GLOBAL WELL-POSEDNESS FOR PERIODIC GENERALIZED
KORTEWEG-DE VRIES EQUATION

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ABSTRACT. In this paper, we show the global well-posedness for periodic gKdV equations in the space $H^s(\mathbb{T})$, $s \geq \frac{1}{2}$ for quartic case, and $s > \frac{5}{9}$ for quintic case. These improve the previous results of Colliander et al in 2004. In particular, the result is sharp in the quartic case.

1. Introduction

In this paper, we consider the global well-posedness of the Cauchy problem for the periodic generalized Korteweg-de Vries equations (gKdV):

\[
\begin{aligned}
\partial_t u + \partial_x^3 u &= F(u)_x, \quad (t, x) \in [0, T] \times \mathbb{T}, \\
u(0, x) &= \phi(x), \quad x \in \mathbb{T},
\end{aligned}
\tag{1.1}
\]

where $u$ is an unknown real function defined on $[0, T] \times \mathbb{T}$, $\phi$ is a given real-valued function, $F$ is a polynomial of degree $k+1$, and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle. For simplicity, we may assume that $F(u) = \mu u^{k+1}$ and $\mu = \pm 1$. When $\mu = 1$, the equation in (1.1) is referred to “defocusing”, while when $\mu = -1$ it is referred to “focusing”. For $k = 1$ and $k = 2$, they are called by the KdV and modified KdV equations, respectively. These two equations are completely integrable. For $k \geq 3$, they are classified as the generalized KdV equations, which are not completely integrable in general. In particular, the quartic case $k = 3$ and the quintic case $k = 4$ are of special interest, which are regarded as the mass-subcritical and mass-critical equations.

The Cauchy problem (1.1) has been widely studied. The periodic KdV and periodic modified KdV equations are globally well-posed in $H^s(\mathbb{T})$ for any $s \geq -\frac{1}{2}$ and $s \geq \frac{1}{2}$ respectively. See Kenig, Ponce and Vega [17] (also [1, 3, 16] and [21] for unconditional well-posedness of modified KdV equation) for local results and Colliander, Keel, Staffilani, Takaoka and Tao (I-team) [8] for the global results. These above ranges of $s$ are sharp in the sense of

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the uniformly continuous dependence of the solution on the data, see [18]. While in the sense of only continuous of the solution map, Kappeler and Topalov [14, 15] (also [29, 32] for previous results) established the global well-posedness of the KdV and the defocusing mKdV equations in $H^s(\mathbb{T})$ for $s \geq -1$ and $s \geq 0$ respectively. These results were shown to be sharp by Molinet [25]. In the non-integrable case, $k \geq 3$, results are relatively short. Local well-posedness was shown by I-team [9] in $H^s(\mathbb{T})$ for any $s \geq \frac{1}{2}$, and for any $k \geq 3$. The authors [9] also showed the analytic ill-posedness in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$. So in this sense, the index $\frac{1}{2}$ is optimal for local well-posedness in Sobolev space $H^s(\mathbb{T})$. Moreover, they established the global well-posedness results in $H^s(\mathbb{T})$ for $s > \frac{13}{14} - \frac{2}{7k}$ in the defocusing case (or for small data in focusing case when $k \geq 4$). In particular, they proved that the quartic and quintic gKdV equations are global well-posed in $H^s(\mathbb{T})$ whenever $s > \frac{5}{6}$ and $s > \frac{5}{7}$ respectively. But there exist some gaps to the local threshold $s = \frac{1}{2}$. In the present paper, we improve the indices and obtain the optimal one for $k = 3$, while for $k = 4$ there is still room to improve to a sharp result. For the related results in real line case, we just refer to [4, 8, 13, 17, 20, 19, 24, 26, 27, 33] for a few of them. Now our main result can be stated as follows.

**Theorem 1.1.** The Cauchy problems of defocusing generalized KdV equations

\[
\begin{align*}
\partial_t u + \partial_x^3 u &= \partial_x (u^{k+1}), \quad (t, x) \in [0, T] \times \mathbb{T}, \\
\quad u(0, x) &= \phi(x), \quad x \in \mathbb{T}
\end{align*}
\]

are globally well-posed in $H^s(\mathbb{T})$ with $s \geq \frac{1}{2}$ for $k = 3$, and $s > \frac{5}{9}$ for $k = 4$.

Similar results as Theorem 1.1 also hold for the focusing equations with the suitable small initial data in the quintic case. Moreover, for general nonlinearity, our method here is also available. However, compared with the local theory, the global result for $k > 4$ falls far short of expectations. Even for the quintic case, it still has gap from the sharp local result.

The main approach used here is I-method introduced by I-team, see [7, 8, 9] for examples. Also, we shall use the resonant decomposition argument given in [2, 10], see also [27, 28, 22, 23] for more related argument. It is known that the problem (1.2) obeys the conserved Hamiltonian

\[E(u) := \int_\mathbb{T} \left( \frac{1}{2} u_x^2 + \frac{1}{k+2} u^{k+2} \right) dx.\]

The scheme of I-method is to construct the “almost conservation law” of the “first” modified energy $E(Iu)$ by introducing the I-operator. Then the global result can be obtained by iteration. Moreover, some suitable “correction-term” may be added to the first modified
energy $E(Iu)$. If this is done, one may define the “second” modified energy. Then the better energy increment and global result could be gotten. For the gKdV equation, one may note that it is hard to define the second modified energy in a naive way, via adding a “correction-term” to $E(Iu)$ directly. The reason is that the multiplier, introduced to obtain the second modified energy, is singular in the sense that its $L^\infty$-norm is infinity in a nontrivial set.

In Subsection 1.1, we give a connection between the I-method and the normal form method. Then we use this connection to explain the difficulties in the study of the non-integrable gKdV equations. To get around the difficulty, we employ the resonant decomposition method, which was essentially given in [27]. More precisely, we will split the multiplier into “resonant piece” and “non-resonant piece”, and then treat them separately. For “non-resonant piece”, we add a “correction-term” to define the second modified energy, while for “resonant piece”, we prove that it is relatively small. This process is somewhat challenging in periodic setting, because of the weaker Strichartz estimate, and thus the problem under study has been left for several years. Actually, as shown in [1], $L^6$ Strichartz estimate has a 0+derivative loss. Due to this, the decomposition should be slightly finer and the estimates have to be sharpen to get a smaller control in “resonant piece” than the real line case, especially to achieve the sharp result in the quartic case.

1.1. **Outline of the proof.**

1.1.1. **Working space.** First, we use the gauge transformation introduced in [1, 31]. Let

$$G_u(t, x) = u(t, x + \int_0^t \int_\mathbb{T} u^k \, dx \, ds).$$

Then we denote functional space $X^s$ as our working space, which is equipped by the norm,

$$\|u\|_{X^s} := \|G_u\|_{Y^s}, \quad (1.3)$$

where $Y^s$ is the standard (but slightly modified) Bourgain space defined in Section 2. Moreover, we denote $X^s(I)$ to be its restricted space on time interval $I$.

In [9, 31], the authors employed this gauge transformation to avoid a nontrivial resonance in the original equation. Under this transform, the function $G_u$ satisfies the equation

$$\partial_t u + \partial_x^3 u = \mathbb{P}[\mathbb{P}(v^k)v_x], \quad (1.4)$$

where $\mathbb{P}$ denotes the orthogonal projection onto mean zero functions,

$$\mathbb{P}f = f - \int_\mathbb{T} f \, dx,$$
that is, \( \mathbb{F}f(0) = 0 \). Then the authors considered \( G u \) instead to prove the sharp local wellposedness via multilinear estimates.

However, to study the global theory, we can not employ the forms of (1.4) because it breaks the symmetries, which gives the bad form of the modified energies and thus against finer multiplier estimates, see Step 3 below. So we still consider the original equation, but use the gauged norm (1.3) which is related to the local theory. We remark that the symmetries are powerful when we use the second modified energy, and thus the situation and the treatment in the present paper are different from those in [9]. Actually, the inharmonious relationship between the equation and the working space causes difficulties in the multiplier estimates. Fortunately, these difficulties can be overcome by using the good properties of the gauge transformation. For this reason, one shall be careful in the usage of the Bourgain norm.

1.1.2. I-operator. Let \( N \gg 1 \) be fixed, and the Fourier multiplier operator \( I_{N,s} \) be defined as

\[
\hat{I}_{N,s}f(\xi) = m_{N,s}(\xi) \hat{f}(\xi).
\]  

(1.5)

Here the multiplier \( m_{N,s}(\xi) \) is a smooth, monotone function satisfying \( 0 < m_{N,s}(\xi) \leq 1 \) and \( m_{N,s}(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ \frac{N^{1-s}|\xi|^{s-1}}{N}, & |\xi| > 2N. \end{cases} \)  

(1.6)

Usually, we denote \( I_{N,s} \) and \( m_{N,s} \) as \( I \) and \( m \) respectively for short if there is no confusion. Then

\[
\|f\|_{H^s} \lesssim \|I_{N,s}f\|_{H^1} \lesssim N^{1-s}\|f\|_{H^s}.
\]  

(1.7)

1.1.3. Sketch the proofs. Now we sketch the proof of Theorem 1.1 in the following steps.

Step 1: Rescaling.

We rescale the problem by writing

\[
u_{\lambda}(t, x) = \lambda^{-\frac{2}{3}} u(t/\lambda^3, x/\lambda); \quad \phi_{\lambda}(x) = \lambda^{-\frac{2}{3}} \phi(x/\lambda),
\]

then \( u_{\lambda} \) satisfies that

\[
\begin{aligned}
\partial_t v_{\lambda} + \partial_x^3 v_{\lambda} &= (u_{\lambda}^{k+1})_x, & (t, x) \in [0, \lambda^3 T] \times [0, \lambda], \\
v_{\lambda}(0, x) &= \phi_{\lambda}(x), & x \in [0, \lambda].
\end{aligned}
\]  

(1.8)

Moreover, the solution of (1.1) \( u \) exists on \([0, T] \) if and only if \( u_{\lambda} \) exists on \([0, \lambda^3 T] \). On the other hand, we get that for any \( q \geq 1 \) and \( s \geq 0 \),

\[
\|\phi_{\lambda}\|_{L^2_q} = \frac{1}{\lambda^{\frac{2}{3}} - \frac{2}{3}} \|\phi\|_{L^2_q}; \quad \|\phi_{\lambda}\|_{\dot{H}^s} = \lambda^{\frac{1}{3} - \frac{2}{3} - s} \|\phi\|_{\dot{H}^s}.
\]  

(1.9)
Hence, by (1.7) and \( m(\xi) \leq 1 \),

\[
E(I\phi_\lambda) = \frac{1}{2} \| \partial_x I\phi_\lambda \|_{L^2}^2 + \frac{1}{k + 2} \| I\phi_\lambda \|_{L^{k+2}}^{k+2} \\
\lesssim N^{2-2s} \| \phi_\lambda \|_{H^s}^2 + \| \phi_\lambda \|_{L^{k+2}}^{k+2} \\
\lesssim N^{2-2s}/\lambda^{k+2s-1} \cdot \| \phi \|_{H^s} + \lambda^{-1-\frac{s}{k}} \| \phi \|_{L^{k+2}}^{k+2}.
\]

