h\} \in L^\infty(\Omega) \) converges weakly to \( \mu \) in the sense of measure and satisfies for each open set \( V \subset \mathbb{R}^n \),

\[
(1.5) \quad \limsup_{h \to \infty} \|\mu_h\|_V \leq |\mu|(V),
\]

with \( \mu_h \) defined in \( \mathbb{R}^n \) by the zero extension of \( \mu_h \) to \( \mathbb{R}^n \setminus \Omega \).

Here we consider \( \mu_h := \mu * \phi_h \), where \( \phi_h \) is the standard mollifier, and then \( \mu_h \in C^\infty(\Omega) \) converges weakly to \( \mu \) in the sense of measure, the following uniform \( L^1 \)-estimate holds:

\[
(1.6) \quad \|\mu_h\|_{L^1(\Omega)} \leq |\mu|(\Omega),
\]

and such a SOLA \( u \) of (1.1) belongs to \( W^{1,q}_0(\Omega) \) such that

\[
(1.7) \quad u_h \to u \text{ in } W^{1,q}_0(\Omega) \text{ for all } q \in \left[ 1, \frac{n}{n-1} \right).
\]

The existence of such a solution is due to Boccardo and Gallouët [2], who proved a priori \( W^{1,q} \)-estimate of solutions for regularized problems with a proper approximation scheme. On the other hand, the uniqueness of a SOLA is still a main open problem except for the linear case, \( a(\xi, x) = a(x)\xi \), see [20] and the references therein. Very recently, however, it was proved that a SOLA \( u \) is unique under the assumption that \( a \) is strongly asymptotically Uhlenbeck, see [4, Corollary 2.2]. We also refer to [1–3, 19, 20] for a thorough discussion regarding the existence and uniqueness of measure data problems.

We next recall a brief overview of variable exponent spaces and log-Hölder continuity, see the monographs [18, 22] for details. Let \( p(\cdot) \) be a measurable function defined on \( \Omega \) with

\[
(1.8) \quad 1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty
\]

for appropriate constants \( \gamma_1 \) and \( \gamma_2 \). The variable exponent Lebesgue space \( L^{p(\cdot)}(\Omega) \) consists of all measurable functions \( f : \Omega \to \mathbb{R} \) such that the modular

\[
\rho_{p(\cdot)}(f) := \int_{\Omega} |f(x)|^{p(x)} \, dx
\]
is finite. If \( f \in L^{p(\cdot)}(\Omega) \), then we define its norm to be

\[
\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left( \frac{f}{\lambda} \right) \leq 1 \right\}.
\]

Then there is the close relationship between the norm and the modular:

\[
\min \left\{ \rho_{p(\cdot)}(f)^{\frac{1}{p(\cdot)}}, \rho_{p(\cdot)}(f)^{\frac{1}{p(\cdot)}} \right\} \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq \max \left\{ \rho_{p(\cdot)}(f)^{\frac{1}{p(\cdot)}}, \rho_{p(\cdot)}(f)^{\frac{1}{p(\cdot)}} \right\}.
\]

The variable exponent Sobolev space \( W^{1,p(\cdot)}(\Omega) \) consists of all functions \( f \in L^{p}(\Omega) \) whose gradient \( Df \) exists in the weak sense and belongs to \( L^{p(\cdot)}(\Omega) \), equipped with the norm

\[
\|f\|_{W^{1,p(\cdot)}(\Omega)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \|Df\|_{L^{p(\cdot)}(\Omega, \mathbb{R}^n)}.
\]

These spaces are all separable reflexive Banach spaces.

We introduce the log-Hölder continuity which plays a very important role in the study of variable exponent spaces, such as potential theory, singular integrals, maximal operators, and regularity theory, etc. Given a function \( p(\cdot) \) satisfying (1.8), we say that \( p(\cdot) \) is log-Hölder continuous in \( \Omega \) if \( p(\cdot) \) has a modulus of continuity, that is, there exists a nondecreasing concave function \( \omega : [0, \infty) \to [0, \infty) \) with \( \omega(0) = 0 \) and

\[
|p(x) - p(y)| \leq \omega(|x - y|) \quad \text{for } x, y \in \Omega,
\]

and moreover

\[
\sup_{0 < r \leq \frac{1}{2}} \omega(r) \log \left( \frac{1}{r} \right) \leq L
\]

for some constant \( L > 0 \).

We now state the main assumption on \( a \) and \( \Omega \) in the paper, see the notation explained in the next section.

**Definition 1.2.** We say \( (a, \Omega) \) is \((\delta, R)\)-vanishing of codimension 1 if for every point \( y \in \Omega \) and number \( r \in (0, \frac{R}{4}] \), the following conditions hold.

(i) If \( \text{dist}(y, \partial \Omega) > r \sqrt{2} \), then there exists a new coordinate system depending only on \( y \) and \( r \), still denoted by \( \{x_1, \ldots, x_n\} \), in which the origin is \( y \) and

\[
\int_{Q_{r, \sqrt{2}}} |\theta(a, Q_{r, \sqrt{2}})(x)| \, dx \leq \delta,
\]

where

\[
\theta(a, Q_r)(x) := \sup_{\xi \in \mathbb{R}^{n} \setminus \{0\}} \frac{|a(\xi, x', x_n) - \bar{a}_{B^+}(\xi, x_n)|}{|\xi|},
\]

and \( \bar{a}_{B^+}(\xi, x_n) \) is the integral average of \( a(\xi, \cdot, x_n) \) over \( B^+_r \subset \mathbb{R}^{n-1} \).

(ii) If \( \text{dist}(y, \partial \Omega) = |y - y_0| \leq r \sqrt{2} \) for some \( y_0 \in \partial \Omega \), then there is a new coordinate system depending only on \( y \) and \( r \), still denoted by \( \{x_1, \ldots, x_n\} \), in which the origin is \( y_0 + 3\delta e_n \), where \( e_n := (0, \cdots, 0, 1) \),

\[
Q^+_{3r} \subset \Omega_{3r} \subset Q_{3r} \cap \{(x', x_n) : x_n > -6\delta \},
\]

and

\[
\int_{Q_{3r}} |\theta(a, Q_{3r})(x)| \, dx \leq \delta.
\]
Remark 1.3.  
(i) The number $\delta$ is a sufficiently small universal constant with $\delta \in (0, \frac{1}{2})$, as determined later in the proof of Theorem 1.4. This number is invariant under the dilation scaling for the problem (1.1). On the other hand, the number $R$ is given arbitrary.
(ii) The numbers $r \sqrt{2}$ and $3r$ above are selected so that the size of an elliptic cylinder $Q_r$ is large enough to contain its rotation in any direction.
(iii) If $(a, \Omega)$ is $(\delta, R)$-vanishing of codimension 1, then for each point and sufficiently small scale, there is a coordinate system for which the vector field $a(\xi, \cdot)$ is merely measurable in the $x_n$ variable and of small $BMO$ in the other variables $x'$. Moreover, the domain $\Omega$ with (1.12) is called a $(\delta, R)$-Reifenberg flat domain, which means that the boundary of $\Omega$ can be locally approximated by two hyperplanes in the new coordinate system under the scale chosen. This domain may have a very rough boundary including $C^1$ domain or Lipschitz domain with a small Lipschitz constant. We refer to [12, 31, 32, 44, 46] and the references therein for a further discussion on Reifenberg flat domains.
(iv) If $\Omega$ is $(\delta, R)$-Reifenberg flat, then there is the following measure density condition
\[
(1.13) \quad \sup_{0 < r < \frac{R}{4}} \sup_{y \in \Omega} \frac{|Q_r(y)|}{|\Omega \cap Q_r(y)|} \leq \left( \frac{2 \sqrt{2}}{1 - \delta} \right)^n \leq \left( \frac{16 \sqrt{2}}{7} \right)^n,
\]
which can be found in [10].

We are ready to present our main results.

Theorem 1.4. Assume that (1.2) and (1.3) are hold, and that $u$ is a SOLA of the problem (1.1). Let $0 < R < R_0$ and let $p(\cdot)$ be log-Hölder continuous satisfying (1.8). Then there is a sufficiently small constant $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, L) \in (0, \frac{1}{8})$ such that if $(a, \Omega)$ is $(\delta, R)$-vanishing of codimension 1, then there exists a constant $c_0 = c_0(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), R, \Omega) > 1$ so that for any $x_0 \in \Omega$ and $R_0 \in \left( 0, \frac{1}{\delta(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) + 1} \right)$, we have
\[
(1.14) \quad \iint_{\Omega \cap B_{R_0}(x_0)} |Du|^p(x) dx 
\leq c \left\{ \left( \int_{\Omega \cap B_{R_0}(x_0)} |Du|^{p_-(x)} dx \right)^{p_-(x)} + \int_{\Omega \cap B_{R_0}(x_0)} M_1(\mu)^p(x) dx + 1 \right\}
\]
for some constant $c = c(n, \Lambda, \gamma_1, \gamma_2, L) > 0$, where $p_- := \inf_{x \in \Omega \cap B_{R_0}(x_0)} p(x)$.
Moreover, we have
\[
(1.15) \quad \int_{\Omega} |Du|^{p(x)} dx \leq c \left\{ \left( \int_{\Omega} M_1(\mu)^p(x) dx \right)^{\frac{\gamma_2(\gamma_2-1)+\gamma_1}{\gamma_1}} + 1 \right\}
\]
and
\[
(1.16) \quad \|Du\|_{L^p(\cdot)(\Omega)} \leq c \|M_1(\mu)\|_{L^{p_-(\cdot)}(\Omega)}
\]
where the constants $c$ depend only on $n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), L, R,$ and $\Omega$. Here $M_1$ is the fractional maximal function of order 1 for $\mu$ defined by
\[
M_1(\mu)(x) := \sup_{r > 0} \frac{r^\gamma |\mu||B_r(x)|}{|B_r(x)|} \quad \text{for} \quad x \in \mathbb{R}^n.
\]
Remark 1.5. We know from (1.7) that \( u \in W_{0}^{1,q}(\Omega) \) for all \( 1 \leq q < \frac{n}{n-1} \), and then the first term of the right-hand side in (1.14) is well defined by selecting \( c_{0} \) sufficiently large with \( \frac{p(x)}{p_{-}} < \frac{n}{n-1} \) for \( x \in \Omega_{R_{0}}(x_{0}) \), see Section 5.1 for details.

