SINGULAR TRACES AND RESIDUES OF THE $\zeta$-FUNCTION

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Abstract. This paper studies the relationship between the singular trace of a weak trace class operator and the asymptotic behaviour of its $\zeta$-function at its leading singularity. For Dixmier measurable and universally measurable operators we describe their measurability in terms of the behaviour of the $\zeta$-function. We use recent advances in singular trace theory and a new approach based on Tauberian theorems other than the familiar Hardy-Littlewood or weak Karamata theorems. The approach finalises the long story of results [9, 3, 32, 54, 4] relating singular traces on the weak trace class ideal, the $\zeta$-function and the leading term of heat semi-group expansion. The results are illustrated by a number of examples, including discussions of pseudo-differential operators and Laplacians on fractals.

1. Introduction

The first examples of singular (that is, non-normal) traces were constructed by J. Dixmier in 1966 [14]. Later A. Connes used Dixmier traces in his noncommutative calculus and geometry [8, 10]. In that setting Dixmier traces provide a direct analogue of integration. Since then Dixmier traces have proved to be a useful tool in different areas of mathematics and its physical applications, ranging from fractals [28, 22], to foliations [2], to spectral theory of pseudo-differential operators [35] and applications in string theory, to the standard model of particle physics [12, 7] and to quantum field theory [7, 8, 11]. Very recently, Dixmier traces have found applications in the investigation of fundamental properties of Banach spaces [40].

The computation of Dixmier traces of operators arising in geometrically or physically inspired problems is a highly non-trivial task. The calculation relies on knowing precisely the asymptotic behaviour of the spectrum of an operator, which is often unknown. In applications the $\zeta$-function or a heat kernel asymptotic expansion of geometric operators is often known or derived. Let $B(H)$ be the algebra of all bounded linear operators on a separable Hilbert space $H$ equipped with the uniform norm and the standard trace $\text{Tr}$. We denote by $\{\lambda(n, A)\}_{n \geq 0}$ a sequence of eigenvalues of a compact operator $A \in B(H)$, ordered in such a way that the sequence $\{\lambda(n, A)\}_{n \geq 0}$ is decreasing. The weak trace class ideal $\mathcal{L}_{1,\infty}$ consists of all compact operators $A \in B(H)$ such that $\sup_n n\lambda(n, |A|) < \infty$. The first result linking the Dixmier trace of $0 \leq A \in \mathcal{L}_{1,\infty}$ with the residue of its $\zeta$-function at its leading singularity was stated by A. Connes and H. Moscovici [9, Proposition A.4]. The following theorem is not exactly the statement proved by Connes and Moscovici [9]. The fact that the value of all Dixmier traces is the same implies that the logarithmic mean of its eigenvalue sequence converges was proved later in [32].

**Theorem 1.1.** Let $0 \leq A \in \mathcal{L}_{1,\infty}$ and $c \geq 0$. The following conditions are equivalent:

(i) All Dixmier traces applied to $A$ produce the same value $c$;

(ii) $\lim_{n \to \infty} \frac{1}{\log(n+1)} \sum_{k=0}^{n} \lambda(k, A) = c$;

(iii) $\lim_{s \to 1^+} (s - 1) \text{Tr}(A^s) = c$.

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If all Dixmier traces agree on an operator, it is said to be Dixmier measurable (see [10] and [32]). Theorem 1.1 says that the common value of all Dixmier traces applied to $A$ is equal to the residue of its $\zeta$-function at the leading singularity. Theorem 1.1 follows from the Hardy-Littlewood Tauberian theorem, [3], and it was extended to the case of the Dixmier-Macaev ideal $M_{1,\infty}$ (of all compact operators for which the sum of first $n$ singular values diverges logarithmically) in [5].

Generally, in noncommutative geometry, we wish to compute traces of operators of the form $V A$, where $0 \leq A \in L_{1,\infty}$ and $V \in B(H)$. Repeating the statement of Theorem 1.1 for $V A$ has proven to be much harder.

Our first main result in this paper is the following general version of Connes-Moscovici’s Theorem 1.1:

**Theorem 1.2.** Let $0 \leq A \in L_{1,\infty}$, $V \in B(H)$ and $c \in \mathbb{C}$. All Dixmier traces applied to $V A$ equal the same value $c$, that is

$$\lim_{n \to \infty} \frac{1}{\log(n+1)} \sum_{k=0}^{n} \lambda(k, V A) = c$$

if and only if

$$\lim_{s \to 1^+} (s-1)\text{Tr}(VA^s) = c.$$

Working with the product $VA$ covers all (not necessary positive) operators in $L_{1,\infty}$. Theorem 1.2 has a long history of proofs of less general and alternative versions. The statement of Theorem 1.2 was proven for projections in [34], using the results of [3]. It is not difficult to construct an operator $A \in L_{1,\infty}$ such that not all Dixmier traces coincide on $A$ (see e.g. [44, Theorem 7.9]). An example of a pseudo-differential operator $A \in L_{1,\infty}$ which is not Dixmier measurable was constructed in [25]. It was proved in a number of papers (see e.g. [2, 6, 54, 52]) that even for Dixmier non-measurable operators the values of Dixmier traces still can be obtained from the asymptotics of the $\zeta$-function (see Theorem 2.5 below) and that of the heat semi-group. These results were further generalised to the case of operators from the Lorentz ideals in [20] for a proper subset of Dixmier traces generated by exponentiation invariant states. All these subsequent efforts were essentially based on [3], where the weak Karamata theorem could be used to obtain the statement of Theorem 1.1 for the heat semi-group expansion, and then the Mellin transform used to relate the heat semi-group behaviour to the leading singularity of the $\zeta$-function. Theorem 1.2 was, essentially, proved for the heat semi-group expansion at the same time as Theorem 1.1, [10, p. 563], [3, Theorem 4.1], [54]. The Mellin transform, however, is not suited to transforming the product $VA$.

We use very recent advances in the theory of singular traces on the weak trace class operator ideal to obtain Theorem 1.2. These results that were not available to previous authors. Our method is not based on the Karamata theorem or Mellin transform at all.

1. We use a new construction of Dixmier (and singular) traces suggested by A. Pietsch [36, 38, 37] and further developed in a recent paper [44]. Pietsch’s construction produces a bijection between the set of all traces on the weak trace class ideal and all shift invariant functionals on bounded sequences. The form of Pietsch’s bijection enable us to employ Tauberian theorems between Abel and Cesàro summability.

2. To establish formulae relating Dixmier traces, the $\zeta$-function residues and the asymptotic expansion of the heat semi-group traditionally uses the classical Hardy-Littlewood, Karamata and Ikehara Tauberian theorems. However, we also go beyond the class of Dixmier traces (see below), and this naturally requires a sharper control on the asymptotic behaviour of the $\zeta$-function and the related heat semi-group. To link these asymptotics to normalised traces we employ a spectrum of results from quantitative Tauberian theory and Tauberian remainder theory (see e.g. [27]).

3. We also employ an intrinsic diagonal representation of operators developed recently (see [34, 25]). Let $A \in L_{1,\infty}$ be a positive operator and $V \in B(H)$. Let $\{e_k\}$ be the set of eigenvectors of $A$ and $\text{diag} : \ell_\infty \to B(H)$ be the diagonal embedding with respect to $\{e_k\}$. Define the diagonal operator $W = \text{diag}(Ve_k, e_k)$. It was
proved in [25] that the difference $VA - WA$ belongs to the commutator subspace of $L_{1,\infty}$. In particular, all traces on $L_{1,\infty}$ vanish on this difference and $\tau(VA) = \tau(WA)$ for every trace $\tau$ on the weak trace class operator ideal. Actually in [25] this result was proven for every trace on every two-sided ideal of compact operators for every positive operator $A$ belonging to that ideal. Thus, to consider the trace of the product $VA$ it suffices to consider the product $WA$ where $W$ is normal and commutes with $A$.

We now state the second main result of this paper. We have mentioned that we go beyond the class of Dixmier traces. Every Dixmier trace is a positive normalised (in the sense that $\tau(A) = 1$ for every $A$ such that $\lambda(n, |A|) = \frac{1}{n^\alpha}$) functional on the weak trace class ideal. They form a proper subset in the set of all positive normalised traces, whereas the set of all normalised traces is much wider (see [44] and Remark 3.4 below for a detailed explanation). In general, by the term trace on the weak trace class operators we mean a linear functional on the weak trace ideal, which may be neither positive nor continuous, that vanishes on operators of the form $AV - VA$ where $A \in L_{1,\infty}$ and $V \in B(H)$. It is of interest to study the class of operators $A \in L_{1,\infty}$ such that all normalised traces on $L_{1,\infty}$ takes the same value on $A$ (see e.g. [18, 15, 25]). Such operators are termed universally measurable. The class of universally measurable operators is strictly smaller than the class of Dixmier measurable operators. An example of a pseudo-differential operator $A \in L_{1,\infty}$ which is Dixmier measurable, but which is not universally measurable may be found in [44, Theorem 8.13]. Recently, in [4] universally measurable operators were employed to study the Hochschild class of the Chern character.

In [4] universal measurability was described in terms of heat semi-groups as follows:

**Theorem 1.3.** Let $0 \leq A \in L_{1,\infty}$, $V \in B(H)$ and $c \in \mathbb{C}$. The following conditions are equivalent:

(i) all normalised traces on $L_{1,\infty}$ applied to $VA$ produce the same value $c$;

(ii) $\text{Tr}(VAe^{-(nA)^{-2}}) = c\log n + O(1)$, $n \to \infty$.

The second main result of this paper complements Theorem 1.3. It describes the universal measurability in terms of the leading singularity of the $\zeta$-function. It is not an equivalence however. In applications $A$ is the inverse power of a Dirac-type operator and the regularity condition (2) below is usually given.

**Theorem 1.4.** Let $0 \leq A \in L_{1,\infty}$, $V \in B(H)$ and $c \in \mathbb{C}$.

(i) If all normalised traces on $L_{1,\infty}$ applied to $VA$ produce the same value $c$, then the following asymptotics hold:

$$\text{Tr}(VA^s) = \frac{c}{s-1} + O(1), \ s \to 1^+.$$ 

(ii) If the function

$$z \mapsto \text{Tr}(VA^z) - \frac{c}{z-1}$$

is regular at $z = 1$

and

$$z \mapsto \text{Tr}(A^z) - \frac{c_1}{z-1}$$

is regular at $z = 1$

then all normalised traces on $L_{1,\infty}$ applied to $VA$ produce the value $c$.

Theorems 1.2 and 1.4 may be viewed as the last pieces of puzzle in the relation between singular traces and the $\zeta$-function of an operator from $L_{1,\infty}$.

Although we assume some extra conditions on the behaviour of the $\zeta$-function in part (ii) of Theorem 1.4, we shall demonstrate in Section 5 that these conditions are satisfied by a large class of geometrically significant operators. In the same section we also discuss relation between Minkowski measurability of the boundary of a fractal string (that is, a fractal) and Dixmier/universal measurability of the Laplacian on the fractal string. It is of interest to observe that Laplacians on fractal strings with Minkowski measurable boundary form a
class of operators, which lies strictly in between the class of all Dixmier measurable operators and that of all universally measurable operators.

2. Preliminaries

Throughout the paper $L_{\infty}(0, \infty)$ denotes the space of all (equivalence classes of) real-valued essentially bounded Lebesgue measurable functions on $(0, \infty)$ equipped with the essential supremum norm.

First we need a notion of extended limits.