To normalize the rescaled initial data, we choose

\[
\lambda \sim N^{\frac{1-s}{k+s-\frac{s}{2}}}. \tag{1.10}
\]

Then,

\[
\| I\phi_\lambda \|_{H^1}, E(I\phi_\lambda) \lesssim 1. \tag{1.11}
\]

**Step 2: Local theory for rescaled solutions.** We need the following local theory,

**Lemma 1.1** ([9]). Let \( s \geq \frac{1}{2} \) and \( \phi \) satisfy \( \| I\phi_\lambda \|_{H^1} \lesssim 1 \), then Cauchy problem (1.8) is locally well-posed on the interval \([0, \delta]\) with the lifetime

\[
\delta \sim \lambda^{-\epsilon} \tag{1.12}
\]

for some small \( \epsilon > 0 \). Furthermore, the solution satisfies the estimate

\[
\| Iu_\lambda \|_{X^1([0, \delta])} \lesssim \| I\phi_\lambda \|_{H^1}. \tag{1.13}
\]

**Step 3: Definition of modified energies.** It will be convenient to define

\[
f_\lambda(t) := e^{t\partial_x^3} u_\lambda(t),
\]

then one may find that

\[
\partial_t f_\lambda = e^{t\partial_x^2} \partial_x (u_\lambda^{k+1}).
\]

Therefore, we have

\[
\partial_t \widehat{f}_\lambda(\xi) = i\xi \int_{\xi_1 + \cdots + \xi_{k+1} = \xi} e^{i(-\xi_1^3+\xi_1^3+\cdots+\xi_{k+1}^3)t} \widehat{f}_\lambda(t, \xi_1) \cdots \widehat{f}_\lambda(t, \xi_{k+1})(d\xi_1)_\lambda \cdots (d\xi_k)_\lambda. \tag{1.14}
\]

We denote \( m_j = m(\xi_j) \), \( \alpha_{k+2} = \xi_1^3 + \cdots + \xi_{k+2}^3 \), \( \Gamma_n \) to be the hyperplane

\[
\Gamma_n = \left\{ (\xi_1, \cdots, \xi_n) \in \left( \frac{Z}{\lambda} \right)^n : \xi_1 + \cdots + \xi_n = 0 \right\}. \tag{1.15}
\]
From Plancherel’s identity (see (2.1) below), it follows that
\[
E(Iu_\lambda) = \frac{1}{2} \int_0^\lambda |\partial_x Iu_\lambda(t, x)|^2 \, dx + \frac{1}{k + 2} \int_0^\lambda |Iu_\lambda(t, x)|^{k+2} \, dx
\]
\[
= \frac{1}{2} \int_{\Gamma_2} m_1^2 \xi_1^2 \hat{f}_\lambda(t, \xi_1) \hat{f}_\lambda(t, \xi_2) (d\xi_1)_\lambda
\]
\[
+ \frac{1}{k + 2} \int_{\Gamma_{k+2}} m_1 \cdots m_{k+2} e^{i\alpha_{k+2} t} \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{k+2}) (d\xi_1)_\lambda \cdots (d\xi_{k+1})_\lambda.
\]

By (1.14), the symmetries of the variables $\xi_j$ in the integration and a direct computation, we have
\[
\frac{d}{dt} E(Iu_\lambda(t)) = \int_{\Gamma_{k+2}} e^{i\alpha_{k+2} t} (M_{k+2} + i\sigma_{k+2} \alpha_{k+2}) \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{k+2}) (d\xi_1)_\lambda \cdots (d\xi_{k+1})_\lambda
\]
\[
+ \int_{\Gamma_{2k+2}} e^{i\alpha_{2k+2} t} M_{2k+2} \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{2k+2}) (d\xi_1)_\lambda \cdots (d\xi_{2k+1})_\lambda.
\]

where
\[
M_{k+2}(\xi_1, \cdots, \xi_{k+2}) := i(m_1^2 \xi_1^3 + \cdots + m_{k+2}^2 \xi_{k+2}^3); \quad \sigma_{k+2} := \frac{1}{k + 2} m_1 \cdots m_{k+2};
\]
\[
M_{2k+2}(\xi_1, \cdots, \xi_{2k+2}) := i(k + 2)[\sigma_{k+2}(\xi_1, \cdots, \xi_{k+1}, \xi_{k+2} + \cdots + \xi_{2k+2})](\xi_{k+2} + \cdots + \xi_{2k+2})_{\text{sym}},
\]
and $[m]_{\text{sym}}$ denotes the symmetrization of a multiplier $m$ (see [8]).

Now we focus our attention on the term (1.16), and consider the quantity
\[
\frac{M_{k+2}}{\alpha_{k+2}}.
\]

If it makes sense, then one may use the identity
\[
e^{i\alpha_{k+2} s} = \frac{1}{i\alpha_{k+2}} \partial_s (e^{i\alpha_{k+2} s}),
\]
and take the derivative in $s$. The process is similar as what in the normal form method. This way gives the definition of the second modified energy, and may improve the tiny increment estimate of $E(Iu)$.

One may find (1.18) is bounded when $k = 1, 2$. But unfortunately, (1.18) is singular and thus does not make sense in general when $k \geq 3$. So it fails to define the second modified energy in this way. Here our argument is the resonance decomposition.

To do this, we first make a convenient reduction. Denote $\xi_1^*, \cdots, \xi_{k+2}^*, \cdots, \xi_{2k+2}^*$ to be the rearrangement of $\xi_1, \cdots, \xi_{k+2}, \cdots, \xi_{2k+2}$, with $|\xi_1^*| \geq \cdots \geq |\xi_{k+2}^*| \geq \cdots \geq |\xi_{2k+2}^*|$.

**Remark 1.1** (A convenient reduction). If $|\xi_1^*| \ll N$, then $M_{k+2}, M_{2k+2} = 0$ which gives the conservation of $E(Iu_\lambda)$. Hence one may restrict $|\xi_1^*| \gtrsim N$ in the support of $\Gamma_{k+2}$ and $\Gamma_{2k+2}$.
Now we define the “non-resonance” set using the spirit in [27], let

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,$$

where

$$\Omega_1 = \{(\xi_1, \ldots, \xi_{k+2}) \in \Gamma_{k+2} : |\xi_4^\ast| \gg |\xi_4^\ast|\};$$
$$\Omega_2 = \{(\xi_1, \ldots, \xi_{k+2}) \in \Gamma_{k+2} : |\xi_1^\ast| \sim |\xi_2^\ast| \gg N \gg |\xi_3^\ast|, |\xi_1^\ast + \xi_2^\ast| \gg |\xi_3^\ast + \cdots + \xi_{k+2}^\ast|\};$$
$$\Omega_3 = \{(\xi_1, \ldots, \xi_{k+2}) \in \Gamma_{k+2} : |\xi_1^\ast| \gg |\xi_3^\ast|, |\xi_1^\ast + \xi_2^\ast||\xi_1^\ast + \xi_3^\ast| \gg |\xi_3^\ast| \xi_1^\ast|^2\};$$
$$\Omega_4 = \{(\xi_1, \ldots, \xi_{k+2}) \in \Gamma_{k+2} : |\xi_1^\ast| \gg |\xi_3^\ast|, |\xi_1^\ast + \xi_2^\ast||\xi_1^\ast + \xi_3^\ast| \gg |\xi_3^\ast||\xi_1^\ast|^2\} + \ldots + m(\xi_1^\ast)^2|\xi_1^\ast| \gg |m(\xi_5^\ast)^2|\xi_5^\ast| + m(\xi_6^\ast)^2|\xi_6^\ast|\},$$

and $\xi_6^\ast = 0$ if $k = 3$.

Compared with the “non-resonance” sets defined in [27], we add the set $\Omega_3$ and slightly change the definition on $\Omega_4$. They are employed to overcome the trouble from the weak Strichartz estimates in the periodic setting.

First, we show that (1.18) is bounded in “non-resonance” set, that is,

**Lemma 1.2.**

$$|M_{k+2}| \lesssim |\alpha_{k+2}|, \quad \text{in } \Omega.$$  \hspace{1cm} (1.20)

Second, we have

**Lemma 1.3.** In $\Gamma_{k+2} \setminus \Omega$,

(1) It holds that

$$|M_{k+2}| \lesssim m(\xi_1^\ast)^2|\xi_1^\ast||\xi_3^\ast|^2.$$  \hspace{1cm} (1.21)

(2) If $|\xi_1^\ast| \sim |\xi_2^\ast| \gg N \gg |\xi_3^\ast| \sim |\xi_4^\ast|$, then

$$|M_{k+2}| \lesssim |\xi_3^\ast||\xi_4^\ast||\xi_5^\ast|.$$  \hspace{1cm} (1.22)

(3) If $|\xi_1^\ast| \gg |\xi_5^\ast|$, then

$$|M_{k+2}| \lesssim m(\xi_1^\ast)^2|\xi_1^\ast|^2|\xi_5^\ast|.\hspace{1cm} (1.23)$$

Lemma 1.3 implies that the bound of $M_{k+2}$ in “resonance” set is less than its natural bound (which is $m(\xi_1^\ast)^2|\xi_1^\ast|^2|\xi_3^\ast|$).
Based on these two lemmas, we rewrite the term (1.16) as
\[
\begin{align*}
(1.16) & = \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} (\chi_{\Omega} M_{k+2} + i\sigma_{k+2} \alpha_{k+2}) \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2})(d\xi_1) \cdots (d\xi_{k+1}) \\
& + \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} (1 - \chi_{\Omega}) M_{k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2})(d\xi_1) \cdots (d\xi_{k+1}) .
\end{align*}
\]
From Lemma 1.2, \( \frac{1}{i\alpha_{k+2}} (\chi_{\Omega} M_{k+2} + i\sigma_{k+2} \alpha_{k+2}) \) is bounded. Thus by (1.19), integration by parts in time, (1.14), and combining with (1.17), we have
\[
\frac{d}{dt} E(I u_\lambda(t)) = \int_{\Gamma_{k+2}} \partial_t (e^{i\alpha_{k+2}t}) \frac{1}{i\alpha_{k+2}} (\chi_{\Omega} M_{k+2} + i\sigma_{k+2} \alpha_{k+2}) \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2}) \\
+ \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} (1 - \chi_{\Omega}) M_{k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2}) \\
+ \int_{\Gamma_{2k+2}} e^{i\alpha_{k+2}t} M_{2k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{2k+2}) \\
= \frac{d}{dt} \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} \chi_{\Omega} M_{k+2} + i\sigma_{k+2} \alpha_{k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2}) \\
+ \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} (1 - \chi_{\Omega}) M_{k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2}) \\
+ \int_{\Gamma_{2k+2}} e^{i\alpha_{k+2}t} M_{2k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{2k+2}), \quad (1.24)
\]
where
\[
\overline{M_{2k+2}} := i(k + 2) \left[ \tilde{\sigma}_{k+2} (\xi_1, \cdots, \xi_{k+1}, \xi_{k+2} + \cdots + \xi_{2k+2})(\xi_{k+2} + \cdots + \xi_{2k+2}) \right]_{\text{sym}},
\]
and \( \tilde{\sigma}_{k+2} = -\chi_{\Omega} M_{k+2} / \alpha_{k+2} \). In particular, we have the bound of \( \overline{M_{2k+2}} \) as follows.