Remark 1.6. The condition on the above vector field \( \mathbf{a} \) is a possibly optimal assumption for the estimates (1.14)–(1.16). In other words, if \( a(\xi, \cdot) \) has two or more measurable coefficients, then these estimates are not generally satisfied even in the constant exponent case \( p(\cdot) \equiv p \), see [34]. For the measurability in one variable, there have been regularity results for linear elliptic equations, see [10, 11, 13, 24]. Recently, Byun and Kim [5] considered nonlinear elliptic equations, without measure data, to obtain global \( L^{p} \) estimates for the gradient of a weak solution under the assumptions (1.2), (1.3), and Definition 1.2. They obtained the desired results by proving Lipschitz regularity for limiting problems. However, the case of problems having \( p \)-growth under the same condition (Definition 1.2) still remains unsolved in the literature.

Theorem 1.4 generalizes the recent result of [5] in two aspects. For one thing, we consider measure data problems. Since the measure data is not in general regular enough, we need a new notion of a suitable solution (see Definition 1.1) and a systematic investigation for uniform regularity estimates (see Section 3 and 4). Indeed, there have been various regularity results concerning measure data problems, see [25, 26, 28–30, 35–39, 41–43]. For another, we obtain the Calderón-Zygmund type estimates in the variable exponent context. Unlike the constant exponent case, it is important to study how the function \( p(\cdot) \) changes as a point varies, and so one needs the log-Hölder continuity in order to control the rate of decrease or increase of \( p(\cdot) \). We refer to [6–9, 22, 23] for regularity results on variable exponent spaces.

A main ingredient in our proof is to derive a power decay estimate of the upper-level sets of \( |Du|^{\frac{p(x)}{p_{-}}} \) for a SOLA \( u \) on a small ball \( B \) with \( p_{-} = \inf_{x \in B} p(x) \). We employ some properties of the SOLA, comparison estimates along with higher integrability of homogeneous problems and the log-Hölder continuity of \( p(\cdot) \), and then the so-called maximal function technique which was introduced in [15, 47]. The difficulty in the present work comes from the measure data \( \mu \) and the presence of the variable exponent \( p(\cdot) \), and so more complicated and finer analysis than that previously made in [5, 8] has to be carefully carried out in the whole process.

This paper is organized as follows. In the next section, we introduce some notation and auxiliary results. Section 3 deduces suitable comparison estimates of the regularized problem (1.4). In Section 4, we discuss the assumptions of the Vitali type covering lemma (Lemma 2.1). Finally, Section 5 is dedicated to the proof of Theorem 1.4.

2. Notation and auxiliary results

We start with some standard notation, which will be used throughout the paper.

1. A point \( x \in \mathbb{R}^{n} \) will be written \( x = (x', x_{n}) \) for \( x' = (x_{1}, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \).
2. \( B_{r}(x') = \{ y' \in \mathbb{R}^{n-1} : |x'-y'| < r \} \) is the open ball in \( \mathbb{R}^{n-1} \) with center \( x' \) and radius \( r > 0 \), and write \( B_{r} = B_{r}(0) \) for simplicity.
3. \( Q_{r}(x) = B_{r}(x') \times (x_{n} - r, x_{n} + r) \) is the elliptic cylinder in \( \mathbb{R}^{n} \) with center \( x \) and size \( r > 0 \), \( Q_{r} = B_{r} \times (-r, r) \), and \( Q^{+}_{r} = Q_{r} \cap \{ x_{n} > 0 \} \).
4. \( \Omega_{r}(x) = \Omega \cap Q_{r}(x) \), and \( \Omega_{r} = \Omega \cap Q_{r} \).
With the estimate

\[ \varrho \]  

For each measurable sets such that

\[ (\xi \in \mathbb{R}^n) \quad \text{(8)} \]

For each set \( U \)

\[ \|M\|_p < p \quad \text{and for} 1 < p \leq \infty \]

Then

\[ \varrho \]  

We denote by \( \varrho \) to mean a universal constant greater than one that can be computed in terms of known quantities, and so may be different from line to line.

We now recall some analytic and geometric properties, which are used later. We begin with the Hardy-Littlewood maximal function. For \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), we define

\[ \mathcal{M}f(y) = \mathcal{M}(f)(y) := \sup_{r > 0} \frac{1}{|B_r(y)|} \int_{B_r(y)} |f(x)| \, dx. \]  

We will use the following weak \((1, 1)\) estimates and strong \((p, p)\) estimates:

\[ |\{ x \in \mathbb{R}^n : \mathcal{M}f(x) > \alpha \}| \leq \frac{c(n)}{\alpha} \int_{\mathbb{R}^n} |f| \, dx \quad \text{for all} \ \alpha > 0, \]  

and for \( 1 < p \leq \infty \),

\[ \|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leq c(n, p) \|f\|_{L^p(\mathbb{R}^n)}. \]  

The next lemma is a Vitali type covering lemma whose proof is similar to that for Theorem 2.8 in [12].

**Lemma 2.1.** Suppose \( \Omega \) is \((\delta, R)\)-Reifenberg flat. Consider the domain \( \Omega_{R_0}(x_0) \)

with \( 0 < R_0 \leq R \) and \( x_0 \in \Omega \). Let \( 0 < \epsilon < 1 \) and \( C \subseteq D \subseteq \Omega_{R_0}(x_0) \) be two measurable sets such that

\[ (i) \quad |C| < \left( \frac{1}{1000} \right)^n \epsilon |B_{R_0}|, \quad \text{and} \]

\[ (ii) \quad \text{for any} \ y \in C \quad \text{and any} \ r_0 \in (0, \frac{R_0}{1000}], \quad \text{if} \ |C \cap B_{r_0}(y)| \geq \epsilon |B_{r_0}(y)|, \quad \text{then} \]

\[ B_{r_0}(y) \cap \Omega_{R_0}(x_0) \subset D. \]

Then

\[ |C| \leq \left( \frac{10}{1 - \delta} \right)^n \epsilon |D| \leq \left( \frac{80}{7} \right)^n \epsilon |D|. \]

We will also use the following measure theoretic property.

**Lemma 2.2.** (See [14, Lemma 7.3].) Let \( f \) be a measurable function in a bounded open set \( U \subseteq \mathbb{R}^n \). Let \( \theta > 0 \) and \( N > 1 \) be constants. Then, for \( 0 < q < \infty \),

\[ f \in L^q(U) \iff S := \sum_{k \geq 1} N^{\frac{q}{k}} \left| \left\{ x \in U : |f(x)| > \theta N^k \right\} \right| < \infty \]

with the estimate

\[ c^{-1} \theta^q S \leq \int_U |f|^q \, dx \leq c \theta^q (|U| + S) \]

for some constant \( c = c(N, q) > 0 \).
3. Comparison estimates for regular problems

In this section we consider the regular problem (1.4), where $\mu_h = \mu \ast \phi_h$ with $\phi_h$ the standard mollifier. We as always assume that $(a, \Omega)$ is $(\delta, R)$-vanishing of codimension $1$. This section develops the comparison $L^1$-estimates for the gradient of the weak solution $u_h$ to (1.4) in localized boundary and interior regions. We denote, for a measurable set $E \subset \mathbb{R}^n$,

$$|\mu_h|(E) := \int_E |\mu_h(x)| \, dx.$$ 

3.1. Boundary comparisons. Let $0 < r \leq \frac{R}{8}$. Assume the following geometric setting:

$$Q^+_r \subset \Omega_{8r} \subset Q_{8r} \cap \{x_n > -16\delta r\},$$

$\delta$ being determined later in a universal way.

Let $w_h \in u_h + W^{1,2}_0(\Omega_{8r})$ be the weak solution of the homogeneous problem

$$\begin{cases}
\text{div } a(Dw_h, x) = 0 & \text{in } \Omega_{8r}, \\
w_h = u_h & \text{on } \partial \Omega_{8r}.
\end{cases}$$

Using the measure density condition (1.13), we can extend the comparison result in [30, Lemma 2] up to the boundary.