**Definition 2.1.** A linear functional $\gamma$ on $L_{\infty}(0, \infty)$ is called an extended limit if it is a Hahn-Banach extension of the ordinary limit (at $\infty$).

Similarly, one can define extended limits on $l_{\infty}$, the Banach space of all real bounded sequences equipped with the uniform norm.

**Definition 2.2.** An extended limit $\omega$ on $L_{\infty}(0, \infty)$ is called dilation invariant if

$$
\omega \circ \sigma_{\beta} = \omega \quad \text{for every } \beta > 0.
$$

Here,

$$(\sigma_{\beta}x)(t) := x(t/\beta), \ t > 0.
$$

We now define the ideals with which we shall work. Let $B(H)$ denote the algebra of all bounded linear operators on a separable Hilbert space $H$. Denote by $\{\mu(n, A)\}_{n \geq 0}$ the sequence of singular values of a compact operator $A \in B(H)$. Define the principal ideal $L_{1, \infty}$ (also termed the weak-$L_{1}$ ideal) of the algebra $B(H)$ by setting

$$
L_{1, \infty} := \left\{ A \in B(H) \text{ is compact} : \sup_{n \geq 0} (1 + n)\mu(n, A) < \infty \right\}
$$

and the Dixmier-Macaev ideal $M_{1, \infty}$ by setting

$$
M_{1, \infty} := \left\{ A \in B(H) \text{ is compact} : \sup_{n \geq 0} \frac{1}{\log(2 + n)} \sum_{k=0}^{n} \mu(k, A) < \infty \right\}.
$$

**Definition 2.3.** A trace $\tau$ on $M_{1, \infty}$ (resp., $L_{1, \infty}$) is called a Dixmier trace if it is a linear extension of a weight

$$
\text{Tr}_{\omega}(A) := \omega \left( t \mapsto \frac{1}{\log(1 + t)} \int_{0}^{t} \mu(s, A)ds \right), \ 0 \leq A \in M_{1, \infty} \ (\text{resp., } 0 \leq A \in L_{1, \infty}),
$$

for some dilation invariant extended limit $\omega$ on $L_{\infty}$.

It was proved in [24] that for every dilation invariant extended limit $\omega$ on $L_{\infty}$ the functional

$$
\text{Tr}_{\omega}(A) = \omega \left( t \mapsto \frac{1}{\log(1 + t)} \int_{0}^{t} \mu(s, A)ds \right), \ 0 \leq A \in M_{1, \infty}
$$

extends to the whole space $M_{1, \infty}$ by linearity. It follows directly from Definition 2.3 that every Dixmier trace on $L_{1, \infty}$ extends to a Dixmier trace $\tau$ on $M_{1, \infty}$. Although, it is not clear whether this extension is unique.

Following [54] we define an extended residue of $\zeta$-function on $M_{1, \infty}$ and on $L_{1, \infty}$.

**Definition 2.4.** A functional $\zeta_{\gamma}$ on $M_{1, \infty}$ (resp., $L_{1, \infty}$) is called an extended $\zeta$-function residue if it is a linear extension of a weight

$$
(3) \quad \zeta_{\gamma}(A) := \gamma \left( t \mapsto \frac{1}{t} \text{Tr} \left( A^{1+1/t} \right) \right), \ 0 \leq A \in M_{1, \infty} \ (\text{resp., } 0 \leq A \in L_{1, \infty}),
$$

for some extended limit $\gamma$ on $L_{\infty}$.
It was proved in [6] that \( \text{Tr}(A^{1+1/t}) = O(t), \) \( t \to \infty \) for every \( 0 \leq A \in \mathcal{M}_{1,\infty} \). So, the functional \( \zeta_\gamma \) given by (3) is well-defined for every \( 0 \leq A \in \mathcal{M}_{1,\infty} \). It was proved in [54] that for every extended limit \( \gamma \) on \( L_\infty \) this functional extends to the whole space \( \mathcal{M}_{1,\infty} \) by linearity. It follows directly from Definition 2.4 that every \( \zeta \)-function residue on \( \mathcal{L}_{1,\infty} \) extends to a \( \zeta \)-function residue on \( \mathcal{M}_{1,\infty} \). Although, it is not clear whether this extension is unique.

The following theorem is a combination of [33, Theorems 8.6.4, 6.4.1 and 8.7.1] (see also [54]).

**Theorem 2.5.** If \( \gamma \) is an extended limit on \( L_\infty(0, \infty) \), then

\[
\zeta_\gamma(A) = \text{Tr}_\omega(A), \quad \forall A \in \mathcal{M}_{1,\infty},
\]

for some dilatation invariant extended limit \( \omega \) on \( L_\infty(0, \infty) \). Moreover, the set of all Dixmier traces on \( \mathcal{M}_{1,\infty} \) is strictly larger than the set of all extended \( \zeta \)-function residues on \( \mathcal{M}_{1,\infty} \).

The first part of the latter theorem can be easily transferred to the case of traces on \( \mathcal{L}_{1,\infty} \). Indeed, if \( \zeta_\gamma \) is a zeta-function residue on \( \mathcal{L}_{1,\infty} \), then it extends to a zeta-function residue on \( \mathcal{M}_{1,\infty} \), which coincides with some Dixmier trace on \( \mathcal{M}_{1,\infty} \) (by Theorem 2.5). So, \( \zeta_\gamma \) equals to the restriction of this Dixmier trace to \( \mathcal{L}_{1,\infty} \), which is a Dixmier trace on \( \mathcal{L}_{1,\infty} \).

In Section 3 below we prove that the set of Dixmier traces on \( \mathcal{L}_{1,\infty} \) is strictly larger than the set of extended \( \zeta \)-function residues on \( \mathcal{L}_{1,\infty} \). In other words, we show that not every Dixmier trace on \( \mathcal{L}_{1,\infty} \) can be represented as an extended residue of \( \zeta \)-function on \( \mathcal{L}_{1,\infty} \). Note that this does not follow directly from Theorem 2.5, since it remains unknown whether the extension of a Dixmier trace from \( \mathcal{L}_{1,\infty} \) to \( \mathcal{M}_{1,\infty} \) is unique.

### 3. Dixmier traces and \( \zeta \)-function residues

In this section we prove the aforementioned specialisation of Theorem 2.5 to weak trace class ideal \( \mathcal{L}_{1,\infty} \). We also lay the technical ground for the main results proved in Section 4. First we give an alternative description of Dixmier traces and \( \zeta \)-function residues in terms of classical Banach limits. This characterisation is extensively based on the A. Pietsch’s theory outlined in [36] and further developed in a joint paper by the present authors with E. Semenov [44].

**Definition 3.1.** A linear functional \( B \) on \( l_\infty \) is called a Banach limit if

1. \( B \geq 0 \), that is \( B(x) \geq 0 \) for \( x \geq 0 \),
2. \( B(\mathbb{1}) = 1 \), where \( \mathbb{1} = (1, 1, 1, \ldots) \),
3. \( B(x_1, x_2, x_3, \ldots) = B(x_0, x_1, x_2, \ldots) \) for all \( x \in l_\infty \).

It follows directly from the latter definition that any Banach limits is an extended limit. Recent studies of the set of Banach limits may be found in [1, 42, 43].

The following theorem is a combination of [44, Corollary 4.2 and Theorem 6.3].

**Theorem 3.2.** There is a canonical bijection between positive normalised traces on \( \mathcal{L}_{1,\infty} \) and Banach limits given by the formulae

\[
\tau(A) = \frac{1}{\log 2} B\left( \left\{ \sum_{k=2^n-1}^{2^n+1-2} \lambda(k, A) \right\}_{n \geq 0} \right), \quad A \in \mathcal{L}_{1,\infty},
\]

\[
B(x) = \log 2 \cdot \tau\left( \text{diag}\left\{ x_0, \frac{x_1}{2^1}, \frac{x_2}{2^2}, \ldots, \frac{x_2}{2^2} \right\} \right), \quad x \in l_\infty.
\]

\( \text{diag}\{ \cdot \} \) denotes the diagonal matrix.
As Dixmier traces clearly form a subclass in the class of all positive normalised traces, they should correspond to some subclass of Banach limits. Consider the Cesàro operator $C : l_\infty \to l_\infty$ defined by the setting

$$(Cx)_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k, \quad x \in l_\infty.$$ 

The following assertion is Theorem 5.8 in [44].

**Lemma 3.3.** Dixmier traces correspond (in the sense of bijection established in Theorem 3.2) to the Banach limits of the form $\theta \circ C$, where $\theta$ is an extended limit on $l_\infty$.

**Remark 3.4.** It follows from Raimi’s result in [41], that the set of “factorisable” Banach limits, that is the functionals of the form $\theta \circ C$, where $\theta$ is an extended limit on $l_\infty$, is a proper subset of the set of all Banach limits. It immediately follows from this result combined with Theorem 3.2, that the set of all Dixmier traces is a proper subset in the set of all positive normalised traces on $L_{1,\infty}$.

Since extended $\zeta$-function residues are Dixmier traces, in view of the latter theorem they should correspond to some subset of the set

$$\{\theta \circ C : \theta \text{ is an extended limit on } l_\infty\}.$$ 

The following lemma specifies the form of a Banach limit corresponding (in the sense of Theorem 3.2) to an extended $\zeta$-function residue on $L_{1,\infty}$.

**Lemma 3.5.** If $\gamma$ is an extended limit on $L_\infty(0, \infty)$, then an extended zeta-function residue $\zeta_\gamma$ corresponds (in the sense of bijection established in Theorem 3.2) to a Banach limit defined by setting

$$(4) \quad B(x) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} x_k 2^{-k/t} \right), \quad x \in l_\infty.$$ 

In view of Theorem 3.2, we have

$$\zeta_\gamma(A) = \gamma \left( t \mapsto \frac{1}{t} \sum_{n=0}^{\infty} a_n 2^{-n/t} \right), \quad A \in L_{1,\infty},$$

where

$$a_n = \sum_{k=2^n-1}^{2^n+1-2} \lambda(k, A), \quad n \geq 0.$$ 

**Proof.** By Theorem 2.5, $\zeta_\gamma$ is a Dixmier trace on $L_{1,\infty}$ and therefore by Theorem 3.2, we have

$$B(x) = \log 2 \cdot \zeta_\gamma \left( \text{diag}\{x_0, x_1, \frac{x_1}{2^1}, x_2, \frac{x_2}{2^2}, \ldots, \frac{x_2}{2^2}, \ldots\} \right), \quad 0 \leq x \in l_\infty.$$ 

Noting that in the latter expression the term $\frac{x_k}{2^k}$ is repeated $2^k$ times for every $k \geq 0$ and appealing to (3), we obtain

$$B(x) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^k \cdot \left(\frac{x_k}{2^k}\right)^{1+t} \right) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t} x_k^{1+1/t} \right),$$

for every $0 \leq x \in l_\infty$. Without loss of generality, $0 \leq x_k \leq 1$ for every $k \geq 0$. For every fixed $k \geq 0$ we have

$$|x_k^{1+1/t} - x_k| \leq \sup_{s \in [0,1]} (s - s^{1+1/t}).$$

Consider the function $f(s) = s - s^{1+1/t}$ on $[0, 1]$, where $0 < t < \infty$. It has a maximum point at $s_0 = \left(\frac{t}{1+t}\right)^{t}$. Since

$$f(s_0) = \left(\frac{t}{1+t}\right)^{t} - \left(\frac{t}{1+t}\right)^{1+t} = \left(\frac{t}{1+t}\right)^{t} \left(1 - \frac{t}{1+t}\right) \leq \frac{1}{1+t} < \frac{1}{t}, \quad t > 0,$$
it follows that
\[
|x_k^{1+1/t} - x_k| < \frac{1}{t}, \quad \forall \; k \geq 0, \; t > 0,
\]
and so
\[
B(x) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t}(x_k + O(1/t)) \right).
\]

Observing that \( \sum_{k=0}^{\infty} 2^{-k/t} = \frac{1}{1 - 2^{-1/t}} \) and
\[
\lim_{t \to \infty} \frac{1/t}{1 - 2^{-1/t}} = \frac{1}{\log 2},
\]
we see that \( \sum_{k=0}^{\infty} 2^{-k/t}O(1/t) = O(1) \). The functional \( \gamma \) is an extended limit and so
\[
\gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t}O(1/t) \right) = \gamma(O(1/t)) = 0,
\]
which implies
\[
B(x) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t} x_k \right), \quad \forall \; 0 \leq x \in l_\infty.
\]

For an arbitrary \( x \in l_\infty \) using the linearity of extended limits, we obtain
\[
B(x) = B(x_+) - B(x_-)
\]
\[
= \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t}(x_+)_k \right) - \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t}(x_-)_k \right)
\]
\[
= \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t}(x_+ - x_-)_k \right)
\]
\[
= \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} 2^{-k/t} x_k \right).
\]

A combination of Lemmas 3.3 and 3.5 tells us that any Banach limit \( B \) of the form (4) can be written as \( B = \theta \circ C \) for some extended limit \( \theta \) on \( l_\infty \). In Lemma 3.7 below we further characterise these Banach limits. We need the following auxiliary result.