**Lemma 1.4.** In \( \Gamma_{2k+2} \),

(1). It holds that
\[
|\overline{M_{2k+2}}| \lesssim |\xi_1^*|. \quad (1.26)
\]

(2). If \( |\xi_1^*| \sim |\xi_2^*| \sim N \gg |\xi_3^*| \sim |\xi_4^*| \), then
\[
|\overline{M_{2k+2}}| \lesssim |\xi_3^*|. \quad (1.27)
\]

According to (1.24), we define
\[
E_1^2(u_\lambda(t)) := E(I u_\lambda(t)) - \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} \chi_{\Omega} M_{k+2} + i\sigma_{k+2} \alpha_{k+2} \tilde{f}_\lambda(t, \xi_1) \cdots \tilde{f}_\lambda(t, \xi_{k+2}), \quad (1.28)
\]
and get
\[
\frac{d}{dt} E^2_I(u_\lambda(t)) = \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} (1 - \chi_\Omega) M_{k+2} \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{k+2}) \\
+ \int_{\Gamma_{2k+2}} e^{i\alpha_{2k+2}t} M_{2k+2} \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{2k+2}).
\]
This gives that
\[
E^2_I(u_\lambda(t)) = E^2_I(u_\lambda(0)) + \int_0^t \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}s} (1 - \chi_\Omega) M_{k+2} \hat{f}_\lambda(s, \xi_1) \cdots \hat{f}_\lambda(s, \xi_{k+2}) \\
+ \int_0^t \int_{\Gamma_{2k+2}} e^{i\alpha_{2k+2}s} M_{2k+2} \hat{f}_\lambda(s, \xi_1) \cdots \hat{f}_\lambda(s, \xi_{2k+2}).
\]
(1.29)

Step 4: Energy increment estimates.

By the preparation in Step 1–Step 3, we can derive the following proposition, which is sufficient to prove Theorem 1.1.

**Proposition 1.1** (Existence of an almost conserved quantity). For a solution $u_\lambda$ to (1.8) which is smooth-in-time, Schwartz-in-space on the time interval $[0, \delta]$ with $\delta$ satisfying (1.12), we have

- **(Fixed-time bound)**
  \[
  \left| E^2_I(u_\lambda(t)) - E(Iu_\lambda(t)) \right| \lesssim N^{-2+} \|Iu_\lambda(t)\|^{k+2}_{H^2_x}. \tag{1.30}
  \]

- **(Almost conservation law)** Let $\|Iu_\lambda\|_{X^1([0,\delta])} \lesssim 1$, then
  \[
  \left| E^2_I(u_\lambda(\delta)) - E^2_I(u_\lambda(0)) \right| \leq K := N^{-3+} + N^{-2+} \lambda^{-\frac{1}{2}}. \tag{1.31}
  \]

Now the paper is organized as follows. In Section 2, we introduce some notations and state some preliminary estimates that will be used throughout this paper. In Section 3, we prove Lemma 1.2, Lemma 1.3 and Lemma 1.4. In Section 4, we give the proof of Proposition 1.1. In Section 5, we show that Proposition 1.1 implies the global well-posedness stated in Theorem 1.1.

2. Notations and Preliminary Estimates

2.1. Basic notations and definitions. We use $A \lesssim B, B \gtrsim A$, or sometimes $A = O(B)$ to denote the statement that $A \leq CB$ for some large constant $C$ which may vary from line to line, and may depend on the data and the index $s$. When it is necessary, we will write the constants by $C_1, C_2, \cdots$ to see the dependency relationship. We use $A \sim B$ to mean $A \lesssim
$B \lesssim A$. We use $A \ll B$, or sometimes $A = o(B)$ to denote the statement $A \leq C^{-1} B$. The notation $a+$ denotes $a + \epsilon$ for any small $\epsilon$, and $a-$ for $a - \epsilon$. $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, $|\nabla|^\alpha = (-\partial_x^2)^{\alpha/2}$ and $J_x^\alpha = (1 - \partial_x^2)^{\alpha/2}$. We use $\|f\|_{L_p^g L_q^i}$ to denote the mixed norm $\left( \int \|f(\cdot, x)\|_{L_q^i}^p dx \right)^{\frac{1}{p}}$, and $\|f\|_{L_p^g} := \|f\|_{L_p^g L_q^i}$.

Throughout this paper, we use $\eta$ to denote a smooth cut-off function such that

$$
\eta(x) = \begin{cases} 
1, & |x| \leq 1, \\
0, & |x| \geq 2.
\end{cases}
$$

For an interval $I \subset \mathbb{R}$, we denote $\chi_I$ as its characteristic function

$$
\chi_I(x) = \begin{cases} 
1, & x \in I, \\
0, & x \notin I.
\end{cases}
$$

Now we introduce some other notations and definitions, some of which are employed from [8]. We define $(d\xi)_\lambda$ to be the normalized counting measure on $\mathbb{Z}/\lambda$ such that

$$
\int a(\xi) (d\xi)_\lambda = \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}/\lambda} a(\xi).
$$

The Fourier transform of a function $f$ on $\mathbb{T}_\lambda = \mathbb{R}/\lambda \mathbb{Z}$ is defined by

$$
\hat{f}(\xi) = \int_0^\lambda e^{-2\pi i x \xi} f(x) \, dx,
$$

and thus the Fourier inversion formula

$$
f(x) = \int e^{2\pi i x \xi} \hat{f}(\xi) \, (d\xi)_\lambda.
$$

Then the following usual properties of the Fourier transform hold,

\begin{align*}
\|f\|_{L^2([0, \lambda])} &= \|\hat{f}\|_{L^2((d\xi)_\lambda)} \quad \text{(Plancherel);} \\
\int_0^\lambda f(x) \overline{g(x)} \, dx &= \int \hat{f}(\xi) \overline{\hat{g}(\xi)} \, (d\xi)_\lambda \quad \text{(Parseval);} \\
\hat{f}(\xi) &= \int \hat{f}(\xi - \xi_1) \overline{\hat{g}(\xi_1)} \, (d\xi_1)_\lambda \quad \text{(Convolution).}
\end{align*}

We define the Sobolev space $H^s([0, \lambda])$ with the norm,

$$
\|f\|_{H^s([0, \lambda])} = \left\| \langle \xi \rangle^s \hat{f}(\xi) \right\|_{L^2((d\xi)_\lambda)}.
$$

For $s, b \in \mathbb{R}$, define the Bourgain space $X_{s,b}$ to be the closure of the Schwartz class under the norm

$$
\|f\|_{X_{s,b}} := \left( \int \int \langle \xi \rangle^{2s} |\tau - \xi|^b |\hat{f}(\tau, \xi)|^2 (d\xi)_\lambda d\tau \right)^{\frac{1}{2}},
$$

(2.4)
for any $\lambda$-periodic function $f$. The space $X_{s, \frac{1}{2}}$ barely fails to control the $L^\infty_\lambda([0,T], H^s(\mathbb{R}))$ norm. To rectify this we define the slightly stronger space $Y^s$ under the norm
\[
\|f\|_{Y^s} := \|f\|_{X_{s, \frac{1}{2}}} + \left\|\langle \xi \rangle^s \hat{f}\right\|_{L^2(\mathbb{R})}.
\]
Moreover, we define the restricted space $Y^s(I)$ as
\[
\|f\|_{Y^s(I)} := \inf\{\|\tilde{f}\|_{Y^s} : \tilde{f} = f, \text{ on } I\},
\]
and as (1.3),
\[
\|f\|_{X^s(I)} := \|Gf\|_{Y^s(I)}.
\]
If there is no confusion, we will not mention the restriction.

2.2. Some linear estimates.

**Lemma 2.1.** Let $0 < \delta < 1$, and $f \in X_{0, \frac{1}{2}}$. Then $\chi_{[0, \delta]}(t)f \in X_{0,b}$ for any $b < \frac{1}{2}$, and
\[
\|\chi_{[0, \delta]}(t)f\|_{X_{0,b}} \lesssim \|f\|_{X_{0, \frac{1}{2}}}.
\]

**Proof.** Note that
\[
\|J_t^s \chi_{[0, \delta]}(t)\|_{L^p_t} + \|\chi_{[0, \delta]}(t)\|_{L^\infty_\lambda} \lesssim 1
\]
for any $s < \frac{1}{p}$. Then by the fractional product role, Hölder’s and Sobolev’s inequalities, we have
\[
\|\chi_{[0, \delta]}(t)e^{it\partial_x^2}f\|_{H^b_t} \lesssim \|J_t^b \chi_{[0, \delta]}(t)\cdot e^{it\partial_x^2}f\|_{L^2_t} + \|\chi_{[0, \delta]}(t)\cdot J_t^b e^{it\partial_x^2}f\|_{L^2_t}
\]
\[
\lesssim \|J_t^b \chi_{[0, \delta]}(t)\|_{L^p_t} \|e^{it\partial_x^2}f\|_{L^q_t} + \|\chi_{[0, \delta]}(t)\|_{L^\infty_\lambda} \|J_t^b e^{it\partial_x^2}f\|_{L^2_t}
\]
\[
\lesssim \|f\|_{H^b_t},
\]
where $p = 2+$, $b < \frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. From this estimate, we have
\[
\|\chi_{[0, \delta]}(t)f\|_{X_{0,b}} = \|\chi_{[0, \delta]}(t)e^{it\partial_x^2}f(t, x)\|_{L^2_t H^b_t}
\]
\[
\lesssim \|e^{it\partial_x^2}f\|_{L^2_t H^{\frac{1}{2}}_t} = \|f\|_{X_{0, \frac{1}{2}}}.
\]
This proves the lemma. \qed

Now we state some preliminary estimates which will be used in the following sections. First we recall some well-known Strichartz estimates (see [1, 9], for examples):
\[
\|f\|_{L^4_t L^2_\lambda} \lesssim \|f\|_{X_{0, \frac{1}{2}}},
\]
and
\[
\|f\|_{L^6_t L^3_\lambda} \lesssim \lambda^{0+} \|f\|_{X_{0+, \frac{1}{2}+}}.
\]
It follows from the interpolation between (2.6) and (2.7) that
\[ \| f \|_{L^q_{xt}} \lesssim \lambda^{0+} \| f \|_{X_{0^+, \frac{1}{2} - \sigma(q)}}, \] (2.8)
for all $4 < q < 6$ and $\sigma(q) < 2\left(\frac{1}{q} - \frac{1}{6}\right)$.

