THE $L_p$ MINKOWSKI PROBLEM FOR POLYTOPES FOR $p < 0$

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Abstract. Existence of solutions to the $L_p$ Minkowski problem is proved for all $p < 0$. For the critical case of $p = -n$, which is known as the centro-affine Minkowski problem, this paper contains the main result in [71] as a special case.

1. Introduction

A convex body in $n$-dimensional Euclidean space, $\mathbb{R}^n$, is a compact convex set that has non-empty interior. If $p \in \mathbb{R}$ and $K$ is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the $L_p$ surface area measure, $S_p(K, \cdot)$, of $K$ is a Borel measure on the unit sphere, $S^{n-1}$, defined for each Borel $\omega \subset S^{n-1}$ by

$$S_p(K, \omega) = \int_{x \in \nu^{-1}_K(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of $K$, defined on $\partial' K$, the set of boundary points of $K$ that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

The $L_p$ surface area measure was introduced by Lutwak [40]. The $L_p$ surface area measure contains three important measures as special cases: the $L_1$ surface area measure is the classic surface area measure; the $L_0$ surface area measure is the cone-volume measure; the $L_{-n}$ surface area measure is the centro-affine surface area measure. Today, the $L_p$ surface area measure is a central notation in convex geometry analysis, and appeared in, e.g., [3, 8, 21–28, 36–51, 53, 55–59, 64–66].

The following $L_p$ Minkowski problem that posed by Lutwak [40] is considered as one of the most important problems in modern convex geometry analysis.

$L_p$ Minkowski problem: Find necessary and sufficient conditions on a finite Borel measure $\mu$ on $S^{n-1}$ so that $\mu$ is the $L_p$ surface area measure of a convex body in $\mathbb{R}^n$.

The associated partial differential equation for the $L_p$ Minkowski problem is the following Mong-Ampère type equation: For a given positive function $f$ on the unit sphere, solve

$$h^{1-p} \det(h_{ij} + h\delta_{ij}) = f,$$

where $h_{ij}$ is the covariant derivative of $h$ with respect to an orthonormal frame on $S^{n-1}$ and $\delta_{ij}$ is the Kronecker delta.

The solutions of the $L_p$ Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [69], Lutwak, Yang and Zhang [45], Ciachi, Lutwak,

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Yang and Zhang [12], Haberl and Schuster [25–27]. The solutions to the $L_p$ Minkowski problem are also related with some important flows (see, e.g., [1, 2, 60, 61]).

When $p = 1$, the $L_p$ Minkowski problem is the classical Minkowski problem. The existence and uniqueness for the solution of this problem was solved by Minkowski, Aleksandrov, and Fenchel and Jessen (see Schneider [56] for references). Regularity of the Minkowski problem was studied by e.g., Caffarelli [7], Cheng and Yau [10], Nirenberg [52] and Pogorelov [54].

For $p \neq 1$, the $L_p$ Minkowski problem was studied by, e.g., Lutwak [40], Lutwak and Oliker [41], Lutwak, Yang and Zhang [46], Chou and Wang [11], Guan and Lin [19], Hug, Lutwak, Yang and Zhang [22], Böröczky, Hegedüs and Zhu [4], Böröczky, Lutwak, Yang and Zhang [5,6], Chen [9], Dou and Zhu [14], Haberl, Lutwak, Yang and Zhang [22], Huang, Liu and Xu [30], Jian, Lu and Wang [32], Jian and Wang [33], Jiang, Wang and Wei [34], Lu and Wang [35], Stancu [60, 61], Sun and Long [62] and Zhu [70–72]. Analogues of the Minkowski problems were studied in, e.g., [13, 15, 16, 18, 20, 29, 67].

The uniqueness of solutions to the $L_p$ Minkowski for $p > 1$ can be shown by applying the $L_p$ Minkowski inequality established by Lutwak [40]. However, little is know about the $L_p$ Minkowski inequality for the case where $p < 1$. This is one of the main reasons that most of the previous work on the $L_p$ Minkowski problem was limited to the case where $p > 1$.

The critical case where $p = -n$ of the $L_p$ Minkowski problem is called the centro-affine Minkowski problem, which describes the centro-affine surface area measure. This problem is especially important due to the affine invariant of the partial differential equation (1.1). It is known that the centro-affine Minkowski problem has connections with several important geometric problems (see, e.g., Jian and Wang [33] for reference). The centro-affine Minkowski problem was explicitly posed by Chow and Wang [11]. Recently, the centro-affine Minkowski problem was studied by Lu and Wang [35] for rotationally symmetric case and was studied by Zhu [71] for discrete measures.

When $p < -n$, very few results are known for the $L_p$ Minkowski problem. So far as the author knows, in $\mathbb{R}^2$, the $L_p$ Minkowski problem for all $p < 0$ was studied by Dou and Zhu [14], Sun and Long [62]. It is the aim of this paper to study the $L_p$ Minkowski problem for all $p < 0$ and $n \geq 2$.

It is know that the Minkowski problem and the $L_p$ Minkowski problem $(p > 1)$ for arbitrary measures can be solved by an approximation argument by first solving the polytopal case (see, e.g., [31] or [56] pp. 392-393). This is one of the reasons why the Minkowski problem and the $L_p$ Minkowski problem for polytopes are of great importance.