**Lemma 3.1.** If $w_h \in u_h + W^{1,2}_0(\Omega_{8r})$ is the weak solution of (3.2) satisfying (3.1), then there exists a constant $c = c(n, \lambda, q) > 0$ such that

$$\int_{\Omega_{8r}} |Du_h - Dw_h|^q \, dx \leq c \left( \frac{|\mu_h|_{L^1(\Omega_{8r})}}{r^{n-1}} \right)^q \text{ for all } q \in \left(0, \frac{n}{n-1}\right).$$

Applying Gehring’s lemma to the weak solution $w_h$ of (3.2), we discover some higher integrability result, see [27, Remark 6.12], as we now state.

**Lemma 3.2.** There exists a constant $\sigma_0 = \sigma_0(n, \lambda, \Lambda) > 0$ such that for any $0 < r \leq \frac{R}{8}$, if $w_h$ is the weak solution of (3.2) satisfying (3.1), then for any $0 < \sigma \leq \sigma_0$ and $\Omega_{2\tilde{r}}(\tilde{x}_0) \subset \Omega_{8r}$ with $\tilde{r} \leq 4r$, there is a constant $c = c(n, \lambda, \Lambda, t) > 0$ such that

$$\left( \int_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw_h|^{2(1+\sigma)} \, dx \right)^{\frac{1}{1+\sigma}} \leq c \left( \int_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw_h|^{2t} \, dx \right)^{\frac{1}{t}}$$

for all $t \in (0, 1]$.

From Hölder’s inequality and Lemma 3.2, we can directly obtain the following estimate.

**Corollary 3.3.** Under the same assumptions and conclusion as in Lemma 3.2, we have

$$\int_{\Omega_{r}(x_0)} |Dw_h|^2 \, dx \leq c \left( \int_{\Omega_{2r}(\tilde{x}_0)} |Dw_h| \, dx \right)^2$$

for some constant $c = c(n, \lambda, \Lambda) > 0$. 

We next consider the homogeneous frozen problem
\begin{equation}
\begin{aligned}
\left\{ \begin{array}{l}
\text{div} \ a_{B_4^*}(Dv_h, x_n) = 0 \quad \text{in } \Omega_{4r}, \\
v_h = w_h \quad \text{on } \partial \Omega_{4r},
\end{array} \right.
\end{aligned}
\tag{3.3}
\end{equation}
where \(w_h\) is the weak solution of (3.2). Then \(v_h \in w_h + W^{1,2}_{0}(\Omega_{3r})\) is the weak solution of (3.3), and the vector field \(a_{B_4^*}\) satisfies (1.2) and (1.3) with \(a(\xi', x_n)\) replaced by \(a_{B_4^*}(\xi, x_n)\). Moreover, we derive the standard energy estimate
\begin{equation}
\int_{\Omega_{4r}} |Dv_h|^2 \, dx \leq c \int_{\Omega_{4r}} |Dw_h|^2 \, dx,
\tag{3.4}
\end{equation}
by substituting the test function \(v_h - w_h\) into the weak formulation of (3.3).

The following lemma demonstrates some comparison result between two problems (3.2) and (3.3).

**Lemma 3.4.** (See [5, Lemma 5.6].) Suppose that \(\Omega_{8r}\) satisfies (3.1). If \(w_h\) and \(v_h\) are the weak solutions of (3.2) and (3.3), respectively, then there is a constant \(c = c(n, \lambda, \Lambda) > 0\) such that
\begin{equation}
\int_{\Omega_{4r}} |Dw_h - Dv_h|^2 \, dx \leq c \delta^{\sigma_0} \left( \int_{\Omega_{4r}} |Dw_h| \, dx \right)^2,
\end{equation}
where \(\sigma_0\) is given in Lemma 3.2.

Let us assume now \(\tilde{v}_h \in W^{1,2}(Q_{4r}^+)\) is a weak solution of the reference problem
\begin{equation}
\begin{aligned}
\left\{ \begin{array}{l}
\text{div} \ a_{B_4^*}(D\tilde{v}_h, x_n) = 0 \quad \text{in } Q_{4r}^*, \\
\tilde{v}_h = 0 \quad \text{on } Q_{3r} \cap \{x_n = 0\}.
\end{array} \right.
\end{aligned}
\tag{3.5}
\end{equation}

We can now state some comparison estimate and Lipschitz regularity result.

**Lemma 3.5.** (See [5, Lemma 5.8].) For any \(\epsilon \in (0, 1)\), there is \(\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0\) such that if \(v_h \in w_h + W^{1,2}_{0}(\Omega_{4r})\) is the weak solution of (3.3) with (3.1), then there exists a weak solution \(\tilde{v}_h \in W^{1,2}(Q_{3r}^+)\) of (3.5) such that
\begin{equation}
\int_{\Omega_{3r}} |Dv_h - D\tilde{v}_h|^2 \, dx \leq c^2 \int_{\Omega_{4r}} |Dv_h|^2 \, dx,
\end{equation}
and
\[
\|D\tilde{v}_h\|_{L^\infty(\Omega_{2r})} \leq c \int_{\Omega_{3r}} |D\tilde{v}_h| \, dx
\]
for some constant \(c = c(n, \lambda, \Lambda) > 0\). Here \(\tilde{v}_h\) is extended by zero from \(Q_{3r}^+\) to \(\Omega_{3r}\).

We finally summarize the comparison \(L^1\)-estimates near a boundary region.

**Lemma 3.6.** Let \(\rho > 1\) and \(0 < r \leq \frac{\delta}{\epsilon}\). Suppose that \(\Omega_{8r}\) satisfies (3.1). Then, for any \(0 < \epsilon < 1\), there exists a small constant \(\delta = \delta(n, \lambda, \Lambda, \epsilon) > 0\) such that if \((a, \Omega)\) is \((\delta, R)\)-vanishing of codimension 1, and if \(u_h \in W^{1,2}_0(\Omega), w_h \in u_h + W^{1,2}_0(\Omega_{8r}),\) and \(v_h \in w_h + W^{1,2}_0(\Omega_{4r})\) are the weak solutions of (1.4), (3.2), and (3.3), respectively, with
\begin{equation}
\int_{\Omega_{8r}} |Du_h| \, dx \leq \rho \quad \text{and} \quad \|u_h(\Omega_{8r})\|_{\frac{1}{p-1}} \leq \delta \rho,
\end{equation}
then there is a weak solution \(\tilde{v}_h \in W^{1,2}(Q_{3r}^+)\) of (3.5) such that
\begin{equation}
\int_{\Omega_{3r}} |Du_h - D\tilde{v}_h| \, dx \leq c\rho \quad \text{and} \quad \|D\tilde{v}_h\|_{L^\infty(\Omega_{2r})} \leq c\rho
\end{equation}
for some constant \(c = c(n, \lambda, \Lambda) > 0\). Here \(\tilde{v}_h\) is extended by zero from \(Q_{3r}^+\) to \(\Omega_{3r}\).
Proof. We first have from Lemma 3.1 \((q = 1)\) that
\[
\int_{\Omega_{8r}} |Du_h - Dw_h| \, dx \leq c\delta \rho \quad \text{and} \quad \int_{\Omega_{8r}} |Dw_h| \, dx \leq c\rho.
\] According to Hölder’s inequality and Lemma 3.4, we observe
\[
\int_{\Omega_{4r}} |Dw_h - Dw_h| \, dx \leq \left( \int_{\Omega_{4r}} |Dw_h - Dw_h|^2 \, dx \right)^{\frac{1}{2}} \leq c\delta \frac{\sigma}{1+\sigma_0} \rho,
\]
and
\[
\int_{\Omega_{4r}} |Dv_h| \, dx \leq c\rho.
\]
According to Lemma 3.5 with \(\epsilon\) replaced by \(\tilde{\epsilon}\), there exists a weak solution \(\tilde{v}_h \in W^{1,2}(Q^+_{8r})\) of (3.5) such that
\[
\int_{\Omega_{3r}} |Dv_h - D\tilde{v}_h|^2 \, dx \leq \tilde{\epsilon}^2 \int_{\Omega_{3r}} |Dv_h|^2 \, dx.
\]
Then we see from this estimate, Hölder’s inequality, (3.4), Corollary 3.3, and (3.6) that
\[
\int_{\Omega_{3r}} |Dv_h - D\tilde{v}_h| \, dx \leq c\tilde{\epsilon} \int_{\Omega_{3r}} |Dw_h| \, dx \leq c\tilde{\epsilon} \rho \leq \frac{\epsilon}{3} \rho
\]
by choosing \(\tilde{\epsilon}\) sufficiently small, and it follows from (3.8) and (3.9) that
\[
\int_{\Omega_{3r}} |D\tilde{v}_h| \, dx \leq c\rho.
\]
Finally, we combine (3.6), (3.7) and (3.9), to obtain
\[
\int_{\Omega_{3r}} |Du_h - D\tilde{v}_h| \, dx \leq \int_{\Omega_{3r}} |Du_h - Dw_h| + |Dw_h - Dv_h| + |Dv_h - D\tilde{v}_h| \, dx
\]
\[
\leq c\delta \rho + c\delta \frac{\sigma}{1+\sigma_0} \rho + \frac{\epsilon}{3} \rho
\]
\[
\leq \epsilon \rho,
\]
by selecting \(\delta\) small enough.
On the other hand, in light of Lemma 3.5 and (3.10), we obtain
\[
\|D\tilde{v}_h\|_{L^\infty(\Omega_{2r})} \leq \epsilon \rho,
\]
which completes the proof. 

