Existence of invariant measures for the stochastic damped KdV equation

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Abstract

We address the long time behavior of solutions of the stochastic Korteweg-de Vries equation
du + (\partial_x^3 u + u\partial_x u + \lambda u)dt = f dt + \Phi dW_t on \mathbb{R}
where \( f \) is a deterministic force. We prove that the Feller property holds and establish the existence of an invariant measure. The tightness is established with the help of the asymptotic compactness, which is carried out using the Aldous criterion.

Mathematics Subject Classification:

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1 Introduction

In this paper, we investigate the long time behavior of solutions of the stochastic damped KdV equation

\[ du + (\partial_x^3 u + u\partial_x u + \lambda u)dt = f dt + \Phi dW_t, \] (1.1)

with a nonzero deterministic force, by establishing the existence of an invariant measure.

Invariant measures play a crucial role in understanding the long time dynamics of solutions of stochastic partial differential equations [7, 14, 22, 19, 20]. In particular, they were constructed for the stochastic Navier-Stokes system [19], the stochastic conservation laws [16], the stochastic primitive equations [22], and for many other equations and systems in mathematical physics. However, as far as we know, the existence of an invariant measure for the stochastic damped KdV equation is open, the difficulties being the non-compactness of the domain and the asymptotic compactness of the semigroup.

The existence and uniqueness of solutions for the stochastic KdV equation has been established by de Bouard and Debussche in [10] (cf. also [8, 11, 12, 13, 15, 33]). However, the existence of invariant measure, which by the Krylov-Bogoliubov procedure requires the Feller property and the tightness property
of the time averages, has not been established except when including additional dissipative terms and in bounded domains [29, 30].

There are two main difficulties in carrying out the Krylov-Bogoliubov procedure for of the stochastic damped KdV equation. The first difficulty is related to establishing the Feller property, whose proof usually follows from a priori estimates on the solutions and the dominated convergence theorem. However, in the case of the KdV equation, there are no a priori estimates up to deterministic times. In order to circumvent this difficulty, we use the results in [10] and establish a priori estimates up to some stopping times, which are then used to show the Feller property of the transition semigroup. The second and the main difficulty in establishing the existence of an invariant measure for (1.1) is the tightness of the time averages. In fact, known approaches fail due to the lack of compactness and dissipation. Indeed, we study the equation on the whole domain which is unbounded with non-compact Sobolev embeddings. For instance the existence of compact embeddings is the assumption of [20] that fails for the equation studied here. Therefore, one cannot use the ideas available in the literature even by adding some perturbation (such as adding an artificial dissipation) to the equation to show the existence of invariant measures for (1.1). In addition, it is not clear how to obtain tightness of the family of the invariant measures for the perturbed equations as the size of perturbation converges to zero. Note that the existence of the invariant measure in an unbounded domain was considered in [17]; however, the strong dissipation in the Ginzburg-Landau linear and cubic terms are not available in the KdV case.

In order to obtain the tightness of averages, we are thus led to an unconventional proof. To show the tightness, we first use the existence results in [10] in order to establish uniform estimates on the solutions of the equation. These bounds give us tightness of measures on the space $L^2_{\text{loc}}(\mathbb{R})$ of locally square integrable functions. To pass from tightness in $L^2_{\text{loc}}(\mathbb{R})$ to tightness in $L^2(\mathbb{R})$, one intuitively needs to show that there is no mass escaping to infinity, which in this stochastic framework means the convergence of the expectation of the square of the $L^2(\mathbb{R})$ norm to the expectation of the square of the $L^2(\mathbb{R})$ norm of the limiting measure. We then use a result in [35] on the convergence in measure in Hilbert spaces, to obtain the tightness in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$.

In the deterministic case, there is a vast literature on the well-posedness of solutions of the KdV equation, starting with the seminal work of Temam [38], who established the global existence of weak solutions in $H^1$. Then the existence and uniqueness in Sobolev spaces was established by Kato (cf. [26]). The well-posedness theory was further studied by Bona-Smith [2, 3], Saut-Temam [37], Bourgain [4], Kenig et al [27, 28], Colliander et al [6], and by many other authors. The long time behavior of the KdV equations was initially studied by Ghidaglia in [21], who also established the existence and $H^2$ regularity of global attractors thus showing compactness at the infinite time. Further works by Moise, Rosa, Goubet, and Laurencot lowered the regularity of the force and showing infinite time compactness in periodic setting as well [23, 24, 25, 31, 32].

The existence and uniqueness of strong solutions of the stochastic KdV equation on the domain $\mathbb{R}$ with additive noise is established in [10]. The authors provide estimates on the solutions of the linear KdV equation and use these estimates to show local in time existence of solutions for the nonlinear equation. Then using the estimates in $H^1(\mathbb{R})$, the authors show global existence of strong solutions. We also mention that the problem was also studied in [34] on weighted Sobolev spaces.
It is well-known that considering martingale solutions brings additional compactness, and we take advantage of it by virtue of the Aldous criterion to prove the tightness of time averages in the Krylov-Bogoliubov procedure. We note that in [5] the authors use a similar compactness argument to compensate for the lack of Feller property for the 2D Navier-Stokes equation.

The paper is organized as follows. In Section 2, we introduce the main notation, while Section 3 contains the main results. Sections 4–7 contain the main steps of the proof: bounds in $L^2$, bounds in $H^1$, the Feller property, and the asymptotic compactness. The Appendix contains the proof of the convergence of norms, which is the crucial step in showing compactness in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$.

2 The notation and the main result

2.1 The stochastic Korteweg-de Vries equation

Fix a stochastic basis $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P})$. With $(e_i)_{i \in \mathbb{N}}$, an orthonormal basis of $L^2(\mathbb{R})$, consisting of smooth compactly supported elements and $(\beta_i)_{i \in \mathbb{N}}$, a sequence of mutually independent one dimensional Brownian motions, denote by

$$W(t) = \sum_{i \in \mathbb{N}} \beta_i(t)e_i$$

(2.1)

a cylindrical Wiener process on $L^2(\mathbb{R})$. Consider the stochastic weakly damped Korteweg-de Vries equation

$$du + (\partial_x^3 u + u\partial_x u + \lambda u)dt = f dt + \Phi dW_t,$$

where $\lambda > 0$, with the initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$  

(2.3)

2.2 Notation

For functions $u, v \in L^2(\mathbb{R})$, denote by $\|u\|_{L^2}$ the $L^2(\mathbb{R})$ norm of $u$ and by $(u, v)$ the $L^2$-inner product of $u$ and $v$. For a Banach space $B$ and with $T > 0$ and $p > 0$, denote by $L^p([0, T]; B)$ the space of functions from $[0, T]$ into $B$ with integrable $p$-th power over $[0, T]$ and by $C([0, T]; B)$ the set of continuous functions from $[0, T]$ into $B$. We write $B(H^1(\mathbb{R}))$ for the set of Borel measurable subsets of $H^1(\mathbb{R})$.

For a Hilbert space $H$, we write $\text{HS}(L^2, H)$ for the space of linear operators $\Phi$ from $L^2(\mathbb{R})$ into $H$ with finite Hilbert-Schmidt norm

$$\|\Phi\|_{\text{HS}(L^2, H)} = \left(\sum_{i=1}^{\infty} \|\Phi e_i\|_H^2\right)^{1/2}. \quad (2.4)$$

Similarly to functional spaces, for $p > 0$ we denote by $L^p(\Omega, B)$ the space of random variables with values in $B$ and finite $p$-th moment.

2.3 Well-posedness of the equation

The equation (2.2) was studied in [10] in the case $\lambda = 0$. The arguments carry over to the case $\lambda > 0$ with only slight modifications.
For all $\lambda \in \mathbb{R}$, we denote by $\{U_\lambda(t)\}_{t \geq 0}$ the solution operator of the partial differential equation
\[
d u_t + (\partial_x^3 u + \lambda u)dt = 0.
\] (2.5)

Note that $U_\lambda(t) = e^{-\lambda t}U_0(t)$. We then write the equation (2.2) in the mild form
\[
 u_t = U_\lambda(t)u_0 - \int_0^t U_\lambda(t-s)u_s \partial_x u_s ds + \int_0^t U_\lambda(t-s)f ds + \int_0^t U_\lambda(t-s)\Phi dW_s.
\] (2.6)

Throughout the paper we assume that $f \in H^3(\mathbb{R})$ (2.7) and $v \mapsto (v, f)$ is continuous in $L^2_{\text{loc}}(\mathbb{R})$. (2.8)

We also require $\Phi \in \text{HS}(L^2(\mathbb{R}); H^{3+}(\mathbb{R}))$. (2.9)

By $\text{HS}(L^2(\mathbb{R}); H^{3+}(\mathbb{R}))$ we mean $\text{HS}(L^2(\mathbb{R}); H^{\sigma}(\mathbb{R}))$ for some $\sigma > 3$. Recall that $u$ is a mild solution of (2.2) if $u$ verifies (2.6) for all $t \geq 0$, $\mathbb{P}$ a.s. The following statement addresses the existence and uniqueness of solutions.

**Theorem 2.1.** Assume that $u_0 \in L^2(\Omega; H^1(\mathbb{R})) \cap L^4(\Omega; L^2(\mathbb{R}))$ is $\mathcal{G}_0$-measurable. Then there exists a unique mild solution of (2.2) with paths almost surely in $C([0, \infty); H^1(\mathbb{R}))$ and with $u \in L^2(\Omega; L^\infty(0, T; H^1(\mathbb{R})))$ for all $T > 0$. Additionally, if $u_0 \in L^2(\Omega; H^3(\mathbb{R}))$ then $u \in L^2(\Omega; L^\infty(0, T; H^3(\mathbb{R})))$ for all $T > 0$.

The theorem follows as in [10] (Theorem 3.1 and Lemma 3.2) which treats the case $\lambda = 0$ and it is thus omitted. The inclusion in $C([0, \infty); H^1(\mathbb{R}))$ is not explicitly mentioned in [10]. However with the assumption (2.9), we can use Theorem 3.2 and Proposition 3.5 in [10] to obtain this inclusion.

### 2.4 The semigroup

Let $u_0 \in H^1(\mathbb{R})$ be a deterministic initial condition, and let $u$ be the corresponding solution of (2.2). For all $B \in \mathcal{B}(H^1(\mathbb{R}))$ we define the transition probabilities of the equation by
\[
P_t(u_0, B) = \mathbb{P}(u_t \in B).
\] (2.10)

For any function $\xi \in C_b(H^1; \mathbb{R})$ and for $t \geq 0$ we denote
\[
P_t \xi(u_0) = \mathbb{E} [\xi(u_t)] = \int_{H^1} \xi(u)P_t(u_0, du).
\] (2.11)

Here and in the sequel, $u_t = u(t)$ denotes the value of $u$ at time $t$; in particular, we do not use subscripts for partial derivatives.
3 The main results

We shall rely on the Krylov-Bogoliubov procedure (cf. [9, Corollary 3.1.2]) to show the existence of an invariant measure for the semigroup. The following statement is our main result.

**Theorem 3.1.** Suppose \( \lambda > 0 \), and assume that \( f \) and \( \Phi \) verify (2.7)-(2.9). Then there exists an invariant measure of the equation (2.2).

The proof is based on the following two lemmas. The first lemma states that the Feller property holds for the stochastic damped KdV equation.

**Lemma 3.2.** (Feller Property) Under the assumptions of Theorem 3.1, the semigroup \( P_t \) is Feller on \( H^1(\mathbb{R}) \). Namely, for \( \xi \in C_b(H^1, \mathbb{R}) \) and with \( u_0^1, u_0^2, \ldots \in H^1(\mathbb{R}) \) satisfying \( \|u_0^n - u_0\|_{H^1} \to 0 \) as \( n \to \infty \), where \( u_0 \in H^1(\mathbb{R}) \), the convergence

\[
P_t\xi(u_0^n) \to P_t\xi(u_0), \quad n \to \infty
\]

holds for all \( t \geq 0 \).