**Lemma 3.6.** If \( \gamma \) is an extended limit on \( L_\infty(0, \infty) \), then the functionals \( B_2 \) and \( B_3 \) defined on \( l_\infty \) by setting
\[
B_2(x) = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} (k + 1)x_k 2^{-k/t} \right), \quad x \in l_\infty,
\]
\[
B_3(x) = \frac{1}{2} \log^3 2 \cdot \gamma \left( t \mapsto \frac{1}{t^3} \sum_{k=0}^{\infty} (k + 1)^2 x_k 2^{-k/t} \right), \quad x \in l_\infty
\]
are Banach limits.

**Proof.** We prove the assertion for \( B_2 \). The proof for \( B_3 \) is similar.

First of all we show that the definition of \( B_2 \) is correct, that is the function
\[
t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} (k + 1)x_k 2^{-k/t}
\]
belongs to $L_\infty(0, \infty)$. To this end it is sufficient to check that it is bounded for a constant sequence $x$, or rather that the function $t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} (k + 1)2^{-k/t}$ belongs to $L_\infty(0, \infty)$. We shall use the following discrete version of the integration by parts formula:

$$
\sum_{k=0}^{n} f_k g_k = f_n \sum_{k=0}^{n} g_k - \sum_{j=0}^{n-1} (f_{j+1} - f_j) \sum_{k=0}^{j} g_k, \; (f_k)_{k=1}, (g_k)_{k=1} \in \mathbb{R}^n.
$$

For fixed $n \in \mathbb{N}$ and $t > 0$ using (7) we obtain

$$
\sum_{k=0}^{\infty} (k + 1)2^{-k/t} = (n + 1) \sum_{k=0}^{n} 2^{-k/t} - \sum_{j=0}^{n-1} \sum_{k=0}^{j} 2^{-k/t}
$$

$$
= (n + 1) \frac{1 - 2^{-(n+1)/t}}{1 - 2^{-1/t}} - \sum_{j=0}^{n-1} \frac{1 - 2^{-(j+1)/t}}{1 - 2^{-1/t}}
$$

$$
= \frac{1}{1 - 2^{-1/t}} \left( (n + 1)(1 - 2^{-(n+1)/t}) - n + 2^{-1/t} \frac{1 - 2^{-n/t}}{1 - 2^{-1/t}} \right).
$$

Now letting $n \to \infty$ we have for every $t > 0$

$$
\sum_{k=0}^{\infty} (k + 1)2^{-k/t} = \frac{1}{1 - 2^{-1/t}} \lim_{n \to \infty} \left( (n + 1)(1 - 2^{-(n+1)/t}) - n + 2^{-1/t} \frac{1 - 2^{-n/t}}{1 - 2^{-1/t}} \right)
$$

$$
= \frac{1}{1 - 2^{-1/t}} \left( 1 + \frac{2^{-1/t}}{1 - 2^{-1/t}} \right) = \frac{1}{(1 - 2^{-1/t})^2}.
$$

Using (6) we obtain

$$
\frac{1}{t^2} \sum_{k=0}^{\infty} (k + 1)x_k 2^{-k/t} = \frac{1}{(1 - 2^{-1/t})^2} \to \frac{1}{\log 2}, \; \text{as} \; t \to \infty.
$$

That is $B_2$ is well-defined on $l_\infty$ and

$$
B_2(\mathbb{I}) = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} (k + 1)2^{-k/t} \right) = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \frac{1}{(1 - 2^{-1/t})^2} \right) = 1.
$$

The positivity of $B_2$ as a linear functional on $l_\infty$ follows directly from the positivity of the extended limit $\gamma$ on $L_\infty(0, \infty)$.

Let $T$ be the backward shift operator on $l_\infty$, that is

$$
T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).
$$

We need to prove that $B_2(x - Tx) = 0$ for every $x \in l_\infty$. To this end, fix $x \in l_\infty$ and consider

$$
\sum_{k=0}^{\infty} (k + 1)(x_k - x_{k+1})2^{-k/t} = \sum_{k=1}^{\infty} x_k \left( (k + 1)2^{-k/t} - k2^{-(k-1)/t} \right) + x_0
$$

$$
= \sum_{k=1}^{\infty} x_k \left( (k + 1)2^{-k/t} - (k + 1)2^{-(k-1)/t} + 2^{-(k-1)/t} \right) + x_0
$$

$$
= (1 - 2^{1/t}) \sum_{k=1}^{\infty} x_k (k + 1)2^{-k/t} + \sum_{k=1}^{\infty} x_k 2^{-(k-1)/t} + x_0
$$

$$
= (1 - 2^{1/t}) \frac{1}{x^2} + O(1).
$$

By (6) we have $1 - 2^{-1/t} = O(1/t)$. Hence,

$$
\sum_{k=0}^{\infty} (k + 1)(x_k - x_{k+1})2^{-k/t} = O(t).
$$
Therefore,
\[ B_2(x - Tx) = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} (k+1)(x_k - x_{k+1})2^{-k/t} \right) \]
\[ = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \cdot O(t) \right) = 0. \]

Summing up, \( B_2 \) is a positive normalised shift-invariant linear functional on \( l_\infty \), that is \( B_2 \) is a Banach limit.

\[ \square \]

In [49] L. Sucheston showed that every Banach limit satisfies the following inequality
\[ B(x) \leq \lim_{n \to \infty} \sup_{m \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{n} x_{i+m}, \ x \in l_\infty. \]

The following result should be compared with his result.

**Lemma 3.7.** If \( \gamma \) is an extended limit on \( L_\infty(0, \infty) \), then the Banach limit \( B \) corresponding to an extended residue \( \zeta_\gamma \) (in the sense of bijection established in Theorem 3.2) satisfies the condition \( B(x) \leq p(x), \ x \in l_\infty \) where the convex functional \( p \) on \( l_\infty \) is defined by the setting
\[ p(x) := \lim_{n \to \infty} \sup_{m \in \mathbb{N}} \frac{2}{(n+1)^2} \sum_{k=0}^{n} (n+1-k)x_k, \ x \in l_\infty. \]

**Proof.** Let \( B \) be the Banach limit corresponding to an extended residue \( \zeta_\gamma \) (in the sense of bijection established in Theorem 3.2). By Lemma 3.5 we have
\[ B(x) = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} x_k 2^{-k/t} \right), \ x \in l_\infty. \]

Consider the operator \( C_2 : l_\infty \to l_\infty \) defined by the setting
\[ (C_2x)_n = \frac{2}{(n+1)^2} \sum_{k=0}^{n} (k+1)x_k, \ x \in l_\infty \]
and observe that
\[ (C_2Cx)_n = \frac{2}{(n+1)^2} \sum_{j=0}^{n} (j+1)(Cx)(j) = \frac{2}{(n+1)^2} \sum_{j=0}^{n} \sum_{k=0}^{j} x_k, \ \forall x \in l_\infty. \]

Changing the order of summation we obtain
\[ (C_2Cx)_n = \frac{2}{(n+1)^2} \sum_{k=0}^{n} x_k \sum_{j=k}^{n} 1 = \frac{2}{(n+1)^2} \sum_{k=0}^{n} (n+1-k)x_k, \ x \in l_\infty. \]

Let \( B_2 \) and \( B_3 \) be Banach limits from Lemma 3.6. We claim that \( B = B_2 \circ C \) and \( B_2 = B_3 \circ C_2 \). To establish the first equality, we fix \( x \in l_\infty \) and consider
\[ \sum_{k=0}^{\infty} (k+1)(Cx)_k 2^{-k/t} = \sum_{k=0}^{\infty} x_l 2^{-k/t}, \ t > 0. \]

Changing the order of summation we obtain
\[ \sum_{k=0}^{\infty} (k+1)(Cx)_k 2^{-k/t} = \sum_{l=0}^{\infty} x_l \sum_{k=l}^{\infty} 2^{-k/t} = \frac{1}{1 - 2^{-1/t}} \sum_{l=0}^{\infty} x_l 2^{-l/t}. \]

Hence,
\[ B_2(Cx) = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1}{t^2} \sum_{k=0}^{\infty} (k+1)(Cx)_k 2^{-k/t} \right) \]
\[ = \log^2 2 \cdot \gamma \left( t \mapsto \frac{1/t^2}{1 - 2^{-1/t}} \sum_{l=0}^{\infty} x_l 2^{-l/t} \right). \]
Using (6) we obtain
\[ B_2(Cx) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{l=0}^{\infty} x_l 2^{-l/t} \right) = B(x). \]

The claim \( B_2 = B_3 \circ C_2 \) can be proved similarly.

By Lemma 3.6 the functional \( B_3 \) is a Banach limit, that is a particular type of extended limits. Since every extended limit is majorized by \( \lim sup \), it follows that
\[ B_3(x) \leq \lim sup_{n \to \infty} x_n, \quad \forall x \in l_\infty. \]

Using (10) we obtain
\[ B(x) = B_3(C_2Cx) \leq \lim sup_{n \to \infty} (C_2Cx)_n = \lim sup_{n \to \infty} \frac{2}{(n+1)^2} \sum_{k=0}^{n} (n + 1 - k)x_k = p(x). \]

In Lemma 3.9 we compute the value of the functional \( p \) for a particular bounded sequence. This computation will be used (along with Lemma 3.7) in the proof of the main result of this section, Theorem 3.10 below.

Define a linear operator \( K : L_\infty(0, \infty) \to L_\infty(0, \infty) \) by setting
\[ (Kx)(t) = \frac{2}{t^2} \int_0^t (t-s)x(s)ds, \quad x \in L_\infty(0, \infty). \]

This is the continuous analogue of the operator \( C_2 \circ C \) considered in the previous lemma. We introduce it to simplify the calculations in Lemma 3.9 below.

