CURVATURE INEQUALITIES FOR OPERATORS IN THE COWEN-DOUGLAS CLASS OF A PLANAR DOMAIN

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ABSTRACT. Fix a bounded planar domain $\Omega$. If an operator $T$, in the Cowen-Douglas class $B_1(\Omega)$, admits the compact set $\Omega$ as a spectral set, then the curvature inequality $\mathcal{K}_T(w) \leq -4\pi^2 S_\Omega(w, w)^2$, where $S_\Omega$ is the Szegő kernel of the domain $\Omega$, is evident. Except when $\Omega$ is simply connected, the existence of an operator for which $\mathcal{K}_T(w) = -4\pi^2 S_\Omega(w, w)^2$ for all $w$ in $\Omega$ is not known. However, one knows that if $w$ is a fixed but arbitrary point in $\Omega$, then there exists a bundle shift of rank 1, say $S$, depending on this $w$, such that $\mathcal{K}_{S^*}(w) = -4\pi^2 S_\Omega(w, w)^2$. We prove that these extremal operators are uniquely determined: If $T_1$ and $T_2$ are two operators in $B_1(\Omega)$ each of which is the adjoint of a rank 1 bundle shift and $\mathcal{K}_{T_1}(w) = -4\pi^2 S_\Omega(w, w)^2 = \mathcal{K}_{T_2}(w)$ for a fixed $w$ in $\Omega$, then $T_1$ and $T_2$ are unitarily equivalent. A surprising consequence is that the adjoints of only some of the bundle shifts of rank 1 occur as extremal operators in domains of connectivity $> 1$. These are described explicitly.

1. Introduction

Let $\Omega$ be a bounded, open and connected subset of the complex plane $\mathbb{C}$. Assume that $\partial \Omega$, the boundary of $\Omega$, consists of $n+1$ analytic Jordan curves. Let $\partial \Omega_1, \partial \Omega_2, \ldots, \partial \Omega_{n+1}$ denote the boundary components of $\Omega$. We shall always let $\partial \Omega_{n+1}$ denote the curve whose interior contains $\Omega$. Set $\Omega^* = \{ \bar{z} | z \in \Omega \}$, which is again a planar domain whose boundary consists of $n+1$ analytic Jordan curves. In this paper we study operators in $B_1(\Omega^*)$, first introduced by Cowen and Douglas in [6], namely, those bounded linear operators $T$ acting on a complex separable Hilbert space $\mathcal{H}$, for which $\Omega^* \subseteq \sigma(T)$ and which meet the following requirements.

1. $\text{ran}(T-w) = \mathcal{H}, \ w \in \Omega^*$,
2. $\bigvee_{w \in \Omega^*} \ker(T-w) = \mathcal{H}$ and
3. $\dim(\ker(T-w)) = 1, \ w \in \Omega^*$.

These conditions ensure that one may choose an eigenvector $\gamma_T(w)$ with eigenvalue $w$, for any operator $T$ in $B_1(\Omega^*)$, such that $w \rightarrow \gamma_T(w)$ is holomorphic on $\Omega^*$ (cf. [6, Proposition 1.11]). This is the holomorphic frame for the operator $T$. Cowen and Douglas also provide a model for the operators in the class $B_1(\Omega^*)$ which is easy to describe:

If $T \in B_1(\Omega^*)$ then $T$ is unitarily equivalent to the adjoint $M^*$ of the operator of multiplication $M$ by the coordinate function on some Hilbert space $\mathcal{H}_K$ consisting of holomorphic function on $\Omega$ possessing a reproducing kernel $K$. Throughout this paper, we let $M$ denote the operator of multiplication by the coordinate function and as usual $M^*$ denotes its adjoint.

The kernel $K$ is a complex valued function defined on $\Omega \times \Omega$, which is holomorphic in the first and anti-holomorphic in the second variable and is positive definite in the sense that $(K(z_i, z_j))$ is positive definite for every subset $\{z_1, \ldots, z_n\}$ of the domain $\Omega$. We will therefore assume, without loss of generality, that an operator $T$ in $B_1(\Omega^*)$ has been realized as the operator $M^*$ on some
reproducing kernel Hilbert space $\mathcal{H}_K$. The curvature $\mathcal{K}_T$ of the operator $T$ is defined as

$$\mathcal{K}_T(z) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log K_T(w, w)|_{w=z} = -\frac{\|K_z\|^2 \|\partial K_z\|^2 - \langle K_z, \partial K_z \rangle^2}{(K(z, z))^2}, \quad z \in \Omega,$$

where $K_z$ and $\partial K_z$ are the vectors

$$K_z(u) := K(u, z), \quad u \in \Omega,$$

$$\partial K_z(u) := \frac{\partial}{\partial w} K(u, w) \big|_{w=z}, \quad u \in \Omega,$$

in $\mathcal{H}_K$. Thus the curvature is a real analytic function on $\Omega^*$.

It turns out that this definition of curvature is independent of the representation of the operator $T$ as the adjoint $M^*$ of a multiplication operator $M$ on some reproducing kernel Hilbert space $\mathcal{H}_K$. Indeed, if $T$ also admits a representation as the adjoint of the multiplication operator on another reproducing kernel Hilbert space $\mathcal{H}_K$, then we must have $K(z, w) = \varphi(z) \tilde{K}(z, w)\tilde{\varphi}(w)$ for some holomorphic function $\varphi$ defined on $\Omega$ (cf. [6, Section 1.15]). This implies $\frac{\partial^2}{\partial w \partial \bar{w}} \log K_T(w, w)|_{w=z} = \frac{\partial^2}{\partial w \partial \bar{w}} \log \tilde{K}_T(w, w)|_{w=z}.$

If $T_1$ and $T_2$ are any two operators in $B_1(\Omega)$, then any intertwining unitary must map the holomorphic frame of one to the other. From this, it follows that the curvatures of these two operators must be equal, as shown in [6, Theorem 1.17] along with the non-trivial converse.

**Theorem 1.1.** Two operators $T_1$ and $T_2$ in $B_1(\Omega^*)$ are unitarily equivalent if and only if their associated curvature functions are equal, that is, $\mathcal{K}_{T_1}(w) = \mathcal{K}_{T_2}(w)$ for all $w \in \Omega^*$.

Recall that a compact subset $X \subseteq \mathbb{C}$ is said to be a spectral set for an operator $A$ in $\mathcal{L}(\mathcal{H})$ if

$$\sigma(A) \subseteq X \quad \text{and} \quad \sup \{\|r(A)\| \mid r \in \text{Rat}(X) \quad \text{and} \quad \|r\|_{\infty} \leq 1\} \leq 1,$$

where $\text{Rat}(X)$ denotes the algebra of rational functions whose poles are off $X$ and $\|r\|_{\infty}$ denotes the sup norm over the compact subset $X$. Equivalently, $X$ is a spectral set for the operator $A$ if the homomorphism $\rho_A : \text{Rat}(X) \to \mathcal{L}(\mathcal{H})$ defined by the formula $\rho_A(r) = r(A)$ is contractive. There are plenty of examples where the spectrum of an operator is a spectral set, for instance, this is the case for subnormal operators (cf. [9, Chapter 21]).

Now assume $\overline{\Omega}^*$, the closure of $\Omega^*$, is a spectral set for the operator $T$ in $B_1(\Omega^*)$. The space $\ker(T - w)^2 = \text{span}\{K_w, \partial K_w\}$ is an invariant subspace for $T$. Representing the restriction of the operator $T$ to this subspace with respect to an orthonormal basis as a $2 \times 2$ matrix, we have

$$T|_{\ker(T-w)^2} = \begin{pmatrix} w & 1 \\ 0 & \sqrt{-\mathcal{K}_T(w)} \end{pmatrix}.$$

It follows that $\overline{\Omega}^*$ is also a spectral set for $T|_{\ker(T-w)^2}$. For any $r$ in $\text{Rat}(\overline{\Omega}^*)$, it is not hard to verify that

$$r(T|_{\ker(T-w)^2}) = \begin{pmatrix} r(w) & r'(w) \\ 0 & \sqrt{-\mathcal{K}_T(w)} \end{pmatrix}.$$

Since

$$\sup \{|r'(w)| \mid \|r\|_{\infty} \leq 1, \quad r \in \text{Rat}(\overline{\Omega}^*)\} = 2\pi(S_{\overline{\Omega}^*}(w, w)), \quad w \in \Omega^*,$$

where $S_{\overline{\Omega}^*}(z, w)$, the Szegő kernel of $\Omega^*$, is the reproducing kernel for the Hardy space $(H^2(\Omega^*), ds)$, a curvature inequality becomes evident (cf. [12, Corollary 1.2]), that is,

$$\mathcal{K}_T(w) \leq -4\pi^2(S_{\overline{\Omega}^*}(w, w))^2, \quad w \in \Omega^*. \quad (1.1)$$
Equivalently, since $S_{\Omega}(z, w) = S_{\Omega^*}(\bar{w}, \bar{z})$, the curvature inequality takes the form

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K_T(w, w) \geq 4\pi^2(S_{\Omega}(w, w))^2, \quad w \in \Omega. \tag{1.2}$$

The operator $M^*$ on the Hardy space $(H^2(\mathbb{D}), ds)$ is in $B_1(\mathbb{D})$. The closed unit disc is a spectral set for the operator $M^*$. The reproducing kernel of the Hardy space, as is well-known, is the Szegő kernel $S_D$ of the unit disc $\mathbb{D}$. It is given by the formula $S_D(z, w) = \frac{1}{2\pi(1-z\bar{w})}$, for all $z, w$ in $\mathbb{D}$. The computation of the curvature of the operator $M^*$ is straightforward and is given by the formula

$$-\mathcal{K}_{M^*}(w) = \frac{\partial^2}{\partial w \partial \bar{w}} \log S_D(w, w) = 4\pi^2(S_D(w, w))^2, \quad w \in \mathbb{D}.$$ 

Since the closed unit disc is a spectral set for any contraction, it follows that the curvature of the operator $M^*$ on the Hardy space $(H^2(\mathbb{D}), ds)$ dominates the curvature of every other contraction in $B_1(\mathbb{D})$.