The second lemma asserts tightness of averages originating from the initial datum \( u_0 = 0 \).

**Lemma 3.3.** (Tightness) Under the assumptions of Theorem 3.1, the family of measures on \( H^1(\mathbb{R}) \)

\[
\mu_n(\cdot) = \frac{1}{n} \int_0^n P_t(0, \cdot) \, dt, \quad n = 1, 2, \ldots
\]

is tight.

The proof of Lemma 3.2 is provided in Section 6, while the proof of tightness is given in Section 7.

4 Uniform bounds in \( L^2(\mathbb{R}) \)

The next statement establishes uniform in \( t \) bounds on \( \mathbb{E}[\|u_t\|_{L^2}^2] \).

**Lemma 4.1.** Under the assumptions of Theorem 3.1, there exists a sequence \( \{C_k\}_{k \geq 1} \) depending on \( f \), \( \Phi \), and \( \lambda \) such that

\[
\sup_{t \geq 0} \mathbb{E} \left[ \|u_t\|_{L^2}^2 \right] \leq C_k(f, \Phi, \lambda)(\mathbb{E} \left[ \|u_0\|_{L^2}^2 \right] + 1)
\]

holds for all \( k \in \mathbb{N} \) for which \( \mathbb{E} \left[ \|u_0\|_{L^2}^2 \right] < \infty \).

From here on, we shall consider \( \lambda \), \( f \), and \( \Phi \) fixed and thus omit indicating the dependence of the constants on these quantities.

**Proof of Lemma 4.1.** We define

\[
\sigma_n = \inf \left\{ t \geq 0 : \int_0^t \|u_s\|_{L^2}^2 \, ds \geq n \right\}
\]

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and apply Ito’s lemma to \(\|u_t\|_{L^2}^2\) in order to obtain
\[
\|u_t\|_{L^2}^2 + 2\lambda \int_0^t \|u_s\|_{L^2}^2 \, ds = \|u_0\|_{L^2}^2 + t\|\Phi\|_{HS(L^2,L^2)}^2 + \int_0^t 2(u_s, f) \, ds + 2M_t
\]  
(4.3)
for \(0 \leq t \leq \sigma_n\), where
\[
M_t = \int_0^t \sum_i u_s \Phi e_i \, dx \, \beta^i_s.
\]  
(4.4)

We compute
\[
E[M^2_{t \wedge \sigma_n}] = E\left[ \int_0^{t \wedge \sigma_n} \sum_i (u_s, \Phi e_i)^2 \, ds \right] \leq \|\Phi\|^2_{HS(L^2,L^2)} E\left[ \int_0^{t \wedge \sigma_n} \|u_s\|^2 \, ds \right]
\]
\[
\leq n \|\Phi\|^2_{HS(L^2,L^2)} t < \infty,
\]  
(4.5)
and thus \(M_{t \wedge \sigma_n}\) is a square integrable martingale. Taking the expectation of both sides of the equation (4.3) at \(t \wedge \sigma_n\), we get
\[
E\|u(t \wedge \sigma_n)\|_{L^2}^2 + 2\lambda E\left[ \int_0^{t \wedge \sigma_n} \|u_s\|_{L^2}^2 \, ds \right]
\]
\[
= E[\|u_0\|_{L^2}^2] + E[t \wedge \sigma_n] \|\Phi\|_{HS(L^2,L^2)}^2 + 2E\left[ \int_0^{t \wedge \sigma_n} (u_s, f) \, ds \right].
\]  
(4.6)

Therefore, we obtain an upper bound
\[
E[\|u(t \wedge \sigma_n)\|_{L^2}^2] + \lambda E\left[ \int_0^{t \wedge \sigma_n} \|u_s\|_{L^2}^2 \, ds \right] \leq E[\|u_0\|_{L^2}^2] + t \|\Phi\|^2_{HS(L^2,L^2)} + \frac{t}{\lambda} \|f\|^2_{L^2}
\]  
(4.7)
which is uniform in \(n\). Note that we have
\[
\int_0^{t \wedge \sigma_n} \|u_s\|_{L^2}^2 \, ds = \int_0^t \|u_s\|_{L^2}^2 \, ds \mathbf{1}_{\{\sigma_n > t\}} + n \mathbf{1}_{\{\sigma_n \leq t\}}
\]  
(4.8)
from where, using (4.7),
\[
\lambda n P(\sigma_n \leq t) \leq \lambda E\left[ \int_0^{t \wedge \sigma_n} \|u_s\|_{L^2}^2 \, ds \right] \leq E[\|u_0\|_{L^2}^2] + t \|\Phi\|^2_{HS(L^2,L^2)} + \frac{t}{\lambda} \|f\|^2_{L^2}.
\]  
(4.9)
Taking the limit as \(n\) goes to infinity, we conclude that the stopping time \(\sigma^* = \lim_{n \to \infty} \sigma_n\) verifies \(P(\sigma^* = \infty) = 1\). We now return to the equality (4.6). Using the dominated convergence theorem and the fact that \(\|u_s\|_{L^2}\) is continuous, we obtain
\[
E[\|u_t\|_{L^2}^2] + 2\lambda E\left[ \int_0^t \|u_s\|_{L^2}^2 \, ds \right] = E[\|u_0\|_{L^2}^2] + t \|\Phi\|^2_{HS(L^2,L^2)} + 2E\left[ \int_0^t (u_s, f) \, ds \right].
\]  
(4.10)

We differentiate this equality
\[
\frac{d}{dt} E[\|u_t\|_{L^2}^2] + 2\lambda E[\|u_t\|_{L^2}^2] = \|\Phi\|^2_{HS(L^2,L^2)} + 2E[(u_t, f)]
\]  
(4.11)
from where we obtain
\[
\frac{d}{dt} E[\|u_t\|_{L^2}^2] + \lambda E[\|u_t\|_{L^2}^2] \leq \|\Phi\|^2_{HS(L^2,L^2)} + \frac{\|f\|^2_{L^2}}{\lambda}.
\]  
(4.12)
Using the Gronwall inequality, we get

\[
E[\|u_t\|_{L^2}^2] \leq e^{-\lambda t} E[\|u_0\|_{L^2}^2] + \left(\|\Phi\|_{HS(L^2,L^2)}^2 + \frac{\|f\|_{L^2}^2}{\lambda}\right) \int_0^t e^{-\lambda(t-s)} ds \leq C(\|u_0\|_{L^2}^2) + 1, \tag{4.13}
\]

where the constant \(C\) depends on \(\Phi\) only through \(\|\Phi\|_{HS(L^2,L^2)}^2\).

In order to use the induction for \(k \geq 1\), we need to control \(E[\sup_{s \leq t} \|u_s\|_{L^2}^2]\). To achieve this, we return to (4.3) and obtain

\[
\sup_{s \in [0,t]} \|u_s\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + \frac{t}{\lambda} \|\Phi\|_{HS(L^2,L^2)}^2 + C \sup_{s \in [0,t]} |M_s|. \tag{4.14}
\]

Then, by the Burkholder-Davis-Gundy inequality

\[
E\left[ \sup_{s \in [0,t]} |M_s| \right] \leq C E\left[ \left( \int_0^t \sum_i (u_s, \Phi e_i)^2 ds \right)^{1/2} \right] \leq C \left( E\left[ \int_0^t \|u_s\|_{L^2}^2 ds \right] \right)^{1/2} \tag{4.15},
\]

which then gives \(E[\sup_{s \in [0,t]} \|u_s\|_{L^2}^2] \leq C(t)\).

In order to use induction, suppose that for some \(k \in \mathbb{N}\) we have

\[
\sup_{t \geq 0} E\left[ \|u_t\|_{L^2}^{2k} \right] \leq C(\|u_0\|_{L^2}^{2k}) + 1 < \infty \tag{4.16}
\]

and

\[
E\left[ \sup_{s \in [0,t]} \|u_s\|_{L^2}^{2k} \right] \leq C_k(t). \tag{4.17}
\]

Assume that

\[
E\left[ \|u_0\|_{L^2}^{2(k+1)} \right] < \infty \tag{4.18}
\]

and denote \(X_t = \|u_t\|_{L^2}^2\). Similarly to the previous case, let

\[
\sigma_n = \inf\left\{ t \geq 0 : \int_0^t X_s^{k+1} ds \geq n \right\}. \tag{4.19}
\]

Applying Ito’s lemma leads to

\[
dX_t^{k+1} = (k+1)X_t^k \left(-2\lambda X_t dt + \|\Phi\|_{HS(L^2,L^2)}^2 dt + 2(u_t, f) dt + 2dM_t \right) + 2(k+1)X_t^{k-1}\|\Phi^* u_t\|_{L^2}^2 dt \\
\leq -\lambda (k+1)X_t^{k+1} dt + C(X_t^k + 1) dt + 2(k+1)X_t^k dM_t. \tag{4.20}
\]

The quadratic variation of the stochastic integral is proportional to

\[
\int_0^{T \wedge \sigma_n} X_t^{2k} d(M_t) \leq C \int_0^{T \wedge \sigma_n} X_t^{2k} \sum_i (u_t, \Phi e_i)^2 dt \\
\leq C \int_0^{T \wedge \sigma_n} X_t^{2k+1} dt \leq Cn \sup_{t \in T \wedge \sigma_n} \|u_t\|_{L^2}^{2k+1}. \tag{4.21}
\]
where the brackets denote the quadratic variation; we used (4.19) in the step. Since the last term is integrable due to (4.17), \( \int_0^{t \land \sigma_n} X_s^k dM_s \) is a square integrable martingale. We take the expectation of (4.20) and obtain an upper bound

\[
\mathbb{E}[X_{t \land \sigma_n}^{k+1}] + 2(k+1)\lambda \mathbb{E} \left[ \int_0^{t \land \sigma_n} X_s^{k+1} ds \right] \leq C(t) \tag{4.22}
\]

which is uniform in \( n \). Similarly to (4.8), we have

\[
\frac{1}{\sqrt{n}} \left( \int_0^{t \land \sigma_n} X_s^{2k+1} ds \right)^{1/2} \geq 1_{\{ \sigma_n \leq t \}} \tag{4.23}
\]

Thus

\[
\mathbb{P}(\sigma_n \leq t) \leq \frac{1}{\sqrt{n}} \mathbb{E} \left[ \left( \int_0^{t \land \sigma_n} X_s^{2k+1} ds \right)^{1/2} \right] \leq \frac{1}{\sqrt{n}} \mathbb{E} \left[ \sup_{s \in [0,t]} X_s^k \right]^{1/2} \left( \int_0^{t \land \sigma_n} X_s^{k+1} ds \right)^{1/2} \leq \frac{C}{\sqrt{n}} \left( \mathbb{E} \sup_{s \in [0,t]} X_s^k + \mathbb{E} \left[ \int_0^{t \land \sigma_n} X_s^{k+1} ds \right] \right) \leq \frac{C}{\sqrt{n}} \tag{4.24}
\]

which converges to 0 as \( n \to \infty \) due to (4.22) and the inductive assumption. Thus the stochastic integral arising from (4.20) is a martingale on \([0, \infty)\). Taking the expectation of (4.20), we get

\[
d\mathbb{E}[X_t^{k+1}] + \lambda (k+1) \mathbb{E}[X_t^{k+1}] dt \leq C dt \tag{4.25}
\]

which implies

\[
\mathbb{E}[X_t^{k+1}] \leq e^{-\lambda(k+1)t} \mathbb{E}[X_0^{k+1}] + C \int_0^t e^{-\lambda(k+1)(t-s)} ds \leq C \left( \mathbb{E} \left[ \|u_0\|_L^{2(k+1)} \right] + 1 \right). \tag{4.26}
\]