Let us consider a convex functional \( q \) on \( L_\infty(0, \infty) \) given by
\[ q(x) := \lim sup_{t \to \infty} (Kx)(t). \]

Let \( \pi \) be an isometric embedding of \( l_\infty \) to \( L_\infty(0, \infty) \) given by
\[ \pi x := \sum_{k=0}^{\infty} x_k \chi_{[k,k+1]}, \quad \forall x \in l_\infty. \]

Here \( \chi_A \) is a characteristic function of a set \( A \subset \mathbb{R} \).

We prove the following technical lemma, which simplifies the evaluation of the functional \( p \) introduced in Lemma 3.7.

**Lemma 3.8.**
\[ p(x) = q(\pi x), \quad \forall x \in l_\infty. \]

**Proof.** For every \( t \in (n, n+1), \ n \geq 0 \) we have
\[ \int_0^t (t-s)(\pi x)(s)ds = \sum_{k=0}^{n-1} x_k \int_k^{k+1} (t-s) \, ds + \int_0^t (t-s) \, ds = \sum_{k=0}^{n-1} x_k (t-k + O(1)) + x_n \cdot O(1) = \sum_{k=0}^{n-1} x_k (n+1-k + O(1)) + x_n (1 + O(1)) = \sum_{k=0}^{n} x_k (n+1-k) + O(n). \]

Therefore,
\[ \lim sup_{t \to \infty} (K\pi x)(t) = \lim sup_{n \to \infty} \frac{2}{(n+1)^2} \sum_{k=0}^{n} (n + 1 - k)x_k. \]
In the following lemma we construct a function \( y \in L_\infty(0, \infty) \) which we use to demonstrate that the classes of Dixmier traces and extended \( \zeta \)-function residues are distinct in Theorem 3.10 below.

**Lemma 3.9.** If \( A = \bigcup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}] \), then \( q(\chi_A) = \frac{5}{9} \).

**Proof.** Denote \( y = \chi_A \). We have

\[
y(t) := \sum_{n=-\infty}^{\infty} \chi([2^{2n}, 2^{2n+1}))(t), \quad t > 0
\]

and

\[
q(\chi_A) = \lim_{t \to \infty} \sup(Ky)(t).
\]

by Lemma 3.8.

Note that for every \( n \in \mathbb{Z} \) one has \( t \in (2^{2n}, 2^{2n+1}] \) if and only if 
\( 4t \in (2^{2(n+1)}, 2^{2(n+1)+1}] \). Hence, \( y(4t) = y(t), \ t > 0 \). Thus,

\[
(Ky)(4t) = \frac{2}{16t^2} \int_0^{4t} (4t-s)y(s)ds = \frac{2}{16t} \int_0^t (t-s)y(4s)ds = (Ky)(t)
\]

for every \( t > 0 \). Hence, by the definition of \( q \) we have \( q(\chi_A) = \sup_{t \in (1,4]}(Ky)(t) \). If \( 0 < t \leq 4 \), then

\[
y = \begin{cases} 
\sum_{n=0}^{\infty} \chi([2^{-2n}, 2^{-2n+1}]) & \text{if } 0 < t \geq 2 \\
0 & \text{if } t > 2.
\end{cases}
\]

We now shall explicitly compute \((Ky)(t), t \in (1,4] \). Let us first evaluate

\[
\int_0^1 (t-s)y(s)ds = \int_0^1 (t-s) \sum_{n=1}^{\infty} \chi([2^{-2n}, 2^{-2n+1}](s)ds
\]

\[
= \sum_{n=1}^{\infty} \int_{2^{-2n}}^{2^{-2n+1}} (t-s)ds = \sum_{n=1}^{\infty} \left( \frac{t 2^{-2n} - \frac{2^{4n+2} - 2^{4n}}{2}}{1 - 1/4} - \frac{3}{2} \frac{2^{-4}}{1 - 1/16} \right) = \frac{t}{3} - \frac{1}{10}.
\]

Now, consider the following cases:

(i) if \( t \in (1,2] \), then using (11) we obtain

\[
(Ky)(t) = \frac{2}{t^2} \left( \int_0^1 (t-s)y(s)ds + \int_1^t (t-s)ds \right) = \frac{2}{t^2} \left( t \frac{t}{3} - \frac{1}{10} + t(t-1) - \frac{t^2 - 1}{2} \right)
\]

\[
= 1 - \frac{4}{5t} + \frac{4}{5t^2}.
\]

The function \( t \to 1 - \frac{4}{5t} + \frac{4}{5t^2} \) has an extremum at \( t_0 = 6/5 \). So,

\[
\sup_{t \in (1,2]} (Ky)(t) = \max \{ \frac{7}{15}, \frac{4}{5}, \frac{8}{15} \} = \frac{8}{15}.
\]

(ii) if \( t \in (2,4] \), then taking into account that \( y(t) = 0 \) for \( t \in (2,4] \), we have

\[
(Ky)(t) = \frac{2}{t^2} \left( \int_0^1 (t-s)y(s)ds + \int_1^2 (t-s)ds \right) = \frac{2}{t^2} \left( t \frac{t}{3} - \frac{1}{10} + t - \frac{4 - 1}{2} \right)
\]

\[
= \frac{8}{3t} - \frac{16}{5t^2}.
\]

The function \( t \to \frac{8}{3t} - \frac{16}{5t^2} \) has an extremum at \( t_0 = 12/5 \). So,

\[
\sup_{t \in (2,4]} (Ky)(t) = \max \{ \frac{8}{15}, \frac{5}{9}, \frac{7}{15} \} = \frac{5}{9}.
\]

Therefore, \( q(\chi_A) = \frac{5}{9} \).

The following theorem is the main result of this section. It shows that on the weak trace class ideal \( \mathcal{L}_{1,\infty} \) the class of extended \( \zeta \)-function residues is a proper subclass in the class of all Dixmier traces.
Theorem 3.10. Not every Dixmier trace on $L_{1,\infty}$ is an extended $\zeta$-function residue.

Proof. Let $\mathfrak{B}_1$ be the class of Banach limits corresponding to the class of all Dixmier traces on $L_{1,\infty}$ and let $\mathfrak{B}_2$ be the class of Banach limits corresponding to the class of all extended $\zeta$-function residues on $L_{1,\infty}$. Let $\mathcal{A}$ be as in Lemma 3.9 and let $\mathcal{E} = \bigcup_{n \in \mathbb{N}} [2^{2n}, 2^{2n+1}]$.

By Lemma 3.3, every $B \in \mathfrak{B}_1$ is a composition of some extended limit on $l_\infty$ and the Cesàro operator $C$. Since $\limsup_{n \to \infty} x_n = \sup_\theta (x)$ for every $x = \{x_n\} \in l_\infty$, where the latter supremum is taken over all extended limits on $l_\infty$, it follows that

$$\sup_{B \in \mathfrak{B}_1} B(\chi_\mathcal{E}) = \limsup_{n \to \infty} (C\chi_\mathcal{E})_n.$$ 

Computing

$$\limsup_{n \to \infty} (C\chi_\mathcal{E})_n \geq \limsup_{n \to \infty} (C\chi_\mathcal{E})_{2^{2n+1}} \geq \limsup_{n \to \infty} \frac{1}{2^{2n+1}+1} \sum_{k=0}^{2^{2n+1}} (\chi_\mathcal{E})_n$$

$$= \limsup_{n \to \infty} \frac{1}{2^{2n+1}+1} \sum_{k=0}^{n} 2^{2k+1}$$

$$= \limsup_{n \to \infty} \frac{1}{2^{2n+1}+1} \sum_{k=0}^{n} 2^{2k} = \limsup_{n \to \infty} \frac{1}{2^{2n+1}+1} \cdot \frac{2^{2(n+1)} - 1}{4 - 1} = \frac{2}{3},$$

we see that

$$\sup_{B \in \mathfrak{B}_1} B(\chi_\mathcal{E}) = \limsup_{n \to \infty} (C\chi_\mathcal{E})_n \geq \frac{2}{3}.$$ 

On the other hand, by Lemmas 3.7 and 3.8 we have

$$\sup_{B \in \mathfrak{B}_2} B(\chi_\mathcal{E}) \leq p(\chi_\mathcal{E}) = q(\pi \chi_\mathcal{E}).$$

Since $\chi_\mathcal{A} - \pi \chi_\mathcal{E}$ is finitely supported, it follows from the definition of $q$ and Lemma 3.9 that

$$q(\pi \chi_\mathcal{E}) = q(\chi_\mathcal{A}) = \frac{5}{9}$$

and so

$$\sup_{B \in \mathfrak{B}_2} B(\chi_\mathcal{E}) \leq \frac{5}{9}.$$ 

Hence,

$$\sup_{B \in \mathfrak{B}_2} B(\chi_\mathcal{E}) \leq \frac{5}{9} < \frac{2}{3} \leq \sup_{B \in \mathfrak{B}_1} B(\chi_\mathcal{E}),$$

and so $\mathfrak{B}_1 \neq \mathfrak{B}_2$.

This implies the fact that the set of all Dixmier traces on $L_{1,\infty}$ is strictly larger that the set of all traces given by extended $\zeta$-function residues. This completes the proof. \hfill \Box

4. Classes of measurable operators in $L_{1,\infty}$

In this section we prove the main results of this paper (Theorems 4.2, 4.5 and 4.13 below). The proof is based on the techniques developed in Section 3 and various Tauberian theorems.
4.1. Dixmier measurable operators. We start this section with a definition of Dixmier measurability.

**Definition 4.1.** An operator $A \in L_{1,\infty}$ is called Dixmier measurable if the values of all Dixmier traces on $L_{1,\infty}$ coincide on $A$.

There are a number of papers studying Dixmier measurability (see e.g. [32, 45, 44, 50, 51]). It was proved in [44, Proposition 7.3] that $A \in L_{1,\infty}$ is Dixmier measurable if and only if

$$\lim_{n \to \infty} \frac{1}{\log(n+1)} \sum_{k=0}^{n} \lambda(k, A)$$

exists.

Despite the fact that extended ζ-function residues form a proper subclass in the class of Dixmier traces, this subclass of large enough to characterise Dixmier measurability. The following result is the first main result of this paper. It extends [9, Proposition A.4], covering the case of non-positive operators.

**Theorem 4.2.** Let $0 \leq A \in L_{1,\infty}$, $V \in B(H)$ and $c \in \mathbb{C}$. The following conditions are equivalent:

(i) all Dixmier traces equal $c$ on $VA$, that is

$$\lim_{n \to \infty} \frac{1}{\log(n+1)} \sum_{k=0}^{n} \lambda(k, VA) = c;$$

(ii) all extended ζ-function residues $\zeta_\gamma$ on $L_{1,\infty}$ equal $c$ on $VA$;

(iii) the limit

$$\lim_{s \to 1^+} (s-1) \text{Tr}(VA^s) = c.$$

**Proof.** (i) → (ii). By Theorem 2.5 every extended ζ-function residue on $L_{1,\infty}$ coincides with some Dixmier trace on $L_{1,\infty}$. Then Dixmier measurability implies measurability with respect to all extended ζ-function residues.

(ii) → (i). Let $0 \leq A \in L_{1,\infty}$, $V \in B(H)$ and $\zeta_\gamma(VA) = c$ for all extended ζ-function residues on $L_{1,\infty}$. Here, $\zeta_\gamma$ of a non-positive operator is understood as

$$\zeta_\gamma(VA) = \zeta_\gamma(\Re(VA)_+) - \zeta_\gamma(\Re(VA)_-) + i \zeta_\gamma(\Im(VA)_+) - i \zeta_\gamma(\Im(VA)_-).$$

Denote the sequence $\left\{ \sum_{k=2^n-1}^{2^{n+1}-2} \lambda(k, VA) \right\}_{n \geq 0}$ by $z$. Hence, Lemma 3.5 yields that $B(z) = c$ for all Banach limit of the form

$$B(x) = \log 2 \cdot \gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} x_k 2^{-k/t} \right), \quad x \in l_\infty,$$

where $\gamma$ is an extended limit on $L_\infty(0, \infty)$.