If the region $\Omega$ is simply connected, then, using the Riemann map and the transformation rule for the Szegő kernel (cf. [3, Theorem 12.3]) together with the chain rule for composition, we see that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\Omega}(w, w) = 4\pi^2(S_{\Omega}(w, w))^2, \quad w \in \Omega. \tag{1.3}$$

This shows that in the case of a bounded simply connected domain with Jordan analytic boundary, the operator $M^*$ on $(H^2(\Omega), ds)$ is an extremal operator.

On the other hand, if the region is not simply connected, then (1.3) fails. Indeed, Suita (cf. [16]) has shown that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\Omega}(w, w) > 4\pi^2(S_{\Omega}(w, w))^2, \quad w \in \Omega. \tag{1.4}$$

Equivalently,

$$\mathcal{K}_{M^*}(z) < -4\pi^2(S_{\Omega^*}(z, z))^2, \quad z \in \Omega^*, \tag{1.5}$$

where $M^*$ is the adjoint of the multiplication by the coordinate function on the Hardy space $(H^2(\Omega), ds)$. We therefore conclude that if $\Omega$ is not simply connected, then the operator $M^*$ fails to be extremal.

We don’t know if there exists an operator $T$ in $B_1(\Omega^*)$ admitting $\overline{\Omega^*}$ as a spectral set for which

$$\mathcal{K}_T(w) = -4\pi^2(S_{\Omega^*}(w, w))^2, \quad w \in \Omega^*. \quad \text{The question of equality at just one fixed but arbitrary point} \ \zeta \ \text{in} \ \Omega^* \ \text{was answered in [12, Theorem 2.1]. An operator} \ T \ \text{in} \ B_1(\Omega^*), \ \text{which admits} \ \overline{\Omega^*} \ \text{as a spectral set would be called extremal at} \ \zeta \ \text{if} \ \mathcal{K}_T(\zeta) = -4\pi^2(S_{\Omega^*}(\zeta, \zeta))^2. \tag{1.6}$$

Equivalently, representing the extremal operator $T$ as the operator $M^*$ on a Hilbert space possessing a reproducing kernel $K_T : \Omega \times \Omega \to \mathbb{C}$, we have that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K_T(w, w)|_{w=\zeta} = 4\pi^2S_{\Omega}(\zeta, \zeta)^2.$$ 

Since the operator $M^*$ on the Hardy space $(H^2(\mathbb{D}), ds)$ is extremal, we have $\mathcal{K}_T(w) \leq \mathcal{K}_{M^*}(w)$, for all $w \in \mathbb{D}$ and for every $T$ in the class of contractions in $B_1(\mathbb{D})$. Since the curvature is a complete unitary invariant, one may ask:

**Question 1.2** (R. G. Douglas). For a contraction $T$ in $B_1(\mathbb{D})$, if $\mathcal{K}_T(w_0) = \mathcal{K}_{M^*}(w_0)$ for some fixed $w_0$ in $\mathbb{D}$, then does it follow that $T$ must be unitarily equivalent to $M^*$?
This question has an affirmative answer if, for instance, $T$ is a homogeneous operator and it is easy to construct examples where the answer is negative (cf. [11]).

The operator $M$ on the Hardy space $(H^2(\mathbb{D}), ds)$ is a pure subnormal operator with the property that the spectrum of the minimal normal extension, designated the normal spectrum, is contained in the boundary of the spectrum of the operator $M$. These properties determine the operator $M$ uniquely up to unitary equivalence. The question of characterizing all pure subnormal operators with spectrum $\overline{\Omega}$ and normal spectrum contained in the boundary of $\Omega$ is more challenging if $\Omega$ is not simply connected. The deep results of Abrahamse and Douglas (cf. [2, Theorem 11]) show that these are exactly the bundle shifts. Moreover, they are in one to one correspondence with the equivalence classes of flat unitary bundles on the domain $\Omega$. Adjoint of a bundle shift of rank 1 lies in $B_1(\Omega^*)$. Since bundle shifts are subnormal with spectra equal to $\overline{\Omega}$, it follows that $\overline{\Omega}^*$ is a spectral set for the adjoint of the bundle shift. In fact, the extremal operator at $\overline{\zeta}$, found in [2], is the adjoint of a bundle shift of rank 1. Therefore, one may ask, following R. G. Douglas, if the curvature $\mathcal{K}_T(\zeta)$ of an operator $T$ in $B_1(\Omega^*)$, admitting $\overline{\Omega}^*$ as a spectral set, equals $-4\pi^2S_{\Omega}(\zeta, \zeta)^2$, whether it follows that $T$ is necessarily unitarily equivalent to the extremal operator at $\overline{\zeta}$ found in [2]. In this paper, we show that an extremal operator must be uniquely determined within $\{[T^*] : T$ is a bundle shift of multiplicity 1 over $\Omega\}$, where $[\cdot]$ denotes the unitary equivalence class.

2. Preliminaries on bundle shifts of rank one

Let $\alpha$ be a character of the fundamental group $\pi_1(\Omega)$, that is, $\alpha$ is an element of Hom$(\pi_1(\Omega), \mathbb{T})$. Each of these characters gives rise to a flat unitary bundle $E_\alpha$ of rank 1 on $\Omega$ (cf. [4, Proposition 2.5]). There is a one to one correspondence (cf. [8, p. 186]) between Hom$(\pi_1(\Omega), \mathbb{T})$ and the set of equivalence classes of flat unitary vector bundle over $\Omega$ of rank 1.

**Theorem 2.1.** Two rank one flat unitary vector bundles $E_\alpha$ and $E_\beta$ are equivalent as flat unitary vector bundles if and only if their inducing characters are equal, that is, $\alpha = \beta$.

Let $\{U_i, \phi_i\}_{i \in I}$ be a trivialization of the flat unitary vector bundle $E_\alpha$. If $f$ is a holomorphic section of the bundle $E_\alpha$, then we have $|((\phi_i^*)^{-1}(f(z)))| = |((\phi_j^*)^{-1}(f(z)))|$ for $z \in U_i \cap U_j$. Thus the function $h_f(z) := |(\phi_i^*)^{-1}(f(z))|$, $z \in U_i$, is well defined on all of $\Omega$ and is subharmonic there. Let $H^2_{E_\alpha}$ be the linear space of those holomorphic sections $f$ of $E_\alpha$ such that the subharmonic function $(h_f)^2$ on $\Omega$ is majorized by a harmonic function on $\Omega$. There is no natural inner product on the space $H^2_{E_\alpha}$. However, Abrahamse and Douglas (cf. [2]) define an inner product on it relative to the harmonic measure with respect to a fixed but arbitrary point $p \in \Omega$. We make the comment in [2, p. 118] explicit in what follows. Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be a regular exhaustion of $\Omega$, that is, it is a sequence of increasing subdomains $\Omega_k$ of $\Omega$ satisfying

(a) $\overline{\Omega}_k \subset \Omega_{k+1}$,
(b) $\cup_k \Omega_k = \Omega$,
(c) $p \in \Omega_1$
(d) The boundary of each $\Omega_k$ consists of finitely many smooth Jordan curves.

The norm of the section $f$ in $H^2_{E_\alpha}(\Omega)$ is then defined by the limit

$$\|f\|^2 = \lim_{k \to \infty} \frac{1}{2\pi} \int_{\partial \Omega_k} (h_f(z))^2 \frac{\partial}{\partial \eta}(g_k(z, p)) ds(z),$$

where $g_k(z, \zeta)$ denotes the Green’s function for the domain $\Omega_k$ at the point $p$ and $\frac{\partial}{\partial \eta}$ denotes the directional derivative along the outward normal direction with respect to the positively oriented boundary of $\Omega_k$. The linear space $H^2_{E_\alpha}$ is complete with respect to this norm making it into a Hilbert space. A bundle shift $T_{E_\alpha}$ is simply the operator of multiplication by the coordinate function on $H^2_{E_\alpha}$. 
Theorem 2.2 (Abrahamse and Douglas). Let $E_\alpha$ and $E_\beta$ be rank one flat unitary vector bundles induced by the homomorphisms $\alpha$ and $\beta$ respectively. Then the bundle shift $T_{E_\alpha}$ is unitarily equivalent to the bundle shift $T_{E_\beta}$ if and only if $E_\alpha$ and $E_\beta$ are equivalent as flat unitary vector bundles.

An operator $T$ in $\mathcal{L}(\mathcal{H})$ is said to be rationally cyclic if there exists a vector $v_0$ in $\mathcal{H}$ such that $\{r(T)(v_0) \mid r \in \text{Rat}(\Omega)\}$ is dense in $\mathcal{H}$. It is not very hard to verify that $T_{E_\alpha}$ is a pure, rationally cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum $\partial\Omega$. In fact, these are the characterizing properties of a rank one bundle shift.

Theorem 2.3 (Abrahamse and Douglas). Every pure, rationally cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum contained in $\partial\Omega$ is unitarily equivalent to a bundle shift $T_{E_\alpha}$ for some character $\alpha$.