A \textit{polytope} in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that it has positive $n$-dimensional volume. The convex hull of a subset of these points is called a \textit{facet} of the polytope if it lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume. Let $P$ be a polytope which contains the origin in its interior with $N$ facets whose outer unit normals are $u_1, ..., u_N$, and such that the facet with outer unit normal $u_k$ has area $a_k$ and distance $h_k$ from the origin for all $k \in \{1, ..., N\}$. Then,

$$S_p(P, \cdot) = \sum_{k=1}^{N} h_k^{1-p} a_k \delta_{u_k}(\cdot).$$

where $\delta_{u_k}$ denotes the delta measure that is concentrated at the point $u_k$. 
A finite subset $U$ of $S^{n-1}$ is said to be in general position if any $k$ elements of $U$, $1 \leq k \leq n$, are linearly independent.

In [71], the author solved the centro-affine Minkowski problem for polytopes whose outer unit normals are in general position:

**Theorem A.** Let $\mu$ be a discrete measure on the unit sphere $S^{n-1}$. Then $\mu$ is the centro-affine surface area measure of a polytope whose outer unit normals are in general position if and only if the support of $\mu$ is in general position and not concentrated on a closed hemisphere.

A linear subspace $X$ ($0 < \dim X < n$) of $\mathbb{R}^n$ is said to be essential with respect to a Borel measure $\mu$ on $S^{n-1}$ if $X \cap \text{supp}(\mu)$ is not concentrated on any closed hemisphere of $X \cap S^{n-1}$.

Obviously, if the support of a discrete measure $\mu$ is in general position, then the set of essential subspaces of $\mu$ is empty. On the other hand, in $\mathbb{R}^n$ ($n \geq 3$), one can easily construct a discrete measure $\mu$ such that $\mu$ does not have essential subspace but the support of $\mu$ is not in general position. Therefore, the set of discrete measures whose supports are in general position is a subset of the set of discrete measures that do not have essential subspaces.

It is the aim of this paper to solve the $L_p$ Minkowski problem for discrete measures that do not have essential subspaces. Obviously, the following main theorem of this paper contains Theorem A as a special case.

**Theorem 1.1.** Let $p < 0$ and $\mu$ be a discrete measure on the unit sphere $S^{n-1}$. Then $\mu$ is the $L_p$ surface area measure of a polytope whose $L_p$ surface area measure does not have essential subspace if and only if $\mu$ does not have essential subspace and not concentrated on a closed hemisphere.

2. Preliminaries

In this section, we standardize some notations and list some basic facts about convex bodies. For general references regarding convex bodies, see, e.g., [17, 56, 63].

The sets in this paper are subsets of the $n$-dimensional Euclidean space $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we write $x \cdot y$ for the standard inner product of $x$ and $y$, $|x|$ for the Euclidean norm of $x$, and $S^{n-1}$ for the unit sphere of $\mathbb{R}^n$.

Suppose $S$ is a subset of $\mathbb{R}^n$, then the positive hull, $\text{pos}(S)$, of $S$ is the set of all positive combinations of any finitely many elements of $S$. Let $\text{lin}(S)$ be the smallest linear subspace of $\mathbb{R}^n$ containing $S$. The diameter of a subset, $S$, of $\mathbb{R}^n$ is defined by

$$d(S) = \max\{|x - y| : x, y \in S\}.$$ 

The convex hull of a subset, $S$, of $\mathbb{R}^n$ is defined by

$$\text{Conv}(S) = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1 \text{ and } x, y \in S\}.$$ 

For convex bodies $K_1, K_2$ in $\mathbb{R}^n$ and $s_1, s_2 \geq 0$, the Minkowski combination is defined by

$$s_1K_1 + s_2K_2 = \{s_1x_1 + s_2x_2 : x_1 \in K_1, x_2 \in K_2\}.$$ 

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body $K$ is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$
Obviously, for $s \geq 0$ and $x \in \mathbb{R}^n$,
\[ h(sK, x) = h(K, sx) = sh(K, x). \]

If $K$ is a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$, then the support set $F(K, u)$ of $K$ in direction $u$ is defined by
\[ F(K, u) = K \cap \{ x \in \mathbb{R}^n : x \cdot u = h(K, u) \}. \]

The Hausdorff distance of two convex bodies $K_1, K_2$ in $\mathbb{R}^n$ is defined by
\[ \delta(K_1, K_2) = \inf\{ t \geq 0 : K_1 \subset K_2 + tB^n, K_2 \subset K_1 + tB^n \}, \]

where $B^n$ is the unit ball.

Let $\mathcal{P}$ be the set of polytopes in $\mathbb{R}^n$. If the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere, let $\mathcal{P}(u_1, ..., u_N)$ be the subset of $\mathcal{P}$ such that a polytope $P \in \mathcal{P}(u_1, ..., u_N)$ if the the set of the outer unit normals of $P$ is a subset of $\{ u_1, ..., u_N \}$. Let $\mathcal{P}_N(u_1, ..., u_N)$ be the subset of $\mathcal{P}(u_1, ..., u_N)$ such that a polytope $P \in \mathcal{P}_N(u_1, ..., u_N)$ if, $P \in \mathcal{P}(u_1, ..., u_N)$, and $P$ has exactly $N$ facets.