In order to complete the proof, we also need to control \( \sup_{t \in [0,T]} X_t^{k+1} \). Note first that \( X_t \geq 0 \). From (4.20), we get

\[
\sup_{t \in [0,T]} X_t^{k+1} \leq C(T) \left( X_0^{k+1} + \sup_{t \in [0,T]} X_t^k \right) + \sup_{t \in [0,T]} \left| \int_0^t 2(k+1) X_s^k dM_s \right|. \tag{4.27}
\]

We use the Burkholder-Davis-Gundy inequality to bound the expectation of the last term with the expectation of the square root of the quadratic variation of the stochastic integral computed in (4.21) to obtain

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} X_t^{k+1} \right] \leq C(T) \tag{4.28}
\]

and the proof is complete. \( \square \)
\section{Uniform bounds in $H^1(\mathbb{R})$}

We now present certain uniform bounds on the $L^2$ norm of $\partial_x u$. Denote by
\[ I(v) = \int_{\mathbb{R}} \left( \partial_x v(x)^2 - \frac{v(x)^3}{3} \right) dx, \quad v \in H^1(\mathbb{R}) \] (5.1)
the second invariant of the deterministic KdV equation. For ease of notation we define
\[ \alpha(t) = \frac{\lambda}{3} \int u_t(x)^3 dx + \| \partial_x \Phi \|^2_{HS(L^2, L^2)} - \sum_i \int u_t(x) |(\Phi e_i)(x)|^2 dx \]
\[ + 2(\partial_x u_t, \partial_x f) - (u_t^2, f). \] (5.2)

**Lemma 5.1.** If $u_0 \in L^4(\Omega; H^1(\mathbb{R})) \cap L^8(\Omega; L^2(\mathbb{R}))$ is $\mathcal{G}_0$-measurable, the evolution of $I(u_t)$ is given by
\[ di(u_t) + 2\lambda I(u_t)dt = \alpha(t)dt + 2(\partial_x u_t, \partial_x \Phi dW_t) - (u_t^2, \Phi dW_t). \] (5.3)

Moreover,
\[ \sup_{t \geq 0} \mathbb{E} \left[ \| \partial_x u_t \|^2_{L^2} \right] \leq C \left( \mathbb{E} \left[ \| u_0 \|^2_{H^1} + \| u_0 \|^2_{L^4} \right] + 1 \right), \quad k = 1, 2 \] (5.4)
where $C$ is a constant.

**Proof of Lemma 5.1.** The identity (5.3) follows by Ito’s formula as in Lemma 3.3 of [10]. The quadratic variations of the stochastic integrals in (5.3) equal
\[ \langle \partial_x u_t, \partial_x \Phi dW_t \rangle = \sum_i \langle \partial_x u_t, \partial_x \Phi e_i \rangle^2 dt \leq \| \partial_x u_t \|^2_{L^2} \| \Phi \|^2_{HS(L^2, H^1)} dt \] (5.5)
(the brackets denoting the quadratic variation) and
\[ \langle u_t^2, \Phi dW_t \rangle = \sum_i \langle u_t^2, \Phi e_i \rangle^2 dt \leq \| u_t \|^4_{L^2} \sum_i \| \Phi e_i \|^2_{L^2} dt \]
\[ \leq \| u_t \|^4_{L^2} \sum_i \| \Phi e_i \|_{L^2} \| \partial_x \Phi e_i \|_{L^2} dt \leq \| u_t \|^4_{L^2} \| \Phi \|^2_{HS(L^2, H^1)} dt. \] (5.6)

Theorem 2.1, combined with the previous inequalities, shows that the stochastic integrals define square integrable martingales.

Now, we estimate $\alpha$. Using Agmon’s inequality
\[ \| u \|_{L^\infty} \leq \| u \|^2_{L^2} \| \partial_x u \|^2_{L^2} \] (5.7)
we obtain
\[ |\alpha(t)| \leq C \left( 1 + \| u \|^{5/2}_{L^2} \| \partial_x u \|^{1/2}_{L^2} + \| u \|_{L^2} + \| \partial_x u \|_{L^2} + \| u \|^{3/2}_{L^2} \| \partial_x u \|^{1/2}_{L^2} \right). \] (5.8)
Then we use the $c$-Young inequality and obtain
\[ |\alpha(t)| \leq \lambda \| \partial_x u \|^2_{L^2} + C(\| u \|^{10/3}_{L^2} + 1) \leq \lambda \| \partial_x u \|^2_{L^2} + C(\| u \|^4_{L^2} + 1). \] (5.9)
Inserting these inequalities into (5.3) and taking the expectation, we obtain
\[ d\mathbb{E}[I(u_t)] + 2\lambda \mathbb{E}[I(u_t)]dt \leq \lambda \mathbb{E}[\| \partial_x u_t \|^2_{L^2}] dt + C \left( 1 + \mathbb{E} \left[ \| u_0 \|^4_{L^2} \right] \right) dt. \] (5.10)
Using Sobolev inequalities, we have
\[ \frac{3}{4} \| \partial_x v \|_{L^2}^2 - C \| v \|_{L^2}^{10/3} \leq I(v) \leq \frac{4}{3} \| \partial_x v \|_{L^2}^2 + C \| v \|_{L^2}^{10/3}. \] (5.11)

Using the left inequality and Lemma 4.1, we get
\[ dE[I(u_t)] + 2 \lambda E[I(u_t)] dt \leq \frac{4 \lambda}{3} E[I(u_t)] dt + C \left( 1 + E \left[ \| u_0 \|_{L^2}^4 \right] \right) dt \] (5.12)
from where
\[ E[I(u_t)] \leq e^{-\lambda t/2} E[I(u_0)] + C \int_0^t e^{-\lambda (t-s)/2} \left( 1 + E \left[ \| u_0 \|_{L^2}^4 \right] \right) ds \]
\[ \leq C (E[I(u_0)] + E[\| u_0 \|_{L^2}^4] + 1). \] (5.13)

By the right inequality in (5.11) we obtain
\[ \sup_{t \geq 0} E[\| \partial_x u_t \|_{L^2}^2] \leq C (E[I(u_0)] + \| u_0 \|_{L^2}^4 + 1). \] (5.14)

Finally, combining all the inequalities above, we conclude
\[ \sup_{t \geq 0} E[\| u_t \|_{H^1}^2] \leq C (E[I(u_0)] + \| u_0 \|_{H^1}^4 + \| u_0 \|_{L^2}^4 + 1) \] (5.15)
which gives (5.4) with \( k = 1 \).

In order to obtain (5.4) for \( k = 2 \), we apply Itô’s Lemma to \( I^2(u_t) \) and get
\[ dI^2(u_t) + 4 \lambda I^2(u_t) dt \]
\[ = 2 I(u_t) \alpha(t) dt + d\tilde{M}_t + \sum_i \left( 2 (\partial_x u_t, \partial_x \Phi e_i) - \langle u_t^2, \Phi e_i \rangle \right)^2 dt, \] (5.16)
where
\[ d\tilde{M}_t = 2 I(t) \sum_i \left( 2 (\partial_x u_t, \partial_x \Phi e_i) - \langle u_t^2, \Phi e_i \rangle \right) dB_i^t. \]

By (5.8), we have
\[ 2 I(u_t) \alpha(t) \leq 2 \lambda I^2(u_t) + \frac{8}{\lambda} \alpha(t)^2 \leq 2 \lambda I^2(u_t) + C(1 + \| \partial_x u_t \|_{L^2}^2 + \| u_t \|_{L^2}^{20/3}). \] (5.17)

We also estimate the quadratic variation term as
\[ \sum_i \left( 2 (\partial_x u_t, \partial_x \Phi e_i) - \langle u_t^2, \Phi e_i \rangle \right)^2 \]
\[ \leq 8 \| \partial_x u_t \|_{L^2}^2 \| \partial_x \Phi \|_{H^2(L^2,L^2)}^2 + 2 \| u_t \|_{L^4}^2 \| \Phi \|_{H^2(L^2,L^2)}^2 \]
\[ \leq C(\| \partial_x u_t \|_{L^2}^2 + \| u \|_{L^2}^6) \] (5.18)
where we used
\[ \| u \|_{L^1} \leq \| u \|_{L^2}^{3/4} \| \partial_x u \|_{L^2}^{1/4}. \] (5.19)
Finally, we compute the quadratic variation of $\tilde{M}$,
\[
d\langle \tilde{M} \rangle_t = 4I^2(u_t) \sum_i \left(2(\partial_x u_t, \partial_x \Phi e_i) - (u_t^2, \Phi e_i)\right)^2 dt
\leq C I^2(u_t)(\|\partial_x u_t\|_{L^2}^2 + \|u_t\|_{L^4}^4) dt
\leq C I^2(u_t)(\|\partial_x u_t\|_{L^2}^2 + \|u_t\|_{L^2}^6) dt
\]
where we used (5.19) and the $\epsilon$-Young inequality. For all $n \in \mathbb{N}$, we define the stopping time
\[
\tau_n = \inf \left\{ t \geq 0 : \int_0^t I^2(u_s) ds \geq n \right\}
\]
and
\[
\tau^* = \lim_n \tau_n.
\]
Integrating the evolution of $I^2(u_t)$ we obtain for all $T > 0$
\[
\sup_{t \in [0,T \wedge \tau_n]} I^2(u_t) - I^2(u_0) + 4\lambda \int_0^{T \wedge \tau_n} I^2(u_s) ds
\leq \sup_{t \in [0,T \wedge \tau_n]} \tilde{M}_t + 2\lambda \int_0^{T \wedge \tau_n} I^2(u_s) ds + C \int_0^{T \wedge \tau_n} \left(\|\partial_x u_s\|_{L^2}^2 + \|u_s\|_{L^2}^{20/3}\right) ds.
\]
We now take the expectation and use the Burkholder-Davis-Gundy inequality to obtain
\[
\mathbb{E} \left[ \sup_{t \in [0,T \wedge \tau_n]} I^2(u_t) - I^2(u_0) + 4\lambda \int_0^{T \wedge \tau_n} I^2(u_s) ds \right]
\leq \mathbb{E} \left[ \left( \int_0^{T \wedge \tau_n} I^2(u_t)(\|\partial_x u_t\|_{L^2}^2 + \|u_t\|_{L^2}^{6}) dt \right)^{1/2} \right] + 2\lambda \mathbb{E} \left[ \int_0^{T \wedge \tau_n} I^2(u_s) ds \right]
+ \mathbb{E} \left[ \int_0^{T \wedge \tau_n} (\|\partial_x u_s\|_{L^2}^2 + \|u_s\|_{L^2}^{20/3}) ds \right].
\]
Applying the $\epsilon$-Young inequality, we get
\[
\mathbb{E} \left[ \left( \int_0^{T \wedge \tau_n} I^2(u_t)(\|\partial_x u_t\|_{L^2}^2 + \|u_t\|_{L^2}^{6}) dt \right)^{1/2} \right]
\leq \lambda \mathbb{E} \left[ \int_0^{T \wedge \tau_n} I^2(u_s) ds \right] + \lambda \mathbb{E} \left[ \sup_{t \in [0,T]} \|\partial_x u_t\|_{L^2}^2 \right] + \lambda \mathbb{E} \left[ \sup_{t \in [0,T]} \|u_t\|_{L^2}^6 \right].
\]
Using this inequality in (5.22), we obtain that for all $T > 0$ we have
\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_n} I^2(u_s) ds \right] \leq C
\]
for a constant independent of $n$ which implies that $\tilde{M}_t$ is a martingale. Using (5.16) and (5.17), we get
\[
\mathbb{E} \left[ I^2(u_t) \right] + 2\lambda \int_0^t \mathbb{E} \left[ I^2(u_s) \right] ds \leq \mathbb{E}[I^2(u(0))] + C \int_0^t \mathbb{E} \left[ \|\partial_x u_s\|_{L^2}^2 + \|u_s\|_{L^2}^{20/3} \right] ds.
\]
Since the function \( s \mapsto E[\|\partial_x u_s\|_{L^2}^2 + \|u_s\|_{L^2}^{20/3}] \) is bounded, we may proceed as above and obtain
\[
\sup_{t \geq 0} E[I^2(u_t)] \leq C E \left[ \|\partial_x u(0)\|_{L^2}^2 + \|u(0)\|_{L^2}^{20/3} + 1 \right] \tag{5.25}
\]
and the lemma is established. \( \square \)

In order to prove the Feller property, stated as Lemma 3.2, we need the following estimate.