Thus, for every extended limit $\gamma$ on $L_\infty(0, \infty)$ we obtain

$$\gamma \left( t \mapsto \frac{1}{t} \sum_{k=0}^{\infty} x_k 2^{-k/t} \right) = \frac{c}{\log 2}.$$

It is well-known that all extended limits agree on a function from $L_\infty(0, \infty)$ if and only if it is convergent (at $+\infty$). So,

$$\lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{\infty} z_k 2^{-k/t} = \frac{c}{\log 2}.$$

Note that $\frac{1/t}{1-2^{-1/t}} \to \frac{1}{\log 2}$ and $2^{-1/t} \to 1^-$, as $t \to \infty$. Setting $r = 2^{-1/t}$, we conclude that the limit

$$\lim_{r \to 1^-} (1-r) \sum_{k=0}^{\infty} z_k r^k = c.$$
Consider
\[ (1 - r) \sum_{k=0}^{\infty} z_k r^k = \sum_{k=0}^{\infty} z_k (r^k - r^{k+1}) = z_0 + \sum_{k=1}^{\infty} (z_k - z_{k-1}) r^k. \]

Setting \( y_0 := z_0 \) and \( y_k := z_k - z_{k-1}, \quad k \geq 1 \) we obtain that
\[ \lim_{r \to 1^-} \sum_{k=0}^{\infty} y_k r^k = c. \]

That is the series \( \sum_{k=0}^{\infty} y_k \) is Abel summable (see e.g. [23, Section 4.7]). Note that, the partial sums of this series
\[ \sum_{k=0}^{n} y_k = z_0 + z_1 - z_0 + \cdots + z_n - z_{n-1} = z_n, \quad n \geq 0 \]
are bounded. It is proved in [13, Theorem 1] that if a sequence of partial sums of a series is bounded, then the Abel summability implies the Cesàro summability.

Therefore, the series \( \sum_{k=0}^{\infty} y_k \) is Cesàro summable to the number \( c \).

That is, the series \( \sum_{k=0}^{\infty} z_k \) is Cesàro summable to the number \( c \).

It follows from [33, Theorem 10.1.3(g)] that
\[ \sum_{k=0}^{n} \lambda(k, A) = c \log(n + 1) + O(1), \quad n \to \infty, \]
whereas, by [33, Theorem 10.1.3(f)] Dixmier measurability of \( A \in \mathcal{L}_{1,\infty} \) is equivalent to the following asymptotics:
\[ \sum_{k=0}^{n} \lambda(k, A) = c \log(n + 1) + o(\log n), \quad n \to \infty. \]

In other words, the estimate of non-leading terms in asymptotics of partial sums of eigenvalues of universally measurable operators \( O(1) \) is more specific than that of Dixmier measurable operators \( o(\log n) \). This is
a consequence of the fact that the class of all Dixmier traces is a proper subset of the class of all positive normalised traces.

In this section we prove a necessary condition for universal measurability in terms of $\zeta$-function asymptotics. We need an auxiliary lemma.

**Lemma 4.4.** For every $0 \leq A_1, A_2 \in \mathcal{L}_{1,\infty}$ the condition

$$\sum_{k=0}^{n} \mu(k, A_1) - \sum_{k=0}^{n} \mu(k, A_2) = c \log(n + 1) + O(1), \ n \to \infty$$

implies

$$\text{Tr}(A_1^s) - \text{Tr}(A_2^s) = \frac{c}{s-1} + O(1), \ s \to 1^+.$$

**Proof.** Without loss of generality we assume $c = 1$. Let

$$\sum_{k=0}^{n} \mu(k, A_1) - \sum_{k=0}^{n} \mu(k, A_2) = \log(n + 1) + O(1), \ n \to \infty.$$

Noting that, $\sum_{k=1}^{n} \frac{1}{k} > \log(n + 1)$ we obtain

$$\sum_{k=0}^{n} \mu(k, A_1) \leq \sum_{k=0}^{n} \mu(k, A_2) + \sum_{k=0}^{n} \frac{1}{k+1} + \alpha$$

for some absolute constant $\alpha$.

Introducing the sequence $x_{2k} = 1/(k+1)$, $x_{2k-1} = \mu(k, A_2)$, $k \geq 1$ and $x_0 = \alpha$ we have

$$\sum_{k=0}^{n} \mu(k, A_1) \leq \sum_{k=0}^{2n+1} x_k.$$

Since $x_k \leq \max\{\|A_2\|_{1,\infty}, 1\} \frac{1}{k+1}$, it follows that $\sum_{k=n+1}^{2n+1} x_k \leq \alpha_1$ for some absolute constant $\alpha_1$. Introducing the sequence $y_k = x_k$, $k \geq 1$ and $y_0 = x_0 + \alpha_1$ we have

$$\sum_{k=0}^{n} \mu(k, A_1) \leq \sum_{k=0}^{n} y_k, \ \forall n \geq 0.$$

Therefore, for every $s > 1$ we have (see e.g. [21, Lemma II.3.4])

$$\sum_{k=0}^{\infty} \mu(k, A_1)^s \leq \sum_{k=1}^{\infty} y_k^s = \sum_{k=0}^{\infty} \mu(k, A_2)^s + \sum_{k=1}^{\infty} \frac{1}{k^s} + (\alpha + \alpha_1)^s$$

$$= \sum_{k=0}^{\infty} \mu(k, A_2)^s + \zeta(s) + (\alpha + \alpha_1)^s,$$

where $\zeta(s)$ is the Riemann $\zeta$-function.

Similarly, noting that $\sum_{k=1}^{n} \frac{1}{k} < \log n + 1$ we obtain

$$\sum_{k=0}^{\infty} \mu(k, A_1)^s \geq \sum_{k=0}^{\infty} \mu(k, A_2)^s + \zeta(s) + (\beta + \beta_1)^s$$

for some absolute constants $\beta$, $\beta_1$ and every $s > 1$. Hence,

$$\sum_{k=0}^{\infty} \mu(k, A_1)^s - \sum_{k=0}^{\infty} \mu(k, A_2)^s = \frac{1}{s-1} + O(1), \ s \to 1^+.$$

This proves the assertion. \qed

The following theorem provides a necessary condition for universal measurability.
Theorem 4.5. If $0 \leq A \in \mathcal{L}_{1,\infty}$ and $V \in B(\mathcal{H})$ such that the operator $VA$ is universally measurable, that is
\begin{equation}
\sum_{k=0}^{n} \lambda(k, VA) = c \log(n+1) + O(1), \ n \to \infty
\end{equation}
for some $c \in \mathbb{C}$, then
\begin{equation}
\text{Tr} (VA^*) = \frac{c}{s-1} + O(1), \ s \to 1 + .
\end{equation}

Proof. We split the proof into a number of steps.

1. Since zero eigenvalues do not affect the value of traces, we may assume that $A$ is strictly positive. Let $\{e_k\}$ be the set of eigenvectors of $0 < A$ ordered such that $Ae_k = \lambda(k, A)e_k$. Consider the operator $W$ given by $We_k = \langle Ve_k, e_k \rangle e_k$. Note that $W$ commutes with $A$, since it is diagonal in the basis $\{e_k\}$. We have
\begin{align*}
\text{Tr} (VA^*) &= \sum_{k=0}^{\infty} \langle VA^* e_k, e_k \rangle = \sum_{k=0}^{\infty} \langle Ve_k, e_k \rangle \lambda(k, A)^s = \sum_{k=0}^{\infty} \langle We_k, e_k \rangle \lambda(k, A)^s \\
&= \sum_{k=0}^{\infty} \langle WA^* e_k, e_k \rangle = \text{Tr} (WA^*), \ \forall s > 1.
\end{align*}

It is a non-trivial result that for every normalised trace $\tau$ on $\mathcal{L}_{1,\infty}$ [33, Theorem 11.2.3] yields
\begin{equation}
\tau(VA) = \tau(\text{diag}(VAe_k, e_k)),
\end{equation}
where $\text{diag}$ is a diagonal operator with respect to $\{e_k\}$. Since we have $\langle VAe_k, e_k \rangle = \langle Ve_k, e_k \rangle \lambda(k, A)$, it follows immediately that
\begin{equation}
\tau(VA) = \tau(\text{diag}(Ve_k, e_k)\lambda(k, A)) = \tau(WA).
\end{equation}
for every normalised trace $\tau$ on $\mathcal{L}_{1,\infty}$.

Therefore, the theorem holds for $V$ if and only if it holds for $W$. Hence, we may assume that $V$ commutes with $A$.

2. Since $A$ is positive and $V$ commutes with $A$, it follows that $VA$ is normal. For a normal operator $VA = \Re(VA) + i\Im(VA)$ [33, Lemma 5.2.10] yields
\begin{equation}
\sum_{k=0}^{n} \left( \lambda(k, VA) - \lambda(k, \Re(VA)) - i\lambda(k, \Im(VA)) \right) \leq 5n\mu(n, VA), \ \forall n \geq 0.
\end{equation}

In view of the assumption $VA \in \mathcal{L}_{1,\infty}$ the preceding estimate implies
\begin{equation}
\sum_{k=0}^{n} \left( \lambda(k, VA) - \lambda(k, \Re(VA)) - i\lambda(k, \Im(VA)) \right) = O(1), \ \forall n \geq 0.
\end{equation}

So, the asymptotic (13) holds if and only if the asymptotics
\begin{equation}
\sum_{k=0}^{n} \lambda(k, \Re(VA)) = \Re(c) \log(n+1) + O(1), \ n \to \infty
\end{equation}
and
\begin{equation}
\sum_{k=0}^{n} \lambda(k, \Im(VA)) = \Im(c) \log(n+1) + O(1), \ n \to \infty
\end{equation}
both hold. Therefore, $VA$ is universally measurable if and only if $\Re(VA)$ and $\Im(VA)$ are. Similarly, the asymptotic (14) in the conclusion of the theorem may be split into its real and imaginary part. Hence, without loss of generality we may assume that $VA$ and $V$ are self-adjoint. We denote by $V_+$ and $V_-$ the positive and the negative parts of a self-adjoint operator $V$.

3. For a compact self-adjoint operator $VA$ by [33, Lemma 5.7.1] we obtain
\begin{equation}
\sum_{k=0}^{n} \lambda(k, VA) = \sum_{k=0}^{n} \mu(k, V_+ A) - \sum_{k=0}^{n} \mu(k, V_- A) + O(1), \ n \to \infty.
\end{equation}
So, the assumption of the theorem guarantees that
\[ \sum_{k=0}^{n} \mu(k, V_{+}A) - \sum_{k=0}^{n} \mu(k, V_{-}A) = c \log(n + 1) + O(1), \quad n \to \infty. \]

Lemma 4.4 combined with (15) yields
\[ \text{Tr}((V_{+}A)^s) - \text{Tr}((V_{-}A)^s) = \frac{c}{s-1} + O(1), \quad s \to 1+. \]

4. Since \( V \) commutes with \( A \) it follows that \( \text{Tr}((V_{+}A)^s) = \text{Tr}(V_{+}^sA^s) \) and \( \text{Tr}((V_{-}A)^s) = \text{Tr}(V_{-}^sA^s) \) and so,
\[ |\text{Tr}((V_{+}A)^s) - \text{Tr}(V_{+}^sA^s)| \leq \text{Tr}(|V_{+}^s - V_{+}|A^s) \leq \|V_{+}^s - V_{+}\|_{\infty}\text{Tr}(A^s). \]

Since \( A \in \mathcal{L}_{1,\infty} \), it follows from [6, Theorem 4.5] that \( \text{Tr}(A^s) = O\left(\frac{1}{s} \right) \) as \( s \to 1+ \). Also
\[ \|V_{+}^s - V_{+}\|_{\infty} = \sup_{x \in \sigma(V_{+})} |x^s - x| \leq k(s - 1), \]
as shown in the proof of Theorem 3.5. Here, \( \sigma(V_{+}) \) stands for the spectrum of \( V_{+} \) and \( k \) is a constant depending on \( \|V_{+}\|_{\infty} \) only.