3. An extremal problem related to the $L_p$ Minkowski problem

Suppose $p < 0$, $\alpha_1, ..., \alpha_N > 0$, the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere, and $P \in \mathcal{P}(u_1, ..., u_N)$. Define the function, $\Phi_P : \text{Int} (P) \to \mathbb{R}$, by
\[ \Phi_P(\xi) = \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi \cdot u_k)^p. \]

In this section, we study the extremal problem
\[ (3.1) \sup_{\xi \in \text{Int} (P)} \inf_{Q \in \mathcal{P}(u_1, ..., u_N) \text{ and } V(Q) = 1} \Phi_Q(\xi) = \Phi_P(\xi). \]

The main purpose of this section is to prove that a dilation of the solution to problem (3.1) solves the corresponding $L_p$ Minkowski problem.

**Lemma 3.1.** If $p < 0$, $\alpha_1, ..., \alpha_N > 0$, the unit vectors $u_1, ..., u_N$ are not concentrated on a closed hemisphere and $P \in \mathcal{P}(u_1, ..., u_N)$, then there exists a unique $\xi(P) \in \text{Int} (P)$ such that
\[ \Phi_P(\xi(P)) = \inf_{\xi \in \text{Int} (P)} \Phi_P(\xi). \]

**Proof.** Since $p < 0$, the function $f(t) = t^p$ is strictly convex on $(0, +\infty)$. Hence, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in \text{Int} (P)$,
\[ \lambda \Phi_P(\xi_1) + (1 - \lambda) \Phi_P(\xi_2) = \lambda \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_1 \cdot u_k)^p + (1 - \lambda) \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_2 \cdot u_k)^p \]
\[ = \sum_{k=1}^{N} \alpha_k [\lambda (h(P, u_k) - \xi_1 \cdot u_k)^p + (1 - \lambda) (h(P, u_k) - \xi_2 \cdot u_k)^p] \]
\[ \geq \sum_{k=1}^{N} \alpha_k [h(P, u_k) - (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot u_k]^p \]
\[ = \Phi_P(\lambda \xi_1 + (1 - \lambda) \xi_2). \]
Equality holds if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \ldots, N$. Since $u_1, \ldots, u_N$ are not concentrated on a closed hemisphere, $\mathbb{R}^n = \text{lin}\{u_1, \ldots, u_N\}$. Thus, $\xi_1 = \xi_2$. Hence, $\Phi_P$ is strictly convex on $\text{Int}(P)$.

From the fact that $P \in \mathcal{P}(u_1, \ldots, u_N)$, we have, for any $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, \ldots, u_N\}$ such that 

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$ 

Thus, $\Phi_P(\xi) \to \infty$ whenever $\xi \in \text{Int}(P)$ and $\xi \to x$. Therefore, there exists a unique interior point $\xi(P)$ of $P$ such that 

$$\Phi_P(\xi(P)) = \inf_{\xi \in \text{Int}(P)} \Phi_P(\xi).$$

\[\Box\]

Obviously, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, \ldots, u_N)$, 

\begin{equation}
\xi(\lambda P) = \lambda \xi(P),
\end{equation}

and if $P_i \in \mathcal{P}(u_1, \ldots, u_N)$ and $P_i$ converges to a polytope $P$, then $P \in \mathcal{P}(u_1, \ldots, u_N)$.

**Lemma 3.2.** If $p < 0$, $\alpha_1, \ldots, \alpha_N > 0$, the unit vectors $u_1, \ldots, u_N$ are not contained in a closed hemisphere, $P_i \in \mathcal{P}(u_1, \ldots, u_N)$, and $P_i$ converges to a polytope $P$, then $\lim_{i \to \infty} \xi(P_i) = \xi(P)$ and

$$\lim_{i \to \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

**Proof.** Since $P_i$ converges to $P$ and $\xi(P_i) \in \text{Int}(P_i)$, $\xi(P_i)$ is bounded. Let $\xi_0$ be the limit point of a subsequence, $\xi(P_{i_j})$, of $\xi(P_i)$. We claim that $\xi_0 \in \text{Int}(P)$. Otherwise, $\xi_0$ is a boundary point of $P$ with $\lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) = \infty$, which contradicts the fact that

\begin{equation}
\overline{\lim_{j \to \infty}} \Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P)) = \Phi(\xi(P)) < \infty.
\end{equation}

We claim that $\xi_0 = \xi(P)$. Otherwise,

$$\lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) = \Phi_{P}(\xi_0) > \Phi_{P}(\xi(P)) = \lim_{j \to \infty} \Phi_{P_{i_j}}(\xi(P)).$$

This contradicts the fact that 

$$\Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \Phi_{P_{i_j}}(\xi(P)).$$

Hence, $\lim_{i \to \infty} \xi(P_i) = \xi(P)$ and

$$\lim_{i \to \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_{P}(\xi(P)).$$

\[\Box\]