3.2. Interior comparisons. In this subsection we derive comparison \(L^1\)-estimates for the interior case in a similar way that we derived their counterparts in the previous subsection. We just outline it for the sake of completeness.

Let \(0 < r \leq \frac{4}{3} \) with \(Q_{8r}(x_0) \subseteq \Omega\). With the weak solution \(u_h \in W^{1,2}_0(\Omega)\) of (1.4), we consider the weak solution \(w_h \in u_h + W^{1,2}(Q_{8r}(x_0))\) of the homogeneous problem
\[
\begin{cases}
\text{div } a(Dw_h, x) = 0 & \text{in } Q_{8r}(x_0), \\
w_h = u_h & \text{on } \partial Q_{8r}(x_0).
\end{cases}
\]
Next let \( v_h \in w_h + W_0^{1,2}(Q_r(x_0)) \) be the weak solution of the homogeneous frozen problem

\[
\begin{align*}
\text{div} \tilde{\mathbf{a}}_{P_r} (Dv_h, x_n) &= 0 \quad \text{in } Q_r(x_0), \\
v_h &= w_h \quad \text{on } \partial Q_r(x_0).
\end{align*}
\]

Then we have \( Dv_h \in L^\infty(Q_{2r}(x_0)) \) with the estimate

\[
\|Dv_h\|_{L^\infty(Q_{2r}(x_0))} \leq c \int_{Q_r(x_0)} |Dv_h| \, dx
\]

for some constant \( c = c(n, \lambda, \Lambda) > 0 \), see [21] for details.

We now state the comparison \( L^1 \)-estimates in an interior region.

**Lemma 3.7.** Let \( \rho > 1 \) and \( 0 < r \leq \frac{\rho}{4} \). Then, for any \( \epsilon \in (0, 1) \), there is a small constant \( \delta = \delta(n, \lambda, \Lambda, \epsilon) > 0 \) such that if \( (\mathbf{a}, \Omega) \) is \((\delta, R)\)-vanishing of codimension 1, and if \( u_h \in W_0^{1,2}(\Omega) \), \( w_h \in u_h + W_0^{1,2}(Q_r(x_0)) \), and \( v_h \in w_h + W_0^{1,2}(Q_r(x_0)) \) are the weak solutions (1.4), (3.11), and (3.12), respectively, with

\[
\int_{Q_{r\delta}(x_0)} |Du_h| \, dx \leq \rho \quad \text{and} \quad \frac{|\mu_h|((Q_{r\delta}(x_0))}{|\mu|} \leq \delta \rho,
\]

then we have

\[
\int_{Q_r(x_0)} |Du_h - Dv_h| \, dx \leq \epsilon \rho \quad \text{and} \quad \|Dv_h\|_{L^\infty(Q_{2r}(x_0))} \leq \epsilon \rho
\]

for some constant \( c = c(n, \lambda, \Lambda) > 0 \).

4. COVERING ARGUMENTS

Now, we consider a SOLA \( u \) of (1.1) and suppose that \( (\mathbf{a}, \Omega) \) is \((\delta, R)\)-vanishing of codimension 1. Moreover, we assume that \( R_0 > 0 \) satisfies

\[
R_0 \leq \min \left\{ \frac{1}{16\sqrt{2}}, \frac{R}{8\sqrt{2}}, \frac{1}{|\Omega|} \right\},
\]

and upper-level sets: for \( k \in \mathbb{N} \cup \{0\} \),

\[
C_k := \left\{ x \in \Omega_{R_0} : \mathcal{M} \left( |Du|^{\frac{\rho}{p_-}} \chi_{\Omega_{4R_0}} \right)(x) > N^{k+1} \lambda_0 \right\},
\]

\[
D_k := \left\{ x \in \Omega_{R_0} : \mathcal{M} \left( |Du|^{\frac{\rho}{p_-}} \chi_{\Omega_{4R_0}} \right)(x) > N^k \lambda_0 \right\}
\]

\[
\cup \left\{ x \in \Omega_{R_0} : [\mathcal{M}_1(\mu)(x)]^{\frac{\rho}{p_-}} > \delta N^k \lambda_0 \right\},
\]

\[
\lambda_0 := \frac{1}{\epsilon} \left\{ \int_{\Omega_{4R_0}} |Du|^{\frac{\rho}{p_-}} \, dx + 1 \right\} > 1
\]

and upper-level sets: for \( k \in \mathbb{N} \cup \{0\} \),

\[
C_k := \left\{ x \in \Omega_{R_0} : \mathcal{M} \left( |Du|^{\frac{\rho}{p_-}} \chi_{\Omega_{4R_0}} \right)(x) > N^{k+1} \lambda_0 \right\},
\]

\[
D_k := \left\{ x \in \Omega_{R_0} : \mathcal{M} \left( |Du|^{\frac{\rho}{p_-}} \chi_{\Omega_{4R_0}} \right)(x) > N^k \lambda_0 \right\}
\]

\[
\cup \left\{ x \in \Omega_{R_0} : [\mathcal{M}_1(\mu)(x)]^{\frac{\rho}{p_-}} > \delta N^k \lambda_0 \right\},
\]

\[
\lambda_0 := \frac{1}{\epsilon} \left\{ \int_{\Omega_{4R_0}} |Du|^{\frac{\rho}{p_-}} \, dx + 1 \right\} > 1
\]
Lemma 4.1. There exists a constant $N_1 = N_1(n) > 1$ such that for any fixed $N \geq N_1$ and $k \in \mathbb{N} \cup \{0\}$,

$$|C_k| < \frac{\epsilon}{(1000)^n}|B_{R_0}|.$$  

Proof. For each $k \in \mathbb{N} \cup \{0\}$, $|C_k| \leq |C_0|$. Thus, it suffices to prove that (4.4) holds for $k = 0$. We have from (2.2) and (4.3) that

$$|C_0| = \left\{ x \in \Omega_{R_0} : \mathcal{M} \left( \left| Du \right|^{\frac{n-1}{p-1}} \chi_{\Omega_{R_0}} \right) (x) > N\lambda_0 \right\} \leq \frac{c}{N\lambda_0} \int_{\Omega_{R_0}} \left| Du \right|^{\frac{n}{p}} dx \leq \frac{c\epsilon}{N}|B_{R_0}| < \frac{\epsilon}{(1000)^n}|B_{R_0}|,$$

by selecting $N_1$ large enough. \hfill \Box

Lemma 4.2. There is a constant $N_2 = N_2(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 1$ so that for any $\epsilon > 0$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, L, \epsilon) > 0$ such that for any fixed $N \geq N_2$, $k \in \mathbb{N} \cup \{0\}$, $y_0 \in C_k$ and $r_0 \leq \frac{R_0}{1000}$, if

$$|C_k \cap B_{r_0}(y_0)| \geq \epsilon |B_{r_0}(y_0)|,$$

then $B_{r_0}(y_0) \cap \Omega_{R_0} \subset D_k$.

Proof. We write $\lambda_k := N^k \lambda_0 > 1$, where $N \geq N_2 > 1$. The proof is by contradiction. Were $B_{r_0}(y_0) \cap \Omega_{R_0} \subset D_k$ false, there exists $y_1 \in B_{r_0}(y_0) \cap \Omega_{R_0}$ such that $y_1 \notin D_k$. Then we have

$$\frac{1}{|B_r(y_1)|} \int_{B_r(y_1) \cap \Omega_{R_0}} |Du|^{\frac{n}{p-1}} dx \leq \lambda_k, \quad \text{and}$$

$$\left| \frac{\mu(B_r(y_1))}{r^n} \right|^{\frac{n}{p-1}} \leq c(n, \gamma_1, \gamma_2) \delta \lambda_k,$$

for all $r > 0$.

Before proving this lemma, we outline the plan of the proof.

(i) We first divide the proof into two cases: $Q_{10r_0}(y_1) \subset \Omega$ and $Q_{10r_0}(y_1) \not\subset \Omega$.

(ii) We transfer the exponent powers in (4.6) from $\frac{n}{p-1}$ to 1, see (4.8) and (4.15).

(iii) We apply Lemma 3.7 (Lemma 3.6) to obtain the $L^1$ comparison estimates for the interior (boundary) case, see (4.9) and (4.16).

(iv) We transfer the exponent powers in the comparison estimates from 1 to $\frac{n}{p-1}$, see (4.11) and (4.17).

(v) We arrive at a contradiction by using standard technique of the covering argument mentioned in [15, 47].