Bundle shifts can also be realized as multiplication operators on certain subspaces of the classical Hardy space $H^2(\mathbb{D})$. Let $\pi : \mathbb{D} \rightarrow \Omega$ be a holomorphic covering map satisfying $\pi(0) = p$. Let $G$ denote the group of deck transformations associated to the map $\pi$, that is,

$$G = \{ A \in \text{Aut}(\mathbb{D}) \mid \pi \circ A = \pi \}.$$

Since $G$ is isomorphic to the fundamental group $\pi_1(\Omega)$ of $\Omega$, it follows that every character $\alpha$ determines a unique element $\hat{\alpha}$ in $\text{Hom}(G, \mathbb{T})$. By a slight abuse of notation, we will not distinguish $\hat{\alpha}$ and $\alpha$. A holomorphic function $f$ on the unit disc $\mathbb{D}$ satisfying $f \circ A = \alpha(A)f$ for all $A \in G$ is called a modulus automorphic function of index $\alpha$. Let

$$H^2(\mathbb{D}, \alpha) = \{ f \in H^2(\mathbb{D}) \mid f \circ A = \alpha(A)f, \text{ for all } A \in G \} \subseteq H^2(\mathbb{D}).$$

and $T_\alpha$ be the multiplication operator by the covering map $\pi$ on $H^2(\mathbb{D}, \alpha)$. Abrahamse and Douglas have shown, in [2, Theorem 5], that the operator $T_\alpha$ is unitarily equivalent to the bundle shift $T_{E_\alpha}$.

There is yet another realization of the bundle shift. Let $h$ be a multi-valued function on $\Omega$ with the property that $|h|$ is single valued on $\Omega$. For each $w \in \Omega$, suppose that there exists a neighborhood $U_w$ and a single valued holomorphic function $g_w$ on $U_w$ with the property $|g_w| = |h|$ on $U_w$. Then $h$ is said to be a multiplicative function. Every modulus automorphic function $f$ on $\mathbb{D}$ induces a multiplicative function on $\Omega$, namely, $f \circ \pi^{-1}$ and conversely (cf. [18, Lemma 3.6]). Let $H^2_\alpha(\Omega)$ be the linear space

$$H^2_\alpha(\Omega) := \{ f \circ \pi^{-1} \mid f \in H^2(\mathbb{D}, \alpha) \}.$$

The space $H^2_\alpha(\Omega)$ consists of those multiplicative functions $h$ on $\Omega$ such that $|h|^2$ admits a harmonic majorant on $\Omega$ (cf. [7, p.101]). Since the covering map $\pi$ lifts the harmonic measure $d\omega_p$ on $\partial\Omega$ at the point $\pi(0) = p$ to the linear Lebesgue measure on the unit circle $\mathbb{T}$, it follows that $H^2_\alpha(\Omega)$, endowed with the norm

$$\|f\|^2_{d\omega_p} = \int_{\partial\Omega} |f(z)|^2 d\omega_p(z),$$

becomes a Hilbert space (cf. [7, p. 101]). To emphasize the dependence on the measure, we denote this space by $(H^2_\alpha(\Omega), d\omega_p)$. In fact, the map $f \mapsto f \circ \pi^{-1}$ is an unitary map from $H^2(\mathbb{D}, \alpha)$ onto $(H^2_\alpha(\Omega), d\omega_p)$ which intertwines the multiplication by $\pi$ on $H^2(\mathbb{D}, \alpha)$ and $M$, the multiplication by the coordinate function on $(H^2_\alpha(\Omega), d\omega_p)$.

We have described three distinct but unitarily equivalent realizations of a bundle shift of rank 1 over the domain $\Omega$. We prefer to work with the third realization. It is well known that the harmonic measure $d\omega_p$ on $\partial\Omega$ at the point $p$ is boundedly mutually absolutely continuous with
respect to the arc length measure $ds$ on $\partial \Omega$. In fact, we have

$$d\omega_p(z) = -\frac{1}{2\pi} \frac{\partial}{\partial \eta_z}(g(z,p)) ds(z), \ z \in \partial \Omega,$$

where $g(z, \zeta)$ denotes the Green’s function for the domain $\Omega$ at the point $p$ and $\frac{\partial}{\partial \eta_z}$ denote the directional derivative along the outward normal direction (with respect to the positively oriented $\partial \Omega$). In this paper, instead of working with the harmonic measure $d\omega_p$ on $\partial \Omega$, we will work with the arc length measure $ds$ on $\partial \Omega$. This is the approach in Sarason [15]. So, we define the norm of a function $f$ in $H^2_\alpha(\Omega)$ by

$$\|f\|_{ds}^2 = \int_{\partial \Omega} |f(z)|^2 ds.$$

Since the outward normal derivative of the Green’s function is negative on the boundary $\partial \Omega$, we have

$$d\omega_p(z) = h^2(z) ds(z), \ z \in \partial \Omega,$$

where $h(z)$ is a positive continuous function on $\partial \Omega$. We also see that

$$c_2 \|f\|_{ds}^2 \leq \|f\|_{d\omega_p}^2 \leq c_1 \|f\|_{ds}^2,$$

where $c_1$ and $c_2$ are the supremum and the infimum of the function $h$ on $\partial \Omega$.

Hence it is clear that $\| \cdot \|_{ds}$ defines an equivalent norm on $H^2_\alpha(\Omega)$. We let $(H^2_\alpha(\Omega), ds)$ be the Hilbert space which is the same as $H^2_\alpha(\Omega)$ as a linear space but is given the new norm $\| \cdot \|_{ds}$. In fact, the identity map from $(H^2_\alpha(\Omega), d\omega_p)$ onto $(H^2_\alpha(\Omega), ds)$ is invertible and intertwines the corresponding multiplication operators by the coordinate functions. It is easily verified that the multiplication operator by the coordinate function on $(H^2_\alpha(\Omega), ds)$ is also a pure, rationally cyclic subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum contained in $\partial \Omega$. By a slight abuse of notation we will denote the multiplication operator by the coordinate function on $(H^2_\alpha(\Omega), ds)$ also by $T_\alpha$.

Using the characterization of all pure, rationally cyclic subnormal operators with spectrum equal to $\overline{\Omega}$ and normal spectrum contained in $\partial \Omega$ given by Abrahamse and Douglas, we conclude that for every character $\beta$, the operator $T_\beta$ on $(H^2_\beta(\Omega), ds)$ is unitarily equivalent to $T_\alpha$ on $(H^2_\alpha(\Omega), d\omega_p)$ for some $\alpha$. In the following section we will establish a bijective correspondence (which respects the unitary equivalence class) between these two kinds of bundle shifts. The following Lemma helps in establishing this bijection.

**Lemma 2.4.** If $v$ is a positive continuous function on $\partial \Omega$, then there exists a character $\gamma$ and a function $F$ in $H^\infty_\gamma(\Omega)$ such that $|F|^2 = v$ almost everywhere (with respect to the arc length measure) on $\partial \Omega$. Moreover, there exists $G$ in $H^\infty_{\gamma^{-1}}(\Omega)$ such that $FG = 1$ on $\Omega$ making $F$ invertible.

**Proof.** Since $v$ is a positive continuous function on $\partial \Omega$, it follows that $\log v$ is continuous on $\partial \Omega$. The boundary $\partial \Omega$ of $\Omega$ consists of analytic Jordan curves, therefore the Dirichlet problem is solvable with continuous boundary data. Now, solving the Dirichlet problem with boundary value $\frac{1}{2} \log v$, we get a harmonic function $u$ on $\Omega$ with continuous boundary value $\frac{1}{2} \log v$. Let $u^*$ be the multiple value conjugate harmonic function of $u$. Let us denote the period of the multiple valued conjugate harmonic function $u^*$ around the boundary component $\partial \Omega_j$ by

$$c_j = -\int_{\partial \Omega_j} \frac{\partial}{\partial \eta_z}(u(z)) ds_z, \text{ for } j = 1, 2, \ldots, n$$

In this equation, a negative sign appears since we have assumed that $\partial \Omega$ is positively oriented, hence the different components of the boundary $\partial \Omega_j$, $j = 1, 2, \ldots, n$, except the outer one are
oriented in the clockwise direction. Now consider the function $F(z)$ defined by

$$F(z) = \exp(u(z) + i u^*(z))$$

Observe that $F$ is a multiplicative holomorphic function on $\Omega$. Hence, following [18, Lemma 3.6], there exists a modulus automorphic function $f$ on the unit disc $\mathbb{D}$ so that $F = f \circ \pi^{-1}$. We find the index of the modulus automorphy for the function $f$ in the following way. Around each boundary component $\partial \Omega_j$ in the anticlockwise direction the value of $F$ gets changed to $\exp(i c_j)$ times its initial value. The index of $f$ is determined by the $n$-tuple of numbers $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ given by

$$\gamma_j = \exp(ic_j), \quad j = 1, 2, \ldots, n.$$ 

For each such $n$-tuple of numbers, there exists a homomorphism $\gamma : \pi_1(\Omega) \to \mathbb{T}$ such that the $n$-tuple of numbers occur as the image of the $n$ generators of the group $\pi_1(\Omega)$ under the map $\gamma$. Also we have $|F(z)|^2 = \exp(2u(z)) = \nu(z)$, $z \in \partial \Omega$. Since $u$ is continuous on $\Omega$, it follows that $|F(z)|$ is bounded on $\Omega$. Hence $F$ belongs to $H^\infty(\Omega)$ with $|F|^2 = \nu$ on $\partial \Omega$.