Since the $L^q$-norm is invariant under the gauge transformation, we have almost the same Strichartz estimates between $X^s$ and $Y^s$. In particular, we have the following two estimates.

**Lemma 2.2.**  
(1) Let $s > \frac{1}{2}$ and $f \in X^s$, then
\[ \| f \|_{L^\infty_{xt}} \lesssim \| f \|_{X^s}. \] (2.9)
(2) Let $s > 0$ and $f \in X^s$, then
\[ \| f \|_{L^6_{xt}} \lesssim \lambda^{0+} \| f \|_{X^s}. \] (2.10)

**Proof.** Let $g = Gf$. For (2.9), by Young’s and Cauchy-Schwartz’s inequalities, we have
\[ \| f \|_{L^\infty_{xt}} = \| g \|_{L^\infty_{xt}} \lesssim \| \hat{g} \|_{L^1((d\xi)(d\tau))} \lesssim \left\| \langle \xi \rangle^{\frac{1}{2} + \rho} \hat{g} \right\|_{L^2((d\xi)(d\tau))}. \]

For (2.10), we note that $\| f \|_{L^6_{xt}} = \| g \|_{L^6_{xt}}$. Then by the dyadic decomposition, we write $g = \sum_{j=0}^{\infty} g_j$, for each dyadic constituents $g_j$ with frequency support $\langle \xi \rangle \sim 2^j$. Then, by (2.8) and (2.9),
\[
\| g \|_{L^6_{xt}} \leq \sum_{j=0}^{\infty} \| g_j \|_{L^6_{xt}} \lesssim \sum_{j=0}^{\infty} \| g_j \|_{L^6_{xt}} \| g_j \|_{L^1_{xt}}^{1-\theta} \\
\lesssim \lambda^{0+} \sum_{j=0}^{\infty} \| g_j \|_{X_{0^+, \frac{1}{2}}} \| g_j \|_{Y^\theta}^{1-\theta} \lesssim \lambda^{0+} \sum_{j=0}^{\infty} 2^{[\theta\epsilon + \rho(1-\theta)]j} \| g_j \|_{X_{0^+, \frac{1}{2}}} \| g_j \|_{Y^\theta}^{1-\theta} \\
\lesssim \lambda^{0+} \sum_{j=0}^{\infty} 2^{[\theta\epsilon + \rho(1-\theta)]j} \| g_j \|_{Y^\theta},
\]
where $\rho > \frac{1}{2}$, and we choose $q = 6$ such that $\epsilon = 0+$, $\theta = 1$. Choosing $q$ close enough to 6 such that $s > \theta\epsilon + \rho(1-\theta)$, then we have the claim by Cauchy-Schwartz’s inequality. \qed

Interpolation between (2.9) and (2.10), we have
\[ \| f \|_{L^q_{xt}} \lesssim \lambda^{0+} \| f \|_{Y^{\beta(q)}}, \] (2.11)
for all $6 < q < \infty$ and $\beta(q) > (\frac{1}{2} - \frac{3}{q})$. 
2.3. Bilinear Strichartz estimate. Now we present the bilinear Strichartz estimates in the periodic version. Let \( S_\lambda(t) \) be the solution map to the free KdV equation
\[
\partial_t u + \partial_x^3 u = 0, \quad \text{in } [0, \lambda^3 T] \times [0, \lambda],
\]
and the bilinear operator \( I_M(f, g) \) satisfy
\[
I_M(f, g)(\xi) = \int_{\xi = \xi_1 + \xi_2} \chi_{\{|\xi_1^2 - \xi_2^2| \geq M\}} \hat{f}(\xi_1)\hat{g}(\xi_2)(d\xi_1)\lambda. \tag{2.12}
\]
Now we recall the following the bilinear Strichartz estimate, which was obtained in [25], see also [6].

**Proposition 2.1.** Let \( u = u(t, x), v = v(t, x) \) be the \( \lambda \)-periodic functions of \( x \), then
\[
\| \eta^2(t) I_M(u, v) \|_{L_{x, t}^2} \lesssim C(M, \lambda) \| u \|_{X_{0, \frac{4}{3}}} \| v \|_{X_{0, \frac{4}{3}}}, \tag{2.13}
\]
where
\[
C(M, \lambda) = \begin{cases} 
1, & M \leq 1, \\
\left(\frac{1}{M} + \frac{1}{\lambda}\right)^\frac{1}{2}, & M > 1.
\end{cases} \tag{2.14}
\]

**Remark 2.1.** In particular, we set \( \lambda \) to be the number in (1.10). Then we see that \( C(N^2, \lambda) \), for which bound we use in this paper, has the similar size of \( \lambda^{-\frac{1}{2}} \) rather than \( N^{-1} \). Indeed, when \( s \geq 1 \), \( k = 3, 4, \)
\[
(1 - s)/(\frac{2}{k} + s - \frac{1}{2}) < 2,
\]
thus \( \lambda^{-\frac{1}{2}} > N^{-1} \). This means that the efficacy of the bilinear Strichartz estimate in the periodic case is exactly weaker than the one in the real line case.

**Corollary 2.1.** Let \( u, v, I_M \) be as Proposition 2.1, and let \( \lambda \) be the number in (1.10), then for \( N \gg 1 \),
\[
\| \eta^2(t) I_{N^2}(u, v) \|_{L_{x, t}^2} \lesssim \lambda^{\frac{1}{2}} \| u \|_{X_{0, \frac{4}{3}}} \| v \|_{X_{0, \frac{4}{3}}}. \tag{2.15}
\]

**Proof.** First, by interpolating between (2.13) and the following estimate
\[
\| \eta^2(t) I_M(u, v) \|_{L_{x, t}^2} \lesssim \| u \|_{L_{x, t}^4} \| v \|_{L_{x, t}^4} \lesssim \| u \|_{X_{0, \frac{4}{3}}} \| v \|_{X_{0, \frac{4}{3}}},
\]
we have
\[
\| \eta^2(t) I_M(u, v) \|_{L_{x, t}^2} \lesssim C(M, \lambda)^{-1} \| u \|_{X_{0, \frac{4}{3}}} \| v \|_{X_{0, \frac{4}{3}}}. \tag{2.16}
\]
In particular, when \( M = N^2 \), by Remark 2.1,
\[
C(N^2, \lambda) = \lambda^{-\frac{1}{2}}, \quad \text{whenever } N \gg 1.
\]
This proves the corollary.
3. Proof of Lemmas 1.2–1.4

3.1. Proof of Lemma 1.2. Note that
\[ \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \]
so we need to prove that in every \( \Omega_j, j = 1, 2, 3, 4, \)
\[ |M_{k+2}| \lesssim |\alpha_{k+2}|. \]

The estimates in \( \Omega_1, \Omega_2 \) and \( \Omega_4 \) are similar as Lemma 4.2 in [27]. To simplify the notations, we set \( \xi_j^* = \xi_j, j = 1, \cdots, k+2. \)

In \( \Omega_1 \), we note that \( \xi_1 \cdot \xi_2 < 0 \), thus,
\[ |\alpha_{k+2}| = |\xi_1^3 + \xi_2^3 + \xi_3^3| + o(|\xi_3^3|) \]
\[ = |(\xi_1 + \xi_2)(\xi_1^2 - \xi_1\xi_2 + \xi_2^2) + \xi_3^3| + o(|\xi_3^3|) \]
\[ = |\xi_3(\xi_1^2 - \xi_1\xi_2 + \xi_2^2 - \xi_3^2)| + o(|\xi_3^3|) \geq |\xi_3\xi_1^2| + o(|\xi_3^3|) \sim |\xi_3||\xi_1|^2. \]

Moreover, by the mean value theorem,
\[ |M_{k+2}| \lesssim |m_{1}\xi_1^3 + m_2\xi_2^3| + |m_3\xi_3^3| + \cdots + |m_{k+2}\xi_{k+2}^3| \]
\[ \lesssim m_2\xi_1^2|\xi_1 + \xi_2| + |\xi_3^3| \]
\[ \lesssim |\xi_1^3||\xi_3|. \]

Thus we obtain the desirable estimate in \( \Omega_1 \).

In \( \Omega_2 \), we have
\[ |\alpha_{k+2}| \sim |\xi_1^3 + \xi_2^3|. \]

Moreover,
\[ |M_{k+2}| \leq |m_{1}\xi_1^3 + m_2\xi_2^3| + |\xi_3^3 + \cdots + \xi_{k+2}^3| \]
\[ \lesssim m_1^2|\xi_1^3 + \xi_2^3| + |\xi_3^3 + \cdots + \xi_{k+2}^3| \lesssim |\xi_1^3 + \xi_2^3|. \]

So these give the desirable estimate in \( \Omega_2 \).

In \( \Omega_3 \), on one hand, since
\[ |\xi_1^3 + \xi_2^3| \sim \xi_1^2|\xi_1 + \xi_2| \gg |\xi_1||\xi_3|^2 \gg |\xi_3|^3, \tag{3.1} \]
thus,
\[ |\alpha_{k+2}| = |(\xi_1^3 + \xi_2^3) + (\xi_3^3 + \cdots + \xi_{k+2}^3)| \sim |\xi_1^3 + \xi_2^3|. \]
On the other hand, by the mean value theorem and (3.1),

\[ |M_{k+2}| \lesssim |m_1^2 \xi_1^3 + m_2^2 \xi_2^3| + |m_3^2 \xi_3^3| + \cdots + |m_{k+2}^2 \xi_{k+2}^3| \]
\[ \lesssim m_2^2 |\xi_1^3 + \xi_2^3| + |\xi_3^3| \]
\[ \lesssim |\xi_1^3 + \xi_2^3|. \]

Together these two estimates, gives the desirable estimate in \( \Omega_3 \).

For \( \Omega_4 \), we set \( \bar{\xi}_4 = \xi_4 + \xi_5 + \xi_6 \), then \( \xi_1 + \xi_2 + \xi_3 + \bar{\xi}_4 = 0 \). Therefore,

\[ \alpha_{k+2} = \xi_1^3 + \xi_2^3 + \xi_3^3 + \bar{\xi}_4^3 + (\xi_4^3 - \bar{\xi}_4^3) + \xi_5^3 + \xi_6^3 \]
\[ = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \bar{\xi}_4) + (\xi_4^3 - \bar{\xi}_4^3) + \xi_5^3 + \xi_6^3 \]
\[ = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4) + O(|\xi_5 + \xi_6| \xi_1^2) \]
\[ = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4) + O(|\xi_5| \xi_1^2). \]

By the definition of \( \Omega_4 \), \( |\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4| \gg |\xi_5| \xi_1^2 \). Thus we have

\[ |\alpha_{k+2}| \sim |\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4|. \quad (3.2) \]

By the similar way and the mean value theorem, we have

\[ M_{k+2} = m(\xi_1)^2 \xi_1^3 + m(\xi_2)^2 \xi_2^3 + m(\xi_3)^2 \xi_3^3 + m(\bar{\xi}_4)^2 \bar{\xi}_4^3 + (m(\xi_4)^3 - m(\bar{\xi}_4)^3) + \xi_5^3 + \xi_6^3 \]
\[ = m(\xi_1)^2 \xi_1^3 + m(\xi_2)^2 \xi_2^3 + m(\xi_3)^2 \xi_3^3 + m(\bar{\xi}_4)^2 \bar{\xi}_4^3 + O(|\xi_5| \xi_1^2). \quad (3.3) \]

Now we claim that

\[ |m(\xi_1)^2 \xi_1^3 + m(\xi_2)^2 \xi_2^3 + m(\xi_3)^2 \xi_3^3 + m(\bar{\xi}_4)^2 \bar{\xi}_4^3| \lesssim |\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4|. \quad (3.4) \]

To prove this, we split it into two cases: either \( |\xi_1| - |\xi_4| = o(|\xi_1|) \), or \( |\xi_1| - |\xi_4| \sim |\xi_1| \). The first case follows from the following double mean value theorem.