Therefore,
\[ \text{Tr}((V_{+}A)^s) - \text{Tr}(V_{+}^sA^s) = O(1), \quad s \to 1+ \]
and similarly,
\[ \text{Tr}((V_{-}A)^s) - \text{Tr}(V_{-}^sA^s) = O(1), \quad s \to 1+. \]

Hence,
\[ \text{Tr}(VA^s) = \text{Tr}(V_{+}A^s) - \text{Tr}(V_{-}A^s) = \text{Tr}((V_{+}A)^s) - \text{Tr}((V_{-}A)^s) + O(1) \]
\[ = \frac{c}{s-1} + O(1), \quad s \to 1+. \]

This proves the assertion. \( \square \)

4.3. **Sufficient condition for universal measurability.** Now we discuss the converse implication in Theorem 4.5.

Observe that the \( \zeta \)-function can be directly written as a generalised Dirichlet series. Indeed, for \( 0 \leq A \in \mathcal{L}_{1,\infty} \) we have
\[ \zeta_A(s) = \sum_{k=0}^{\infty} \lambda(k, A)^s = \sum_{k=0}^{\infty} \lambda(k, A)e^{(s-1)\log \lambda(k, A)}, \quad \Re(s) > 1. \]

Setting \( x = e^{1-s}, a_k = \lambda(k, A) \) and \( d_k = -\log \lambda(k, A) \to +\infty \), we obtain
\[ \zeta_A(s) = \sum_{k=0}^{\infty} a_kx^{d_k}, \quad \Re(s) > 1 \text{ (or } |x| < 1). \]

We shall show that the \( \zeta \)-function can be written as a sum of a *power* series and a bounded term.

**Theorem 4.6.** For \( 0 \leq A \in \mathcal{L}_{1,\infty} \) we have
\[ \zeta_A(s) = \sum_{n=0}^{\infty} 2^{n(1-s)}b_n + O(1), \quad s \to 1+, \]
where \( b_n = \sum_{k=2^n-1}^{2^{n+1}-2} \lambda(k, A), \quad n \geq 0. \)

**Proof.** Without loss of generality we assume \( \|A\|_{\mathcal{L}_{1,\infty}} \leq 1 \). Denote \( a_k = \lambda(k, A), \quad k \geq 0. \)

Since \( 0 \leq a_n \leq 1/(n + 1) \), it follows that \( 0 \leq b_n \leq 1 \). Therefore,
\[ 0 \leq b_n - b_n^s < s - 1, \quad \forall \ n \geq 0 \]
as shown in the proof of Theorem 3.5.
Therefore
\[ \sum_{n=0}^{\infty} 2^{n(1-s)} b_n = \sum_{n=0}^{\infty} 2^{n(1-s)} b_n^s + O(1), \quad s \to 1^+, \]
since \( \sum_{n=0}^{\infty} 2^{n(1-s)} (s-1) = \frac{s-1}{1-2^{1-s}} \to \frac{1}{\log 2}, \) as \( s \to 1^+ . \)

The equation (17) yields
\[ \sum_{n=0}^{\infty} a_n^s - \sum_{n=0}^{\infty} 2^{n(1-s)} b_n = \sum_{n=0}^{\infty} \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k^s - 2^{n(1-s)} \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k \right)^s \right) + O(1), \quad s \to 1^+. \]

Hence, it is sufficient to show that
\[ f(s) := \sum_{n=0}^{\infty} \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k^s - 2^{n(1-s)} \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k \right)^s \right) \]
is \( O(1), \) as \( s \to 1^+ . \)

For a fixed \( n \geq 0 \) we consider the sequence \( \{a_n\} \) as a step function on \( I := [2^n - 1, 2^{n+1} - 2] . \) We have
\( \|a\|_{L_s(I)} \geq \|a\|_{L^1(I)} \) for every \( s \geq 1 . \) That is,
\[ \sum_{k=2^n-1}^{2^{n+1}-2} a_k^s \geq \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k \right)^s \geq 2^{n(1-s)} \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k \right)^s \]
for every \( s \geq 1 . \) So, \( f(s) \geq 0 . \)

Since \( a_n \) is a decreasing sequence, it follows that \( b_{n-1} \geq 2^n a_{2^n-2} . \) Next for every \( 2^n - 1 \leq k \leq 2^{n+1} - 2 \) we obtain
\[ a_k^s \leq a_k a_{2^n-2}^s \leq a_k \left( \frac{b_{n-1}}{2^n} \right)^{s-1} . \]

Therefore,
\[ 0 \leq f(s) \leq \sum_{n=0}^{\infty} \left( \sum_{k=2^n-1}^{2^{n+1}-2} a_k \left( \frac{b_{n-1}}{2^n} \right)^{s-1} - 2^{n(1-s)} b_n^s \right) = \sum_{n=0}^{\infty} b_n \left( \left( \frac{b_{n-1}}{2^n} \right)^{s-1} - \left( \frac{b_n}{2^n} \right)^{s-1} \right) . \]

Since \( 0 \leq a_n \leq 1/(n + 1) , \) it follows that \( 0 \leq b_n \leq 1 \) and \( 2b_{n-1} \geq b_n . \) Hence, \( \left( \frac{b_{n-1}}{2^n} \right)^{s-1} \geq \left( \frac{b_n}{2^n} \right)^{s-1} \) and
\[ 0 \leq f(s) \leq \sum_{n=0}^{\infty} \left( \left( \frac{b_{n-1}}{2^n} \right)^{s-1} - \left( \frac{b_n}{2^n} \right)^{s-1} \right) = b_0^{s-1} . \]

Hence, \( f(s) = O(1) , \quad s \to 1^+ . \)

Having a \( \zeta \)-function written as a power series it is desirable to use Tauberian theorems to deduce the asymptotic behaviour of coefficients \( a_k \) from the behaviour of the series \( \sum a_k x^k . \)

The following result may be viewed as a refined version of the classical Hardy-Littlewood theorem [23, Theorem 90]. It was proved in [19]) (see also [27, III, Section 18]).

**Theorem 4.7.** If \( \{a_n\} \in l_\infty \) and
\[ \sum_{k=0}^{\infty} a_k x^k = \frac{1}{1-x} + O(1), \quad x \to 1^-, \]
then
\[ \sum_{k=0}^{n} a_k = n + O\left( \frac{n}{\log n} \right), \quad n \to \infty. \]

Where the estimate \( O\left( \frac{n}{\log n} \right) \) for the remainder is optimal.
The remainder $O\left(\frac{n}{\log n}\right)$ in the latter expression can be improved by making an additional assumption on the behaviour of the series $\sum a_k x^k$ in the complex plane. The following important result was proved in [48, Theorem 2.3.1] (see also [27, III, Theorem 18.1]).

**Theorem 4.8.** If $\{a_n\} \in \ell_\infty$ and
$$\sum_{k=0}^\infty a_k z^k = \frac{1}{1 - z} + O(1)$$
for all $z \in \mathbb{C}$ such that $0 \leq \Re z < 1$, $|\Im z| \leq c(1 - \Re z)^\beta$ for some $c > 0$ and $\beta < 1$, then
$$\sum_{k=0}^n a_k = n + O(\log n), \ n \to \infty.$$ The estimate for the remainder $O(\log n)$ is optimal (see [39, §21]).

We now explain the relevance of Theorems 4.7 and 4.8 to $\zeta$-functions.

**Theorem 4.9.** For every $0 \leq A \in \mathcal{L}_{1,\infty}$ if
$$\operatorname{Tr} (A^n) = \frac{1}{s - 1} + O(1), \ s \to 1+,$$
then
$$\sum_{k=0}^n \lambda(k, A) = \log(n + 1) + O\left(\frac{\log n}{\log \log n}\right), \ n \to \infty.$$ 

**Proof.** By Theorem 4.6 we have
$$\sum_{n=0}^\infty 2^{n(1-s)} b_n = \frac{1}{s - 1} + O(1), \ s \to 1+,$$
where $b_n = \sum_{k=2^n}^{2^{n+1}-2} \lambda(k, A)$, $n \geq 0$. Setting $x = 2^{1-s}$ yields
$$\sum_{n=0}^\infty b_n x^n = \frac{\log 2}{1 - x} + O(1), \ x \to 1-,$$
since $\frac{s-1}{1-2^{1-s}} \to \log 2, \ s \to 1+$. 

By Theorem 4.7 one has
$$\sum_{k=0}^n b_k = n \log 2 + O\left(\frac{n}{\log n}\right), \ n \to \infty.$$ 

That is,
$$\sum_{k=0}^{2^{n+1}-2} \lambda(k, A) = n \log 2 + O\left(\frac{n}{\log n}\right), \ n \to \infty.$$ 

If we write $\log_2 n$ instead of $n$ in the preceding asymptotic, we obtain
$$\sum_{k=0}^n \lambda(k, A) = \log n + O\left(\frac{\log n}{\log \log n}\right), \ n \to \infty.$$ 

If we assume “nice behaviour” of the $\zeta$-function in a region of the complex plane around $s = 1$, then the remainder is improved as follows:

**Theorem 4.10.** For every $0 \leq A \in \mathcal{L}_{1,\infty}$ if
$$\operatorname{Tr} (A^n) = \frac{1}{s - 1} + O(1)$$
for all $s \in \mathbb{C}$ such that $0 \leq \Re(2^{1-s}) < 1$, $|\Im(2^{1-s})| \leq c(1 - \Re(2^{1-s}))^\beta$ for some $c > 0$ and $\beta < 1$ then
$$\sum_{k=0}^n \lambda(k, A) = \log(n + 1) + O(\log \log n), \ n \to \infty.$$
The proof of Theorem 4.10 uses Theorem 4.8. It is similar to the proof of Theorem 4.9 and therefore omitted.

Note that the remainder $O(\log \log n)$ in the preceding theorem is still disappointingly large compared to the desired asymptotics $O(1)$ as in Theorem 4.5. Also, the remainder $O(\log \log n)$ cannot be further improved due to the example in [39, §21].

Next we shall prove the converse implication in Theorem 4.5 under some additional conditions. We shall do so in two ways ending up with assumptions which are almost identical. The first method is based on the Tauberian remainder theory, whereas the second method is based on a recent result [17] concerning the classical relation between the $\zeta$-function and the heat semi-group via the inverse Mellin transform. As the assumptions in the second approach are slightly stronger we explain it in Appendix.