The function $\frac{1}{\nu}$ is also positive and continuous on $\partial \Omega$. So, as before, there exists a character $\delta$ and a function $G$ in $H^\infty(\Omega)$ with $|G|^2 = \frac{1}{\nu}$ on $\partial \Omega$. Since $\log \frac{1}{\nu} = -\log \nu$, it is easy to verify that the index of $G$ is exactly $(\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \ldots, \frac{1}{\gamma_n})$ and hence $\delta$ is equal to $\gamma^{-1}$. Evidently $FG = 1$ on $\Omega$. $\square$

Now we establish a bijective correspondence between bundle shifts defined using the harmonic measure and the ones defined using the arc length measure. From (2.1), we know that the harmonic measure $d\omega_p$ is of the form $h^2 ds$ for some positive continuous function on $\partial \Omega$. Combining this with the preceding Lemma, we see that there is $F$ in $H^\infty(\Omega)$ with $|F|^2 = h^2$ on $\partial \Omega$ and $G$ in $H^\infty(\Omega)$ with $|G|^2 = h^{-2}$ on $\partial \Omega$. Let $M_F : (H^2_\alpha(\Omega), d\omega_p(z)) \to (H^2_{\alpha \gamma}(\Omega), ds)$ be the map defined by

$$M_F(g) = Fg, \quad g \in (H^2_\alpha(\Omega), d\omega_p(z)).$$

Clearly, $M_F$ is an unitary operator and its inverse is the operator $M_G$. The multiplication operator $M_F$ intertwines the operator of multiplication by the coordinate function on the Hilbert spaces $(H^2_\alpha(\Omega), d\omega_p(z))$ and $(H^2_{\alpha \gamma}(\Omega), ds)$, establishing a bijective correspondence of the unitary equivalence classes of bundle shifts. As a consequence, we have the following theorem, first proved by Abrahamse and Douglas (cf. [2, Theorems 5 and 6]) using the harmonic measure $d\omega_p$ instead of the arc length measure $ds$.

**Theorem 2.5.** The bundle shift $T_\alpha$ on $(H^2_\alpha(\Omega), ds)$ is unitarily equivalent to the bundle shift $T_\beta$ on $(H^2_\beta(\Omega), ds)$ if and only if $\alpha = \beta$.

For any character $\alpha$, the adjoint of the rank 1 bundle shift $T_\alpha$ lies in $B_1(\Omega^*)$, see [2, Theorem 3]. Since the bundle shift $T_\alpha$ is subnormal, it follows that the adjoint of the bundle shift $T_\alpha$ admits $\Omega^*$ as a spectral set. Consequently, we have an inequality for the curvature of the bundle shifts, namely,

$$\mathcal{K}_{T_\alpha^*}(w) \leq -4\pi^2 (S_{\Omega^*}(w, w))^2, \quad w \in \Omega^*.$$ 

Given any fixed but arbitrary point $\zeta$ in $\Omega$, in the following section, we recall the proof (slightly different from the original proof given in [12]) of the existence of a bundle shift $T_\alpha$ for which equality occurs at $\zeta$ in the curvature inequality. However, the main theorem of this paper is the “uniqueness” of such an operator. We defer the proof until section 3.4.

**Theorem 2.6 (Uniqueness).** If the bundle shift $T_\alpha$ on $(H^2_\alpha(\Omega), ds)$ and the bundle shift $T_\beta$ on $(H^2_\beta(\Omega), ds)$ are extremal at the point $\zeta$, that is, if they satisfy

$$\mathcal{K}_{T_\alpha^*}(\zeta) = -4\pi^2 (S_{\Omega^*}(\zeta, \zeta))^2 = \mathcal{K}_{T_\beta^*}(\zeta)$$

then the bundle shifts $T_\alpha$ and $T_\beta$ are unitarily equivalent, which is the same as $\alpha = \beta$. 

Each \(f\) in \((H^2(\Omega), d\omega_p)\) has a non tangential boundary value almost everywhere. In the usual way \((H^2(\Omega), d\omega_p)\) is identified with a closed subspace of \(L^2(\partial\Omega, d\omega_p)\) (cf. [14, Theorem 3.2]). Let \(\lambda\) be a positive continuous function on \(\partial\Omega\). As the measure \(\lambda ds\) and the harmonic measure \(d\omega_p\) on \(\partial\Omega\) are boundedly mutually absolutely continuous one can define an equivalent norm on \(H^2(\Omega)\) in the following way

\[
\|f\|_{\lambda ds}^2 = \int_{\partial\Omega} |f(z)|^2 \lambda(z) ds(z).
\]

Let \((H^2(\Omega), \lambda ds)\) denote the linear space \(H^2(\Omega)\) endowed with the norm \(\|\cdot\|_{\lambda ds}\). Since the harmonic measure \(d\omega_p\) is boundedly mutually absolutely continuous with respect to the arc length measure \(ds\) and \(\lambda\) is a positive continuous function on \(\partial\Omega\), it follows that the identity map \(id : (H^2(\Omega), d\omega_p) \to (H^2(\Omega), \lambda ds)\) is bounded and invertible. It also intertwines the associated multiplication operators \(M\). Thus \((H^2(\Omega), \lambda ds)\) acquires the structure of a Hilbert space and the operator \(M\) on it is a pure, rationally cyclic, subnormal operator with spectrum equal to \(\bar{\Omega}\) and normal spectrum equal to \(\partial\Omega\). Consequently, the operator \(M\) on \((H^2(\Omega), \lambda ds)\) must be unitarily equivalent to the bundle shift \(T_\alpha\) on \((H^2_\alpha(\Omega), ds)\) for some character \(\alpha\). Now, we compute this character \(\alpha\).

Since \(\lambda\) is a positive continuous function on \(\partial\Omega\), using Lemma 2.4, we have the existence of a character \(\alpha\) and a function \(F\) in \(H^\infty_\alpha(\Omega)\) satisfying \(|F|^2 = \lambda\) on \(\partial\Omega\). The function \(F\) is also invertible in the sense that there exists a function \(G\) in \(H^\infty_{\alpha^{-1}}(\Omega)\) such that \(FG = 1\) on \(\Omega\). It is straightforward to verify that the linear map \(M_F : (H^2(\Omega), \lambda ds) \to (H^2_\alpha(\Omega), ds)\) defined by

\[
M_F(g) = Fg, \quad g \in (H^2(\Omega), \lambda ds)
\]

is unitary. Also \(M_F\), being a multiplication operator, intertwines the corresponding multiplication operators by the coordinate function on the respective Hilbert spaces. From Lemma 2.4, it is clear that the character \(\alpha\) is determined by the following \(n\)-tuple of numbers:

\[
(2.2) \quad c_j(\lambda) = -\int_{\partial\Omega_j} \frac{\partial}{\partial n_z} (u_\lambda(z)) ds(z), \quad \text{for } j = 1, 2, \ldots, n,
\]

where \(u_\lambda\) is the harmonic function on \(\Omega\) with continuous boundary value \(\frac{1}{2}\log \lambda\). Using this information, along with Theorem 2.5, we deduce the following Lemma which describes the unitary equivalence class of the multiplication operator \(M\) on \((H^2(\Omega), \lambda ds)\).

Lemma 2.7. Let \(\lambda, \mu\) be two positive continuous functions on \(\partial\Omega\). Then the operators \(M\) on the Hilbert spaces \((H^2(\Omega), \lambda ds)\) and \((H^2(\Omega), \mu ds)\) are unitarily equivalent if and only if

\[
\exp (ic_j(\lambda)) = \exp (ic_j(\mu)), \quad j = 1, \ldots, n.
\]

It also follows from a result of Abrahamse (cf. [1, Proposition 1.15]) that, given a character \(\alpha\), there exists an invertible element \(F\) in \(H^\infty_\alpha(\Omega)\) such that

\[
|F(z)|^2 = \begin{cases} 1, & \text{if } z \in \partial\Omega_{n+1} \\ p_j, & \text{if } z \in \partial\Omega_j, \quad j = 1, \ldots, n, \end{cases}
\]

where each \(p_j\) is a positive constant. Thus, we have proved the following theorem.

Theorem 2.8. Given any character \(\alpha\), there exists a positive continuous function \(\lambda\) defined on \(\partial\Omega\) such that the operator \(M\) on \((H^2(\Omega), \lambda ds)\) is unitarily equivalent to the bundle shift \(T_\alpha\) on \((H^2_\alpha(\Omega), ds)\).
3. Weighted kernel and extremal Operator at a fixed point

Let \( \lambda \) be a positive continuous function on \( \partial \Omega \). Since \( (H^2(\Omega), d\omega_p) \) is a reproducing kernel Hilbert space and the norm on \( (H^2(\Omega), d\omega_p) \) is equivalent to the norm on \( (H^2(\Omega), \lambda ds) \), it follows that \( (H^2(\Omega), \lambda ds) \) is also a reproducing kernel Hilbert space. Let \( K^{(\lambda)}(z, w) \) denote the kernel function for \( (H^2(\Omega), \lambda ds) \). These kernels have been studied extensively in the past (cf.\[5\]).

The case \( \lambda \equiv 1 \) gives us the Szegö kernel \( S^\Omega(z, w) \) for the domain \( \Omega \). In what follows, we will also denote the Szegö kernel \( S^\Omega(z, w) \) by \( S(z, w) \). Associated to the Szegö kernel, there exists a conjugate kernel \( L(z, w) \), called the Garabedian kernel, which is related to the Szegö kernel via the following identity:

\[
\overline{S(z, w)} ds = \frac{1}{i} L(z, w) dz, \quad w \in \Omega \text{ and } z \in \partial \Omega.
\]

We recall several well known properties of these two kernels when \( \partial \Omega \) consists of analytic Jordan curves. For each fixed \( w \) in \( \Omega \), the function \( S_w(z) \) is holomorphic in a neighborhood of \( \Omega \) and \( L_w(z) \) is holomorphic in a neighborhood of \( \Omega - \{w\} \) with a simple pole at \( w \). \( L_w(z) \) is non vanishing on \( \Omega \) and has exactly \( n \) zeros in \( \Omega \) (cf. \[3, Theorem 13.1\]). In \[13, Theorem 1\] Nehari has extended these results for the kernel \( K^{(\lambda)}(z, w) \).