**Lemma 5.2.** For all \( R_0, T > 0 \) there exists a constant \( C(R_0, T) \geq 0 \) such that
\[
E\left[ \sup_{t \in [0, T]} \|u_t\|_{H^1}^2 \right] \leq C(R_0, T) \tag{5.26}
\]
for all deterministic initial conditions \( u_0 \in H^1(\mathbb{R}) \) verifying \( \|u_0\|_{H^1} \leq R_0 \).

**Proof of Lemma 5.2.** We write (5.3) in the form
\[
I(u_t) = I(u_0) + M_t - 2\lambda \int_0^t I(u_s) ds + \int_0^t \left( \frac{\lambda}{3} \int_{\mathbb{R}} |u^3(s, x)| dx + \|\partial_x \Phi\|_{HS(L^2, L^2)}^2 + \sum_i \int_{\mathbb{R}} u(s, x) |(\Phi e_i)(x)|^2 dx 
+ 2(\partial_x u_s, \partial_x f) - (u^2(s), f) \right) ds \tag{5.27}
\]
where \( M_t \) is the martingale term. Therefore,
\[
E\left[ \sup_{0 \leq t \leq T} |I(u_t)| \right] \\
\leq |I(u_0)| + C \int_0^T E[|I(u_s)|] \, ds \\
+ C \int_0^T E \left[ \int_{\mathbb{R}} |u^3(s, x)| dx + \|\partial_x \Phi\|_{HS(L^2, L^2)}^2 + \sum_i \int_{\mathbb{R}} u(s, x) |(\Phi e_i)(x)|^2 dx \right] \, ds \\
+ C \int_0^T E \left[ |\partial_x u| |\partial_x f| dx + \int_{\mathbb{R}} u^2|f| \, dx \right] + E \left[ \sup_{0 \leq t \leq T} |M_t| \right] \\
\leq |I(u_0)| + C T (\|u_0\|_{H^1} + \|u_0\|_{L^2} + 1) + E \left[ \sup_{0 \leq t \leq T} |M_t| \right] \tag{5.28}
\]

The Burkholder-Davis-Gundy inequality, together with (5.5) and (5.6), gives the required bound for \( E[\sup_{0 \leq t \leq T} |M_t|] \). \( \square \)

### 6 Proof of the Feller property

For all \( T \geq 0 \) we, as in [10, 27, 28], define
\[
X_1(T) = \left\{ u \in C(0, T; H^1(\mathbb{R})) \cap L^2(\mathbb{R}; L^\infty([0, T])) : \right. \\
\left. \partial_x u \in L^4([0, T]; L^\infty(\mathbb{R})), \partial_{xx} u \in L^\infty(\mathbb{R}, L^2([0, T])) \right\} \tag{6.1}
\]
and denote
\[ \bar{\pi}_s = \int_0^s U_\lambda(s-r)\Phi dW_r, \quad s \geq 0. \tag{6.2} \]

Observe that \( \bar{\alpha}_s \) solves
\[ d\bar{\alpha}_s + \lambda \bar{\alpha} ds + \partial_{xxx} \bar{\alpha} ds = \Phi dW_s \tag{6.3} \]
with \( \bar{\alpha}_0 = 0 \).

Due to the nonlinearity in the equation, we have explicit continuous dependence on the initial data only on small time intervals. In order to prove the Feller property, we define a sequence of hitting times between which this continuous dependence is known. We first prove that we only need a finite number of these hitting times. Then we use this result and continuous dependence between the hitting times to prove the Feller property.

We start with the following auxiliary result.

**Lemma 6.1.** Under the assumption (2.9), \( \|\pi\|_{L^1(s)} \) is a continuous process.

**Proof of Lemma 6.1.** Since \( \bar{\pi} \) is a solution of a linear equation, it is sufficient to prove that
\[ \|\bar{\pi}\|_{X_1(s)} \to 0, \quad \text{P-a.s.} \tag{6.4} \]
as \( s \to 0 \). Now, \( X_1(s) \) is defined in (6.1) as the intersection of four spaces and thus we need to show convergence to 0 in all four norms.

Note that \( \|\bar{\pi}_s\|_{H^1} \), for \( s \in [0,T] \), is a continuous uniformly integrable semi-martingale and thus
\[ \mathbb{E} \left[ \sup_{0 \leq r \leq s} \|\bar{\pi}_r\|_{H^1} \right] \to 0 \quad \text{as} \quad s \to 0. \tag{6.5} \]
For the norms associated to \( L^2(\mathbb{R};L^\infty([0,s])) \) and \( L^4([0,s];L^\infty(\mathbb{R})) \), the convergence follows by the monotone convergence theorem. The only issue is for the \( L^\infty(\mathbb{R},L^2([0,s])) \) norm. In order to show convergence, we modify the proof of [10, Proposition 3.3] and obtain
\[ \mathbb{E} \left[ \sup_{x \in \mathbb{R}} \int_0^s |\partial_{xx} \bar{\pi}_r|^2 dr \right] \leq C(\lambda, s) \|\Phi\|_{L^2(H^3)}^2 \tag{6.6} \]
where \( C(\lambda, s) \to 0 \) as \( s \to 0 \), which completes the proof. \( \square \)

We now prove the Feller property. Fix \( u_0 \in H^1(\mathbb{R}) \) and \( t > 0 \). Also, let \( \xi \in C_b(H^1,\mathbb{R}) \). We claim that
\[ \mathbb{E} [\|\xi(u_t) - \xi(v_t)\|] \to 0 \tag{6.7} \]
as \( v_0 \to u_0 \) in \( H^1 \).

Let \( \epsilon > 0 \). Also, denote \( R_0 = \|u_0\|_{H^1} + 1 \), and let
\[ M = \sup_{v \in H^1(\mathbb{R})} |\xi(v)|. \tag{6.8} \]
Step 1: For all \( v_0 \) such that \( \|v_0 - u_0\|_{H^1} \leq 1 \) with the associated solution \( v \) of (2.2), Lemma 5.2 gives

\[
\mathbb{P} \left( \max \left\{ \sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1}^2 \right\} \geq R \right) \leq \frac{1}{R} \mathbb{E} \left[ \sup_{s \in [0,t]} \|u_s\|_{H^1} + \sup_{s \in [0,t]} \|v_s\|_{H^1}^2 \right] \leq \frac{C(R_0)}{R}.
\] (6.9)

(Note that \( t > 0 \) is fixed and thus the dependence of all constants on \( t \) is not indicated.) Fix \( R > 0 \) so that we have

\[
\frac{C(R_0)}{R} \leq \frac{\epsilon}{6M}.
\] (6.10)

Step 2: By combining (5.26) and [10, Proposition 3.5], we obtain the existence of a non-decreasing function \( \tilde{C}(T) \geq 1 \) (6.11) such that we have the inequalities

\[
\mathbb{E}[\|\pi\|_{X_1(T)}] + \left\| \int_0^T U_\lambda(\cdot - r)f \, dr \right\|_{X_1(T)} \leq \tilde{C}(T)
\] (6.12)

and

\[
\left\| \int_0^T U_\lambda(\cdot - s)h(s)\partial_x g(s) \, ds \right\|_{X_1(T)} \leq \tilde{C}(T)T^{1/2}\|h\|_{X_1(T)}\|g\|_{X_1(T)}
\] (6.13)

for all \( h, g \in X_1(T) \) and

\[
\|U_\lambda(\cdot) v_0\|_{X_1(T)} \leq \tilde{C}(T)\|v_0\|_{H^1}
\] (6.14)

for \( v_0 \in H^1(\mathbb{R}) \). Since \( \tilde{C} \) is non-decreasing, we may increase it so that it is also continuous. For all \( s \leq t \) the solutions \( u \) and \( v \) verify

\[
u_s = \mathcal{T}_u(0)(s)
\] (6.15)

and

\[
v_s = \mathcal{T}_v(0)(s)
\] (6.16)

where

\[
\mathcal{T}_h(g)(s) = U_\lambda(s)h - \int_0^s U_\lambda(s - r)g(r)\partial_x g(r) \, dr + \int_0^s U_\lambda(s - r)f \, dr + \pi_s
\] (6.17)

for \( h \in H^1(\mathbb{R}) \) and \( g \in X_1(t) \). Define

\[
\tau = \inf \left\{ s \geq 0 : 8\tilde{C}(t)s^{1/2} \left( \tilde{C}(t)R + \|\pi\|_{X_1(s)} + \left\| \int_0^s U_\lambda(\cdot - r)f \, dr \right\|_{X_1(s)} \right) > 1 \right\}.
\] (6.18)
By Lemma 6.1, τ is a stopping time. Note that for all \( s \leq t \)

\[
P(\tau < s) \leq P \left( 8\tilde{C}(t)s^{1/2} \left( \tilde{C}(t)R + \|\pi\|_{X_1(s)} + \left\| \int_0^\tau U_\lambda(\cdot - r) f \, dr \right\|_{X_1(s)} \right) > 1 \right)
\]

\[
\leq E \left[ 8\tilde{C}(t)s^{1/2} \left( \tilde{C}(t)R + \|\pi\|_{X_1(s)} + \left\| \int_0^\tau U_\lambda(\cdot - r) f \, dr \right\|_{X_1(s)} \right) \right]
\]

\[
\leq 8\tilde{C}^2(t)s^{1/2}(R + 1)
\]

\[
\leq C(R)s^{1/2}
\]

(6.19)

where we used (6.12) in the third inequality. Therefore,

\[
E[\tau] = \int_0^\infty P(s \leq \tau) \, ds = \int_0^\infty (1 - P(\tau < s)) \, ds
\]

\[
\geq \int_0^t (1 - P(\tau < s)) \, ds \geq \int_0^t \left( 1 - 1 \wedge (C(R)s^{1/2}) \right) \, ds
\]

\[
\geq \frac{1}{C_0(R)}
\]

(6.20)

where \( C_0(R) \) is a constant depending on \( R \) (the dependence on \( t \), which is fixed, is understood).

**Step 3**: In order to control the noise term, we now inductively define a sequence of stopping times.

We start with

\[
\tau_0 = \tau
\]

(6.21)

and then for \( k = 0, 1, \ldots \) set

\[
\tau_{k+1} = \inf \left\{ s \geq \tau_k : 8\tilde{C}(t)(s - \tau_k)^{1/2} \left( \tilde{C}(t)R + \|\pi^{\tau_k}_s\|_{X_1(\tau_k, s)} + \left\| \int_{\tau_k}^s U_\lambda(\cdot - r) f \, dr \right\|_{X_1(\tau_k, s)} \right) > 1 \right\}
\]

(6.22)

where for a stopping time \( \tau \) we introduce the shifted process

\[
\pi^{\tau}_s = \int_\tau^s U_\lambda(s - r) \Phi \, dW_r, \quad \tau \leq s.
\]

(6.23)

and

\[
\pi^{\tau}_s = 0 \text{ for } \tau \geq s.
\]

(6.24)

Also, similarly to \( X_1(T) \), we define \( X_1(\tau_k, s) \) for a shifted process (defined on \([\tau_k, s]\)). For simplicity of notation, we set \( \tau_{-1} = 0 \).