**Lemma 3.3.** If $p < 0$, $\alpha_1, \ldots, \alpha_N > 0$, the unit vectors $u_1, \ldots, u_N$ are not concentrated on a closed hemisphere and $P \in \mathcal{P}(u_1, \ldots, u_N)$, then

$$\sum_{k=1}^{N} \alpha_k [{h(P, u_k) - \xi(P) \cdot u_k}]^{1-p} = 0.$$
Proof. Define $f : \text{Int}(P) \to \mathbb{R}^n$ by

$$f(x) = \sum_{k=1}^{N} \alpha_k (h(P, u_k) - x \cdot u_k)^p.$$ 

By conditions, 

$$f(\xi(P)) = \inf_{x \in \text{Int}(P)} f(x).$$

Thus, 

$$\sum_{k=1}^{N} \alpha_k \frac{u_k,\xi}{[h(P, u_k) - \xi \cdot u_k]^{1-p}} = 0,$$

for all $i = 1, \ldots, n$, where $u_k = (u_{k,1}, \ldots, u_{k,n})^T$. Therefore,

$$\sum_{k=1}^{N} \alpha_k \frac{u_k}{[h(P, u_k) - \xi \cdot u_k]^{1-p}} = 0.$$ 

\[ \square \]

Lemma 3.4. Suppose $p < 0$, $\alpha_1, \ldots, \alpha_N > 0$, the unit vectors $u_1, \ldots, u_N$ are not concentrated on a closed hemisphere, and there exists a $P \in \mathcal{P}_N(u_1, \ldots, u_N)$ with $\xi(P) = o$, $V(P) = 1$ such that 

$$\Phi_P(o) = \sup \left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1 \right\}.$$

Then,

$$S_p(P_0, \cdot) = \sum_{k=1}^{N} \alpha_k \delta_{u_k}(\cdot),$$

where $P_0 = \left( \sum_{j=1}^{N} \alpha_j h(P, u_j)^p/n \right)^{1/p} \mathrm{P}.$

Proof. By conditions, there exists a polytope $P \in \mathcal{P}_N(u_1, \ldots, u_N)$ with $\xi(P) = o$ and $V(P) = 1$ such that 

$$\Phi_P(o) = \sup \left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1 \right\},$$

where $\Phi_Q(\xi) = \sum_{k=1}^{N} \alpha_k (h(Q, u_k) - \xi \cdot u_k)^p$.

For $\tau_1, \ldots, \tau_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope $P_t$ defined by 

$$P_t = \bigcap_{i=1}^{N} \{ x : x \cdot u_i \leq h(P, u_i) + t\tau_i \}$$

has exactly $N$ facets. By [56] (Lemma 7.5.3),

$$\frac{\partial V(P_t)}{\partial t} = \sum_{i=1}^{N} \tau_i a_i,$$
where \( a_i \) is the area of \( F(P, u_i) \). Let \( \lambda(t) = V(P_t)^{-\frac{1}{n}} \), then \( \lambda(t)P_t \in \mathcal{P}_N^n(u_1, \ldots, u_N) \), \( V(\lambda(t)P_t) = 1 \) and

\[
\lambda'(0) = -\frac{1}{n} \sum_{i=1}^{N} \tau_i S_i.
\]

Define \( \xi(t) := \xi(\lambda(t)P_t) \), and

\[
\Phi(t) := \min_{\xi \in \lambda(t)P_t} \sum_{k=1}^{N} \alpha_k (\lambda(t)h(P_t, u_k) - \xi \cdot u_k)^p
\]

\[
= \sum_{k=1}^{N} \alpha_k (\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k)^p.
\]

It follows from Lemma 3.3 that

\[
\sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k]^{1-p}} = 0,
\]

for \( i = 1, \ldots, n \), where \( u_k = (u_{k,1}, \ldots, u_{k,n})^T \). In addition, since \( \xi(P) \) is the origin,

\[
\sum_{k=1}^{N} \alpha_k \frac{u_k}{h(P, u_k)^{1-p}} = 0.
\]

Let \( F = (F_1, \ldots, F_n) \) be a function from an open neighbourhood of the origin in \( \mathbb{R}^{n+1} \) to \( \mathbb{R}^n \) such that

\[
F_i(t, \xi_1, \ldots, \xi_n) = \sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \ldots + \xi_n u_{k,n})]^{1-p}}
\]

for \( i = 1, \ldots, n \). Then,

\[
\frac{\partial F_i}{\partial t} \bigg|_{(t, \xi_1, \ldots, \xi_n)} = \sum_{k=1}^{N} \frac{(p-1)\alpha_k u_{k,i}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \ldots + \xi_n u_{k,n})]^{2-p}},
\]

\[
\frac{\partial F_i}{\partial \xi_j} \bigg|_{(t, \xi_1, \ldots, \xi_n)} = \sum_{k=1}^{N} \frac{(1-p)\alpha_k u_{k,i} u_{k,j}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \ldots + \xi_n u_{k,n})]^{2-p}}
\]

are continuous on a small neighbourhood of \((0, 0, \ldots, 0)\) with

\[
\left( \frac{\partial F}{\partial \xi} \bigg|_{(0, \ldots, 0)} \right)_{n \times n} = \sum_{k=1}^{N} \frac{(1-p)\alpha_k u_k}{h(P_t, u_k)^{2-p}} u_k^T,
\]