**Theorem 3.1** (Nehari). Let \( \Omega \) be a bounded domain in the complex plane whose boundary consists of \( n+1 \) analytic Jordan curves and let \( \lambda \) be a positive continuous function on \( \partial \Omega \). Then there exist two analytic functions \( K^{(\lambda)}(z, w) \) and \( L^{(\lambda)}(z, w) \) with the following properties: For each fixed \( w \) in \( \Omega \), the function \( K_w^{(\lambda)}(z) \) and \( L_w^{(\lambda)}(z) - (2\pi(z-w))^{-1} \) are holomorphic in \( \Omega \); \( |K_w^{(\lambda)}(z)| \) is continuous on \( \overline{\Omega} \) and \( |L_w^{(\lambda)}(z)| \) is continuous in \( \overline{\Omega} - C_e \), where \( C_e \) denotes a small open disc about \( w \). \( K_w^{(\lambda)}(z) \) and \( L_w^{(\lambda)}(z) \) are connected by the identity

\[
K_w^{(\lambda)}(z) \lambda(z) ds = \frac{1}{i} L_w^{(\lambda)}(z) dz, \quad w \in \Omega \text{ and } z \in \partial \Omega.
\]

These properties determine both functions uniquely.

From (3.1), we have that \( \frac{1}{i} K_w^{(\lambda)}(z)L_w^{(\lambda)}(z) dz \geq 0 \). The boundary \( \partial \Omega \) consists of analytic Jordan curves. Therefore, from the Schwartz reflection principle, it follows that the functions \( K_w^{(\lambda)} \) and \( L_w^{(\lambda)} - (2\pi(z-w))^{-1} \) are holomorphic in a neighborhood of \( \overline{\Omega} \).

We have shown that the operator \( M \) on \( (H^2(\Omega), \lambda ds) \) is unitarily equivalent to a bundle shift of rank 1. Consequently the adjoint operator \( M^* \) lies in \( B_1(\Omega^*) \) admitting \( \overline{\Omega^*} \) as a spectral set from which a curvature inequality follows:

\[
\mathcal{K}_T(w) \leq -4\pi^2(S_{\Omega^*}(w, w))^2, \quad w \in \Omega^*.
\]

Or, equivalently,

\[
\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) \geq 4\pi^2(S_{\Omega}(w, w))^2, \quad w \in \Omega.
\]

Fix a point \( \zeta \) in \( \Omega \). The following lemma provides a criterion for the adjoint operator \( M^* \) on \( (H^2(\Omega), \lambda ds) \) to be extremal at \( \zeta \), that is,

\[
\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) \big|_{w=\zeta} = 4\pi^2(S_{\Omega}(\zeta, \zeta))^2.
\]

**Lemma 3.2.** The operator \( M^* \) on the Hilbert space \( (H^2(\Omega), \lambda ds) \) is extremal at \( \zeta \) if and only if \( L^{(\lambda)}(\zeta) \) and the Szegö kernel at \( \zeta \), namely \( S_{\zeta}(\zeta) \), have the same set of zeros in \( \Omega \).
Proof. Consider the closed convex set $M_1$ in $(H^2(\Omega), \lambda(z)ds)$ defined by

$$M_1 := \{ f \in (H^2(\Omega), \lambda(z)ds) : f(\zeta) = 0, f'(\zeta) = 1 \}.$$

Now consider the extremal problem of finding

$$\inf \{ \| f \| : f \in M_1 \}. \quad (3.2)$$

Since $M_1$ is a closed convex set, there exists a unique function $F$ in $M_1$ which solves the extremal problem. It has been shown in [12] that the function $F$ in $(H^2(\Omega), \lambda(z)ds)$ is a solution to the extremal problem if and only if $F \in M_1$ and $F$ is orthogonal to the subspace

$$H_1 = \{ f \in (H^2(\Omega), \lambda(z)ds) : f(\zeta) = 0, f'(\zeta) = 0 \} = (\text{span}\{ K^{(\lambda)}_\zeta, \partial K^{(\lambda)}_\zeta \})^\perp.$$

A solution to this extremal problem can be found in terms of the kernel function as in [12]:

$$\inf \{ \| f \| : f \in M_1 \} = \left\{ K^{(\lambda)}(\zeta, \zeta) \left( \frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) |_{w=\zeta} \right) \right\}^{-1}.$$

Now consider the function $g$ in $(H^2(\Omega), \lambda(z)ds)$ defined by

$$g(z) := \frac{K^{(\lambda)}_\zeta(z) F^{(\lambda)}_\zeta(z)}{2\pi S(\zeta, \zeta) K^{(\lambda)}(\zeta, \zeta)}, \quad z \in \Omega,$$

where $F^{(\lambda)}_\zeta(z) = \frac{S(\zeta)}{L^{(\lambda)}(\zeta)}$ denotes the Ahlfors map for the domain $\Omega$ at the point $\zeta$ (cf. [3, Theorem 13.1]). Using the reproducing property for the kernel function $K^{(\lambda)}$ and the fact that $|F^{(\lambda)}_\zeta| \equiv 1$ on $\partial \Omega$, it is straightforward to verify that

$$\| g \|^2_{ds} = \left( K^{(\lambda)}(\zeta, \zeta) 4\pi^2 S(\zeta, \zeta)^2 \right)^{-1}.$$

Since $F^{(\lambda)}_\zeta(\zeta) = 0$ and $F^{(\lambda)}_\zeta'(\zeta) = 2\pi S(\zeta, \zeta)$, it follows that $g \in M_1$. Consequently we have

$$\left( K^{(\lambda)}(\zeta, \zeta) 4\pi^2 S(\zeta, \zeta)^2 \right)^{-1} \geq \left\{ K^{(\lambda)}(\zeta, \zeta) \left( \frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) |_{w=\zeta} \right) \right\}^{-1},$$

equivalently,

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) |_{w=\zeta} \geq 4\pi^2 (S_\Omega(\zeta, \zeta))^2.$$

Thus equality holds if and only if $g$ solves the extremal problem in (3.2), that is, $g$ is orthogonal to the subspace $H_1$. Hence, we conclude that the operator $M^*$ on the Hilbert space $(H^2(\Omega), \lambda(z)ds)$ is extremal at $\zeta$ if and only if $g$ is orthogonal to the subspace $H_1$. Now consider the following integral

$$I_f = \int_{\partial \Omega} f(z) \overline{K^{(\lambda)}_\zeta(z)} \overline{F^{(\lambda)}_\zeta(z)} \lambda(z) ds$$

$$= \frac{1}{i} \int_{\partial \Omega} f(z) \overline{F^{(\lambda)}_\zeta(z)} L^{(\lambda)}_\zeta(z) dz \quad (3.1)$$

$$= \left( \frac{2\pi}{2\pi i} \int_{\partial \Omega} f(z) \overline{F^{(\lambda)}_\zeta(z)} dz \right)$$

$$= \left( \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)L^{(\lambda)}_\zeta(z)}{S_\zeta(z)} dz \right)$$

Since $H_1 \cap \text{Rat}(\overline{\Omega})$ is dense in $H_1$, $g$ is orthogonal to $H_1$ if and only if $I_f$ vanishes for all $f \in H_1 \cap \text{Rat}(\overline{\Omega})$. Observe that $L^{(\lambda)}_\zeta(z)L^{(\lambda)}_\zeta(z)$ is holomorphic in $\Omega - \{ \zeta \}$ with a pole of order 2 at $\zeta$. As $\partial \Omega$ consists of analytic Jordan curves, both the functions $L^{(\lambda)}_\zeta(z)$ and $L^{(\lambda)}_\zeta(z)$ are also holomorphic.
in a neighborhood of \( \partial \Omega \). Recall that \( L_\zeta(z) \) has no zero in \( \overline{\Omega} - \{ \zeta \} \) and \( S_\zeta(z) \) has exactly \( n \) zeros, say \( a_1, a_2, \ldots, a_n \), in \( \Omega \) (cf. [3, Theorem 13.1]).

Now we claim that \( I_f \) vanishes for all \( f \in H_1 \cap \text{Rat}(\overline{\Omega}) \) if and only if the set of zeros of the function \( L_\zeta^{(\lambda)}(z) \) in \( \Omega \) is \( \{ a_1, a_2, \ldots, a_n \} \).

First, if we assume that \( L_\zeta^{(\lambda)}(z) \) has \( \{ a_1, a_2, \ldots, a_n \} \) as the zero set in \( \Omega \), then the integrand in \( I_f \) is holomorphic in a neighborhood of \( \overline{\Omega} \) for every \( f \) in \( H_1 \cap \text{Rat}(\overline{\Omega}) \) and, consequently, \( I_f \) vanishes for every \( f \) in \( H_1 \cap \text{Rat}(\overline{\Omega}) \). Conversely, if \( L_\zeta^{(\lambda)}(z) \) doesn’t vanish at one of \( a_j \)’s, without loss of generality, say at \( a_1 \), then the function

\[
f = (z - \zeta)^2 \prod_{k=2}^{n} (z - a_k)
\]

is in \( H_1 \cap \text{Rat}(\overline{\Omega}) \). Observe that the integrand in \( I_f \), with this choice of the function \( f \), is holomorphic in a neighborhood of \( \overline{\Omega} \) except at the point \( a_1 \), where it has a simple pole. Hence the integral \( I_f \) equals the residue of the integrand at \( a_1 \) which is not zero, completing the proof. \( \square \)

3.1. Existence of extremal operator. We provide below two different descriptions of an extremal operator at \( \zeta \) using the criterion obtained in Lemma 3.2. Let \( a_1, a_2, \ldots, a_n \) be the zeros of the Szegő kernel \( S_\zeta(z) \) in \( \Omega \).