Note that the log-Hölder continuity, from the steps (ii) and (iv), is an essential ingredient in correcting the exponent powers.
Since $y_1 \in B_{r_0}(y_0) \cap \Omega_{r_0}$, we see that

$$Q_{8r_0}(y_0) \subset B_{10r_0 \sqrt{2}}(y_1) \subset Q_{10r_0 \sqrt{2}}(y_1) \subset \Omega_{4r_0}.$$  

We set

$$p_1 := \inf_{x \in Q_{8r_0}(y_0)} p(x) \quad \text{and} \quad p_2 := \sup_{x \in Q_{8r_0}(y_0)} p(x).$$

Then it follows that $p_2 - p_1 \leq \omega(16r_0 \sqrt{2})$, and for $x \in Q_{8r_0}(y_0)$,

$$1 \leq \gamma_1 \leq p_- \leq p_1 \leq p(x) \leq p_2 \leq p_+ \leq \gamma_2 < \infty.$$  

Using Hölder’s inequality, (4.1), (4.6), and (1.11), we have

$$\int_{Q_{8r_0}(y_0)} |Du| \, dx = \left( \int_{Q_{8r_0}(y_0)} |Du| \, dx \right)^{\frac{p-1}{p}} \left( \int_{Q_{8r_0}(y_0)} |Du| \, dx \right)^{\frac{1}{p}} \leq c \left( \frac{1}{r_0} \right)^{(n+1)\omega(16r_0 \sqrt{2})} \left( \int_{B_{10r_0 \sqrt{2}}(y_1)} |Du| \, dx + 1 \right)^{\frac{p-1}{p}} \leq c(n, \gamma_1, \gamma_2, L) \lambda_k^{\frac{p-1}{p}},$$

and it follows from (1.7) that for any $\epsilon_h \in (0,1)$ and $q \in \left[1, \frac{n}{n-1}\right)$,

$$\int_{Q_{8r_0}(y_0)} |Du - Du_h|^q \, dx \leq \epsilon_h$$

for $h$ large enough. Then these estimates imply

$$\int_{Q_{8r_0}(y_0)} |Du_h| \, dx \leq c_1 \lambda_k^{\frac{p-1}{p}}$$

for some constant $c_1 = c_1(n, \gamma_1, \gamma_2, L) > 0$.

On the other hand, we compute from (4.1), (4.6), and (1.11) that

$$\frac{|\mu|(Q_{8r_0}(y_0))}{r_0^{n-1}} = \left[ \frac{|\mu|(Q_{8r_0}(y_0))}{r_0^{n-1}} \right]^{\frac{p-1}{p}} \left[ \frac{|\mu|(Q_{8r_0}(y_0))}{r_0^{n-1}} \right]^{\frac{1}{p}} \leq \left( \frac{1}{r_0} \right)^{(n-1)\omega(16r_0 \sqrt{2})} \left( |\mu|(\Omega) + \omega(16r_0 \sqrt{2}) \right) \left[ \frac{|\mu|(B_{10r_0 \sqrt{2}}(y_1))}{r_0^{n-1}} \right]^{\frac{p-1}{p}} \leq c(n, \gamma_1, \gamma_2, L) \delta^{\frac{\gamma_1}{\gamma_2}} \lambda_k^{\frac{p-1}{p}},$$

and so, we have from (1.5) that

$$\frac{|\mu_h|(Q_{8r_0}(y_0))}{r_0^{n-1}} \leq \frac{|\mu_h|(Q_{8r_0}(y_0))}{r_0^{n-1}} + \epsilon_h \frac{c_1}{r_0} \leq c_1 \lambda_k^{\frac{p-1}{p}} + c_1 \delta \delta^{\frac{\gamma_1}{\gamma_2}} \lambda_k^{\frac{p-1}{p}}.$$
for some constant $c_0 = c_0(n, \gamma_1, \gamma_2, L) > 0$, by selecting $\epsilon_h$ sufficiently small with $\epsilon_h \leq r_0^{-1} \delta^{\frac{n}{n-2}}$.

Consequently, we obtain

$$
\int_{Q_{r_0}(y_0)} |Du_h| \, dx \leq c_3 \lambda_k^{\frac{p_n}{p_n-2}} \quad \text{and} \quad \frac{|\mu_h|(Q_{r_0}(y_0))}{r_0^{n-1}} \leq c_3 \delta^{\frac{n}{n-2}} \lambda_k^{\frac{p_n}{p_n-2}},
$$

where $c_3 := \max\{c_1, c_2\}$. Applying Lemma 3.7 with $x_0, \rho, r, \delta, \epsilon$ replaced by $y_0, c_3 \lambda_k^{\frac{p_n}{p_n-2}}, r_0, \delta^{\frac{n}{n-2}}$, and $\eta$, respectively, we find that there is $\delta = \delta(n, \lambda, \gamma_1, \gamma_2, \eta) > 0$ such that

$$
\int_{Q_{r_0}(y_0)} |Du_h - Dv_h| \, dx \leq c_4 \eta \lambda_k^{\frac{p_n}{p_n-2}} \quad \text{and} \quad \|Dv_h\|_{L^\infty(Q_{r_0}(y_0))} \leq c_4 \lambda_k^{\frac{p_n}{p_n-2}}
$$

for some constant $c_4 = c_4(n, \lambda, \gamma_1, \gamma_2, L) > 0$. Then (4.7) and (4.9) imply

$$
\int_{Q_{r_0}(y_0)} |Du - Dv_h| \, dx \leq (4^n + 2^n) c_4 \eta \lambda_k^{\frac{p_n}{p_n-2}} =: c_5 \eta \lambda_k^{\frac{p_n}{p_n-2}}
$$

by choosing $\tilde{\epsilon}_h$ sufficiently small with $\tilde{\epsilon}_h \leq c_3 \eta$.

We next claim that

$$
\int_{Q_{r_0}(y_0)} |Du - Dv_h|^{\frac{p_n}{p_n-2}} \, dx \leq c_6 \eta^\frac{2}{k} \lambda_k \quad \text{and} \quad \left\| Dv_h \right\|_{L^\infty(Q_{r_0}(y_0))}^{\frac{p_n}{p_n-2}} \leq c_6 \lambda_k
$$

for some constant $c_6 = c_6(n, \lambda, \gamma_1, \gamma_2, L) > 0$.

Clearly, we compute from (4.9) that

$$
\left\| Dv_h \right\|_{L^\infty(Q_{r_0}(y_0))}^{\frac{p_n}{p_n-2}} \leq \sup_{x \in Q_{r_0}(y_0)} \left( |Dv_h(x)| + 1 \right)^{\frac{p_n}{p_n-2}} \leq c \left( \|Dv_h\|_{L^\infty(Q_{r_0}(y_0))} + 1 \right) \leq c \lambda_k.
$$

Returning to (4.11), we have from Hölder’s inequality and (4.10) that

$$
\int_{Q_{r_0}(y_0)} |Du - Dv_h|^{\frac{p_n}{p_n-2}} \, dx = \int_{Q_{r_0}(y_0)} |Du - Dv_h|^{\frac{p_n}{p_n-2} - 1} \, dx \leq \left( \int_{Q_{r_0}(y_0)} |Du - Dv_h| \, dx \right)^{\frac{p_n}{p_n-2} - 1} \left( \int_{Q_{r_0}(y_0)} |Du - Dv_h|^{\frac{p_n}{p_n-2} - 1} \, dx \right)^{\frac{p_n}{p_n-2}} \leq c \eta^\frac{2}{k} \lambda_k \left( \int_{Q_{r_0}(y_0)} |Du - Dv_h|^{\frac{p_n}{p_n-2} - 1} \, dx \right)^{\frac{1}{2}} =: c \eta^\frac{2}{k} \lambda_k^{\frac{p_n}{p_n-2}} I_1.
$$
It follows from (4.9) that

\[ I_1^2 \leq \int_{Q_{2r_0}(y_0)} (|Du| + |Dv_h| + 1)^{2p_\frac{2-\gamma}{p-1}} \, dx \]

\[ \leq c \left\{ \int_{Q_{2r_0}(y_0)} |Du|^{2p_\frac{2-\gamma}{p-1}} \, dx + \int_{Q_{2r_0}(y_0)} |Dv_h|^{2p_\frac{2-\gamma}{p-1}} \, dx + 1 \right\} \]

\[ \leq c \left\{ \int_{Q_{2r_0}(y_0)} |Du|^{2p_\frac{2-\gamma}{p-1}} \, dx + c \right\} \]

Continuously, we discover from (4.7), Lemma 3.1, Lemma 3.2, and (4.8) that

\[ I_2 \leq c \int_{Q_{2r_0}(y_0)} |Du - Du_h|^{2p_\frac{2-\gamma}{p-1}} \, dx + c \int_{Q_{2r_0}(y_0)} |Du_h - Dw_h|^{2p_\frac{2-\gamma}{p-1}} \, dx \]

\[ + c \int_{Q_{2r_0}(y_0)} |Dw_h|^{2p_\frac{2-\gamma}{p-1}} \, dx \]

\[ \leq c \left\{ \varepsilon_h \lambda_k^{2p_\frac{2-\gamma}{p-1}} + \left[ \frac{\|t_h\|}{(Q_{2r_0}(y_0))} \right]^{2p_\frac{2-\gamma}{p-1}} \right\} \]

\[ \leq c \lambda_k^{2p_\frac{2-\gamma}{p-1}} \]

since \( 2p_\frac{2-\gamma}{p-1} - 1 \leq 1 + 2^{\frac{p-1}{p-\gamma}} \leq 1 + 2^{-\frac{\gamma(1-\n)}{\gamma_1}} \leq 1 + \min \left\{ \frac{1}{2(n-1)}, \sigma_0 \right\} \), according to (4.2). Then we combine these estimates above, to obtain

\[ \int_{Q_{2r_0}(y_0)} |Du - Dv_h|^{\frac{p(x)}{p-\gamma}} \, dx \leq c_0 \eta^{\frac{\gamma}{p}} \lambda_k \]

for some \( c_0 = c_0(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0 \). We thereby establish the claim (4.11).