**Lemma 3.1** (Double mean value theorem, Lemma 4.1 in [27]). Let \( f(\xi) = m(\xi)^2 \xi^3 \), then for \( |\eta|, |\lambda| \ll |\xi| \),

\[ |f(\xi + \eta + \lambda) - f(\xi + \eta) - f(\xi + \lambda) + f(\xi)| \lesssim |f''(\xi)||\eta||\lambda|. \quad (3.5) \]

In the first case, we may set \( \xi_1 > 0 \) by symmetries, then it can be split into the following three subcases,

\begin{itemize}
  \item (1), \( \xi_1 > 0, \xi_2 < 0, \xi_3 < 0, \xi_4 > 0 \);
  \item (2), \( \xi_1 > 0, \xi_2 < 0, \xi_3 > 0, \xi_4 < 0 \);
  \item (3), \( \xi_1 > 0, \xi_2 > 0, \xi_3 < 0, \xi_4 < 0 \).
\end{itemize}
For (1), we take \( \xi = \xi_1, \eta = -(\xi_1 + \xi_2), \lambda = -(\xi_1 + \xi_3) \) in Lemma 3.1, then \( |\eta| \lesssim |\lambda| \ll |\xi| \).
Hence using (3.5), we have
\[
|m(\xi_1)^2\xi_1^3 + m(\xi_2)^2\xi_2^3 + m(\xi_3)^2\xi_3^3 + m(\overline{\xi_1})^2\overline{\xi_1}| \lesssim m_1^2|\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \overline{\xi_1}|
\]
\[
\lesssim |\xi_1 + \xi_2| |\xi_1 + \xi_3||\xi_1 + \xi_4| + O(|\xi_5|^2) \sim |\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4|,
\]
where we have used \( |(m(\xi_1)^2\xi_1^3)| \lesssim m_1^2|\xi_1| \).

For (2), we take \( \xi = \xi_1, \eta = -(\xi_1 + \xi_2), \lambda = -(\xi_1 + \xi_4); \) For (3), we take \( \xi = \xi_1, \eta = -(\xi_1 + \xi_3), \lambda = -(\xi_1 + \xi_4) \). Then by (3.5), we obtain the same estimate as above.

In the second case, we also have \( |\xi_1| - |\xi_3| \sim |\xi_1| \), and thus
\[
|\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4| \sim |\xi_1^2|\xi_1 + \xi_2|.
\]
Therefore,
\[
|m(\xi_1)^2\xi_1^3 + m(\xi_2)^2\xi_2^3 + m(\xi_3)^2\xi_3^3 + m(\overline{\xi_1})^2\overline{\xi_1}| \lesssim |m(\xi_1)^2\xi_1^3 + m(\xi_2)^2\xi_2^3| + |m(\xi_3)^2\xi_3^3 + m(\overline{\xi_1})^2\overline{\xi_1}| \lesssim m_1^2|\xi_1 + \xi_2| + m_3^2|\xi_3 + \xi_4| \lesssim |\xi_1 + \xi_2| + |\xi_3 + \xi_4|.
\]
This proves (3.4). Now combining with (3.3), we have
\[
|M_{k+2}| \lesssim |\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4| + O(|\xi_5|^2) \sim |\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4|.
\]
Together with (3.2), we obtain \( |M_{k+2}| \lesssim |\alpha_{k+2}| \), which is the desirable estimate in \( \Omega_4 \). This completes the proof of the lemma.

3.2. Proof of Lemma 1.3. We may assume that \( |\xi_1| \geq \cdots \geq |\xi_{k+2}| \) by symmetries, and set \( \xi_6 = 0 \) if \( k = 3 \). Recall that
\[
\Gamma_{k+2} \setminus \Omega = (\Gamma_{k+2} \setminus \Omega_1) \cap (\Gamma_{k+2} \setminus \Omega_2) \cap (\Gamma_{k+2} \setminus \Omega_3) \cap (\Gamma_{k+2} \setminus \Omega_4).
\]
First, we consider (1.21). If \( |\xi_1| \sim |\xi_2| \sim |\xi_3| \), then
\[
|M_{k+2}| \lesssim m_1^2|\xi_1^3| \sim m_1^2|\xi_1||\xi_3|^2.
\]
If \( |\xi_1| \sim |\xi_2| \gg |\xi_3| \), then by the definition of \( \Omega_1 \) and \( \Omega_3 \), we have in \( \Gamma_{k+2} \setminus \Omega \),
\[
|\xi_1| \sim |\xi_2| \gg |\xi_3| \sim |\xi_4|, \quad \text{and} \quad |\xi_1||\xi_1 + \xi_2| \lesssim |\xi_3|^2.
\]
Then by the mean value theorem and the inequality \( \frac{m(\xi)^2}{\eta} \leq \frac{m(\eta)^2}{\eta} \) if \( |\xi| \leq |\eta| \), we have

\[
|M_{k+2}| \leq |m_1^2 \xi_1^3 + m_2^2 \xi_2^3| + |m_3^2 \xi_3^3 + \cdots + m_{k+2}^2 \xi_{k+2}^3|
\]

\[
\lesssim m_1^2 |\xi_1^3 + \xi_2^3| + m_1^2 |\xi_3^3 + \cdots + \xi_{k+2}^3|
\]

\[
\lesssim m_1^2 |\xi_1|^3 + m_1^2 |\xi_3|^3
\]

\[
\lesssim m_1^2 |\xi_1||\xi_3|^2 + m_1^2 |\xi_3|^3
\]

\[
\lesssim m_1^2 |\xi_1||\xi_3|^2.
\]

This proves (1.21).

Now we consider (1.22). By the definition of \( \Omega_2 \), we have \( |\xi_1^3 + \xi_2^3| \lesssim |\xi_3^3 + \cdots + \xi_{k+2}^3| \) in \( \Gamma_{k+2} \setminus \Omega_2 \). This together with \( |\xi_1| \sim |\xi_2| \gg |\xi_3| \) implies that \( |\xi_1 + \xi_2| \ll |\xi_3| \sim |\xi_4| \). Now we claim that

\[
|\xi_1 + \xi_2| \ll |\xi_5|.
\]  

(3.6)

Indeed, it is trivial if \( |\xi_5| \sim |\xi_3| \); if \( |\xi_5| \ll |\xi_3| \) but \( |\xi_1 + \xi_2| \gg |\xi_5| \), then \( |\xi_3 + \xi_4| \ll |\xi_1 + \xi_2| \)

and thus

\[
|\xi_3^3 + \cdots + \xi_{k+2}^3| \lesssim |\xi_3^3 + \xi_4^3| + |\xi_3^3|
\]

\[
\lesssim \xi^3_3 |\xi_3 + \xi_4| + \xi^2_3 |\xi_5| \lesssim \xi^3_3 |\xi_1 + \xi_2| \ll \xi^2_3 |\xi_1 + \xi_2|.
\]

This contradicts with \( |\xi_1^3 + \xi_2^3| \lesssim |\xi_3^3 + \cdots + \xi_{k+2}^3| \). So we have (3.6). Using (3.6), we obtain

\[
|\xi_3 + \xi_4| \lesssim |\xi_1 + \xi_2| + |\xi_5| \lesssim |\xi_5|.
\]

Then by the mean value theorem, we get

\[
|M_{k+2}| \leq |m_1^2 \xi_1^3 + m_2^2 \xi_2^3| + |\xi_3^3 + \cdots + \xi_{k+2}^3|
\]

\[
\lesssim m_1^2 |\xi_1^3 + \xi_2^3| + |\xi_3^3 + \cdots + \xi_{k+2}^3|
\]

\[
\lesssim |\xi_3^3 + \cdots + \xi_{k+2}^3| \lesssim |\xi_3^3 + \xi_4^3| + |\xi_3^3|
\]

\[
\lesssim \xi_3^3 |\xi_3 + \xi_4| + |\xi_5^3| \lesssim |\xi_3||\xi_4||\xi_5|.
\]

We turn to consider (1.23). According to the definition of \( \Omega_4 \), we split it into the following two subsets,

\[
A_1 = \{(\xi_1, \ldots, \xi_6) \in \Gamma_{k+2} \setminus \Omega : |\xi_4| \gg |\xi_5|, \ |m_4^2 \xi_1^3 + \cdots + m_4^2 \xi_4^3| \lesssim |m_5^2 \xi_5^3 + m_6^2 \xi_6^3| \};
\]

\[
A_2 = \{(\xi_1, \ldots, \xi_6) \in \Gamma_{k+2} \setminus \Omega : |\xi_4| \gg |\xi_5|, \ |(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)| \lesssim |\xi_1|^2 |\xi_5| \}.
\]

In \( A_1 \), we have

\[
|M_{k+2}| \leq |m_1^2 \xi_1^3 + \cdots + m_4^2 \xi_4^3| + |m_5^2 \xi_5^3 + m_6^2 \xi_6^3|
\]

\[
\lesssim m_5^2 |\xi_5|^3 \lesssim m_1^2 |\xi_5|^3.
\]
In $A_2$, we may assume that $|\xi_1 + \xi_3| \ll |\xi_1|$ or $|\xi_1 + \xi_4| \ll |\xi_1|$; otherwise, if $|\xi_1 + \xi_3| \gtrsim |\xi_1|$ and $|\xi_1 + \xi_4| \gtrsim |\xi_1|$, then from the relation in $A_2$, we have $|\xi_1 + \xi_2| \lesssim |\xi_5|$ and thus $|\xi_3 + \xi_4| \lesssim |\xi_5|$. Then

$$|M_{k+2}| \lesssim |m_1^2\xi_1^3 + m_1^2\xi_1^3| + |m_1^2\xi_1^3 + m_1^2\xi_1^3| + |m_1^2\xi_1^3 + m_1^2\xi_1^3|$$

$$\lesssim m_1^2\xi_1^3|\xi_5| + m_2^2\xi_5^3|\xi_5| + m_2^2|\xi_5|^3 \lesssim m_1^2\xi_1^3|\xi_5|.$$ 

Therefore, $|\xi_1 + \xi_3| \ll |\xi_1|$ or $|\xi_1 + \xi_4| \ll |\xi_1|$. From the definition of $\Omega_1$, we have $|\xi_1| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4|$ in $A_2$. Further, we set $\xi_1 > 0$ by symmetries, and then have three cases as follows,

(1), $\xi_1 > 0, \xi_2 < 0, \xi_3 < 0, \xi_4 > 0$; (2), $\xi_1 > 0, \xi_2 < 0, \xi_3 > 0, \xi_4 < 0$;

(3), $\xi_1 > 0, \xi_2 > 0, \xi_3 < 0, \xi_4 < 0$.