3.1.1. Realization of the extremal operator at \( \zeta \). Consider the function \( \lambda \) on \( \partial \Omega \) defined by

\[
\lambda(z) := \prod_{k=1}^{n} |z - a_k|^2, \quad z \in \partial \Omega.
\]

Then, for \( z \in \partial \Omega \), we have

\[
\frac{\overline{S_\zeta(z)}}{\prod_{j=1}^{n} (\overline{z} - \overline{a_j})(\zeta - a_j)} \lambda(z) ds = \frac{\prod_{k=1}^{n} (z - a_k)}{\prod_{k=1}^{n} (\zeta - a_k)} S_\zeta(z) ds = \frac{\prod_{k=1}^{n} (z - a_k)}{i \prod_{k=1}^{n} (\zeta - a_k)} L_\zeta(z) dz.
\]

Note that the function \( S_\zeta(z) \left( \prod_{j=1}^{n} (z - a_j)(\overline{\zeta} - \overline{a_j}) \right)^{-1} \) is holomorphic in a neighborhood of \( \overline{\Omega} \) and the function \( L_\zeta(z) \left( \prod_{k=1}^{n} (z - a_k) \right)^{-1} \left( \prod_{k=1}^{n} (\zeta - a_k) \right)^{-1} \) is meromorphic in a neighborhood of \( \overline{\Omega} \) with a simple pole at \( \zeta \). Hence, using the uniqueness portion of the Theorem 3.1, we get

\[
K_\zeta^{(\lambda)}(z) = \frac{S_\zeta(z)}{\prod_{j=1}^{n} (z - a_j)(\overline{\zeta} - \overline{a_j})}, \quad z \in \overline{\Omega} \quad \text{and} \quad L_\zeta^{(\lambda)}(z) = \frac{\prod_{k=1}^{n} (z - a_k)}{\prod_{k=1}^{n} (\zeta - a_k)} L_\zeta(z), \quad z \in \overline{\Omega} - \{ \zeta \}.
\]

Clearly, \( \{ a_1, a_2, \ldots, a_n \} \) is the zero set of the function \( L_\zeta^{(\lambda)}(z) \). Therefore, the adjoint operator \( M^* \) on \( (H^2(\Omega), \lambda(z) ds) \) is an extremal operator at \( \zeta \).
3.1.2. A second realization of the extremal operator at $\bar{\zeta}$. This realization of the extremal operator was obtained earlier in [12]. Consider the measure

$$\lambda(z)ds = \frac{|S_\zeta(z)|^2}{S(\zeta, \zeta)}ds, \ z \in \partial \Omega,$$

on the boundary $\partial \Omega$. Using the reproducing property of the Szegő kernel, it is easy to verify that

$$\langle f, 1 \rangle_{(H^2(\Omega), \lambda ds)} = f(\zeta).$$

This gives us $K_\zeta^{(\lambda)}(z) = 1$ for all $z \in \bar{\Omega}$. So we have

$$\lambda(z)ds = \frac{S_\zeta(z)}{S(\zeta, \zeta)}S_\zeta(z)ds, \ z \in \partial \Omega$$

$$= \frac{1}{i} \frac{S_\zeta(z)}{S(\zeta, \zeta)}L_\zeta(z)dz, \ z \in \partial \Omega$$

Now the function $S_\zeta(z)L_\zeta(z)(S(\zeta, \zeta))^{-1}$ is a meromorphic function in a neighborhood of $\bar{\Omega}$ with a simple pole at $\zeta$. Again, using the uniqueness guaranteed in Theorem 3.1, we get

$$L_\zeta^{(\lambda)}(z) = S_\zeta(z)L_\zeta(z)(S(\zeta, \zeta))^{-1}, \ z \in \bar{\Omega} - \{\zeta\}.$$ Agai

Again, the zero set of the function $L_\zeta^{(\lambda)}(z)$ is $\{a_1, a_2, \ldots, a_n\}$. Hence the operator $M^*$ on the Hilbert space $(H^2(\Omega), \lambda ds)$ is an extremal operator at $\bar{\zeta}$.

We shall prove that any extremal operator which is also the adjoint of a bundle shift is uniquely determined up to unitary equivalence. An interesting consequence of this uniqueness is that the two realizations of the extremal operators given above must coincide up to unitary equivalence.

3.2. Index of the Blaschke product. To facilitate the proof of the uniqueness, we need to recall basic properties of the multiplicative Blaschke product on $\Omega$ and its index of automorphy. This is also going to be a crucial ingredient in determining the character $\alpha$ of the extremal operator at $\bar{\zeta}$.

Let $g(z, a)$ be the Green’s function for the domain $\Omega$, with pole at $a \in \Omega$. The multiplicative Blaschke factor with zero at $a$, is defined as follows:

$$B_a(z) = \exp(-g(z, a) - ig^*(z, a)), \ \text{for all } z \in \Omega,$$

where $g^*(z, a)$ is the multivalued conjugate of the Green’s function $g(z, a)$ which is harmonic on $\Omega \setminus \{a\}$. So, $B_a(z)$ is a multiplicative function on $\Omega$, which vanishes only at the point $a$ with multiplicity 1 and on $\partial \Omega$ its absolute value is identically 1. Note that the period of the conjugate harmonic function $g^*(z, a)$ around the boundary component $\partial \Omega_j$ is equal to

$$p_j(a) = -\int_{\partial \Omega_j} \frac{\partial}{\partial \eta_z}(g(z, a))ds_z, \ \text{for } j = 1, 2, \ldots, n.$$ The negative sign appearing in the equation for the periods is a result of the assumption that $\partial \Omega$ is positively oriented, that is, the boundaries $\partial \Omega_j$, $j = 1, 2, \ldots, n$ are oriented in the clockwise direction and the boundary $\partial \Omega_{n+1}$ is oriented in the counterclockwise direction.

Since the Blaschke factor $B_a(z)$ is a multiplicative function on $\Omega$, it is induced by a modulus automorphic function on the unit disc, say $b_\alpha$, for some character $\alpha$. In fact, the function $B_a = b_\alpha \circ \pi^{-1}$ lies in $H^\infty_\alpha$. The character $\alpha$ uniquely determines an $n$-tuple of complex numbers of unit modulus. These are the images under $\alpha$ of the generators of the group $G$ of deck transformations relative to the covering map $\pi : \mathbb{D} \to \Omega$. This $n$-tuple, called the index of the Blaschke factor $B_a(z)$, is of the form

$$\{\exp(-ip_1(a)), \exp(-ip_2(a)), \ldots, \exp(-ip_n(a))\}.$$
We recall below the well known relationship of the period \( p_j(a) \) to the harmonic measure \( \omega_j(z) \) of
the boundary component \( \partial \Omega_j \), namely,
\[
\omega_j(a) = -\frac{1}{2\pi} \int_{\partial \Omega_j} \frac{\partial}{\partial n_z} (g(z, a)) \, ds_z = \frac{1}{2\pi} p_j(a), \quad \text{for } j = 1, 2, \ldots, n,
\]
where the harmonic measure \( \omega_j(z) \) is the function which is harmonic in \( \Omega \), has the boundary value 1 on \( \partial \Omega_j \) and is 0 on all the other boundary components. Hence, the index of the Blaschke factor \( B_a(z) \) is
\[
\text{ind}(B_a(z)) = \{ \exp(-2\pi i \omega_1(a)), \exp(-2\pi i \omega_2(a)), \ldots, \exp(-2\pi i \omega_n(a)) \}
\]
The index of the finite Blaschke product \( B(z) = \prod_{k=1}^{m} B_{a_k}(z) \), \( a_k \in \Omega \), is equal to
\[
(3.3) \quad \text{ind}(B(z)) = \left\{ \exp\left( -2\pi i \sum_{k=1}^{m} \omega_1(a_k) \right), \ldots, \exp\left( -2\pi i \sum_{k=1}^{m} \omega_n(a_k) \right) \right\}
\]

### 3.3. Zeros of the Szegő kernel \( S_{\zeta}(z) \)

Fixing \( \zeta \) in \( \Omega \), which is \((n+1)\) connected, as pointed out earlier, the Szegő kernel \( S_{\zeta}(z) \) has exactly \( n \) zeros (counting multiplicity) in \( \Omega \). Let \( a_1, a_2, \ldots, a_n \) be the zeros of \( S_{\zeta}(z) \). Hence the Ahlfors function \( F_{\zeta}(z) \) at the point \( \zeta \) has exactly \( n+1 \) zeros in \( \Omega \), namely \( \zeta, a_1, a_2, \ldots, a_n \). Now an interesting relation between the points \( a_1, \ldots, a_n \) and \( \zeta \) becomes evident.