Note that \( \pi^{\tau_k}_s = 0 \) for \( s \leq \tau_k \). For \( s \geq \tau_k \), we have that \( \pi^{\tau_k}_s \) is \( \sigma(\{W_r - W_{\tau_k}, r \in [\tau_k, s]\}) \)-measurable. Therefore, \( \tau_{k+1} - \tau_k \) is independent from \( \mathcal{G}_{\tau_k} \); also, \( \tau_{k+1} - \tau_k \) has the same distribution as \( \tau \). By the law of large numbers, a.s.

\[
\frac{\tau_n}{n} = \frac{1}{n} \sum_{i=0}^n (\tau_i - \tau_{i-1}) \to E[\tau] \geq \frac{1}{C_0(R)}
\]

(6.25)
as \( n \to \infty \), where the constant is as in (6.20). Thus the sequence of random variables

\[ 1\{\tau_n \leq t\} = 1\{\tau_n/n - E[\tau] \leq t/n - E[\tau]\} \tag{6.26} \]

converges \( \mathbb{P} \)-a.s. to 0 as \( n \to \infty \). By the dominated convergence theorem \( \mathbb{P}(\tau_n \leq t) \to 0 \) as \( n \to \infty \). Hence, there exists \( n > 0 \) depending only on \((R, \epsilon, M)\) such that

\[ \mathbb{P}(\tau_n \leq t) \leq \frac{\epsilon}{6M}. \tag{6.27} \]

Therefore, for all \( v_0 \) satisfying \( \|u_0 - v_0\|_{H^1} \leq 1 \), we have

\[
\mathbb{E}[|\xi(u_t) - \xi(v_t)|] \\
\leq \mathbb{E}[|\xi(u_t) - \xi(v_t)|1_{\{\max\{\sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1}\} \geq R\}}]
+ \mathbb{E}[|\xi(u_t) - \xi(v_t)|1_{\{\max\{\sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1}\} \leq R\}}1\{\tau_n \leq t\}]
+ \mathbb{E}[|\xi(u_t) - \xi(v_t)|1_{\{\max\{\sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1}\} \leq R\}}1\{\tau_n \leq t\}]
= T_1 + T_2 + \mathbb{E}[|\xi(u_t) - \xi(v_t)|1_{\{\max\{\sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1}\} \leq R\}}1\{\tau_n \geq t\}]. \tag{6.28} \]

Note that by the choice of \( R \), (6.10), we have \( T_1 \leq \epsilon/3 \). Similarly, by (6.27), \( T_2 \leq \epsilon/3 \). Thus for all \( v_0 \) such that \( \|u_0 - v_0\|_{H^1} \leq 1 \), we have

\[
\mathbb{E}[|\xi(u_t) - \xi(v_t)|] \\
\leq \frac{2\epsilon}{3} + \mathbb{E}[|\xi(u_t) - \xi(v_t)|1_{\{\max\{\sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1}\} \leq R\}}1\{\tau_n \geq t\}]. \tag{6.29} \]

In order to continue, we need the following lemma.

**Lemma 6.2.** Denote the event

\[ A = \left\{ \max\left\{ \sup_{s \in [0,t]} \|u_s\|_{H^1}, \sup_{s \in [0,t]} \|v_s\|_{H^1} \right\} \leq R \right\}. \tag{6.30} \]

Then for all \( k \in \mathbb{N}_0 \) we have the inequality

\[ \sup_{s \in [0,t]} \|u_t - v_t\|_{H^1} \leq (2\tilde{C}(t))^{k+1} \|u_0 - v_0\|_{H^1} \tag{6.31} \]

on the event \( A \cap \{\tau_k \geq t\} \).

**Proof of Lemma 6.2.** We start the induction with \( k = 0 \). Fix \( s \in [0,t] \). Then on the set \( \{\tau_0 > s\} \), we have

\[ \sup_{r \in [0,s]} \|u_r - v_r\|_{H^1} \leq 2\tilde{C}(t)\|u_0 - v_0\|_{H^1}. \tag{6.32} \]

Indeed, let

\[ L_0 = \tilde{C}(t)R + \|\pi\|_{X_1(s)} + \left\| \int_0^t U_\lambda(t - r)f \, dr \right\|_{X_1(s)}. \tag{6.33} \]
and let $\textbf{g} \in \mathbf{X}_1(s)$ be such that $\|\textbf{g}\|_{\mathbf{X}_1(s)} \leq 2L_0$. Then on the set $\{\tau_0 > s\}$

$$
\|\mathcal{T}_{u_0}(\textbf{g})\|_{\mathbf{X}_1(s)} \leq \tilde{C}(s)\|\textbf{u}_0\|_{H^1} + \left\| \int_0^\tau U_\lambda(s-r) g(r) \partial_x g(r) \, dr \right\|_{\mathbf{X}_1(s)} + \left\| \int_0^\tau U_\lambda(s-r) f \, dr \right\|_{\mathbf{X}_1(s)} + \|\mathbf{\Pi}\|_{\mathbf{X}_1(s)}
$$

$$
\leq \tilde{C}(t)R + \|\mathbf{\Pi}\|_{\mathbf{X}_1(s)} + \left\| \int_0^\tau U_\lambda(s-r) f \, dr \right\|_{\mathbf{X}_1(s)} + \tilde{C}(t) s^{1/2} \|\textbf{g}\|_{\mathbf{X}_1(s)}^2 \leq L_0 + 4L_0^2\tilde{C}(t)s^{1/2}
$$

$$
\leq L_0 + \frac{L_0}{2} \leq 2L_0
$$

(6.34)

where the last inequality holds due to the inclusion $\{\tau_0 > s\} \subseteq \{L_0 \leq 1/\tilde{C}(t)s^{1/2}\}$.

Note that $u$ is a fixed point of $\mathcal{T}_{u_0}$ and that $\mathcal{T}_{u_0}$ maps the ball of radius $2L_0$ of $\mathbf{X}_1(s)$ into itself. Then $\|u\|_{\mathbf{X}_1(s)} \leq 2L_0$. Similarly, $\|v\|_{\mathbf{X}_1(s)} \leq 2L_0$. Now observe that $u$ (resp. $v$) is a fixed point of $\mathcal{T}_{v_0}$ (resp. $\mathcal{T}_{v_0}$). Therefore, on the set $A \cap \{\tau_0 > s\}$,

$$
\|u - v\|_{\mathbf{X}_1(s)} = \|\mathcal{T}_{u_0}(u) - \mathcal{T}_{v_0}(v)\|_{\mathbf{X}_1(s)}
$$

$$
= \|U_\lambda(s)\|_{\mathbf{X}_1(s)} + \left\| \int_0^\tau U_\lambda(s-r) ((u_r - v_r) \partial_x u_r + v_r \partial_x (u_r - v_r)) \, dr \right\|_{\mathbf{X}_1(s)}
$$

$$
\leq \tilde{C}(t)\|u_0 - v_0\|_{H^1} + \tilde{C}(t)s^{1/2}\|u - v\|_{\mathbf{X}_1(s)}(\|u\|_{\mathbf{X}_1(s)} + \|v\|_{\mathbf{X}_1(s)})
$$

$$
\leq \tilde{C}(t)\|u_0 - v_0\|_{H^1} + 4\tilde{C}(t)s^{1/2}L_0\|u - v\|_{\mathbf{X}_1(s)}
$$

$$
\leq \tilde{C}(t)\|u_0 - v_0\|_{H^1} + \frac{1}{2}\|u - v\|_{\mathbf{X}_1(s)}
$$

(6.35)

which implies that on the set $\{\tau_0 > s\}$,

$$
\sup_{r \in [0,s]} \|u_r - v_r\|_{H^1} \leq \|u - v\|_{\mathbf{X}_1(s)} \leq 2\tilde{C}(t)\|u_0 - v_0\|_{H^1}.
$$

(6.36)

By the continuity in time of the processes, we have

$$
\sup_{r \in [0,\tau_0]} \|u_r - v_r\|_{H^1} \leq 2\tilde{C}(t)\|u_0 - v_0\|_{H^1}.
$$

(6.37)

We finish the proof by induction. Assume that for some $k \leq n-1$, we have on the set $A \cap \{\tau_k \geq t\}$

$$
\sup_{r \in [0,\tau_k]} \|u_r - v_r\|_{H^1} \leq (2\tilde{C}(t))^{k+1}\|u_0 - v_0\|_{H^1}.
$$

(6.38)

In order to obtain the heredity, we need to give the upper bound on $A \cap \{\tau_{k+1} \geq t \geq \tau_k\}$. Note that by the strong Markov property for all $s \geq \tau_k$

$$
u_s = U_\lambda(s - \tau_k)u_{\tau_k} - \int_{\tau_k}^s U_\lambda(s-r)u_r \partial_x u_r \, dr + \int_{\tau_k}^s U_\lambda(s-r)f \, dr + \mathbf{\Pi}^{\tau_k}_s
$$

(6.39)

and

$$
v_s = U_\lambda(s - \tau_k)v_{\tau_k} - \int_{\tau_k}^s U_\lambda(s-r)v_r \partial_x v_r \, dr + \int_{\tau_k}^s U_\lambda(s-r)f \, dr + \mathbf{\Pi}^{\tau_k}_s.
$$

(6.40)

On the set $A \cap \{\tau_{k+1} \geq t \geq \tau_k\}$, we have $\|u_{\tau_k}\|_{H^1} \leq R$ and $\|v_{\tau_k}\|_{H^1} \leq R$. Hence, we may define

$L_{k+1} = \tilde{C}(t)R + \|\mathbf{\Pi}^{\tau_k}\|_{\mathbf{X}_1(\tau_k,s)} + \left\| \int_{\tau_k}^s U_\lambda(s-r) f \, dr \right\|_{\mathbf{X}_1(\tau_k,s)}$ for $s \in [\tau_k,t]$,
Similarly, we can prove that on the set \( A \cap \{ s < \tau_{k+1} \} \) we have \( \| u \|_{X_1(\tau_k, s)} \leq 2L_{k+1} \) and \( \| u \|_{X_1(\tau_k, s)} \leq 2L_{k+1} \) and proceed as in the proof for \( k = 0 \) that

\[
\sup_{r \in [\tau_k, \tau_{k+1}]} \| u_r - v_r \|_{H^1} \leq (2\tilde{C}(t))^{k+2}\| u_0 - v_0 \|_{L^2}
\]

(6.41)

and the lemma is established.

We now continue the proof of the Feller property, starting from the inequality (6.29).

**Proof of Lemma 3.2.** Using the previous lemma, we have

\[
\| u_t - v_t \|_{H^1} \leq (2\tilde{C}(t))^{n+1}\| u_0 - v_0 \|_{H^1} \text{ on } A \cap \{ \tau_n \geq t \}.
\]

(6.42)

Therefore, as \( v_0 \to u_0 \) we have

\[
|\xi(u_t) - \xi(v_t)| \{ \max \{ \sup_{s \in [0,t]} \| u_s \|_{H^1}, \sup_{s \in [0,t]} \| v_s \|_{H^1} \} \leq R \} \{ \tau_n \geq t \} \to 0 \text{ a.s.}
\]

(6.43)

Using the dominated convergence theorem, we get

\[
E \left[ |\xi(u_t) - \xi(v_t)| \{ \max \{ \sup_{s \in [0,t]} \| u_s \|_{H^1}, \sup_{s \in [0,t]} \| v_s \|_{H^1} \} \leq R \} \{ \tau_n \geq t \} \right] \to 0
\]

(6.44)

as \( v_0 \to u_0 \) in \( H^1 \). Note that the choices of \( R \) and \( n \) do not depend on \( v_0 \).

\[ \square \]

7 Asymptotic compactness of the semi-group

We use the distributional convergence over various Sobolev spaces. In order to fix the ideas, we first recall the definition.

**Definition 7.1.** Let \( \Gamma \) be a topological vector space, and let \( \{ X_n \}_{n \geq 0} \) and \( X_\infty \) be random variables taking values in \( \Gamma \), possibly defined in different probability spaces. We say that \( X_n \) converges to \( X_\infty \) in distribution in \( \Gamma \) if

\[
E[F(X_n)] \to E[F(X_\infty)] \text{ as } n \to \infty
\]

(7.1)

for all bounded continuous functions \( F: \Gamma \to \mathbb{R} \).