where \( u_k u_k^T \) is an \( n \times n \) matrix.
Since \( u_1, \ldots, u_N \) are not contained in a closed hemisphere, \( \mathbb{R}^n = \text{lin}\{u_1, \ldots, u_N\} \). Thus, for any \( x \in \mathbb{R}^n \) with \( x \neq 0 \), there exists a \( u_{i_0} \in \{u_1, \ldots, u_N\} \) such that \( u_{i_0} \cdot x \neq 0 \). Then,

\[
x^T \cdot \left( \sum_{k=1}^{N} \frac{(1-p)\alpha_k}{h(P, u_k)2-p} u_k \cdot u_k^T \right) \cdot x = \sum_{k=1}^{N} \frac{(1-p)\alpha_k}{h(P, u_k)2-p} (x \cdot u_k)^2 \\
\geq \frac{(1-p)\alpha_{i_0}}{h(P, u_{i_0})2-p} (x \cdot u_{i_0})^2 > 0.
\]

Therefore, \( (\frac{\partial F}{\partial \xi}|_{(0, \ldots, 0)}) \) is positive defined. By this, the fact that \( F_i(0, \ldots, 0) = 0 \) for all \( i = 1, \ldots, n \), the fact that \( \frac{\partial F}{\partial \xi_j} \) is continuous on a neighbourhood of \( (0, 0, \ldots, 0) \) for all \( 0 \leq i, j \leq n \) and the implicit function theorem, we have

\[
\xi'(0) = (\xi'_1(0), \ldots, \xi'_n(0))
\]

exists.

From the fact that \( \Phi(0) \) is an extreme value of \( \Phi(t) \) (in Equation (3.5)), Equation (3.4) and Equation (3.6), we have

\[
0 = \Phi'(0)/p \\
= \sum_{k=1}^{N} \alpha_k h(P, u_k)^{p-1} (\lambda'(0)h(P, u_k) + \tau_k - \xi'(0) \cdot u_k) \\
= \sum_{k=1}^{N} \alpha_k h(P, u_k)^{p-1} \left[ -\frac{1}{n} \left( \sum_{i=1}^{N} a_i \tau_i \right) h(P, u_k) + \tau_k \right] - \xi'(0) \cdot \left[ \sum_{k=1}^{N} \frac{u_k}{h(P, u_k)^{1-p}} \right] \\
= \sum_{k=1}^{N} \alpha_k h(P, u_k)^{p-1} \tau_k - \left( \sum_{i=1}^{N} a_i \tau_i \right) \frac{\sum_{k=1}^{N} \alpha_k h(P, u_k)^p}{n} \\
= \sum_{k=1}^{N} \left( \alpha_k h(P, u_k)^{p-1} - \frac{\sum_{j=1}^{N} \alpha_j h(P, u_j)^p}{n} a_k \right) \tau_k.
\]

Since \( \tau_1, \ldots, \tau_N \) are arbitrary,

\[
\sum_{j=1}^{N} \alpha_j h(P, u_j)^p \frac{n}{n} h(P, u_k)^{1-p} a_k = \alpha_k,
\]

for all \( k = 1, \ldots, N \). By letting

\[
P_0 = \left( \frac{\sum_{j=1}^{N} \alpha_j h(P, u_j)^p}{n} \right)^\frac{1}{p-1} P,
\]

we have

\[
S_p(P_0, \cdot) = \sum_{k=1}^{N} \alpha_k \delta_{u_k}(\cdot).
\]
4. The proof of the main theorem

In this section, we prove the main theorem of this paper. The following lemmas will be needed.

**Lemma 4.1.** Let \( \{h_{1j}\}_{j=1}^{\infty}, \ldots, \{h_{Nj}\}_{j=1}^{\infty} \) be \( N (N \geq 2) \) sequences of real numbers. Then, there exists a subsequence, \( \{j_n\}_{n=1}^{\infty} \), of \( \mathbb{N} \) and a rearrangement, \( i_1, \ldots, i_N \), of \( 1, \ldots, N \) such that

\[
h_{i_1j_n} \leq h_{i_2j_n} \leq \ldots \leq h_{i_Nj_n},
\]

for all \( n \in \mathbb{N} \).

**Proof.** For each fixed \( j \), the number of the possible order (from small to big) of \( h_{1j}, \ldots, h_{Nj} \) is \( N! \). Therefore, there exists a subsequence, \( \{j_n\}_{n=1}^{\infty} \), of \( \mathbb{N} \) and a rearrangement, \( i_1, \ldots, i_N \), of \( 1, \ldots, N \) such that

\[
h_{i_1j_n} \leq h_{i_2j_n} \leq \ldots \leq h_{i_Nj_n},
\]

for all \( n \in \mathbb{N} \).

**Lemma 4.2.** Suppose the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a closed hemisphere, and for any subspace, \( X \), of \( \mathbb{R}^n \) with \( 1 \leq \dim X \leq n-1 \), \( \{u_1, \ldots, u_N\} \cap X \) is concentrated on a closed hemisphere of \( S^{n-1} \cap X \). If \( P_m \) is a sequence of polytopes with \( V(P_m) = 1 \), \( o \in \text{Int}(P_m) \) and \( P_m \in \mathcal{P}(u_1, \ldots, u_N) \), then \( P_m \) is bounded.