First consider the Blaschke product \( B(z) = B_{\zeta}(z) \prod_{k=1}^{n} B_{a_k}(z) \). The index of the Blaschke product \( B(z) \), using (3.3), is easily seen to be of the form
\[
\beta = (\beta_1, \beta_2, \ldots, \beta_n), \quad \text{where } \beta_j = \left\{ \exp\left( -2\pi i (\omega_j(\zeta) + \sum_{k=1}^{n} \omega_j(a_k)) \right) \right\} \quad \text{for } j = 1, 2, \ldots, n.
\]

The Ahlfors function \( F_{\zeta}(z) \) is in \( H^\infty(\Omega) \) and it is holomorphic in a neighborhood of \( \bar{\Omega} \) as long as the boundary \( \partial \Omega \) is analytic. Therefore, in the inner outer factorization of \( F_{\zeta}(z) \), there is no singular inner function and it follows that
\[
|F_{\zeta}(z)| = |B(z)||\psi(z)|, \quad z \in \Omega,
\]
where \( \psi(z) \) is a multiplicative outer function of index
\[
\beta^{-1} = (\frac{1}{\beta_1}, \frac{1}{\beta_2}, \ldots, \frac{1}{\beta_n}).
\]

Now consider the linear map \( L : (H^2(\Omega), ds(z)) \rightarrow (H^2_{\beta^{-1}}(\Omega), ds) \), defined by
\[
L f = \psi f, \quad f \in (H^2(\Omega), ds(z)).
\]
Note that \( \psi(z) \) is outer and it is bounded in absolute value (since \( F_{\zeta}(z) \) is bounded) on \( \Omega \). It is straightforward to verify that \( L \) is a unitary operator. Also, since \( L \) is a multiplication operator, it intertwines any two multiplication operators on the respective Hilbert spaces.

As a corollary of Theorem 2.5, we must have \( \beta^{-1} = (1, 1, \ldots, 1) \). This implies
\[
(3.4) \quad \exp\left( -2\pi i (\omega_j(\zeta) + \sum_{k=1}^{n} \omega_j(a_k)) \right) = 1, \quad j = 1, 2, \ldots, n,
\]
relating the point \( \zeta \) to the zeros \( a_1, a_2, \ldots, a_n \) of the Szegő kernel \( S_{\zeta}(z) \).
3.4. Uniqueness of the extremal operator.

Proof of Theorem 2.6. Assume that, for a positive continuous function $\lambda$ on $\partial \Omega$, the operator $M^*$ on the Hilbert space $(H^2(\Omega), \lambda(z)ds)$ is extremal at $\bar{\zeta}$. Recall that the function $K_\zeta^{(\lambda)}(z)$ is analytic in a neighborhood of $\bar{\Omega}$ and the conjugate kernel $L_\zeta^{(\lambda)}(z)$ is meromorphic in a neighborhood of $\bar{\Omega}$ with a simple pole only at the point $\zeta$. Also, from Lemma 3.2, we have that the zero set of $L_\zeta^{(\lambda)}(z)$ is the set $\{a_1, a_2, \ldots, a_n\}$, where $\{a_1, a_2, \ldots, a_n\}$ are the zeros of $S_\zeta(z)$ in $\Omega$. We have, using equation (3.1), that

$$|K_\zeta^{(\lambda)}(z)|^2 \lambda(z)ds = \frac{1}{i} K_\zeta^{(\lambda)}(z) L_\zeta^{(\lambda)}(z)ds, \quad z \in \partial \Omega.$$ 

An application of the generalized argument principle shows that the total number of zeros (counting multiplicity) of $K_\zeta^{(\lambda)}(z)$ and $L_\zeta^{(\lambda)}(z)$ in $\bar{\Omega}$, where a zero on the boundary is counted as $\frac{1}{2}$, is equal to $n$. Hence it follows that $a_1, a_2, \ldots, a_n$ are all the zeros of $L_\zeta^{(\lambda)}(z)$ in $\bar{\Omega}$ and $K_\zeta^{(\lambda)}(z)$ has no zero in $\bar{\Omega}$.

Nehari [13, Theorem 4] has shown that the meromorphic function

$$R(z) = \frac{K_\zeta^{(\lambda)}(z)}{L_\zeta^{(\lambda)}(z)}, \quad z \in \bar{\Omega},$$

with exactly one zero at $\zeta$ and poles exactly at $a_1, a_2, \ldots, a_n$, solves the extremal problem

$$\sup \{|f'(\zeta)| : f \in B_\lambda\},$$

where $B_\lambda$ denotes the class of meromorphic functions on $\Omega$. Each $f$ in $B_\lambda$ is required to vanish at $\zeta$ and it is assumed that the set of poles of $f$ is a subset of $\{a_1, a_2, \ldots, a_n\}$. The radial limits of the functions $f$ at $z_0 \in \partial \Omega$, from within $\Omega$, in the class $B_\lambda$ are uniformly bounded:

$$\limsup_{z \to z_0} |f(z)| \leq \frac{1}{\lambda(z_0)}, \quad z_0 \in \partial \Omega.$$

The proof includes the verification

$$|R(z)| = \frac{1}{\lambda(z)}, \quad z \in \partial \Omega.$$

Now consider the multiplicative function $G$ on $\Omega$ defined by

$$G(z) = \frac{B_\zeta(z)}{R(z) \prod_{j=1}^n B_{a_j}(z)}, \quad z \in \bar{\Omega}.$$

$G$ is a multiplicative function in a neighborhood of $\bar{\Omega}$. Also, by construction $|G|$ has no zero in $\bar{\Omega}$. Using the inner outer factorization for multiplicative functions (cf. [19, Theorem 1]), we see that $G$ is a bounded multiplicative outer function. Also note that

$$|G(z)| = \lambda(z), \quad z \in \partial \Omega.$$

The index of $G$ is given by

$$\left\{ \exp \left( 2\pi i (\omega_1(\zeta) + \sum_{j=1}^n \omega_1(a_j)) \right), \ldots, \exp \left( 2\pi i (\omega_n(\zeta) + \sum_{j=1}^n \omega_n(a_j)) \right) \right\}.$$

Using equation (3.4), we infer that the index of $G(z)$ must be equal to

$$\left\{ \exp \left( -4\pi i \omega_1(\zeta) \right), \ldots, \exp \left( -4\pi i \omega_n(\zeta) \right) \right\}.$$
The function $G$ is outer and hence the function $F := \sqrt{G}$ is well defined. It is a bounded multiplicative outer function with $|F(z)|^2 = \lambda(z)$ for all $z$ in $\partial \Omega$. The index of $F$ is given by
\[
\left\{ \exp \left( -2\pi i \omega_1(\zeta) \right), \ldots, \exp \left( -2\pi i \omega_n(\zeta) \right) \right\}.
\]

Now consider the linear map $V : (H^2(\Omega), \lambda(z) \, ds) \to (H^2_0(\Omega), ds)$ defined by
\[
Vf = Ff, \quad f \in (H^2(\Omega), \lambda(z) \, ds).
\]

It is easily verified that $V$ is a unitary multiplication operator which intertwines the corresponding multiplication operators on the respective Hilbert spaces. Hence, the character $\alpha$ for the bundle shift $T_\alpha$ on $(H^2_0(\Omega), ds)$, which is extremal at $\bar{\zeta}$, is uniquely determined by the following $n$-tuple of complex numbers of unit modulus:
\[
\left\{ \exp \left( -2\pi i \omega_1(\zeta) \right), \ldots, \exp \left( -2\pi i \omega_n(\zeta) \right) \right\} = \left\{ \exp \left( 2\pi i (1 - \omega_1(\zeta)) \right), \ldots, \exp \left( 2\pi i (1 - \omega_n(\zeta)) \right) \right\}.
\]

Hence, if the adjoint of a bundle shift is extremal at $\bar{\zeta}$, then it is uniquely determined. This completes the proof of Theorem 2.6. \qed

Since $G$, the group of the deck transformations for the covering $\pi : \mathbb{D} \to \Omega$, is isomorphic to the free group on $n$ generators, any character $\alpha$ of the group $G$ is unambiguously determined, up to a permutation in the choice of generators for the group $G$, by the $n$-tuple $\{x := (x_1, x_2, \ldots, x_n) : x_1, \ldots, x_n \in [0,1)\}$, namely,
\[
\alpha(g_k) = \exp(2\pi i x_k), \quad x_k \in [0,1), \quad 1 \leq k \leq n,
\]
where $g_k$, $1 \leq k \leq n$, are generators of the group $G$. The unitary equivalence class of the bundle shift $T_\alpha$ of rank 1 is therefore determined by the $n$-tuple $x$ in $[0,1)^n$ corresponding to the character $\alpha$.

For $\zeta$ in $\Omega$, the character corresponding to the $n$-tuple $\left( (1 - \omega_1(\zeta)), (1 - \omega_2(\zeta)), \ldots, (1 - \omega_n(\zeta)) \right)$ defines the bundle shift which is extremal at $\bar{\zeta}$. Let $\phi : \Omega \to [0,1)^n$ be the induced map, that is,
\[
\phi(\zeta) = \left( (1 - \omega_1(\zeta)), (1 - \omega_2(\zeta)), \ldots, (1 - \omega_n(\zeta)) \right).
\]

Suita’s result [16] shows that the map $\phi$ is not onto since $(0, \ldots, 0)$, which corresponds to the operator $M^*$ on the usual Hardy space, cannot be in its range. However, we show below that many other bundle shifts are missing from the range of the map $\phi$ when $n \geq 2$.