The following statement provides asymptotic compactness of solutions.

**Lemma 7.2.** For any sequence of deterministic initial conditions \( u^n_0 \) satisfying

\[
R := \sup_n \left\{ \| u^n_0 \|_{H^1}^2 \right\} < \infty
\]

and a sequence of nonnegative numbers \( t_1, t_2, \ldots \) such that \( \lim_{n \to \infty} t_n = \infty \), the set of probabilities \( \{ P_n(u^n_0, \cdot) : n \in \mathbb{N} \} \) is tight in \( H^1 \).

The proof of the lemma depends on the Appendix.
Proof of Lemma 7.2. Without loss of generality, we may assume that \( t_1, t_2, t_3, \ldots \) is increasing. Let \( \{u^n_0\}_{n=1}^{\infty} \) be a sequence of initial conditions as above. We denote by \( \{u^n_0\}_{n=1}^{\infty} \) the respective solutions of (2.2). We intend to show that there is a subsequence of \( \{u^n_0\}_{n=1}^{\infty} \) that converges in distribution in \( H^1(\mathbb{R}) \).

We first show that we have a distributional tightness in \( L \) distribution in \( L \). This is sufficient to obtain that the distributional convergence holds in \( L \). We then use the evolution of the energy of the equation to show successively that the expectations of the \( L^2(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) norms of the elements of the subsequence converge respectively to the expectation of the \( L^2(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) norms of limiting distribution. By a theorem by Prokhorov, this is sufficient to obtain that the distributional convergence holds in \( H^1(\mathbb{R}) \).

We start with Lemma 5.1, to obtain the bound

\[
\sup_n \mathbb{E}[\|u^n(t_n)\|_{H^1}^2] \leq C(R). \tag{7.2}
\]

**Step 1: convergence in distribution in \( L^2_{\text{loc}}(\mathbb{R}) \)**

Bounded sets of \( H^1(\mathbb{R}) \) are relatively compact in \( L^2_{\text{loc}}(\mathbb{R}) \). Thus the inequality (7.2) and Prokhorov's theorem in \( L^2_{\text{loc}}(\mathbb{R}) \) allow us to conclude that there exists an \( L^2_{\text{loc}}(\mathbb{R}) \) valued random variable \( \xi \) (possibly defined on another probability space) and a subsequence of \( \{u^n_0\} \) such that

\[
u^n_0 \to \xi \text{ in distribution in } L^2_{\text{loc}}(\mathbb{R}) \text{ as } n \to \infty. \tag{7.3}
\]

Let \( \{f_i\}_{i=1}^{\infty} \) be an orthonormal basis of \( H^1(\mathbb{R}) \) with \( f_i \) smooth and compactly supported. For all \( i \in \mathbb{N} \) and \( M > 0 \) we define the mapping

\[
v \mapsto \psi_{i,M}(v) := |(v, f_i)_{H^1}| \wedge M = |(v, (1 - \partial_x^2)f_i)| \wedge M
\]

which is continuous and bounded in \( L^2_{\text{loc}}(\mathbb{R}) \). Therefore, the distributional \( L^2_{\text{loc}}(\mathbb{R}) \) convergence implies

\[
\mathbb{E} \left[ \sum_{i=1}^{N} ((u^n_i, f_i)_{H^1}^2 \wedge M^2) \right] \to \mathbb{E} \left[ \sum_{i=1}^{N} ((\xi, f_i)_{H^1}^2 \wedge M^2) \right],
\]

and thus for all \( N \in \mathbb{N} \) and \( M > 0 \)

\[
\mathbb{E} \left[ \sum_{i=1}^{N} ((\xi, f_i)_{H^1}^2 \wedge M^2) \right] \leq C(R). \tag{7.4}
\]

Sending \( M \) to infinity and using the Monotone Convergence Theorem, we obtain

\[
\mathbb{E} \left[ \sum_{i=1}^{N} (\xi, f_i)_{H^1}^2 \right] \leq C(R) \tag{7.5}
\]

and thus, by using the Monotone Convergence Theorem again,

\[
\mathbb{E}[\|\xi\|_{H^1}^2] = \lim_{N} \mathbb{E} \left[ \sum_{i=1}^{N} (\xi, f_i)_{H^1}^2 \right] \leq C(R). \tag{7.6}
\]

This shows that \( \xi \) is \( H^1(\mathbb{R}) \)-valued.

**Step 2: convergence in distribution in \( L^2 \).** In order to prove this, we use

\[
\lim_n \mathbb{E}[\|u^n_0\|_{L^2}^2] = \mathbb{E}[\|\xi\|_{L^2}^2]. \tag{7.7}
\]
The proof of this fact is given in the Appendix.

Recall from Section 2 that \( \{e_i\} \) denotes an orthonormal basis of \( L^2(\mathbb{R}) \) consisting of smooth compactly supported functions. For all \( N \in \mathbb{N} \) and \( M > 0 \), the convergence in distribution in \( L^2_{\text{loc}}(\mathbb{R}) \) of \( u_{tn}^n \) implies that

\[
E \left[ \sum_{i=1}^{N} (u_{tn}^n, e_i)^2 \wedge M^2 \right] \to E \left[ \sum_{i=1}^{N} (\xi, e_i)^2 \wedge M^2 \right].
\]

(7.8)

Using the inequality (4.1), we obtain that the family \( \|u_{tn}^n\|_{L^2}^2 \) is uniformly integrable. Thus we can send \( M \) to infinity and obtain

\[
E \left[ \sum_{i=1}^{N} (u_{tn}^n, e_i)^2 \right] \to E \left[ \sum_{i=1}^{N} (\xi, e_i)^2 \right].
\]

(7.9)

Therefore, combined with (7.7), we get

\[
E \left[ \infty \sum_{i=N+1}^{\infty} (u_{tn}^n, e_i)^2 \right] \to E \left[ \infty \sum_{i=N+1}^{\infty} (\xi, e_i)^2 \right].
\]

(7.10)

Now, fix \( \epsilon > 0 \). There exists \( N_0 \in \mathbb{N} \) such that

\[
E \left[ \infty \sum_{i=N_0+1}^{\infty} (\xi, e_i)^2 \right] \leq \frac{\epsilon}{2}.
\]

(7.11)

Then, using (7.10), there exists \( n_{\epsilon} \in \mathbb{N} \) such that

\[
\sup_{n \geq n_{\epsilon}} E \left[ \sum_{i=N_0+1}^{\infty} (u_{tn}^n, e_i)^2 \right] \leq \epsilon.
\]

(7.12)

By the uniform second moment bound (7.2) and since \( \{1, 2, \ldots, n_{\epsilon} - 1\} \) is finite, we have

\[
\lim_{N \to \infty} E \left[ \sum_{i=N+1}^{\infty} (u_{tn}^n, e_i)^2 \right] = 0, \quad n \leq n_{\epsilon} - 1.
\]

(7.13)

Using (7.12) and (7.13), there exists \( N_1 \geq N_0 \) such that

\[
\sup_{n \in \mathbb{N}} E \left[ \sum_{i=N_1+1}^{\infty} (u_{tn}^n, e_i)^2 \right] \leq \epsilon.
\]

(7.14)

Since \( \epsilon > 0 \) is arbitrary, we get

\[
\lim_{N \to \infty} \sup_n E \left[ \sum_{i=N+1}^{\infty} (u_{tn}^n, e_i)^2 \right] = 0
\]

(7.15)

which by [35, Theorem 1.13] implies tightness in distribution in \( L^2 \) of measures of \( \{u_{tn}^n\} \). Note that any limiting measure can only be the measure of \( \xi \). Thus

\[
u_{tn}^n \to \xi
\]

in distribution in \( L^2 \).

We emphasize that we have not taken any further subsequence to pass from (7.3) to (7.16). We have proven that any limit in distribution in \( L^2_{\text{loc}}(\mathbb{R}) \) of \( \{u_{tn}^n\} \) is also its limit in distribution in \( L^2(\mathbb{R}) \).
Step 3: Convergence in distribution in $H^1$

This follows from the important fact

$$\mathbb{E}[I(\xi)] = \lim_{n} \mathbb{E}[I(u^n_{s,n})]$$  \hspace{1cm} (7.17)

the proof of which is given in the Appendix. Note that we have the uniform bounds (7.2) and convergence in distribution in $L^2$. Using Agmon’s inequality, we obtain that the mapping $v \to \int v^3(x)dx$ is continuous in $L^2$ on bounded sets of $H^1$. Thus, using again the uniform integrability of the families $\|\partial_x u\|_{L^2}^2$ and $\|u\|_{L^2}^2$,

$$\mathbb{E} \left[ \int (u^n_{s,n})^3(x)dx \right] \to \mathbb{E} \left[ \int \xi^3(x)dx \right].$$  \hspace{1cm} (7.18)

Combined with (7.17) this implies

$$\lim_{n} \mathbb{E}[\|\partial_x u^n_{s,n}\|_{L^2}^2] = \mathbb{E}[\|\partial_x \xi\|_{L^2}^2].$$  \hspace{1cm} (7.19)

Note that the inequality (5.4) for $k = 2$ gives the uniform integrability of $\|u^n_{s,n}\|_{H^1}^2$. Repeating for the space $H^1$ the same ideas that allowed us to obtain the convergence in distribution in $L^2$ we obtain

$$u^n_{s,n} \to \xi$$  \hspace{1cm} (7.20)

in distribution in $H^1$. \hspace{1cm} \Box

Before proving Lemma 3.3, we establish the following statement.

**Lemma 7.3.** Let $K$ be a compact subset of $H^1(\mathbb{R})$. Then the set of measures on $H^1(\mathbb{R})$

$$\{P_s(v, \cdot) : s \in [0, 1], v \in K\}$$

is tight.

**Proof of Lemma 7.3.** We intend to take a countable subset $\{P_s(v, \cdot) : s \in [0, 1], v \in K\}$ and show that it has a convergent subsequence. Let $(s^n, v^n) \in [0, 1] \times K$. By compactness of the sets, there exists a subsequence of $(s^n, v^n)$ (still denoted $(s^n, v^n)$) that converges to $(s, v) \in [0, 1] \times K$. Denote by $u^n$ the solution of (1.1) with initial data $v^n$ and by $u$ the solution of (1.1) with initial data $v$.