**Proof.** We only need to prove that if the diameter, \( d(P) \), of \( P \) is not bounded, then there exists a subspace, \( X \), of \( \mathbb{R}^n \) with \( 1 \leq \dim(X) \leq n-1 \) and \( \{u_1, \ldots, u_N\} \cap X \) is not concentrated on a closed hemisphere of \( S^{n-1} \cap X \).

Let \( \mu \) be a discrete measure on the unit sphere such that \( \text{supp}(\mu) = \{u_1, \ldots, u_N\} \), \( \mu(u_i) = \alpha_i > 0 \) for \( 1 \leq i \leq N \). Obviously, we only need to prove the lemma under the condition that \( \xi(P_m) = o \) for all \( m \in \mathbb{N} \).

By Lemma 4.1, we may assume that

\[
h(P_m, u_1) \leq \ldots \leq h(P_m, u_N).
\]

By this and the condition that \( V(P_m) = 1 \) and \( \lim_{m \to \infty} d(P_m) = \infty \),

\[
\lim_{m \to \infty} h(P_m, u_1) = 0 \quad \text{and} \quad \lim_{m \to \infty} h(P_m, u_N) = \infty.
\]

By this and (4.0), there exists an \( i_0 \) \( (1 \leq i_0 \leq N) \) such that

\[
\lim_{m \to \infty} \frac{h(P_m, u_{i_0})}{h(P_m, u_1)} = \infty,
\]

and for \( 1 \leq i \leq i_0 - 1 \)

\[
\lim_{m \to \infty} \frac{h(P_m, u_i)}{h(P_m, u_1)}
\]

exists and equals to a positive number.

Let

\[
\Sigma = \text{pos}\{u_1, \ldots, u_{i_0-1}\}
\]

and

\[
\Sigma^* = \{x \in \mathbb{R}^n : x \cdot u_i \leq 0 \text{ for all } 1 \leq i \leq i_0 - 1\}.
\]
Let $1 \leq j \leq i_0 - 1$ and $x \in \Sigma^* \cap S^{n-1}$. From the condition that $\xi(P_m)$ is the origin and Lemma 3.3, we have
\[
\sum_{i=0}^{N} \frac{\alpha_i(x \cdot u_i)}{[h(P_m, u_i)]^{1-p}} = 0.
\]
By this and the fact that $x \in \Sigma^* \cap S^{n-1}$,
\[
0 \geq \alpha_j(x \cdot u_j)
\]
\[
= - \sum_{i \neq j} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i(x \cdot u_i)
\]
\[
\geq \sum_{i \geq i_0} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i(x \cdot u_i)
\]
\[
\geq - \sum_{i \geq i_0} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i(x \cdot u_i)
\]
By this, (4.0), (4.1) and (4.2), $\alpha_j(x \cdot u_j)$ is no bigger than 0 and no less than any negative number. Hence,
\[
x \cdot u_j = 0
\]
for all $j = 1, ..., i_0 - 1$ and $x \in \Sigma^* \cap S^{n-1}$. Thus,
\[
(4.3) \quad \Sigma^* \cap \text{lin}\{u_1, ..., u_{i_0-1}\} = \{0\}.
\]
Obviously, $\{u_1, ..., u_{i_0-1}\}$ is not concentrated on a closed hemisphere of $S^{n-1} \cap \text{lin}\{u_1, ..., u_{i_0-1}\}$. Otherwise, there exists an $x_0 \in \text{lin}\{u_1, ..., u_{i_0-1}\}$ with $x_0 \neq 0$ such that $x_0 \cdot u_i \leq 0$ for all $1 \leq i \leq i_0 - 1$. This contradicts with (4.3).

We next prove that
\[
\text{lin}\{u_1, ..., u_{i_0-1}\} \neq \mathbb{R}^n.
\]
Otherwise, from the fact that $u_1, ..., u_{i_0-1}$ are not concentrated on a closed hemisphere of
\[
\text{lin}\{u_1, ..., u_{i_0-1}\} \cap S^{n-1},
\]
we have, the convex hull of $\{u_1, ..., u_{i_0-1}\}$ (denoted by $Q$) is a polytope in $\mathbb{R}^n$ and contains the origin as an interior. Let $F$ be a facet of $Q$ such that $\{su_i : s > 0\} \cap F \neq \emptyset$. Since $F$ is the union of finite $(n-1)$-dimensional simplexes and the vertexes of these simplexes are subsets of $\{u_1, ..., u_{i_0-1}\}$, there exists a subset, $\{u_{i_1}, ..., u_{i_n}\}$, of $\{u_1, ..., u_{i_0-1}\}$ such that
\[
u_{i_0} \in \text{pos}\{u_{i_1}, ..., u_{i_n}\}.
\]
Since $o \in \text{Int}(Q)$, there exists $r > 0$ such that $rB^n \subset Q$. Choose $t > 0$ such that $tu \in F \cap \text{pos}\{u_{i_1}, ..., u_{i_n}\}$. Then,
\[
tu = \beta_{i_1}u_{i_1} + ... + \beta_{i_n}u_{i_n},
\]
where $\beta_{i_1}, ..., \beta_{i_n} \geq 0$ with $\beta_{i_1} + ... + \beta_{i_n} = 1$. If we let $a_{ij} = \beta_{ij}/t$ for $j = 1, ..., n$, we have
\[
u = a_{i_1}u_{i_1} + ... + a_{i_n}u_{i_n}.
\]
Obviously, $a_{ij} \geq 0$ with
\[
a_{ij} = \beta_{ij}/t \leq 1/r
\]
for all \( j = 1, \ldots, n \). Hence,
\[
 h(P_m, u_i) = h(P_m, a_i u_i + \ldots + a_n u_n) \\
\leq a_i h(P_m, u_i) + \ldots + a_n h(P_m, u_n) \\
\leq \frac{1}{r} [h(P_m, u_i) + \ldots + h(P_m, u_n)],
\]
for all \( m \in \mathbb{N} \). This contradicts (4.1) and (4.2). Therefore,
\[
 \mathrm{lin}\{u_1, \ldots, u_{i_0-1}\} \neq \mathbb{R}^n.
\]