Next we claim that

\[ C_k \cap B_{r_0}(y_0) = \left\{ x \in B_{r_0}(y_0) : \mathcal{M} \left( \left| Du \right|^{\frac{p(x)}{p-\gamma}} \chi_{Q_{2r_0}(y_0)} \right) > N \lambda_k \right\} \]

(4.12)

\[ \subset \left\{ x \in B_{r_0}(y_0) : \mathcal{M} \left( \left| Du - Dv_h \right|^{\frac{p(x)}{p-\gamma}} \chi_{B_{2r_0}(y_0)} \right) > \lambda_k \right\} =: J_k, \]

provided \( N \geq N_2 \geq \max \left\{ 2^{\frac{2-\gamma}{p-\gamma}} - 1, 3^n \right\} \).

Let \( y \notin J_k \). If \( y \notin B_{r_0}(y_0) \), then (4.12) is done. Suppose \( y \in B_{r_0}(y_0) \). If \( r_0 < r_0 \), then \( B_r(y) \subset B_{2r_0}(y_0) \subset \Omega_{4r_0} \). It follows from (4.11) that

\[ |Du(x)|^{\frac{p(x)}{p-\gamma}} \leq 2^{\frac{2-\gamma}{p-\gamma}} \left( |Du(x) - Dv_h(x)|^{\frac{p(x)}{p-\gamma}} + |Dv_h(x)|^{\frac{p(x)}{p-\gamma}} \right) \]

\[ \leq 2^{\frac{2-\gamma}{p-\gamma}} \left( |Du(x) - Dv_h(x)|^{\frac{p(x)}{p-\gamma}} + c_0 \lambda_k \right) \]

for almost every \( x \in B_r(y) \). Integrating over \( B_r(y) \) gives

\[ \int_{B_r(y)} |Du|^{\frac{p(x)}{p-\gamma}} \, dx \leq 2^{\frac{2-\gamma}{p-\gamma}} \left\{ \mathcal{M} \left( \left| Du - Dv_h \right|^{\frac{p(x)}{p-\gamma}} \chi_{B_{2r_0}(y_0)} \right) (y) + c_0 \lambda_k \right\} \]

\[ \leq 2^{\frac{2-\gamma}{p-\gamma}} (1 + c_0) \lambda_k, \]
If \( \tilde{r} \geq r_0 \), then \( B_\tilde{r}(y) \subset B_{2\tilde{r}}(y_0) \subset B_{3\tilde{r}}(y_1) \). We have from (4.6) that
\[
\frac{1}{|B_r|} \int_{B_r(y) \cap \Omega \setminus r_0} |Du| \chi_B \frac{p(x)}{\rho(x)} \, dx \leq \frac{3^n}{|B_{3\tilde{r}}|} \int_{B_{3\tilde{r}}(y_1) \cap \Omega \setminus r_0} |Du| \chi_B \frac{p(x)}{\rho(x)} \, dx \leq 3^n \lambda_k.
\]
Consequently, we obtain
\[
\mathcal{M}(|Du| \chi_B \chi_{\Omega \setminus r_0})(y) \leq \max \left\{ 2 \frac{n}{n-1} (1 + c_0) \lambda_k, 3^n \lambda_k \right\}.
\]
Choosing \( N_2 \geq \max \left\{ 2 \frac{n}{n-1} (1 + c_0), 3^n \right\} \), we have \( y \not\in C_k \cap B_{r_0}(y_0) \), that is, the claim (4.12) holds.

We finally conclude, using (4.12), (2.2), and (4.11), that
\[
|C_k \cap B_{r_0}(y_0)| \leq \left\{ x \in B_{r_0}(y_0) : \mathcal{M} \left( |Du - Du_h| \chi_B \chi_{B_{2r_0}(y_0)} \right)(x) > \lambda_k \right\}
\leq \frac{c}{\lambda_k} \int_{B_{2r_0}(y_0)} |Du - Du_h| \chi_B \frac{p(x)}{\rho(x)} \, dx \leq c c_0 \lambda_k^2 |B_{r_0}(y_0)| < \epsilon |B_{r_0}(y_0)|,
\]
by selecting \( \eta \) and \( \delta \) that satisfy the last inequality above, which is a contradiction to (4.5).

**Case 2.** The boundary case \( Q_{10r_0, \sqrt{2}}(y_1) \not\subset \Omega \).

We find a boundary point \( \tilde{y}_1 \in \partial \Omega \cap Q_{8r_0, \sqrt{2}}(y_1) \). Since \( 540r_0 \leq R_0 \leq \frac{R}{8\sqrt{2}} \) and the domain \( \Omega \) is \((\delta, R)\)-Reifenberg flat of codimension 1, there exists a coordinate system, which we still denote \( x = (x_1, \cdots, x_n) \), with the origin at \( \tilde{y}_1 + 480\delta r_0 e_n \), such that
\[(4.13) \quad Q_{8r_0}(0) \subset \Omega_{8r_0}(0) \subset Q_{480r_0}(0) \cap \{ x_n > -960\delta r_0 \}.
\]
We select \( \delta \) so small with \( 0 < \delta < \frac{1}{27} \). Then we have
\[(4.14) \quad \Omega_{2r_0}(y_0) \subset \Omega_{3r_0}(y_1) \subset \Omega_{12r_0}(0), \quad \text{and} \quad \Omega_{480r_0}(0) \subset \Omega_{540r_0}(y_1) \subset \Omega_{R_0}(y_0) \subset \Omega_{4r_0},
\]
since \( |y_1| \leq |y_1 - \tilde{y}_1| + |\tilde{y}_1| \leq 20r_0 + 480\delta r_0 \leq 40r_0 \) in the new coordinate. We denote
\[
p_1 := \inf_{x \in \Omega_{480r_0}(0)} p(x) \quad \text{and} \quad p_2 := \sup_{x \in \Omega_{480r_0}(0)} p(x).
\]
Then \( p_2 - p_1 \leq \omega(960r_0) \).

We next obtain the estimates in the boundary case corresponding to (4.8). From (1.11), (4.1), (4.6), (4.13), and (4.14), we have
\[(4.15) \quad \int_{\Omega_{480r_0}(0)} |Du| \, dx \leq c_7 \lambda_k^\frac{p}{p-2} \quad \text{and} \quad \frac{|\mu|_{\Omega_{480r_0}(0)}}{r_0^{n-1}} \leq c_7 \delta^{\frac{n}{2}} \lambda_k^\frac{p}{p-2},
\]
for some constant \( c_7 = c_7(n, \gamma_1, \gamma_2, L) > 0 \). Furthermore, it follows from (1.7), (1.5) and (4.15) that
\[
\int_{\Omega_{480r_0}(0)} |Du_h| \, dx \leq c_8 \lambda_k^{\frac{p}{p-2}} \quad \text{and} \quad \frac{|\mu_h|_{\Omega_{480r_0}(0)}}{r_0^{n-1}} \leq c_8 \delta^{\frac{n}{2}} \lambda_k^{\frac{p}{p-2}}
\]
for \( h \) large enough and some constant \( c_8 = c_8(n, \gamma_1, \gamma_2, L) > 0 \). Applying Lemma 3.6 with \( \rho, r, \delta \), and \( \epsilon \) replaced by \( c_8 \lambda_k^{\frac{p}{p-2}}, 60r_0, \delta^{\frac{n}{2}}, \) and \( \eta \), respectively, we deduce
that there exists $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \eta) > 0$ such that
\[
\int_{\Omega_{120r_0}(0)} |Du_h - D\bar{v}_h| \, dx \leq c_8 \eta \lambda_k^{\frac{p-2}{2}} \quad \text{and} \quad ||D\bar{v}_h||_{L^\infty(\Omega_{120r_0}(0))} \leq cc_8 \lambda_k^{\frac{p-2}{2}} =: c_9 \lambda_k^{\frac{p-2}{2}}
\]
for some constant $c_9 = c_9(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0$. Then (1.7) and (4.16) imply
\[
\int_{\Omega_{120r_0}(0)} |Du - D\bar{v}_h| \, dx \leq c_{10} \eta \lambda_k^{\frac{p-2}{2}}.
\]
Proceeding as in Case 1, we infer
\[
\int_{\Omega_{120r_0}(0)} |Du - D\bar{v}_h|^{\frac{p(\chi)}{p-2}} \, dx \leq c_{11} \eta^\frac{1}{2} \lambda_k \quad \text{and} \quad ||D\bar{v}_h||_{L^\infty(\Omega_{120r_0}(0))}^{\frac{p(\chi)}{p-2}} \leq c_{11} \lambda_k
\]
for some constant $c_{11} = c_{11}(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0$, and moreover
\[
C_k \cap B_{r_0}(y_0) = \left\{ x \in \Omega_{r_0}(y_0) : M \left( |Du|^{\frac{p(\chi)}{p-2}} \chi_{\Omega_{4r_0}} \right) (x) > N \lambda_k \right\}
\]
\[
\subset \left\{ x \in \Omega_{r_0}(y_0) : M \left( |Du - D\bar{v}_h|^{\frac{p(\chi)}{p-2}} \chi_{\Omega_{2r_0}(y_0)} \right) (x) > \lambda_k \right\}
\]
provided $N \geq N_2 \geq \max \left\{ 2\frac{3^n}{2} - 1, 3^n \right\}$.