We now prove that there exists a subsequence $(s^{n_k}, v^{n_k})$ such that

$$\mathbb{P}\text{-a.s.} \lim_k \left( \|u^{n_k}_s - u_s\|_{H^1} + \|u^{n_k}_s - u^*_s\|_{H^1} \right) \to 0 \text{ as } k \to \infty.$$  \hspace{1cm} (7.21)

The almost sure convergence $\|u^{n_k}_s - u^*_s\|_{H^1} \to 0$ is a direct consequence of $u \in C([0, 1]; H^1)$. Fix $\epsilon > 0$ and $\delta > 0$ and, similarly to the proof of the Feller property, denote $R_0 = \sup_{s \in K} \|v\|_{H^1} + 1$. We choose $R > 0$ (independent of $n$) such that

$$\mathbb{P} \left( \max \left\{ \sup_{s \in [0, t]} \|u_s\|_{H^1}^2, \sup_{s \in [0, t]} \|u^n(s)\|_{H^1}^2 \right\} \geq R \right) \leq \frac{1}{R} \mathbb{E} \left[ \sup_{s \in [0, t]} \|u_s\|_{H^1}^2 + \sup_{s \in [0, t]} \|u^n(s)\|_{H^1}^2 \right] \leq \frac{C(R_0)}{R} \leq \frac{\epsilon}{2}.$$
Given the choice of $R$, we define the hitting times $\tau_k$ as in (6.21), with $t = 1$. We choose $N$ such that
\[ \mathbb{P}(\tau_N \leq 1) \leq \frac{\epsilon}{2} \]  
(7.22)

We define the events
\[ A^n = \left\{ \max \left\{ \sup_{s \in [0,t]} \| u_s \|_{H^1}, \sup_{s \in [0,t]} \| u^n(s) \|_{H^1} \right\} \leq R \right\} . \]

Due to Lemma 6.2, on the set $A^n \cap \{ \tau_N \geq 1 \}$ we have
\[ \sup_{s \in [0,1]} \| u_t - u^n_t \|_{H^1} \leq (2\tilde{C}(1))^{N+1} \| v - v^n \|_{H^1}. \]

We choose $n$ sufficiently large so that to have $(2\tilde{C}(1))^{N+1} \| v - v^n \|_{H^1} \leq \delta$. This implies that for all $n$ sufficiently large
\[ \mathbb{P} \left( \sup_{s \in [0,1]} \| u_t - u^n(t) \|_{H^1} \leq \delta \right) \geq \mathbb{P} (A^n \cap \{ \tau_N \geq 1 \}) \geq 1 - \epsilon \]
which is exactly $\sup_{s \in [0,1]} \| u_t - u^n_t \|_{H^1} \to 0$ as $n \to \infty$ in probability. Hence, there exists a subsequence that converges almost surely. Let $\xi$ be a real valued uniformly continuous function on $H^1(\mathbb{R})$. Then
\[ |P_{s^n_k} \xi(v^{n_k}) - P_s \xi(v)| \leq \mathbb{E} [ |\xi(u^{n_k}_{s^n_k}) - \xi(u_s^{n_k})|] + \mathbb{E} [ |\xi(u^{n_k}_{s^n_k}) - \xi(u_s)|]. \]

The dominated convergence theorem, the convergence (7.21), and the uniform continuity of $\xi$ imply that the right hand side converges to 0. \hfill \Box

Proof of Lemma 3.3. Fix $\epsilon > 0$. The asymptotic compactness of the equation implies that the set of probabilities on $H^1(\mathbb{R})$
\[ \{ P_n(0, \cdot); n \geq 0 \} \]  
(7.23)
is tight. We choose a compact set $K_\epsilon \subseteq H^1(\mathbb{R})$ such that
\[ \sup_n P_n(0, K_\epsilon^c) \leq \frac{\epsilon}{2} \]  
(7.24)
Additionally, by Lemma 7.3, the set of probabilities on $H^1(\mathbb{R})$
\[ \{ P_s(v, \cdot); s \in [0,1], v \in K_\epsilon \} \]
is also tight. We pick another compact $A_\epsilon \subseteq H^1(\mathbb{R})$ such that
\[ \sup_{s \in [0,1], v \in K_\epsilon} P_s(v, A_\epsilon^c) \leq \frac{\epsilon}{2} \]  
(7.25)
By a direct computation
\[
\mu_n(A^c_\varepsilon) = \frac{1}{n} \int_0^n \mathbb{P}(u_t \in A^c_\varepsilon) \, dt \\
= \frac{1}{n} \sum_{i=0}^{n-1} \int_{i}^{i+1} \int_{H^1} P_t(0; dy) P_{t-i}(y; A^c_\varepsilon) \, dt \\
= \frac{1}{n} \sum_{i=0}^{n-1} \int_{i}^{i+1} \left\{ \int_{K_\varepsilon} P_t(0; dy) P_{t-i}(y; A^c_\varepsilon) + \int_{K_\varepsilon^c} P_t(0; dy) P_{t-i}(y; A^c_\varepsilon) \right\} \, dt \\
\leq \frac{1}{n} \sum_{i=0}^{n-1} \int_{i}^{i+1} \left\{ \frac{\varepsilon}{2} \int_{K_\varepsilon} P_t(0; dy) + P_t(0; K_\varepsilon^c) \right\} \, dt \leq \varepsilon. \tag{7.26}
\]

Thus \(\mu_n(A^c_\varepsilon) \leq \varepsilon\), and the proof is concluded. \(\square\)

Proof of Theorem 3.1. The statement follows immediately from Lemmas 3.2 and 3.3 using the Krylov-Bogoliubov procedure applied to the measures resulting from the deterministic initial datum 0 (cf. [9, 19] for instance). We omit further details. \(\square\)

A Appendix

In this section, let \(u^0_n\) be a sequence of deterministic initial conditions satisfying \(R = \sup_n \{\|u^0_n\|_{H^1}\} < \infty\), and let \(t_1, t_2, \ldots\) be an increasing sequence such that \(\lim_{n \to \infty} t_n = \infty\). Denoting by \(u^n_t\) solutions corresponding to \(u^0_n\), we intend to show that \(\lim_n \mathbb{E}[\|u^n_{t_n}\|_{L^2}] \to \mathbb{E}[\|\xi\|_{L^2}]\) and \(\mathbb{E}[I(\xi)] = \lim_n \mathbb{E}[I(u^n_{t_n})]\).

A.1 Proof of \(\lim_n \mathbb{E}[\|u^n_{t_n}\|_{L^2}^2] \to \mathbb{E}[\|\xi\|_{L^2}^2]\)

Note that the inequality \(\liminf_n \mathbb{E}[\|u^n_{t_n}\|_{L^2}^2] \geq \mathbb{E}[\|\xi\|_{L^2}^2]\) can be shown easily. In order to prove the reverse inequality, assume, contrary to the assertion, that there exists \(\varepsilon > 0\) and a subsequence of \(\{u^n_{t_n}\}\) (still indexed by \(n\)) such that for all \(n \geq 0\)
\[
\mathbb{E}[\|u^n_{t_n}\|_{L^2}^2] - \mathbb{E}[\|\xi\|_{L^2}^2] \geq \varepsilon. \tag{A.1}
\]

Fix \(T > 0\) such that \(3C(R)e^{-2\lambda T} \leq \varepsilon\) where \(C(R)\) is the constant in (7.2). Note that the sequence \(\{u^n_{t_n-T}\}\) satisfies the same assumptions as \(\{u^n_{t_n}\}\) and thus there exists a further subsequence (still indexed by \(n\)) and \(\xi-T\), an \(H^1\) valued random variable, such that we also have
\[
u^n_{t_n-T} \to \xi-T \tag{A.2}
\]
in distribution in \(L^2_{\text{loc}}(\mathbb{R})\).

We shall work on the space \(Z = C([0, T]; L^2_{\text{loc}}(\mathbb{R}))\). Denote by \(z\) the canonical process on this space and \(\mathcal{D}\) its right continuous filtration.
**Definition A.1.** A measure $\nu$ on $\mathcal{Z}$ is a solution of the equation (2.2) if for all $\phi$ smooth and compactly supported functions

$$M^\phi_t = (z_t - z_0, \phi) + \int_0^t (\partial_x^2 z_s + z_s \partial_x z_s + \lambda z_s - f, \phi) ds$$

(A.3)

and

$$(M^\phi_t)^2 - t \sum_i (\Phi \phi_i)^2$$

(A.4)

are $\nu$ local-martingales.

Define the sequence of measures

$$\nu^n(dz) = \int_{\Omega} \delta_{\{u^n_t - T_n, (\omega) \in [0, T]\}}(dz) \mathbb{P}(d\omega)$$

(A.5)

on $\mathcal{Z}$. We shall prove by the Aldous criterion ([1, Theorem 16.10]) that the sequence $\{\nu^n\}_{n=1}^{\infty}$ is tight in distribution in $\mathcal{Z}$. The first step is the following estimate.

**Lemma A.2.** We have $\mathbb{E}[\|u^n_{T_n+d_n} - u^n_{T_n}\|^2_{L^2}] \to 0$ for all stopping times $T_n$ and for all $d_n \to 0$.

**Proof of Lemma A.2.** Denote $A = (1 - \partial_x^2)$ and $U^n_s = A^{-1}(u^n_s - u^n_{T_n})$. Applying Ito’s lemma, we get

$$\|u^n_{T_n+d_n} - u^n_{T_n}\|^2_{L^2} = \int_{T_n}^{T_n+d_n} \left(-2(\partial_x^3 u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, u^n_s - u^n_{T_n}) + \|\Phi\|^2_{HS(L^2, L^2)}\right) ds$$

$$+ 2 \int_{T_n}^{T_n+d_n} (\Phi dW_s, u^n_s - u^n_{T_n})$$

$$= \int_{T_n}^{T_n+d_n} \left(-2(\partial_x^3 u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, u^n_s) + \|\Phi\|^2_{HS(L^2, L^2)}\right) ds$$

$$+ 2 \left(\int_{T_n}^{T_n+d_n} \Phi dW_s, u^n_{T_n}\right)$$

$$+ 2 \int_{T_n}^{T_n+d_n} (\Phi dW_s, u^n_s - u^n_{T_n}).$$

(A.6)

Now, we proceed to bound the terms on the far right side of the above equality. We first use $u^n_s \in H^3(\mathbb{R})$ and $(\partial_x^3 u^n_s + u^n_s \partial_x u^n_s, u^n_s) = 0$ to get

$$|\left(\partial_x^3 u^n_s + u^n_s \partial_x u^n_s + \lambda u^n_s - f, u^n_s\right)| = |\left(\lambda u^n_s - f, u^n_s\right)| \leq C(\|u^n_s\|^2_{L^2} + 1)$$

(A.7)

which is bounded in expectation. The difficult term is

$$\left|\left(\int_{T_n}^{T_n+d_n} \Phi dW_s, u^n_{T_n}\right)\right|$$

$$\leq \left(\left\|\int_{T_n}^{T_n+d_n} \Phi dW_s\right\|_{H^{-1}} + \left\|\int_{T_n}^{T_n+d_n} \Phi dW_s\right\|_{H^{-1}}\right) \left\|u^n_{T_n}\right\|_{H^1}.$$ (A.8)

This shows that in order to obtain an estimate in $L^2$ one only needs the estimate in $H^{-1}$. Note that the bound on

$$\mathbb{E}\left[\left\|\int_{T_n}^{T_n+d_n} \Phi dW_s\right\|_{H^{-1}}\right]$$

(A.9)
can be easily obtained by the Burkholder-Davis-Gundy inequality. In order to control \( \| u_{T_n+d_n}^n - u_{T_n}^n \|_{H^{-1}} \), we apply Ito’s lemma and obtain
\[
\| u_{T_n+d_n}^n - u_{T_n}^n \|_{H^{-1}}^2 = \int_{T_n}^{T_n+d_n} \left( -2(\partial_x^3 u_x^n + u_x^n \partial u_x^n + \lambda u_x^n - f, U_x^n) + \| A^{-1/2} \Phi \|_{HS(L^2, L^2)}^2 \right) ds \\
+ 2 \int_{T_n}^{T_n+d_n} (\Phi dW_x, U_x^n). 
\] (A.10)
First note that \( \mathbb{E} \left[ |(u_x^n \partial u_x^n + \lambda u_x^n - f, U_x^n)| \right] \) is bounded due to uniform \( L^2(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) bounds in (4.1) and (5.4). One may also control \( \mathbb{E} \left[ \int_{T_n}^{T_n+d_n} |(\Phi dW_x, U_x^n)| \right] \) by the Burkholder-Davis-Gundy-inequality. The main term is
\[
\mathbb{E} \left[ \int_{T_n}^{T_n+d_n} (\partial_x^3 u_x^n, U_x^n) ds \right] 
\] (A.11)
which we estimate as
\[
\mathbb{E} \left[ |(\partial_x^3 u_x^n, U_x^n)| \right] = \mathbb{E} \left[ |(A^{-1} \partial_x^3 u_x^n, u_x^n - u_{T_n+d_n}^n)| \right] \leq \mathbb{E} \left[ \| A^{-1} \partial_x^3 u_x^n \|_{L^2}^2 \right] + \mathbb{E} \left[ \| u_x^n - u_{T_n}^n \|_{L^2}^2 \right] \\
\leq \mathbb{E} \left[ \| u_x^n \|_{H^{-1}}^2 \right] + \mathbb{E} \left[ \| u_x^n - u_{T_n}^n \|_{L^2}^2 \right] \leq 4C(R). 
\] (A.12)
Finally, combining all the estimates, we obtain that \( \mathbb{E} \left[ \| u_{T_n+d_n}^n - u_{T_n}^n \|_{L^2}^2 \right] \leq C d_n \to 0 \) as \( n \to \infty \). \( \square \)

Lemma A.3. The family of measures \( \nu^n \) is tight over \( Z \) and any limiting measure \( \nu \) of this sequence is a solution of (2.2). Additionally, the distribution of \( z_0 \) (resp. \( z_T \)) under \( \nu \) is the same as the distribution of \( \xi_{-T} \) (resp. \( \xi \)).