Let \( X = \mathrm{lin}\{u_1, \ldots, u_{i_0-1}\} \). Then, \( 1 \leq \dim X \leq n-1 \) but \( \{u_1, \ldots, u_N\} \cap X = \{u_1, \ldots, u_{i_0-1}\} \) is not concentrated on a closed hemisphere of \( S^{n-1} \cap X \), which contradicts the conditions of this lemma. Therefore, \( d(P_m) \) is bounded. \( \square \)

The following lemmas will be needed (see, e.g., [72]).

**Lemma 4.3.** If \( P \) is a polytope in \( \mathbb{R}^n \) and \( v_0 \in S^{n-1} \) with \( V_{n-1}(F(P, v_0)) = 0 \), then there exists a \( \delta_0 > 0 \) such that for \( 0 \leq \delta < \delta_0 \)
\[
 V(P \cap \{x : x \cdot v_0 \geq h(P, v_0) - \delta\}) = c_0 \delta^n + \ldots + c_2 \delta^2,
\]
where \( c_n, \ldots, c_2 \) are constants that depend on \( P \) and \( v_0 \).

**Lemma 4.4.** Suppose \( p < 0, a_1, \ldots, a_N > 0 \), and the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a hemisphere. If for any subspace \( X \) with \( 1 \leq \dim X \leq n-1 \), \( \{u_1, \ldots, u_N\} \cap X \) is always concentrated on a closed hemisphere of \( S^{n-1} \cap X \), then there exists a \( P \in \mathcal{P}(u_1, \ldots, u_N) \) such that \( \xi(P) = o, V(P) = 1, \) and
\[
 \Phi_P(o) = \sup\{ \inf_{\xi \in \mathrm{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1\},
\]
where \( \Phi_Q(\xi) = \sum_{k=1}^N a_k (h(Q, u_k) - \xi \cdot u_k)^p \).

*Proof.* Obviously, for \( P, Q \in \mathcal{P}(u_1, \ldots, u_N) \), if there exists a \( x \in \mathbb{R}^n \) such that \( P = Q + x \), then
\[
 \Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).
\]

Thus, we can choose a sequence of polytopes \( P_i \in \mathcal{P}(u_1, \ldots, u_N) \) with \( \xi(P_i) = o \) and \( V(P_i) = 1 \) such that \( \Phi_{P_i}(o) \) converges to
\[
 \sup\{ \inf_{\xi \in \mathrm{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1\}.
\]

By the conditions of this lemma and Lemma 4.2, \( P_i \) is bounded. From the Blaschke selection theorem, there exists a subsequence of \( P_i \) that converges to a polytope \( P \) such that \( P \in \mathcal{P}(u_1, \ldots, u_N), V(P) = 1, \xi(P) = o \) and
\[
(4.4) \quad \Phi_P(o) = \sup\{ \inf_{\xi \in \mathrm{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \ldots, u_N) \text{ and } V(Q) = 1\}.
\]

We claim that \( F(P, u_i) \) are facets for all \( i = 1, \ldots, N \). Otherwise, there exists an \( i_0 \in \{1, \ldots, N\} \) such that
\[
 F(P, u_{i_0})
\]
is not a facet of \( P \).

Choose \( \delta > 0 \) small enough so that the polytope
\[
 P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, \ldots, u_N),
\]
and (by Lemma 4.3)
\[ V(P_0) = 1 - (c_0\delta^n + \ldots + c_2\delta^2), \]
where \( c_0, \ldots, c_2 \) are constants that depend on \( P \) and direction \( u_{i_0} \).