The key result of the first approach is the quantitative variant of the Fatou theorem for Dirichlet series. The original Fatou theorem (for power series) asserts the convergence of $\sum_{k \geq 0} b_k z^k$ provided the power series $\sum_{k \geq 0} b_k z^k$ defined for $|z| < 1$ is regular at $z = 1$ (that is, analytic and single-valued in any neighbourhood of $z = 1$) and $b_k \to 0$. Variants of this theorem (for power series) estimating the rate of convergence (that is, the remainder) may be found in [39, §17] and [27, Chapter III]. In particular, it is proved in [39, §17, Theorem 4] that for a series $\sum_{k \geq 0} b_k z^k$ with the $b_k \geq 0$ and unit radius of convergence, the regularity of

$$\sum_{k \geq 0} b_k z^k - \frac{1}{1-z}$$

at $z = 1$ implies the following:

$$\sum_{k=0}^{n} b_k = n + O(1), n \to \infty.$$

By Theorem 4.6 the Dirichlet series (16) corresponding to the $\zeta$-function can be written as a power series plus a bounded term. Despite this fact the Fatou theorem for power series can not be directly used for this Dirichlet series, since the bounded term is not necessarily regular.

The analogue of the quantitative Fatou theorem for Dirichlet series (in fact, for Laplace-Stieltjes transform) was established by M. Subkhankulov and appeared in his book [48].

The following extension of the Fatou theorem proved by Subkhankulov [48, Theorem 2.3.2]. The theorem is proved in great generality. We state below a special case of [48, Theorem 2.3.2].

**Theorem 4.11.** Let $\phi : [0, \infty) \to [0, \infty)$ be a positive function such that:

(i) $\phi(t) \to 0$ as $t \to 0$;

(ii) for some $h > 0$ the function $\phi$ is of a bounded variation on $[0, h]$ and increasing on $[h, \infty)$.

If the function

$$z \mapsto \int_0^\infty e^{-zt} d\phi(t) - \frac{1}{z}$$

is regular at $z = 0$, then $\phi(t) = t + O(1), t \to \infty$.

We first prove the converse of Theorem 4.5 in the case $V = 1$.

**Theorem 4.12.** Let $0 \leq A \in \mathcal{L}_{1,\infty}$ and $c \geq 0$. If the function

$$z \mapsto \zeta_A(z) - \frac{c}{z-1}$$

is regular at $z = 1$, then $\tau(A) = c$ for every normalised trace $\tau$ on $\mathcal{L}_{1,\infty}$.

**Proof.** Without loss of generality we assume $\|A\|_{\mathcal{L}_{1,\infty}} = 1$ and $c = 1$.

Consider the function

$$\phi(t) := \sum_{\lambda(k,A) > e^{-t}} \lambda(k, A).$$


It is easy to see that \( \phi(t) \to 0 \) as \( t \to 0 \), since \( \| A \|_{L_1, \infty} = 1 \).

Writing

\[
\phi(t) = \sum_{-\log \lambda(k, A) < t} \lambda(k, A), \quad t \geq 0.
\]

we have that \( \phi \) is increasing on \([0, \infty)\) and of bounded variation on any bounded interval.

Next,

\[
\int_0^\infty e^{-zt} d\phi(t) = \int_0^\infty e^{-zt} \left( \sum_{-\log \lambda(k, A) < t} \lambda(k, A) \right) e^{z \log \lambda(k, A)}
= \sum_{k=0}^\infty \lambda^{1+z}(k, A) = \zeta_A(1 + z).
\]

By Theorem 4.11 we have

\[
\phi(t) = t + O(1), \quad t \to \infty.
\]

Using (19) we obtain

\[
\sum_{\lambda(k, A) > 1/t} \lambda(k, A) = \log t + O(1), \quad t \to \infty.
\]

Since \( \| A \|_{L_1, \infty} = 1 \), it follows that \( \lambda(k, A) \leq \frac{1}{t^{1/k}} \). So, \( \lambda(k, A) > 1/t \) implies that \( k + 1 < t \). Therefore, we have

\[
\sum_{k+1 < t} \lambda(k, A) = \log t + O(1), \quad t \to \infty.
\]

Hence,

\[
\sum_{k=0}^n \lambda(k, A) = \log(n + 1) + O(1), \quad n \to \infty,
\]

that is \( \tau(A) = 1 \) for every normalised trace \( \tau \) on \( L_{1, \infty} \).

Now we extend the preceding result to the case of an arbitrary \( V \in B(H) \).

**Theorem 4.13.** Let \( 0 \leq A \in L_{1, \infty}, \ V \in B(H), \ c_1 \geq 0 \) and \( c \in \mathbb{C} \). If the function

\[
z \mapsto \text{Tr}(VA^z) - \frac{c}{z-1}
\]

is regular at \( z = 1 \)

and

\[
z \mapsto \zeta_A(z) - \frac{c_1}{z-1}
\]

is regular at \( z = 1 \),

then \( \tau(VA) = c \) for every normalised trace \( \tau \) on \( L_{1, \infty} \), that is

\[
\sum_{k=0}^n \lambda(k, VA) = c \log(n + 1) + O(1), \quad n \to \infty.
\]

**Proof.** Without loss of generality we assume \( \| A \|_{L_{1, \infty}} = 1 \). We split the proof into a number of steps.

1. Let \( \{ e_k \} \) be the set of eigenvectors of \( A \) ordered such that \( Ae_k = \lambda(k, A)e_k \). Direct repetition of parts 1 and 2 of the proof of Theorem 4.5 yields that without loss of generality we may assume that \( VA \) and \( V \) are self-adjoint and that \( V \) is diagonal in the basis \( \{ e_k \} \).

2. It follows from the conditions of the theorem that

\[
z \mapsto \text{Tr}((V + \| V \|_\infty)A^z) - \frac{c + c_1 \| V \|_\infty}{z-1}
\]

is regular at \( z = 1 \).

The conclusion of the theorem may be written as

\[
\tau((V + \| V \|_\infty)A) = \tau(VA) + \| V \|_\infty \tau(A) = \tau(VA) + c_1 \| V \|_\infty
\]

by Theorem 4.12. So, the assumptions and the conclusion of this theorem hold for \( V \) if and only if they hold for \( V + \| V \|_\infty \). Therefore, we may assume that \( V \geq 0 \).
3. Denote \( v_k = \langle Ve_k, e_k \rangle \). Consider the function

\[
\phi(t) := \sum_{\lambda(k,A) > e^{-t}} v_k \lambda(k,A)
\]

It is easy to see that \( \phi(t) \to 0 \) as \( t \to 0 \), since \( \|A\|_{L_1,\infty} = 1 \).

Writing

\[
\phi(t) = \sum_{-\log \lambda(k,A) < t} -\log \lambda(k,A) v_k \lambda(k,A),
\]

we have that \( \phi \) is increasing on \([0, \infty)\) and of bounded variation on any bounded interval.

Since \( \phi \) is a step function, it follows that

\[
\int_0^\infty e^{-zt}d\phi(t) = \int_0^\infty \sum_{-\log \lambda(k,A) < t} v_k \lambda(k,A) e^{-z \log \lambda(k,A)} dt
= \sum_{k=0}^\infty v_k \lambda^{1+z}(k,A).
\]

Therefore,

\[
\int_0^\infty e^{-zt}d\phi(t) = \text{Tr}(VA^{1+z}).
\]

By Theorem 4.11 we obtain

\[
\phi(t) = ct + O(1), \ t \to \infty.
\]

Similarly to the proof of Theorem 4.12 we have

\[
\sum_{k=0}^n v_k \lambda(k,A) = c \log(n+1) + O(1), \ n \to \infty.
\]

So, \( \tau(\text{diag}(v_k \lambda(k,A))) = c \) for every normalised trace \( \tau \) on \( L_{1,\infty} \). Here diag is a diagonal operator with respect to \( \{e_k\} \). Using the definition of \( v_k \), we obtain

\[
v_k \lambda(k,A) = \lambda(k,A) \langle Ve_k, e_k \rangle = \langle V Ae_k, e_k \rangle.
\]

Therefore, for every normalised trace \( \tau \) on \( L_{1,\infty} \) we obtain

\[
c = \tau(\text{diag}(v_k \lambda(k,A))) = \tau(\text{diag}(V Ae_k, e_k)) = \tau(VA),
\]

where the latter equality follows from [33, Theorem 11.2.1]. This proves the assertion.

\[
\square
\]

5. Examples and remarks

In this section we demonstrate that conditions of Theorem 4.13 are satisfied by large classes of operators.

Example 5.1. Let \( M \) be a closed Riemannian manifold of dimension \( d \). Let \( A \) be a self-adjoint positive elliptic operator on \( M \). Assume also that \( A \) is a compactly based classical pseudo-differential operator of order \(-d\).

We have \( A \in L_{1,\infty} \) (see e.g. [8]). By [46, Theorem 13.1] one has that \( \zeta_A \) defined as follows:

\[
\zeta_A(z) = \sum_{k=0}^\infty \lambda(k,A)^z, \ \Re(z) > 1
\]

extends meromorphically to the entire complex plane with at most 1-st order poles, which can be possibly located at the points \( \{1 - \frac{j}{d}\}_{j \geq 0} \setminus (-\mathbb{N}) \). So, every classical pseudo-differential operator of order \(-d\) satisfies assumptions of Theorem 4.13.

The following example is based on results established in Section 4 of [17].
Example 5.2. Let \( p \) be a polynomial, such that roots of \( p \) are not in \( 0, 1, 2, \ldots \). Let \( A \) be an operator with eigenvalues \( \lambda(n, A) = p(n) \) and multiplicities one. Denote \( d := \deg(p) \). It is easy to see that an operator \( A^{-d} \) belongs to \( L_{1,\infty} \). Its \( \zeta \)-function extends meromorphically to the whole complex plane with at most 1-st order pole, which can be possibly located at 1 (see [17, Theorem 6]).

The following result is a corollary of Theorem 4.13.

Corollary 5.3. Let \( A \) be an operator with eigenvalues \( \lambda(n, A) = p(n) \) and multiplicities one, where \( p \) is a polynomial, such that roots of \( p \) are not in \( 0, 1, 2, \ldots \). Let \( d := \deg(p) \).

The operator \( A^{-d} \) is universally measurable and

\[
\tau(A^{-d}) = \lim_{s \to 1^+} (s - 1) \text{Tr}(A^{-ds}),
\]

for every normalised trace \( \tau \) on \( L_{1,\infty} \).

Example 5.4. Let \( L \) be a self-similar Laplacian on a Sierpinski gasket [26]. Let \( d = \log 3 / \log 2 \) be the Hausdorff dimension of a Sierpinski gasket. It is shown in [47] that a \( \zeta \)-function of \( L^{-d/2} \) extends meromorphically to \( \Re(z) > 1 - \varepsilon \) with at most 1-st order poles, which can be possibly located at the points \( 1 + i \frac{2\pi n}{\log 2} \), \( n \in \mathbb{Z} \).

Therefore, the operator \( L^{-d/2} \) satisfies assumptions of Theorem 4.13. Hence, by Theorem 4.13 the operator \( L^{-d/2} \) is universally measurable and the common value that all normalised traces on \( L_{1,\infty} \) take on \( L^{-d/2} \) equals to the residue of its \( \zeta \)-function at \( s = 1 \).

Example 5.5. A fractal string \( \mathcal{L} \) is an open bounded subset \( \Omega \subset \mathbb{R} \), that is a disjoint (countable) union of open intervals of not necessarily distinct lengths \( \{l_j\}_{j \geq 1} \), ordered in a non-increasing order (see [31]). Note that \( \sum_{j=1}^{\infty} l_j \) is finite and equals the Lebesgue measure of \( \Omega \).