Similar as the situation in the proof of Lemma 1.2, we only consider the case (1). Then we take $\xi = \xi_1, \eta = -(\xi_1 + \xi_2), \lambda = -(\xi_1 + \xi_3)$ in Lemma 3.1, to obtain that

$$|m_1^2\xi_1^3 + \cdots + m_4^2\xi_4^3| \lesssim m_1^2|\xi_1 + \xi_2||\xi_1 + \xi_4||\xi_1 + \xi_4| \lesssim m_1^2|\xi_1^2||\xi_5|.$$ 

Thus,

$$|M_{k+2}| \lesssim |m_1^2\xi_1^3 + \cdots + m_4^2\xi_4^3| + |m_1^2\xi_1^3 + m_2^2\xi_5^3|$$

$$\lesssim m_1^2|\xi_1 + \xi_2||\xi_1 + \xi_3||\xi_1 + \xi_4| + |m_2^2\xi_5^3 + m_2^2\xi_6^3|$$

$$\lesssim m_1^2|\xi_1^2||\xi_5| + m_2^2|\xi_5|^3 \lesssim m_1^2\xi_1^3|\xi_5|.$$ 

This proves the lemma.

3.3. Proof of Lemma 1.4. First, from Lemma 1.2, we have $|\overline{\sigma}_{k+2}| \lesssim 1$. Therefore, $|M_{2k+2}| \lesssim |\xi_1|$. Now we prove (1.27). For simplicity, we assume that $\xi_1 = \xi_1^*, \xi_2 = \xi_2^*$ by symmetries. Moreover, we denote $\eta_3, \cdots, \eta_{k+2}$ to be the rearrangement of $\xi_3, \cdots, \xi_{k+2}$, that is

$$\{\eta_3, \cdots, \eta_{k+2}\} = \{\xi_3, \cdots, \xi_{k+2}\}.$$ 

Then from (1.25), we rewrite $\overline{M_{2k+2}}$ as

$$\overline{M_{2k+2}} = i \left[ \overline{\sigma}_{k+2}(\xi_1, \xi_2, \eta_{k+3}, \cdots, \eta_{2k+2})\xi_1^* + \sigma_{k+2}(\xi_1, \xi_2, \eta_{k+3}, \cdots, \eta_{2k+2})\xi_2^* \right]_{\text{sym}(\eta)}$$

$$+ ik \left[ \overline{\sigma}_{k+2}(\xi_1, \xi_2, \eta_{k+3}, \eta_{k+4}, \cdots, \eta_{2k+2})\eta_{k+3} \right]_{\text{sym}(\eta)},$$

where the notation $\overline{a} = a + \eta_3 + \cdots + \eta_{k+2}$, and $[m]_{\text{sym}(\eta)}$ denotes the symmetrization of the multiplier $m$ only on the variables $\eta_3, \cdots, \eta_{k+2}$. 

First, we assume that
\[ |\overline{\eta_j}| \lesssim |\xi_3^*| \] for any \( j = k + 3, \ldots, 2k + 2 \), and \( |\sigma_{k+2}| \lesssim 1 \), we have
\[ |[\sigma_{k+2}(\xi_1, \xi_2, \eta_{k+3}, \eta_{k+4}, \ldots, \eta_{2k+2})\eta_{k+3}]_{sym(\eta)}| \lesssim |\xi_3^*|. \]
Furthermore, by the mean value theorem (see Lemma 4.5 in [27] for a similar proof),
\[ |[\sigma_{k+2}(\xi_1, \xi_2, \eta_{k+3}, \ldots, \eta_{2k+2})\xi_1 + \sigma_{k+2}(\xi_1, \overline{\xi_2}, \eta_{k+3}, \ldots, \eta_{2k+2})\overline{\xi_2}]_{sym(\eta)}| \lesssim |\xi_1 + \xi_2| \lesssim |\xi_3^*|. \]
This proves the lemma.

4. Proof of Proposition 1.1

We first give the fixed-time bound in Proposition 1.1. By (1.28), it reduces to the following lemma.

**Lemma 4.1.** For any \( 1/2 \leq s < 1 \),
\[ \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} \frac{\chi M_{k+2} + \sigma_{k+2} \alpha_{k+2}}{i\alpha_{k+2}} \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{k+2}) \left| \lesssim N^{-2+\|Iu_\lambda(t)\|^2} \right. \]

*Proof.* First, we assume that \( \hat{u}_\lambda \) is positive, otherwise one may replace it by \( |\hat{u}_\lambda| \). Second, we also assume that \( |\xi_1| \geq \cdots \geq |\xi_{k+2}| \) by symmetries. Moreover, by the reduction in Remark 1.1, we further assume \( |\xi_1| \sim |\xi_2| \gtrsim N \). Now by Lemma 1.2, we have
\[ \left| \frac{\chi M_{k+2} + \sigma_{k+2} \alpha_{k+2}}{i\alpha_{k+2}} \right| \lesssim 1. \]
Therefore, by Hölder’s and Sobolev’s inequalities, we have
\[ \int_{\Gamma_{k+2}} e^{i\alpha_{k+2}t} \frac{\chi M_{k+2} + \sigma_{k+2} \alpha_{k+2}}{i\alpha_{k+2}} \hat{f}_\lambda(t, \xi_1) \cdots \hat{f}_\lambda(t, \xi_{k+2}) \left| \lesssim N^{2(s-1)} \int_{\Gamma_{k+2}} |\xi_1|^{-s+k}|\xi_2|^{-s} \left| \nabla |Iu_\lambda(t, \xi_1)| \nabla |Iu_\lambda(t, \xi_2)| \nabla |\nabla^{-1+u_\lambda(t, \xi_3)}| \cdots |\nabla^{-1+u_\lambda(t, \xi_{k+2})}| \right| \right. \]
\[ \lesssim N^{-2+k} \left| \int_{\Gamma_{k+2}} |\nabla |Iu_\lambda(t, x)| \nabla |Iu_\lambda(t, x)| \nabla^{-1+u_\lambda(t, x)}|^k \right| \]
\[ \lesssim N^{-2+k} \left| Iu_\lambda \right|_{L^\infty_t H^1_x}^2 \left| \nabla^{-1+u_\lambda} \right|_{L^\infty_t H^1_x}^k \]
\[ \lesssim N^{-2+k} \left| Iu_\lambda \right|_{L^\infty_t H^1_x}^{k+2}, \]
where we have used that for any $\frac{1}{2} \leq s \leq 1$,

$$||\nabla^{-r} u_\lambda ||_{L^\infty_t H^s_x} \lesssim || u_\lambda ||_{L^\infty_t H^s_x} \lesssim || I u_\lambda ||_{L^\infty_t H^s_x}.$$ 

This proves the lemma. 

Lemma 4.1 proves (1.30). To prove (1.31), by (1.29), we need to estimate

$$\int_0^\delta \int_{\Gamma_{k+2}} e^{i\alpha k^2 s} (1 - \chi_\Omega) M_{k+2} \hat{f}_\lambda(s, \xi_1) \cdots \hat{f}_\lambda(s, \xi_{k+2}),$$

and

$$\int_0^\delta \int_{\Gamma_{2k+2}} e^{i\alpha 2k^2 s} \hat{f}_\lambda(s, \xi_1) \cdots \hat{f}_\lambda(s, \xi_{2k+2}).$$

These are included in the following two lemmas.

**Lemma 4.2.** Let $s \geq \frac{1}{2}$, and $|| I u_\lambda ||_{X^1([0,\delta])} \lesssim 1$, then

$$\left| \int_0^\delta \int_{\Gamma_{k+2}} e^{i\alpha k^2 s} (1 - \chi_\Omega) M_{k+2} \hat{f}_\lambda(s, \xi_1) \cdots \hat{f}_\lambda(s, \xi_{k+2}) \right| \lesssim K,$$

where $K = N^{-3+} + N^{-2+} \lambda^{-\frac{1}{2}}$.

**Proof.** Before estimation, we give several reductions. First, let $v = G u$, then $|| I u_\lambda ||_{X^1([0,\delta])} = || I v_\lambda ||_{Y^1([0,\delta])}$. Moreover, we note that

$$\hat{v}(t, \xi) = e^{2\pi i \xi \int_0^t u^k dx ds} \hat{u}(t, \xi).$$

So for $\xi_1 + \cdots + \xi_{k+2} = 0$,

$$\hat{u}(t, \xi_1) \cdots \hat{u}(t, \xi_{k+2}) = \hat{v}(t, \xi_1) \cdots \hat{v}(t, \xi_{k+2}).$$

After rescaling, this gives that

$$\hat{u}_\lambda(t, \xi_1) \cdots \hat{u}_\lambda(t, \xi_{k+2}) = \hat{v}_\lambda(t, \xi_1) \cdots \hat{v}_\lambda(t, \xi_{k+2}).$$

Thus,

$$\int_0^\delta \int_{\Gamma_{k+2}} e^{i\alpha k^2 s} (1 - \chi_\Omega) M_{k+2} \hat{f}_\lambda(s, \xi_1) \cdots \hat{f}_\lambda(s, \xi_{k+2})$$

$$= \int_0^\delta \int_{\Gamma_{k+2}} (1 - \chi_\Omega) M_{k+2} \hat{u}_\lambda(s, \xi_1) \cdots \hat{u}_\lambda(s, \xi_{k+2})$$

$$= \int_0^\delta \int_{\Gamma_{k+2}} (1 - \chi_\Omega) M_{k+2} \hat{v}_\lambda(s, \xi_1) \cdots \hat{v}_\lambda(s, \xi_{k+2}).$$
Second, to extend the integration domain from \([0, \delta]\) to \(\mathbb{R}\), we insert the non-smooth cutoff function \(\chi_{[0,\delta]}(t)\) into one of \(v_\lambda\) and use the estimate in Lemma 2.1. This allows us to turn to show

\[
\left| \int_{\mathbb{R}} \int_{\Gamma_{k+2}} (1 - \chi_{\Omega}) M_{k+2} \widehat{v}_\lambda(s, \xi_1) \cdots \widehat{v}_\lambda(s, \xi_{k+2}) \right| \lesssim K \| Iv_\lambda \|_{X_{1, \frac{1}{2}}} \| Iv_\lambda \|_{Y^{k+1}_1},
\]

where \(v_\lambda\) is time supported on \([0, \delta]\). But the \(0^+\) loss is not essential and will be recorded by \(N_{0^+}\), thus it will not be mentioned. Then by Plancherel’s identity, it turns to show

\[
\left| \int_{\Gamma_{k+2} \times \Gamma_{k+2}} (1 - \chi_{\Omega}) M_{k+2} \widehat{v}_\lambda(\tau_1, \xi_1) \cdots \widehat{v}_\lambda(\tau_{k+2}, \xi_{k+2}) \right| \lesssim K \| Iv_\lambda \|^{k+2}_{Y^1}, \tag{4.2}
\]

where the set \(\Gamma_{k+2} \times \Gamma_{k+2} = \{(\vec{\xi}, \vec{\tau}) : \xi_1 + \cdots + \xi_{k+2} = 0, \tau_1 + \cdots + \tau_{k+2} = 0\}\) and we write \(\vec{\xi} = (\xi_1, \cdots, \xi_{k+2}), \vec{\tau} = (\tau_1, \cdots, \tau_{k+2})\) for short.