Let $\omega_{n+1}(z)$ be the harmonic measure for the outer boundary component $\partial \Omega_{n+1}$. Thus $\omega_{n+1}$ is the harmonic function on $\Omega$ which is 1 on $\partial \Omega_{n+1}$ and is 0 on all the other boundary components. We have
\[
\sum_{j=1}^{n+1} \omega_j \equiv 1 \quad \text{and} \quad 0 < \omega_{n+1}(z) < 1, \quad z \in \Omega.
\]

Therefore
\[
(n - 1) < \sum_{j=1}^{n} (1 - \omega_j(\zeta)) < n.
\]

From this, for $n \geq 2$, it follows that the set of extremal operators does not include the adjoints of many of the bundle shifts. For instance, if the index of a bundle shift is $(x_1, \ldots, x_n)$ in $[0,1)^n$ such that $x_1 + \cdots + x_n < n - 1$, then it cannot be an extremal operator at any $\bar{\zeta}$, $\zeta \in \Omega$. 
4. The special case of the Annulus

Let $\Omega$ be an annular domain $A(0; R, 1)$ with inner radius $R$, $0 < R < 1$, and outer radius 1. In this case we have an explicit expression for the harmonic measure corresponding to the boundary component $\partial\Omega_1$, namely,

$$\omega_1(z) = \frac{\log |z|}{\log R}.$$  

So, for a fixed point $\zeta$ in $A(0; R, 1)$, the character of the unique bundle shift which happens to be an extremal operator at $\zeta$ is determined by the number

$$\alpha(\zeta) = \exp \left( 2\pi i (1 - \omega_1(\zeta)) \right).$$

From this expression for the index, it is clear, in the case of an annular domain $A(0; R, 1)$, that the adjoint of every bundle shift except the trivial one, is an extremal operator at some point $\bar{\zeta}$ in $\Omega^\ast$. In fact, this is true of any doubly connected bounded domain $\Omega$ with Jordan analytic boundary since for such domains we have $\omega_1(\Omega) = (0, 1)$, where $\omega_1$ is the harmonic measure corresponding to the inner boundary component $\partial\Omega_1$.

We now give a different proof of Theorem 2.6 in the case of $\Omega = A(0; R, 1)$. In the course of this proof we see the effect of the weights on the zeros of the weighted Hardy kernels $K^{(\alpha)}$. This question was raised in [10].

For a fixed real number $\alpha$, consider the measure $\mu_\alpha ds$ on the boundary of the annulus, where the function $\mu_\alpha$ is defined by

$$\mu_\alpha(z) = \begin{cases} 1 & \text{if } |z| = 1, \\ R^{2\alpha} & \text{if } |z| = R. \end{cases}$$

It is straightforward to verify that the function $\{f_n(z)\}_{n \in \mathbb{Z}}$ defined by

$$f_n(z) = \frac{z^n}{\sqrt{2\pi(1 + R^{2n+2n+1})}}, \quad n \in \mathbb{Z},$$

forms an orthonormal basis for the Hilbert space $(H^2(\Omega), \mu_\alpha ds)$. The function

$$K^{(\alpha)}(z, w) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{(z\bar{w})^n}{1 + R^{2\alpha+2n+1}}, \quad z, w \in \Omega,$$

is uniformly convergent on compact subsets of $\Omega$. $K^{(\alpha)}$ is the reproducing kernel of the Hilbert space $(H^2(\Omega), \mu_\alpha ds)$. For each fixed $w$ in $\Omega$, the kernel function $K^{(\alpha)}(z, w)$ is defined on $\Omega$. However, it extends analytically to a larger domain. To describe this extension, recall that the Jordan Kronecker function, introduced by Venkatachaliengar (cf. [17, p.37]), is given by the formula

$$f(b, t) = \sum_{k \in \mathbb{Z}} \frac{t^n}{1 - bR^{2n}}.$$  

This series converges for $R^2 < |t| < 1$ and for all $b \neq R^{2k}, k \in \mathbb{Z}$. Venkatachaliengar, using Ramanujan’s $\psi_1$ summation formula, has established the following identity (cf. [17, p. 40])

$$f(b, t) = \prod_{j=0}^{\infty} \left( 1 - t R^{2j} \right) \prod_{j=0}^{\infty} \left( 1 - \frac{R^{2j+2}}{t} \right) \prod_{j=0}^{\infty} \left( 1 - \frac{R^{2j+2}}{b} \right).$$

This formula provides a useful tool for the analysis of the kernel function $K^{(\alpha)}$. In the next section, we will use this identity to study the zeros of $K^{(\alpha)}$.
This extends the definition of $f(b, t)$, as a meromorphic function, to all of the complex plane with simple poles at $b = R^{2k}$, $t = R^{2k}$, $k \in \mathbb{Z}$. For fixed $w$ in $\Omega$, since the function $f(-R^{2\alpha+1}, z\bar{w})$ coincides with $2\pi K^{(\alpha)}(z, w)$ for all $z$ in $\Omega$ and $f$ is meromorphic on the entire complex plane, it follows that $K^{(\alpha)}_w$ also extends to all of $\mathbb{C}$ as a meromorphic function. The poles of $K^{(\alpha)}_w$ are exactly at $\frac{R^{2k}}{w}$, $k \in \mathbb{Z}$. The zeros of the kernel function $K^{(\alpha)}_w(z)$ in $\Omega$ can also be computed using the equation 4.1. The zeros $(b, t)$ of the function $f$ must satisfy one of the following identities

$$bt = R^{-2j}, j = 0, 1, 2, \ldots$$

or

$$bt = R^{2j+2}, j = 0, 1, 2, \ldots$$

For example, when $\alpha = 0$, the kernel $K^{(\alpha)}(z, w)$ is the Szego kernel $S(z, w)$. It follows that if $w$ is a fixed but arbitrary point in $\Omega$, then the zero set of the Szego kernel function $S_w(z)$ in $\Omega$ is equal to $\{\frac{-R}{w}\}$.

The operator $M$ on the Hilbert space $(H^2(\Omega), \mu_\alpha ds)$ is a bilateral weighted shift with weight sequence

$$\omega_n^{(\alpha)} = \sqrt{\frac{1 + R^{2\alpha + 2n + 3}}{1 + R^{2\alpha + 2n + 1}}}, \quad n \in \mathbb{Z}.$$

The identity

$$\omega_n^{(\alpha + 1)} = \omega_n^{(\alpha)}, \quad n \in \mathbb{Z},$$

makes the operators $M$ on $(H^2(\Omega), \mu_\alpha ds)$ and $(H^2(\Omega), \mu_{\alpha+1} ds)$ unitarily equivalent. Thus there is a natural map from the unitary equivalence classes of these bilateral shifts onto $[0, 1)$. In the case of the annulus $A(0; R, 1)$, we find that $u_{\mu_\alpha}$ and $c_1(\mu_\alpha)$, as defined in equation (2.2), are equal to $\alpha \log|z|$ and $2\pi \alpha$ respectively. Applying Lemma 2.7, we see that the operators $M$ on $(H^2(\Omega), \mu_\alpha ds)$ and $(H^2(\Omega), \mu_\beta ds)$ are unitarily equivalent if and only if $\alpha - \beta$ is an integer.

Thus we have a bijective correspondence between the unitary equivalence classes of these bilateral weighted shifts and $[0, 1)$, and we may assume without loss of generality that $\alpha \in [0, 1)$.

For each $\alpha \in [0, 1)$, the operator $M$ on $(H^2(\Omega), \mu_\alpha ds)$ is unitarily equivalent to the bundle shift $T_\beta$ on $(H^2_\beta(\Omega), ds)$, where the character $\beta$ is determined by the unimodular scalar $\exp(2\pi i \alpha)$.

Now fix a point $\zeta$ in $\Omega$. It is known that $S_\zeta(z)$, the Szego kernel at $\zeta$ for the domain $\Omega$ has exactly one zero at $-\frac{R}{\zeta}$. The existence of a conjugate kernel $L^{(\alpha)}(z, w)$ is established in [13]. Then using the characterization for the extremal operator at $\zeta$, it follows that the operator $M^*$ on $(H^2(\Omega), \mu_\alpha ds)$ is extremal at $\zeta$ if and only if $L^{(\alpha)}_\zeta(-\frac{R}{\zeta}) = 0$. From the identity

$$zL^{(\alpha)}(z, w) = K^{(\alpha)}_\bar{z}(\frac{1}{\zeta}, \bar{w})$$

proved in [10, p.1118], and recalling that

$$K^{(\alpha)}_\zeta(-\frac{R}{\zeta}) = \sum_{k \in \mathbb{Z}} \left(\frac{-|\zeta|^2}{1 + R^{2\alpha + 2n + 1}}\right)^n,$$

we conclude that the operator $M^*$ is extremal at $\zeta$ if and only if

$$\sum_{k \in \mathbb{Z}} \left(\frac{-|\zeta|^2}{1 + R^{2\alpha + 2n + 1}}\right)^n = 0.$$

Consequently, the operator $M^*$ on $(H^2(\Omega), \mu_\alpha ds)$ is extremal at $\zeta$ if and only if the Jordan Kronecker function $f$ satisfies

$$f(-R^{2\alpha+1}, -\frac{|\zeta|^2}{R}) = 0.$$

So for a fixed $\zeta$, the real number $\alpha \in [0, 1)$ must satisfy at least one of these identities

$$R^{2\alpha}|\zeta|^2 = R^{-2j}, j = 0, 1, 2, \ldots$$

or

$$R^{2\alpha}|\zeta|^2 = R^{2j+2}, j = 0, 1, 2, \ldots$$
In any case, one must have
\[
\alpha = \left(1 - \frac{\log |\zeta|}{\log R}\right) \pmod{1}.
\]

So the unitary equivalence class of the adjoint of a bundle shift which is extremal at $\bar{\zeta}$ is uniquely determined. Hence we have proved the Theorem stated below.

**Theorem 4.1.** The operator $M^*$ on the Hilbert space $(H^2(\Omega), \mu_\alpha ds)$ is extremal at $\bar{\zeta}$ if and only if \[
\alpha = \left(1 - \frac{\log |\zeta|}{\log R}\right) \pmod{1}.
\]

**Acknowledgement.** The author is grateful to G. Misra for his patient guidance and suggestions in the preparation of this paper. The author thanks the referee for several comments which helped in substantial improvement of the exposition. He would also like to thank the Math Stack Exchange community for providing an excellent opportunity for many stimulating discussions.

**References**


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