Finally, we conclude from (4.18), (2.2), (4.14) and (4.17) that
\[
|C_k \cap B_{r_0}(y_0)| \leq \left| \left\{ x \in \Omega_{r_0}(y_0) : M \left( |Du - D\bar{v}_h|^{\frac{p(\chi)}{p-2}} \chi_{\Omega_{2r_0}(y_0)} \right) (x) > \lambda_k \right\} \right|
\]
\[
\leq \frac{c}{\lambda_k} \int_{\Omega_{2r_0}(y_0)} |Du - D\bar{v}_h|^{\frac{p(\chi)}{p-2}} \, dx
\]
\[
\leq \frac{c||\Omega_{120r_0}(0)||}{\lambda_k} \int_{\Omega_{120r_0}(0)} |Du - D\bar{v}_h|^{\frac{p(\chi)}{p-2}} \, dx
\]
\[
\leq cc_{11} \eta^\frac{1}{2} |B_{r_0}(y_0)| < \epsilon |B_{r_0}(y_0)|
\]
by taking $\eta$ sufficiently small. As a consequence $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, L, \epsilon)$ is also determined. This is a contradiction to (4.5). \hfill \square

Choosing $N = \max\{N_1, N_2\}$ from Lemma 4.1 and 4.2, we can apply Lemma 2.1 to derive the following power decay estimates:

**Corollary 4.3.** Under the same assumptions and conclusions as in Lemma 4.1 and 4.2, we have
\[
|C_k| \leq \left( \frac{80}{7} \right)^n \epsilon |D_k| =: c_1 |D_k| \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}.
\]
In addition, by iteration, we obtain
\[
\left\{ x \in \Omega_{R_0} : \mathcal{M} \left( \frac{|Du|}{\chi_{4R_0}} \right) \left( x \right) > N^k \lambda_0 \right\} \leq \epsilon_1 \left\{ x \in \Omega_{R_0} : \mathcal{M} \left( \frac{|Du|}{\chi_{4R_0}} \right) \left( x \right) > \lambda_0 \right\} + \sum_{i=1}^{k} \epsilon_1 \left\{ x \in \Omega_{R_0} : [\mathcal{M}_1(\mu)(x)]^{\frac{1}{\mu}} > \delta N^{k-i} \lambda_0 \right\}.\tag{4.19}
\]

5. Calderón-Zygmund type estimates

We first obtain standard energy type estimate for the problem (1.1), which will be used to prove our main theorem (Theorem 1.4).

Lemma 5.1. Assume that (1.2) and (1.3), and let $1 \leq q < \frac{n}{n-\tau}$. If $u$ is a SOLA of (1.1), then there exist positive constants $\bar{c}, \hat{c}, k_0$, and $\hat{\Omega}$, such that
\[
\int_{\Omega} |Du|^q \, dx \leq \bar{c}|\mu|^{1/q} \leq \hat{c} \int_{\Omega} \mathcal{M}_1(\mu) \, dx.\tag{5.1}
\]

Proof. Since $|\mu|^{1/q} \leq \text{diam}(\Omega)^{n-1} \mathcal{M}_1(\mu)$ for all $x \in \Omega$, the second inequality of (5.1) holds. We set
\[
\tilde{u}(y) = \frac{u(x_0 + ry)}{Ar}, \quad \tilde{\mu}(y) = \frac{r\mu(x_0 + ry)}{A}, \quad \text{and} \quad \tilde{a}(A\xi, y) = \frac{a(A\xi, x_0 + ry)}{A},
\]
where $r := \text{diam}(\Omega)$, $x_0 \in \Omega$, $A := \left| \mu(\Omega) \right| r^{n-1}$, and $\tilde{\Omega} := \{y \in \mathbb{R}^n : x_0 + ry \in \Omega \} \subset B_1$.

Here we extend $u$ and $\mu$ by zero to $\mathbb{R}^n \setminus \Omega$. Then we see that $\tilde{a}$ satisfies (1.2) and (1.3), and $\tilde{u}$ is a SOLA of the following problem
\[
\left\{ \begin{array}{ll}
-\text{div } \tilde{a}(D\tilde{u}, y) &= \tilde{\mu} & \text{in } \tilde{\Omega}, \\
\tilde{u} &= 0 & \text{on } \partial \tilde{\Omega}.
\end{array} \right. \tag{5.2}
\]

Fix $q \in \left[ 1, \frac{n}{n-\tau} \right)$. If $\int_{\tilde{\Omega}} |D\tilde{u}| \, dy = \int_{B_1} |D\tilde{u}| \, dy \leq c$, then $\int_{\Omega} |Du| \, dx \leq cr^n A$, that is, the first inequality of (5.1) holds. Thus, it suffices to show that $\int_{B_1} |D\tilde{u}| \, dy \leq c$.

Consider the regularized problem (1.4) with $u_h$ replaced by $\tilde{u}_h$. We denote, for $k \in \mathbb{N}$,
\[
D_k := \{y \in B_1 : |\tilde{u}_h(y)| \leq k\} \quad \text{and} \quad C_k := \{y \in B_1 : k < |\tilde{u}_h(y)| \leq k+1\}.
\]

Then (1.6) implies
\[
\int_{D_k} |D\tilde{u}_h|^2 \, dy \leq ck \quad \text{and} \quad \int_{C_k} |D\tilde{u}_h|^2 \, dy \leq c
\]
by substituting test functions $T_k(\tilde{u}_h)$ and $\Phi_k(\tilde{u}_h)$, respectively, into the weak formulation of (5.2). Here the functions $T_k$ and $\Phi_k$ are defined as
\[
T_k(t) := \max \{-k, \min\{k, t\}\}, \quad \Phi_k(t) := T_1 \left( t - T_k(t) \right) \quad \text{for } t \in \mathbb{R}.
\]

We discover
\[
\int_{D_k} |D\tilde{u}_h|^q \, dy \leq \int_{D_k} (|D\tilde{u}_h| + 1)^q \, dy \leq c(k+1).
\]
From the definition of $C_k$, we see

$$|C_k| = \int_{C_k} 1 \, dy \leq \int_{C_k} \left( \frac{|\tilde{u}_h|}{k} \right)^{\frac{n+q}{n}} \, dy = k^{-\frac{n+q}{n}} \int_{C_k} |\tilde{u}_h|^{\frac{n+q}{n}} \, dy.$$ 

It therefore follows from Hölder’s inequality that

$$\int_{C_k} |D\tilde{u}_h|^q \, dy \leq c k^{-\frac{n(q(2-q))}{n+q}} \left( \int_{C_k} |\tilde{u}_h|^{\frac{n+q}{n}} \, dy \right)^{\frac{q}{2}}.$$ 

Then we discover from Hölder’s and Sobolev’s inequality that for $k_0 \in \mathbb{N}$,

$$\int_{B_1} |D\tilde{u}_h|^q \, dy \leq c(k_0 + 1) + c \sum_{k=k_0}^{\infty} k^{-\frac{n(q(2-q))}{n+q}} \left( \int_{C_k} |\tilde{u}_h|^{\frac{n+q}{n}} \, dy \right)^{\frac{q}{2}}$$

$$\leq c(k_0 + 1) + c \left( \sum_{k=k_0}^{\infty} k^{-\frac{n(q(2-q))}{n+q}} \right)^{\frac{q}{2}} \left( \int_{C_k} |\tilde{u}_h|^{\frac{n+q}{n}} \, dy \right)^{\frac{q}{2}}$$

$$\leq c(k_0 + 1) + cH(k_0) \left( \int_{B_1} |D\tilde{u}_h|^q \, dy \right)^{\frac{n(q(2-q))}{2(n+q)}},$$

where $H(k_0) := \left( \sum_{k=k_0}^{\infty} k^{-\frac{n(2-q)}{n+q}} \right)^{\frac{q}{2}}$. Note that $\frac{n(q(2-q))}{2(n+q)} > 1$, since $q < \frac{n}{n+1}$.