Third, by symmetry we may assume that

\[
|\xi_1| \geq |\xi_2| \geq \cdots \geq |\xi_{k+2}|.
\]

Also, by dyadic decomposition, we may write

\[
|\xi_j| \sim N_j, \quad \text{for } j = 1, \cdots, k + 2.
\]

According to the reduction in Remark 1.1, we further assume \(|N_1| \sim |N_2| \gtrsim N\). After replacing \(\widehat{v}_\lambda(\tau, \xi)\) by \(|\widehat{v}_\lambda(\tau, \xi)|\) if necessary, we further assume that \(\widehat{v}_\lambda(\tau, \xi)\) is positive.

Now we divide it into four regions:

\[
\begin{align*}
A_1 &= \{(\vec{\xi}, \vec{\tau}) \in (\Gamma_{k+2} \setminus \Omega) \times \Gamma_{k+2} : |\xi_2| \gtrsim N \gg |\xi_3|\}; \\
A_2 &= \{(\vec{\xi}, \vec{\tau}) \in (\Gamma_{k+2} \setminus \Omega) \times \Gamma_{k+2} : |\xi_3| \gtrsim N \gg |\xi_4|\}; \\
A_3 &= \{(\vec{\xi}, \vec{\tau}) \in (\Gamma_{k+2} \setminus \Omega) \times \Gamma_{k+2} : |\xi_4| \gtrsim N \gg |\xi_5|\}; \\
A_4 &= \{(\vec{\xi}, \vec{\tau}) \in (\Gamma_{k+2} \setminus \Omega) \times \Gamma_{k+2} : |\xi_5| \gtrsim N\}.
\end{align*}
\]

**Estimate in** \(A_1\). By Lemma 1.3 (2), we have

\[
|M_{k+2}| \lesssim |\xi_3| |\xi_4| |\xi_5|.
\]
Therefore, by Lemma 2.2 and Corollary 2.1, we have

\[ \text{LHS of (4.2)} \lesssim \int_{A_1} |\xi_1| |\xi_2| |\xi_5| |\tilde{\omega}_\lambda(\tau_1, \xi_1) \cdots \tilde{\omega}_\lambda(\tau_{k+2}, \xi_{k+2}) \]

\[ \lesssim N^{-2s+} \int_{A_1} |\xi_1|^s |\xi_2|^s |\xi_5|^s |\tilde{\omega}_\lambda(\tau_1, \xi_1) \cdots \tilde{\omega}_\lambda(\tau_{k+2}, \xi_{k+2}) \]

\[ \lesssim N^{-2s+} \int \left( |\nabla|P_{N_1} I_{\nu, \lambda}(t, x)\right) \left( |\nabla|^{-1} P_{N_2} I_{\nu, \lambda}(t, x)\right) \left( |\nabla|P_{N_3} I_{\nu, \lambda}(t, x)\right) \right) \cdot \left( |\nabla |^{-1} P_{L \infty} I_{\nu, \lambda}(t, x)\right)^{k-3}(t, x) \ dx \ dt
\]

\[ \lesssim N^{-2\gamma} \|\eta(t)^2 I_{N_2} \left( |\nabla|P_{N_1} I_{\nu, \lambda}, |\nabla|P_{N_3} I_{\nu, \lambda} \right) \right\|_{L^\infty_t} \left\| |\nabla|^{-1} P_{N_2} I_{\nu, \lambda} \right\|_{L^6_t} \left\| v_\lambda \right\|_{L^\infty_t}^{k-3}
\]

\[ \lesssim N^{-2+\lambda^{-\frac{1}{2}}}. \]

**Estimate in \( A_2 \).** By the definition of \( \Omega_1 \), \( A_2 = \emptyset \).

**Estimate in \( A_3 \).** We split it into two parts again, and define

\[ A_{31} = \{ (\xi, \bar{\xi}) \in A_3 : |\xi| \sim |\xi_2| \sim |\xi_3| \sim |\xi_4| \}; \]

\[ A_{32} = \{ (\xi, \bar{\xi}) \in A_3 : |\xi| \sim |\xi_2| \gg |\xi_3| \sim |\xi_4| \}. \]

**Estimate in \( A_{31} \).** By Lemma 1.3 (3), we have

\[ |\chi_{A_{31}} M_{k+2}| \lesssim m(\xi_1)^2 |\xi_2|^2 |\xi_5| \sim m(\xi_1) m(\xi_2) |\xi_1| |\xi_2|^{-1} |\xi_3|^{0+} |\xi_5|. \]

Therefore, by Lemma 2.2 and Corollary 2.1,

\[ \text{LHS of (4.2)} \lesssim \int_{A_3} m(\xi_1) m(\xi_2) |\xi_1| |\xi_2|^{-1} |\xi_3|^{0+} |\xi_5| |\tilde{\omega}_\lambda(\tau_1, \xi_1) \cdots \tilde{\omega}_\lambda(\tau_{k+2}, \xi_{k+2}) \]

\[ \lesssim \int \left( |\nabla|P_{N_1} I_{\nu, \lambda}(t, x)\right) \left( |\nabla|^{-1} P_{N_2} I_{\nu, \lambda}(t, x)\right) \left( |\nabla|P_{N_3} I_{\nu, \lambda}(t, x)\right) \right) \cdot \left( |\nabla |^{-1} P_{L \infty} I_{\nu, \lambda}(t, x)\right)^{k-3}(t, x) \ dx \ dt
\]

\[ \lesssim \left\| |\nabla|^{-1} P_{N_2} I_{\nu, \lambda} \right\|_{L^6_t}^{k-3}(t, x) \ dx \ dt
\]

\[ \lesssim N^{-2+\lambda^{-\frac{1}{2}}}. \]

**Estimate in \( A_{32} \).** Note that both the estimates in Lemma 1.3 (1) and (3) hold in \( A_{32} \), so for any \( \epsilon > 0 \),

\[ |\chi_{A_{32}} M_{k+2}| \lesssim \left[ m(\xi_1)^2 |\xi_2|^2 |\xi_5| \right]^{1-\epsilon} \left[ m(\xi_1)^2 |\xi_1|^2 |\xi_3|^2 \right]^\epsilon
\]

\[ = m(\xi_1)^2 |\xi_1|^{2-\epsilon} |\xi_3|^2 |\xi_5|^{1-\epsilon}
\]

\[ \lesssim m(\xi_1)^2 |\xi_1||\xi_2|^{1-\epsilon} |\xi_3|^{\epsilon} |\xi_4|^\epsilon |\xi_5|. \]
Therefore, by Lemma 2.2 and Corollary 2.1,

\[
\text{LHS of (4.2)} \lesssim \int_{A_3} \int m(\xi_1)^2 |\xi_1||\xi_2|^2|\xi_3|^2|\xi_4|^2|\xi_5|^{\epsilon}|\xi_6|^{\epsilon} v(\tau_1, \xi_1) \cdots v(\tau_k, \xi_1) \right) dxdt
\]

\[
\lesssim \int \left( |\nabla| P_{N_1} I v(\lambda) (t, x) |\nabla|^{1-\epsilon} P_{N_2} I v(\lambda) (t, x) |\nabla|^{\epsilon} P_{N_3} v(\lambda) \right)^2(t, x) dxdt
\]

\[
\lesssim \eta(t)^2 I N^2 \left( |\nabla| P_{N_1} I v(\lambda), |\nabla| P_{N_2} v(\lambda) \right) \left( |\nabla|^{1-\epsilon} P_{N_2} I v(\lambda) \right) \left( |\nabla|^{\epsilon} P_{N_3} v(\lambda) \right) |||\nabla|^{\epsilon} P_{N_3} v(\lambda) ||^2_{L^2_{xt}} |||\nabla|^{1-\epsilon} P_{N_2} I v(\lambda) ||^2_{L^2_{xt}} |||\nabla|^{1-\epsilon} P_{N_2} I v(\lambda) ||_{L^2_{xt}} \left( |\nabla|^{\epsilon} P_{N_3} v(\lambda) \right) \left( |\nabla|^{1-\epsilon} P_{N_2} I v(\lambda) \right) \left( |\nabla|^{\epsilon} P_{N_3} v(\lambda) \right)
\]

\[
\lesssim N^{-2+\lambda^{-\frac{1}{2}}}
\]

**Estimate in A_4.** Moreover, we split A_4 into two subregions:

\[
A_{41} = \{ (\xi', \tau) \in A_5 : |\xi_4| \gg |\xi_5| \};
\]

\[
A_{42} = \{ (\xi', \tau) \in A_5 : |\xi_4| \approx |\xi_5| \}.
\]

The estimate in A_{41} can be treated as the estimate in A_3, since they have the same bound on M_{k+2}. So we omit the details. Now we consider the estimate in A_{42}. In this part, by Lemma 1.3 (1) and the relationship |\xi_3| \sim |\xi_4| \sim |\xi_5|, we have

\[
|M_{k+2}| \lesssim m(\xi_1)^2 |\xi_1||\xi_3|^2 \lesssim m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2|^{\frac{1}{2}+\epsilon}|\xi_3|^{\frac{1}{2}-\epsilon}|\xi_4|^{\frac{1}{2}-\epsilon}|\xi_5|^{\frac{1}{2}-\epsilon}
\]

\[
\lesssim N^{-\frac{1}{2}} m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2|^{\frac{1}{2}} \cdot m(\xi_3)|\xi_3|^{\frac{1}{2}} \cdot m(\xi_4)|\xi_4|^{\frac{1}{2}} \cdot m(\xi_5)|\xi_5|
\]

\[
\lesssim N^{-2+\epsilon} m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2|^{\frac{1}{2}} \cdot m(\xi_3)|\xi_3|^{\frac{1}{2}} \cdot m(\xi_4)|\xi_4|^{\frac{1}{2}} \cdot m(\xi_5)|\xi_5|
\]

Further, we claim that

\[
|\xi_1| - |\xi_5| \gtrsim |\xi_1| \gtrsim N. \tag{4.3}
\]

Indeed, if |\xi_j| = |\xi_1| + o(|\xi_1|), for all j = 1, \cdots, 5, then there exist \mu_j \in \{-1, 1\} such that

\[
\xi_j = \mu_j \xi_1 + o(|\xi_1|).
\]