From Lemma 3.2, for any \( \delta_i \to 0 \) it always true that \( \xi(P_\delta_i) \to o \). We have,
\[ \lim_{\delta \to 0} \xi(P_\delta) = o. \]
Let \( \delta \) be small enough so that \( h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta \) for all \( k \in \{1, \ldots, N\} \), and let
\[ \lambda = V(P_\delta)^{-\frac{1}{p}} = (1 - (c_0\delta^n + \ldots + c_2\delta^2))^{-\frac{1}{p}}. \]
From this and Equation (3.2), we have
\[ (4.5) \]
\[ \Phi_{\lambda P_\delta}(\xi(\lambda P_\delta)) = \sum_{k=1}^{N} \alpha_k \left( h(P_\delta, u_k) - \xi(\lambda P_\delta) \cdot u_k \right)^p \]
\[ = \lambda^p \sum_{k=1}^{N} \alpha_k \left( h(P_\delta, u_k) - \xi(P_\delta) \cdot u_k \right)^p \]
\[ = \lambda^p \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p - \alpha_{i_0} \lambda^p \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \]
\[ + \alpha_{i_0} \lambda^p \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p \]
\[ = \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p + (\lambda^p - 1) \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p \]
\[ + \alpha_{i_0} \lambda^p \left[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \right] \]
\[ = \Phi_P(\xi(P_\delta)) + B(\delta), \]
where
\[ B(\delta) = (\lambda^p - 1) \left( \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p \right) \]
\[ + \alpha_{i_0} \lambda^p \left[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \right] \]
\[ = \left[ 1 - (c_0\delta^n + \ldots + c_2\delta^2) \right]^{-\frac{1}{p}} \left( \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_\delta) \cdot u_k \right)^p \right) \]
\[ + \alpha_{i_0} \lambda^p \left[ \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta \right)^p - \left( h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} \right)^p \right]. \]

From the facts that \( d_0 = d(P) > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta > 0 \), \( p < 0 \) and the fact that \( f(t) = t^p \) is convex on \((0, \infty)\), we have
\[ (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p > (d_0 - \delta)^p - d_0^p > 0. \]
Hence,

\[
B(\delta) = (\lambda^p - 1) \left( \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_0) \cdot u_k \right)^p \right) + \alpha_{i_0} \lambda^p \left[ (h(P, u_{i_0}) - \xi(P_0) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_0) \cdot u_{i_0})^p \right]
\]

(4.6)

\[
> \left[ (1 - (c_\delta^n + \ldots + c_2 \delta^2)) - \frac{p}{n} - 1 \right] \left( \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_0) \cdot u_k \right)^p \right)
\]

\[
+ \alpha_{i_0} \lambda^p \left[ (d_0 - \delta)^p - d_0^p \right].
\]

On the other hand,

(4.7) \[ \lim_{\delta \to 0} \sum_{k=1}^{N} \alpha_k \left( h(P, u_k) - \xi(P_0) \cdot u_k \right)^p = \sum_{k=1}^{N} \alpha_k h(P, u_k)^p, \]

(4.8) \[ (d_0 - \delta)^p - d_0^p > 0, \]

and

(4.9) \[ \lim_{\delta \to 0} \frac{1 - (c_\delta^n + \ldots + c_2 \delta^2)) - \frac{p}{n} - 1}{(d_0 - \delta)^p - d_0^p} = \lim_{\delta \to 0} \frac{(-\frac{p}{n})(1 - (c_\delta^n + \ldots + c_2 \delta^2)) - \frac{p}{n} - 1(-nc_\delta^{n-1} - \ldots - 2c_2 \delta)}{p(d_0 - \delta)^{p-1}(-1)} = 0. \]

From Equations (4.6), (4.7), (4.8), (4.9), and the fact that \( p < 0 \), we have \( B(\delta) > 0 \) for small enough \( \delta > 0 \). From this and Equation (4.5), there exists a \( \delta_0 > 0 \) such that \( P_{\delta_0} \in \mathcal{P}(u_1, \ldots, u_N) \) and

\[
\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) > \Phi_P(\xi(P_{\delta_0})) \geq \Phi_P(\xi(P)) = \Phi_P(o),
\]

where \( \lambda_0 = V(P_{\delta_0})^{-\frac{1}{p}} \). Let \( P_0 = \lambda_0 P_{\delta_0} - \lambda(\lambda_0 P_{\delta_0}) \), then \( P_0 \in \mathcal{P}(u_1, \ldots, u_N) \), \( V(P_0) = 1, \) \( \xi(P_0) = o \) and

(4.10) \[ \Phi_{P_0}(o) < \Phi_P(o). \]

This contradicts Equation (4.4). Therefore, \( P \in \mathcal{P}(u_1, \ldots, u_N) \). \( \square \)

Now we have prepared enough to prove the main theorem of this paper. We only need to prove the following:

**Theorem 4.5.** Suppose \( p < 0 \), \( \alpha_1, \ldots, \alpha_N > 0 \), and the unit vectors \( u_1, \ldots, u_N \) are not concentrated on a hemisphere. If for any subspace \( X \) with \( 1 \leq \dim X \leq n-1 \), \( \{u_1, \ldots, u_N\} \cap X \) is always concentrated on a closed hemisphere of \( S^{n-1} \cap X \), then there exists a polytope \( P_0 \in \mathcal{P}(u_1, \ldots, u_N) \) such that

\[
S_p(P_0, \cdot) = \sum_{k=1}^{N} \alpha_k \delta_{u_k}(\cdot).
\]

**Proof.** Theorem 4.5 can be directly got by Lemma 3.4 and Lemma 4.4. \( \square \)
References

[19] P. Guan, C.-S. Lin, On equation $\det(u_{ij} + \delta_{ij} u) = u^p f$ on $S^n$, (preprint).
THE $L_p$ MINKOWSKI PROBLEM FOR POLYTOPES FOR $p < 0$


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