Proof of Lemma A.3. The tightness follows directly from Lemma A.2 and the Aldous criterion [1, Theorem 16.10].

We first show the solution property of the limiting measure. Let \( \phi \) be a smooth compactly supported function, let \( 0 \leq s_1 \leq \cdots \leq s_k \leq s < t \), and assume that \( g: \mathbb{R}^k \to \mathbb{R} \) continuous and bounded. Since \( u^n \) is a solution under \( \nu^n \), we have
\[
\mathbb{E}^{\nu^n} \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_t^\phi \right] = \mathbb{E}^{\nu^n} \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_t^\phi \right] 
\] (A.13)
and
\[
\mathbb{E}^{\nu^n} \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) \left( (M_t^\phi)^2 - t \sum_i (\Phi e_i, \phi)^2 \right) \right] \\
= \mathbb{E}^{\nu^n} \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) \left( (M_t^\phi)^2 - s \sum_i (\Phi e_i, \phi)^2 \right) \right]. 
\] (A.14)
The mappings that we are integrating are continuous in \( z \) under the topology of \( Z \), but they are not bounded. However, the bound (7.2) allows us to truncate them, obtain a uniform estimate on the remainder, and pass to the limit in \( n \). We obtain
\[
\mathbb{E}^\nu \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_t^\phi \right] = \mathbb{E}^\nu \left[ g(M_{s_1}^\phi, \ldots, M_{s_k}^\phi) M_t^\phi \right] 
\] (A.15)
and
\[ \mathbb{E}^\nu \left[ g(M^\phi_{s_1}, \ldots, M^\phi_{s_k})((M^\phi_t)^2 - t \sum_i (\Phi e_i, \phi)^2) \right] = \mathbb{E}^\nu \left[ g(M^\phi_{s_1}, \ldots, M^\phi_{s_k})((M^\phi_s)^2 - s \sum_i (\Phi e_i, \phi)^2) \right]. \] (A.16)

Thus \( \nu \) is a solution of (2.2). We have already obtained the convergence in distribution in \( L^2_{loc}(\mathbb{R}) \). Now,
\[ u^n_{t_n} \to \xi \] (A.17)
and
\[ u^n_{t_n-T} \to \xi_T. \] (A.18)

The distribution of \( z_0 \) (resp. \( z_T \)) under \( \nu \) is the distribution of \( \xi_T \) (resp. \( \xi \)), which completes the proof of Lemma A.3.

We now finish the proof of (7.7). Under \( \nu \), given the quadratic variation of \( z_t - z_0 + \int_0^t (\partial_x^2 z_s + z_s \partial_x z_s + \lambda z_s - f) ds \), there exists a sequence of \( \nu \) Brownian motions \( \tilde{B}^i \) such that \( z_t - z_0 + \int_0^t \partial_x^2 z_s + z_s \partial_x z_s + \lambda z_s - f ds = \sum_i \Phi e_i \tilde{B}^i \).

Under \( \nu \), the process \( z \) is \( H^3(\mathbb{R}) \)-valued. Let \( z^k_0 \) be a sequence in \( H^3(\mathbb{R}) \) converging to \( z_0 \) in \( H^3(\mathbb{R}) \) and let \( z^k \) be the associated solutions of (2.2) in the probability space \((\mathcal{Z}, \mathcal{D}, \nu)\). By [10, Lemma 3.2], \( z^k \in H^3(\mathbb{R}) \) for all \( s \in [0, T] \). Applying Ito’s Lemma to \( z^k \), we get that the difference
\[ \| z_t^k \|_{L^2_t}^2 - \| z_0^k \|_{L^2_t}^2 + 2\lambda \int_0^t \| z^k_s \|_{L^2_t}^2 ds - \int_0^t (z^k_s, f) ds - \| \Phi \|_{H^2}^2 \] (A.19)
defines a martingale. Taking the expectation under \( \nu \) we find that
\[ \mathbb{E}^\nu[\| z_t^k \|_{L^2_t}^2] - e^{-2\lambda T} \mathbb{E}^\nu[\| z_0^k \|_{L^2_t}^2] = \int_0^T e^{-2\lambda (T-s)} (\mathbb{E}^\nu[(z^k_s, f)] + \| \Phi \|_{H^2}^2) ds. \] (A.20)

By the Feller property and the convergence of \( z^k_0 \), taking the limit as \( k \) goes to \( \infty \), we obtain that
\[ \mathbb{E}^\nu[\| z_T \|_{L^2_t}^2] - e^{-2\lambda T} \mathbb{E}^\nu[\| z_0 \|_{L^2_t}^2] = \int_0^T e^{-2\lambda (T-s)} (\mathbb{E}^\nu[(z_s, f)] + \| \Phi \|_{H^2}^2) ds. \] (A.21)

By an assumption, the mapping \( v \to (v, f) \) is continuous in \( L^2_{loc}(\mathbb{R}) \). Thus
\[ \mathbb{E}^\nu[\| \xi \|_{L^2_t}^2] - e^{-2\lambda T} \mathbb{E}^\nu[\| \xi_T \|_{L^2_t}^2] = \mathbb{E}^\nu[\| z_T \|_{L^2_t}^2] - e^{-2\lambda T} \mathbb{E}^\nu[\| z_0 \|_{L^2_t}^2] \]
\[ = \int_0^T e^{-2\lambda (T-s)} (\mathbb{E}^\nu[(z_s, f)] + \| \Phi \|_{H^2}^2) ds \]
\[ = \lim_n \int_0^T e^{-2\lambda (T-s)} (\mathbb{E}[(u^n_{t_n-T+s}, f)] + \| \Phi \|_{H^2}^2) ds \]
\[ = \lim_n \mathbb{E}[\| u^n_{t_n-T-s} \|_{L^2_t}^2 - e^{-2\lambda T} \mathbb{E}^\nu[\| u^n_{t_n-T} \|_{L^2_t}^2]]. \] (A.22)
Note that \(e^{-2\lambda T}\mathbb{E}[\|\xi-T\|_{L^2}^2] + e^{-2\lambda T}\mathbb{E}[\|u_{t_n}^n-T\|_{L^2}^2] \leq 2e^{-2\lambda T}C(R)\). Thus using the previous inequality and the choice of \(T\)

\[
\frac{2\epsilon}{3} \geq 2C(R)e^{-2\lambda T} \geq \lim \sup_n \mathbb{E}[\|u_{t_n}^n\|_{L^2}^2] - \mathbb{E}[\|\xi\|_{L^2}^2] \geq \epsilon
\]  

(A.23)

which is a contradiction, thus concluding the proof of (7.7).

**A.2 Proof of** \(\mathbb{E}[I(\xi)] = \lim_n \mathbb{E}[I(u_{t_n}^n)]\)

Recall that

\(u_{t_n}^n \to \xi\)  

(A.24)

in distribution in \(L^2\). We assume again that the convergence \(\mathbb{E}[I(\xi)] = \lim_n \mathbb{E}[I(u_{t_n}^n)]\) does not hold. As in the previous section, there exists a further subsequence (denoted similarly) and \(\epsilon > 0\) such that

\[\|\mathbb{E}[I(\xi)] - \mathbb{E}[I(u_{t_n}^n)]\| \geq \epsilon.\]  

(A.25)

Given the uniform estimates, one can prove that there exists a constant dependent on \(R\) such that

\[\sup_t \|\mathbb{E}[I(u_t)]\| + \|\mathbb{E}[\|u_t\|_{L^2}^2]\| \leq C.\]  

(A.26)

We fix \(T\) such that \(3Ce^{-2\lambda T} \leq \epsilon\).

Using the same notation and results in the first part of the appendix, we have that \(\nu^n \to \nu\) in distribution in \(\mathcal{Z}\).

Thus for all \(s \in [0, T]\), we have \(u_{t_n}^n - T \to z_s\) in distribution in \(L^2_{loc}(\mathbb{R})\). Note that to pass from convergence (7.3) to the convergence (7.16) we haven’t needed to pass to a subsequence. Thus one may show similarly that for all \(s \in [0, T]\) the convergence of \(u_{t_n}^n - T \to z_s\) is in fact in distribution in \(L^2(\mathbb{R})\). Additionally, the mapping \(v \to (\int v^3(x)dx, (\partial_x v, \partial_x f) - (v^2, f), \int v(x) \sum_i |\Phi e_i|^2 dx)\) is continuous in \(L^2(\mathbb{R})\) on bounded sets of \(H^1(\mathbb{R})\).

Therefore, as \(n\) goes to infinity,

\[
\int_0^T e^{-2\lambda(T-s)} \left( \mathbb{E} \left[ \frac{\lambda}{3} \int u^n(t_n - T + s, x)^3 dx + 2(\partial_x u^n(t_n - T + s) \partial_x f) - ((u^n(t_n - T + s))^2, f) \right] + \|\partial_x \Phi\|_{HS(L^2, L^2)}^2 - \sum_i \int \mathbb{E}[u^n(t_n - T + s, x)] |\Phi e_i|^2 dx \right) ds
\]

converges to

\[
\int_0^T e^{-2\lambda(T-s)} \left( \mathbb{E}^{-} \left[ \frac{\lambda}{3} \int z(s, x)^3 dx + 2(\partial_x z(s) \partial_x f) - (z^2(s), f) \right] + \|\partial_x \Phi\|_{HS(L^2, L^2)}^2 - \sum_i \int \mathbb{E}^{-}[z(s, x)] |\Phi e_i|^2 dx \right) ds.
\]

We have that \(\nu\) is a solution of (2.2) and similarly to the previous section we can approximate the process.
\{z_t\} by \{z^k_t\}, appeal to the Feller property in $H^1(\mathbb{R})$, and show that the equality

$$\mathbb{E}^\nu[I(z_T)] - e^{-2\lambda T}\mathbb{E}^\nu[I(z_0)]$$

$$= \int_0^T e^{-2\lambda(T-s)} \left( \mathbb{E}^\nu \left[ \frac{\lambda}{3} \int_{\mathbb{R}} z(z,x)^3 \, dx + 2(\partial_x z(s), \partial_x f) - (z^2(s), f) \right] \right) \left\| \partial_x \Phi \right\|^2_{HS(L^2,L^2)} - \sum_i \int \mathbb{E}^\nu[z(s,x)] |\Phi e_i|^2 \, dx \right) \, ds$$  \hspace{1cm} (A.27)

holds. This finally gives

$$\mathbb{E}[I(\xi)] - e^{-2\lambda T}\mathbb{E}^\nu[I(\xi-T)]$$

$$= \mathbb{E}^\nu[I(z_T)]) - e^{-2\lambda T}\mathbb{E}^\nu[I(z_0)] = \lim_n \mathbb{E}[I(u^n_{T_n})] - e^{-2\lambda T}\mathbb{E}^\nu[I(u^n_{T_n-T})].$$ \hspace{1cm} (A.28)

We finish similarly to the previous section. Namely,

$$\epsilon \leq \lim inf \| \mathbb{E}[I(u^n_{T_n})] - \mathbb{E}[I(\xi)]\| \leq e^{-2\lambda T} \lim inf \| \mathbb{E}[I(u^n_{T_n-T})] - \mathbb{E}[I(\xi-T)]\|$$

$$\leq e^{-2\lambda T} 2C \leq \frac{2\epsilon}{3}$$ \hspace{1cm} (A.29)

which gives the contradiction.

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### References