The geometric \( \zeta \)-function of a fractal string \( \mathcal{L} \) is defined as follows

\[
\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s, \quad \Re(s) > 1.
\]

The geometric \( \zeta \)-function of \( \mathcal{L} \) is related to the spectral (usual) \( \zeta \)-function of a Laplacian \( L \) on \( \Omega \) as follows [31, Theorem 1.19]:

\[
\zeta_L(-s) = \pi^{-s} \zeta_{\mathcal{L}}(s) \zeta(s),
\]

where \( \zeta \) is the Riemann \( \zeta \)-function. Here, we do not normalise the eigenvalue of the Laplacian as in [31], so we take \( \lambda(j, L) = l_j^{-1}, \ j \geq 1 \).

Consider the Cantor string, that is a complement of the Cantor set fractal in \([0, 1]\). It is shown in [31, Sections 1.2.2 and 2.3.1] that the geometric \( \zeta \)-function \( \zeta_{\mathcal{L}} \) of the Cantor string extends meromorphically to the whole complex plane with first order poles located at \( d + inp, \ n \in \mathbb{Z} \), where \( d = \log 2 / \log 3 \) is the Hausdorff dimension of the Cantor set and \( p = 2\pi / \log 3 \).

Therefore, the \( \zeta \)-function of \( L^{-d} \) extends meromorphically to the whole complex plane with first order poles located at \( 1 + i2\pi n / \log 2, \ n \in \mathbb{Z} \). Therefore, the operator \( L^{-d} \) satisfies assumptions of Theorem 4.13. Hence, by Theorem 4.13 the operator \( L^{-d} \) universally measurable and the common value that all normalised traces on \( L_{1,\infty} \) take on \( L^{-d} \) equals to the residue of its \( \zeta \)-function at \( s = 1 \).

We finish this section with discussion of interrelations between Minkowski measurability of the boundary of a fractal string \( \mathcal{L} \) (that is, a fractal) and measurability of the Laplacian \( L \) on \( \mathcal{L} \). For the detailed discussion of Minkowski measurability, Minkowski dimension and Minkowski content, we refer to [31].

We start with the following result first appeared in [30] (announced in [29]).
Theorem 5.6. Let $\mathcal{L} = \{l_j\}_{j \geq 1}$ be a fractal string and let $d \in (0, 1)$. The boundary $\partial \mathcal{L}$ of $\mathcal{L}$ has Minkowski dimension $d$ if and only if
\[ l_j = O(j^{-1/d}), \quad j \to \infty, \]
that is, the operator $L^{-d} \in \mathcal{L}_{1,\infty}$.

It is quite natural to compare Minkowski measurability of a fractal string with (Dixmier or universal) measurability of the operator $L^{-d}$. We provide such a comparison below. To this end, we need [30, Theorem 2.2].

Theorem 5.7. Let $\mathcal{L} = \{l_j\}_{j \geq 1}$ be a fractal string of Minkowski dimension $d \in (0, 1)$. The boundary $\partial \mathcal{L}$ of $\mathcal{L}$ is Minkowski measurable if and only if
\[ \lim_{j \to \infty} j^{1/d} l_j = M, \]
for some positive constant $M$. In addition, the Minkowski content of $\partial \mathcal{L}$ equals $\frac{2^{1-d}}{d} M^d$.

Combining Theorem 5.7 with the Dixmier measurability criterion (see e.g. Theorem 1.1) we obtain the following result relating Minkowski measurability of the boundary of an arbitrary fractal string and Dixmier measurability of the Laplacian on this fractal string.

Corollary 5.8. Let $\mathcal{L} = \{l_j\}_{j \geq 1}$ be a fractal string of Minkowski dimension $d \in (0, 1)$. If the boundary of $\mathcal{L}$ is Minkowski measurable, then the operator $L^{-d}$ is Dixmier measurable and the Minkowski content of $\partial \mathcal{L}$ equals to $\frac{2^{1-d}}{d} \text{Tr}_\omega(L^{-d})$ for every Dixmier trace $\text{Tr}_\omega$ on $\mathcal{L}_{1,\infty}$.

Remark 5.9. Note that in the latter corollary Dixmier measurability can not be replaced by universal measurability, since there are operators whose eigenvalues $\lambda_j$-s satisfies $\lim_{j \to \infty} j^{1/d} \lambda_j = M$ and which are not universally measurable. Take $\lambda_j = \frac{1}{j+1} + \frac{1}{\log(j+2)}, \quad j \geq 0$ as an example.

Also note that the converse of the latter corollary is false in general, since the asymptotic $\lim_{j \to \infty} j^{1/d} \lambda_j = M$ is stronger than that provided by Dixmier measurability, which is
\[ \sum_{j=0}^{n} \lambda_j = c \log(n+1) + o(\log n), \quad n \to \infty. \]
Take $\lambda_j = \frac{1}{j+1} + \frac{1}{(j+2)\sqrt{\log(j+2)}}, \quad j \geq 0$ as an example.

Therefore, Laplacians on fractal strings with Minkowski measurable boundary form a class of operators, which is strictly contained in the class of all Dixmier measurable operators and which is strictly wider than the class of all universally measurable operators.

We finish this section with discussing a more recent result which relates Minkowski measurability of the boundary of a fractal string with complex dimensions of this fractal string.

Conditions on a $\zeta$-function assumed below are somewhat similar to that in Theorem 6.3. Indeed, in both cases we assume the meromorphic extension beyond the leading singularity and mild growth conditions.

Let $\mathcal{L}$ be a fractal string of Minkowski dimension $d \in (0, 1)$. Assume that the geometric $\zeta$-function $\zeta_\mathcal{L}$ extends meromorphically to some neighbourhood of the region $\{\Re(z) > d\}$ and the boundary of this neighbourhood is given by $\{r(t) + it : t \in \mathbb{R}\}$ for some bounded function $r$. We also assume that there exist constants $\alpha \geq 0$ and $C > 0$ and a sequence $\{y_k\}_{k \in \mathbb{Z}}$, with
\[ \lim_{k \to \pm \infty} y_k = \pm \infty, \quad y_{-k} < 0 < y_k, k \geq 1 \quad \text{and} \quad \lim_{k \to +\infty} y_k / |y_{-k}| = 1 \]
such that
(i) For all $k \in \mathbb{Z}$ and all $\sigma \geq r(y_k)$ we have $|\zeta_\mathcal{L}(\sigma + iy_k)| \leq C |y_k|^\alpha$,
(ii) For all $t \in \mathbb{R}$, $|t| \geq 1$ we have $|\zeta_\mathcal{L}(r(t) + it)| \leq C |t|^\alpha$. 

Then by [31, Theorem 8.15] the boundary of $\mathcal{L}$ is Minkowski measurable if and only if the only complex dimension with real part $d$ is $d$ itself and $d$ is simple.

6. Appendix

In this section we provide another approach to the proof of Theorem 1.4. It is based on a Theorem 6.1 below which relates asymptotics of a $\zeta$-function and that of a trace of the heat semi-group. Then we use a generalisation of Theorem 1.3 to deduce universal measurability from the asymptotics of the heat trace.

The following theorem is proved in [17, Theorem 3] (see also [16, Proposition 4.3] for a similar result).

**Theorem 6.1.** Let $0 \leq A \in \mathcal{L}_{1,\infty}$ be such that:

(i) $\zeta_A$ has a meromorphic continuation to the half-plane $\Re(s) > M$ for some $M < 1$;

(ii) The function

$$s \mapsto \Gamma(s)\zeta_A(s)$$

is regular and Lebesgue integrable on the vertical lines $\Re(s) = l$ and $\Re(s) = m$ for some $M < l < 1 < m$;

(iii) There exists a monotone sequence $\{y_k\}_{k \in \mathbb{Z}}$, with $y_0 = 0$ and $\lim_{k \to \pm\infty} y_k = \pm\infty$ such that

$$\sup_{x \in [l,m]} |\Gamma(x + iy_k)\zeta_A(x + iy_k)| < \infty, \forall k \in \mathbb{Z}$$

and

$$\sup_{x \in [l,m]} |\Gamma(x + iy_k)\zeta_A(x + iy_k)| \to 0, k \to \pm \infty.$$

Let $D_k$ denote a rectangle $\{x + iy : l \leq x \leq m, y - k \leq y \leq y_k\} \subset \mathbb{C}$ for $k \geq 0$, with $D_0 = \emptyset$ and let $S_k$ be the set of poles of the function $s \mapsto \Gamma(s)\zeta_A(s)$ contained in $D_k \setminus D_{k-1}$.

Then, for $t > 0$, we have

$$\text{Tr} \left( e^{tA^{-1}} \right) = \sum_{k=1}^{\infty} \sum_{s \in S_k} \text{Res}_{s=s'} \Gamma(s')\zeta_A(s')(s')^{-\gamma} + o(t^{-1}), \quad t \to 0.$$

The following generalisation of Theorem 1.3 was proved in [53, Lemma 3.2].

**Lemma 6.2.** Let $0 \leq A \in \mathcal{L}_{1,\infty}$ and let $V \in B(\mathcal{H})$. Let $\Phi : [0,\infty) \to (0,1]$ be a convex, decreasing function such that $\Phi(0) = 1$ and

$$\int_1^{\infty} \Phi(t) \frac{dt}{t} < \infty, \quad \int_0^1 \frac{1}{(1 - t)(1 - \Phi(t))} dt < \infty.$$

The following conditions are equivalent

(i) $\tau(VA) = c$ for every normalised trace $\tau$ on $\mathcal{L}_{1,\infty}$;

(ii) the following asymptotics hold:

$$\text{Tr}(VA\Phi((tA)^{-1})) = c\log(t) + O(1), \quad t \to \infty.$$

We state and prove the following theorem for positive operators only. The general case can be obtained in the way similar to the proof of Theorem 4.13.

**Theorem 6.3.** Let $0 \leq A \in \mathcal{L}_{1,\infty}$ be such that all conditions of Theorem 6.1 are fulfilled and, additionally, for some $\varepsilon > 0$ the $\zeta$-function of $A$ is meromorphic for $\Re(s) > 1 - \varepsilon$ with $s = 1$ be the only pole. Let $c \geq 0$ be the residue of the $\zeta$-function at $s = 1$. If

$$\text{Tr} \left( A^{s} \right) = \frac{c}{s - 1} + O(1), \quad s \to 1 +.$$

then

$$\sum_{k=0}^{n} \lambda(k, A) = c\log(n + 1) + O(1), \quad n \to \infty.$$
Proof. By Theorem 6.1 for \( t > 0 \) we have
\[
\text{Tr} \left( e^{-tA^{-1}} \right) = \text{Res}_{s=1} \Gamma(s) \zeta_A(s) t^{-1} + o(t^{-1+\varepsilon}) = \frac{c}{t} + o(t^{-1+\varepsilon}), \quad t \to 0.
\]
Integrating both sides yields
\[
\text{Tr} \left( -Ae^{-tA^{-1}} \right) = c \log t + o(t), \quad t \to 0.
\]
Substituting \( t = 1/z \) we obtain
\[
\text{Tr} \left( A e^{-(zA)^{-1}} \right) = c \log z + O(1), \quad z \to \infty.
\]
Note that the function \( \Phi(t) = e^{-t} \) satisfies all conditions of Lemma 6.2. Therefore, by Lemma 6.2 the operator \( A \) is universally measurable.

\[\square\]

Remark 6.4. Although, the conditions of Theorem 6.3 are stronger than those of Theorem 4.13, all of them are satisfied by large classes of geometrically significant operators as shown in the previous section.

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References

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