For $n > 2$, we know $0 < \frac{n(q(2-q))}{2(n+q)} < 1$, and then the above estimate and Young’s inequality yield

$$\int_{B_1} |D\tilde{u}_h|^q \, dy \leq c(n, \lambda, q, \Omega)$$

by putting $k_0 = 1$. For $n = 2$, we know $\frac{n(q(2-q))}{2(n+q)} = 1$. We take an integer $k_0 > 1$ so that $cH(k_0) < \frac{1}{q}$. Then (5.3) also holds. Using this estimate and letting $h$ go to zero, we conclude from (1.7) that $\int_{B_1} |D\tilde{u}|^q \, dy \leq c$, which completes the proof. □

5.1. Local estimates. We first obtain local estimates for the problem (1.1).

Proof of (1.14). We first recall (4.1) and (4.2). Fix any $R_0 \in \left[ 0, \frac{1}{c_0(n, \Lambda, \gamma_1, \omega(\cdot), R, \Omega)} \right]$ with

$$\frac{1}{c_0(n, \lambda, \Lambda, \gamma_1, \omega(\cdot), R, \Omega)} := \min \left\{ \frac{1}{16\sqrt{2}}, \frac{R}{8\sqrt{2}}, \frac{1}{c_1}, \frac{\omega^{-1}(d)}{8\sqrt{2}} \right\},$$

where the constant $c_1$ is given in Lemma 5.1 ($q = 1$),

$$d := \min \left\{ \frac{\sigma_0 \gamma_1}{2}, \frac{\gamma_1}{4(n-1)} \right\}, \text{ and } \omega^{-1}(t) := \sup \{ r \in (0, 1) : \omega(r) \leq t \} \text{ for } t > 0.$$ 

Note that the function $\omega^{-1}$ is well defined by the definition of $\omega$. Then we see from Lemma 5.1 that this $R_0$ above satisfies (4.1) and (4.2), and one can apply all the results obtained in Section 4 as follows.

Set

$$S := \sum_{k=1}^{\infty} N^{-k^p} \left\{ x \in \Omega_{R_0}(x_0) : |Du|^{q(1)} \chi_{\Omega_{R_0}(x_0)}(x) > N^k \lambda_0 \right\},$$
where \( \lambda_0 \) and \( N \) are given in (4.3) and (4.19), respectively, and \( p_- := \inf_{x \in \Omega_{4R_0}(x_0)} p(x) \). Then we deduce from (4.19), Fubini’s theorem, and Lemma 2.2 that

\[
S \leq \sum_{i=1}^{\infty} (N^{7^2} \epsilon_1)^i \left\{ 2|\Omega_{R_0}(x_0)| + \frac{c}{(\delta \lambda_0)^{p_-}} \int_{\Omega_{R_0}(x_0)} M_1(\mu) \frac{\mu}{\gamma} \, dx \right\}.
\]

Now we select \( \epsilon_1 \) with \( N^{7^2} \epsilon_1 = \frac{1}{4} \), and then we can take \( \epsilon \) and a corresponding \( \delta = \delta(n, \lambda, \gamma_1, \gamma_2, L) > 0 \). Consequently, we obtain

\[
S \leq 2|\Omega_{R_0}(x_0)| + \frac{c}{\lambda_0} \int_{\Omega_{R_0}(x_0)} M_1(\mu) \frac{\mu}{\gamma} \, dx.
\]

Finally, according to Lemma 2.2, (5.5), and (4.3), we conclude

\[
\begin{align*}
\int_{\Omega_{R_0}(x_0)} |Du|^{p(x)} \, dx &\leq \int_{\Omega_{R_0}(x_0)} M\left( |Du|^{\frac{\mu}{\gamma}} \chi_{\Omega_{R_0}(x_0)} \right)^{p_-} \, dx \\
&\leq c \lambda_0^{p_-} \left( 1 + \frac{S}{|\Omega_{R_0}(x_0)|} \right) \left\{ \lambda_0^{p_-} + \int_{\Omega_{R_0}(x_0)} M_1(\mu) \frac{\mu}{\gamma} \, dx \right\} \\
&\leq c \left\{ \left( \int_{\Omega_{R_0}(x_0)} |Du|^{\frac{\mu}{\gamma}} \, dx + 1 \right)^{p_-} + \int_{\Omega_{R_0}(x_0)} M_1(\mu)^{p(x)} \, dx + 1 \right\}
\end{align*}
\]

for some constant \( c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, L) > 0 \). This completes the proof. \( \square \)

### 5.2. Global estimates

Now we extend the local estimates (1.14) in Section 5.1 up to the boundary by a standard covering argument.

**Proof of (1.15).** Let \( R_0 = \frac{1}{c_0(\mu(n)/(q+1))} \), where \( c_0 \) is given in (5.4). Since \( \tilde{\Omega} \) is compact, we can cover \( \tilde{\Omega} \) by a collection of finitely many balls, each of which has radius \( \frac{R_0}{4} \) and center in \( \Omega \). By the Vitali covering lemma, there exists a finite family of disjoint open balls \( \left\{ B_{R_0}(y_i) \right\}_{i=1}^m \) such that \( \tilde{\Omega} \subset \bigcup_{i=1}^m B_{R_0}(y_i) \). Note that there is a constant \( c \) depending only on the dimension \( n \) so that

\[
\sum_{i=1}^m \int_{\Omega_{R_0}(y_i)} f \, dx \leq c \int_{\Omega} f \, dx.
\]

Then our applying the estimate (1.14) with \( y_i \, (i \in \mathbb{N}) \) in place of \( x_0 \) yields

\[
\begin{align*}
\int_{\Omega} |Du|^{p(x)} \, dx &\leq \sum_{i=1}^m \int_{\Omega_{R_0}(y_i)} |Du|^{p(x)} \, dx \\
&\leq c \sum_{i=1}^m \left\{ R_0 \left( \int_{\Omega_{R_0}(y_i)} ||Du|| + 1 \right)^{\frac{p(x)}{pi-}} \, dx \right\}^{pi-} + \int_{\Omega_{R_0}(y_i)} M_1(\mu)^{p(x)} + 1 \, dx \\
&\leq c \left\{ R_0^{n(1-\gamma_2)} \left( \int_{\Omega} ||Du|| + 1 \right)^{\frac{p(x)}{pi-}} \, dx \right\}^{\gamma_2} + \int_{\Omega} M_1(\mu)^{p(x)} + 1 \, dx \right\},
\end{align*}
\]

since

\[
\frac{pi+}{pi-} = 1 + \frac{pi+ - pi-}{pi-} \leq 1 + \frac{\omega(8R_0\sqrt{2})}{\gamma_1} \leq 1 + \frac{1}{4(n-1)} < \frac{n}{n-1},
\]

where \( p_i- := \inf_{x \in \Omega_{4R_0}(y_i)} p(x) \), and \( p_i+ := \sup_{x \in \Omega_{4R_0}(y_i)} p(x) \).
Finally, we obtain, using Lemma 5.1 and Hölder’s inequality, that
\[
\int_\Omega |D\bar{u}|^{p(x)} \, dx \\
\leq c \left\{ \left( \int_\Omega M_1(\mu) \, dx \right)^{\alpha_2} \left( \int_\Omega M_1(\mu) \, dx \right)^{\alpha_2} + \int_\Omega M_1(\mu)^{p(x)} \, dx + 1 \right\} \\
\leq c \left\{ \left( \int_\Omega M_1(\mu) \, dx \right)^{\frac{\alpha_2}{\alpha_2+\alpha_2}} + \int_\Omega M_1(\mu)^{p(x)} \, dx + 1 \right\} \\
\leq c \left\{ \left( \int_\Omega |M_1(\mu) + 1|^{p(x)} \, dx \right)^{\frac{\alpha_2}{\alpha_2+\alpha_2}} + 1 \right\}
\]
for some constant \( c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), L, R, \Omega) > 0. \)

We are now in a position to prove the desired estimate (1.16).

Proof of (1.16). First let us define
\[
\bar{u} = \frac{u}{A}, \quad \bar{\mu} = \frac{\mu}{A}, \quad \text{and} \quad \bar{a}(\xi, x) = \frac{a(A\xi, x)}{A},
\]
for some positive constant \( A > 0 \). Then it readily check that \( \bar{a} \) satisfies (1.2) and (1.3), \( (\bar{a}, \Omega) \) is \( (\delta, R) \)-vanishing of codimension 1, and \( \bar{u} \) is a SOLA of the following problem
\[
\begin{cases}
-\text{div} \bar{a}(D\bar{u}, x) = \bar{\mu} \quad \text{in } \Omega, \\
\bar{u} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]
Set \( A := \|M_1(\mu)\|_{L^{p(\cdot)}(\Omega)}. \) We may as well assume \( \|M_1(\mu)\|_{L^{p(\cdot)}(\Omega)} > 0. \) Then we know \( \|M_1(\mu)\|_{L^{p(\cdot)}(\Omega)} = 1. \)

On the other hand, (1.9) implies that \( \int_\Omega M_1(\bar{\mu})^{p(x)} \, dx = 1. \) Furthermore, in light of (1.9) and (1.15), we have \( \|D\bar{u}\|_{L^{p(\cdot)}(\Omega)} \leq c \frac{\alpha_2}{\alpha_2}. \) Consequently we conclude that
\[
\|D\bar{u}\|_{L^{p(\cdot)}(\Omega)} \leq c \|M_1(\mu)\|_{L^{p(\cdot)}(\Omega)}
\]
for some constant \( c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), L, R, \Omega) > 0. \)

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