Therefore,

\[
|\xi_6| = |\xi_1 + \cdots + \xi_5| = |\mu_1 + \cdots + \mu_5| |\xi_1| + o(|\xi_1|).
\]

Note that |\mu_1 + \cdots + \mu_5| \geq 1, we have |\xi_6| \sim |\xi_1|, but this is not the case in A_4. So we have

\[
|\xi_1| - |\xi_5| \gtrsim |\xi_1| \text{ and thus}
\]

\[
|\xi_1^2 - \xi_5^2| \gtrsim N^2.
\]
Therefore, we have
\[
LHS \text{ of } (4.2) \lesssim N^{-2+} \int_{\Lambda_4} m(\xi_1) |\xi_1| \cdot m(\xi_2) |\xi_2|^{1-} \cdot m(\xi_3) |\xi_3|^{1-} \cdot m(\xi_4) |\xi_4|^{1-} \cdot m(\xi_5) |\xi_5| \\
\quad \cdot \tilde{v}_\lambda(\tau_1, \xi_1) \cdots \tilde{v}_\lambda(\tau_{k+2}, \xi_{k+2}) \\
\lesssim N^{-2+} \int_{\Lambda_4} \eta^2(t) I_{N^2} (|\nabla| I v_\lambda, |\nabla| I v_\lambda)(t, x) \\
\quad \cdot (|\nabla|^{1-} - P_{\leq N} I v_\lambda)^3(t, x) (P_{\leq N} v_\lambda)^{k-3}(t, x) \, dx \, dt \\
\lesssim N^{-2+} \|\eta(t)^2 I_{N^2} (|\nabla| I v_\lambda, |\nabla| I v_\lambda)\|_{L^2_t} \|\nabla|^{1-} I v_\lambda\|_{L^2_t}^3 \|v_\lambda\|_{L^2_t}^{k-3} \\
\lesssim N^{-2+} \lambda^{-\frac{1}{2}}.
\]

Collecting the estimates above, we prove the lemma.

\[\square\]

**Lemma 4.3.** Let \( s \geq \frac{1}{2} \), and \( \|Iu\|_{X^1([0,\delta])} \lesssim 1 \), then
\[
\left| \int_0^t \int_{\Gamma_{2k+2}} e^{i\omega t_{2k+2}^s} \overline{M_{2k+2}} \tilde{f}_\lambda(s, \xi_1) \cdots \tilde{f}_\lambda(s, \xi_{2k+2}) \right| \lesssim K' \|Iu\|_{Y^1}^{2k+2}.
\] (4.4)

where \( K' = N^{-3+} + N^{-2+} \lambda^{-1} \).

**Proof.** By the reductions at the beginning of the proof of Lemma 4.2, it suffices to show
\[
\left| \int_0^t \int_{\Gamma_{2k+2} \times \Gamma_{2k+2}} \overline{M_{2k+2}} \tilde{v}_\lambda(\tau_1, \xi_1) \cdots \tilde{v}_\lambda(\tau_{2k+2}, \xi_{2k+2}) \right| \lesssim K' \|Iu\|_{Y^1}^{2k+2},
\] (4.5)

where the hyperplane \( \Gamma_{2k+2}^2 = \{ (\xi_1, \cdots, \xi_2k+2, \tau_1, \cdots, \tau_{2k+2}) : \xi_1 + \cdots + \xi_{2k+2} = 0, \tau_1 + \cdots + \tau_{2k+2} = 0 \} \). Also, we may assume that
\[
|\xi_1| \geq |\xi_2| \geq \cdots \geq |\xi_{2k+2}|, \quad |\xi_j| \sim N_j, \text{ for } j = 1, \cdots, 2k + 2,
\]
and \( \tilde{v}_\lambda(\tau, \xi) \) is positive.

Now we consider the following three subregions separately:

\[
B_1 = \{ (\xi_1, \cdots, \xi_{2k+2}, \tau_1, \cdots, \tau_{2k+2}) \in \Gamma_{2k+2}^2 : |\xi_1| \sim |\xi_2| \gtrsim N \gg |\xi_4| \};
\]
\[
B_2 = \{ (\xi_1, \cdots, \xi_{2k+2}, \tau_1, \cdots, \tau_{2k+2}) \in \Gamma_{2k+2}^2 : |\xi_1| \sim |\xi_2| \gtrsim |\xi_3| \gtrsim N \gg |\xi_4| \};
\]
\[
B_3 = \{ (\xi_1, \cdots, \xi_{2k+2}, \tau_1, \cdots, \tau_{2k+2}) \in \Gamma_{2k+2}^2 : |\xi_4| \gtrsim N \}.
\]
Estimate in $B_1$. By Lemma 1.4 (2), we have $|M_{2k+2}| \lesssim |\xi_3|$. Then, by (2.15), we have
\[
\text{LHS of (4.5)} \lesssim \int_{B_1} |\xi_3| \hat{v}_\lambda(\tau_1, \xi_1) \cdots \hat{v}_\lambda(\tau_{2k+2}, \xi_{2k+2}) \lesssim N^{-2} \int_{B_1} m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2| \cdot m(\xi_3)|\xi_3| \cdot m(\xi_4)|\xi_4| \hat{v}_\lambda(\tau_1, \xi_1) \cdots \hat{v}_\lambda(\tau_{2k+2}, \xi_{2k+2}) \\
\lesssim N^{-2} \|\eta(t)^2I_{N^2}(\nabla|P_{N_1}Iv_\lambda, \nabla|P_{N_3}Iv_\lambda)\|_L^2 \times \|\eta(t)^2I_{N^2}(\nabla|P_{N_1}Iv_\lambda, \nabla|P_{N_4}Iv_\lambda)\|_L^2 \|v_\lambda\|_{L^\infty}^{2k-2} \\
\lesssim N^{-2+\lambda^{-1}}.
\]

Estimate in $B_2$. Similar to the proof of (4.3), we have
\[
|\xi_1| - |\xi_3| \gtrsim |\xi_1| \gtrsim N. \quad (4.6)
\]
Moreover, from (1.26), we have $|M_{2k+2}| \lesssim |\xi_1|$. Then
\[
|M_{2k+2}| \lesssim |\xi_1| \lesssim N^{-2}m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2| \cdot m(\xi_3)|\xi_3| \cdot m(\xi_4)|\xi_4|.
\]
Therefore, we have the same estimate as what in $B_1$, and get also
\[
\text{LHS of (4.5)} \lesssim N^{-2+\lambda^{-1}}.
\]

Estimate in $B_3$. In this part,
\[
|M_{2k+2}| \lesssim |\xi_1| \lesssim N^{-3+s}N^{-s}_4 m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2| \cdot m(\xi_3)|\xi_3| \cdot m(\xi_4)|\xi_4|.
\]
Therefore, we have
\[
\text{LHS of (4.5)} \lesssim N^{-3+s}N^{-s}_4 \int_{B_3} m(\xi_1)|\xi_1| \cdot m(\xi_2)|\xi_2| \cdot m(\xi_3)|\xi_3| \cdot m(\xi_4)|\xi_4| \\
\cdot \hat{v}_\lambda(\tau_1, \xi_1) \cdots \hat{v}_\lambda(\tau_{2k+2}, \xi_{2k+2}) \\
\lesssim N^{-3+s}N^{-s}_4 \|\nabla|P_{N_1}Iv_\lambda\|_{L^2} \cdots \|\nabla|P_{N_4}Iv_\lambda\|_{L^2} \|v_\lambda\|_{L^\infty}^{2k-2} \\
\lesssim N^{-3+s}N^{-s}_4^{s+} \\
\lesssim N^{-3+}. \quad \square
\]

This gives the proof of the lemma.

Since $K' \leq K$, combining with the results on Lemma 4.1–Lemma 4.3, we prove Proposition 1.1.
5. Proposition 1.1 implies Theorem 1.1

Suppose that
\[
\sup_{t \in [0, (j - 1)\delta]} E(Iu_\lambda(t)) \leq 2E(I\phi_\lambda) \quad \text{for some } j \in \mathbb{N}. \tag{5.1}
\]
Then by local theory in Lemma 1.1, we have
\[
\|Iu_\lambda\|_{X^1(\{[i-1, i, \delta]\})} \lesssim 1, \quad \text{for any } 1 \leq i \leq j.
\]
So by (1.31) in Proposition 1.1 and a simple iteration, we have
\[
|E_\lambda^2(t) - E_\lambda^2(0)| \leq jK
\leq C_0 := \frac{1}{2}E(I\phi_\lambda) \tag{5.2}
\]
for any \(t \leq j\delta, j \leq C_0K^{-1}\). By (1.30), we have
\[
|E(Iu(t)) - E_\lambda^2(t)| \lesssim N^{-2}\|Iu(t)\|_{H_\delta^2}^{k+2} \lesssim N^{-2}E(Iu(t))^{\frac{k+2}{2}}.
\]
This combining with (5.2), gives us that for any \(t \leq j\delta, j\) satisfying (5.1) and \(\leq C_0K^{-1}\),
\[
E(Iu(t)) \leq E_\lambda^2(t) + CN^{-2}E(Iu(t))^{\frac{k+2}{2}}
\leq E_\lambda^2(0) + \frac{1}{2}E(I\phi_\lambda) + CN^{-2}E(Iu(t))^{\frac{k+2}{2}}
\leq \frac{3}{2}E(I\phi_\lambda) + CN^{-2}(E(Iu(t))^{\frac{k+2}{2}} + E(I\phi_\lambda)^{\frac{k+2}{2}}).
\]
So by continuity argument, we have for any \(t \leq j\delta\),
\[
E(Iu(t)) \leq 2E(I\phi_\lambda).
\]
This extends (5.1) to \([0, j\delta]\). Thus by finite induction, we obtain that
\[
\sup_{t \in [0, C_0\delta K^{-1}]} E(Iu_\lambda(t)) \leq 2E(I\phi_\lambda).
\]
This proves that \(u_\lambda\) exists on \([0, C_0\delta K^{-1}]\), and thus implies that \(u\) exists on \([0, C_0\delta\lambda^{-3}K^{-1}]\).

Suppose that
\[
\delta\lambda^{-3}K^{-1} \geq N^{0+},
\]
then \(u\) exists for arbitrary time by choosing large \(N\).

Since
\[
\delta \sim \lambda^{0+}, \quad \lambda \sim N^{\frac{1-s}{2+s-\frac{1}{2}}}, \quad K = N^{-3+} + N^{-2+}\lambda^{-\frac{1}{2}},
\]
we have \(\delta\lambda^{-3}K^{-1} \geq N^{0+}\) as long as
\[
2 > \frac{5}{2} \cdot \frac{1-s}{\frac{2}{s} + \frac{1}{2}}; \quad 3 > \frac{3(1-s)}{\frac{2}{s} + \frac{1}{2}}.
\]
Particularly, when $k = 3$, it holds for any $s \geq \frac{1}{2}$; when $k = 4$, it holds for any $s > \frac{5}{9}$. This completes the proof of Theorem 1.1.

**References**


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