A SURGERY FORMULA FOR THE ASYMPTOTICS OF THE HIGHER DIMENSIONAL REIDEMEISTER TORSION AND SEIFERT FIBERED SPACES

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Abstract. We give a surgery formula for the asymptotic behavior of the sequence given by the logarithm of the higher dimensional Reidemeister torsion. Applying the resulting formula to Seifert fibered spaces, we show that the growth of the sequences has the same order as the indices and we give the explicit values for the limits of the leading coefficients. There are finitely many possibilities as the limits of the leading coefficients for a Seifert fibered space. We also show that the maximum is given by $-\chi \log 2$ where $\chi$ is the Euler characteristic of the base orbifold for a Seifert fibered space. These limits of the leading coefficients give a locally constant function on a character variety. This function takes the maximum $-\chi \log 2$ only on the top-dimensional components of the SU(2)-character variety for a Seifert fibered homology sphere.

1. Introduction

This paper is devoted to the study of the asymptotics of the higher dimensional Reidemeister torsion for Seifert fibered spaces. The higher dimensional Reidemeister torsion is defined for a 3-manifold and a sequence of homomorphisms from the fundamental group into special linear groups. This sequence is given by the composition of an $\text{SL}_2(\mathbb{C})$-representation $\rho$ of a fundamental group with $n$-dimensional irreducible representations of $\text{SL}_2(\mathbb{C})$. It is of interest to observe the asymptotic behavior of the higher dimensional Reidemeister torsion on the index $n$. We will show the following theorem.

Asymptotic behaviors for Seifert fibered spaces (Theorem 4.5 and Corollary 4.7). Let $M(\frac{1}{b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m})$ denote a closed Seifert fibered space with the Seifert index:

$$\{b, (\alpha, g); (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}.$$ Then we can express the asymptotic behavior of the sequence given by the Reidemeister torsion $|\text{Tor}(M(\frac{1}{b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|$ as follows:

(i) $\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{1}{b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{(2N)^2} = 0,$

(ii) $\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{1}{b}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}); \rho_{2N})|}{2N} = -(2 - 2g - \sum_{j=1}^m \frac{A_j - 1}{A_j}) \log 2$

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where $2\lambda_j$ is the order of $\rho(\ell_j)$ for the exceptional fiber $\ell_j$. In particular, the second equality can be written as

$$
\lim_{N \to \infty} \log |\text{Tor}(M(1-\delta, \alpha_1^1, \beta_1^1, \ldots, \alpha_m^1, \beta_m^1); \rho_{2N})| = \frac{2}{2N} = -\left(2 - 2\gamma - \sum_{j=1}^{m} \frac{\alpha_j - 1}{\alpha_j}\right) \log 2 - \left(\sum_{j=1}^{m} \left(\frac{1}{\lambda_j} - \frac{1}{\alpha_j}\right)\right) \log 2.
$$

The first term in the right hand side is equal to $-\chi \log 2$ where $\chi$ is the Euler characteristic of the base orbifold. Each $\lambda_j$ turns out to be a divisor of $\alpha_j$. Hence we can see that the maximum of the above limit is given by $-\chi \log 2$. Note that we only require that the original $\text{SL}_2(\mathbb{C})$-representation sends a regular fiber in a Seifert fibered space to $-I$.

This study is motivated by the works of W. Müller [Mü12] and P. Menal-Ferrer and J. Porti [MFP14], which revealed the relationship between the asymptotic behaviors of the Ray–Singer and the Reidemeister torsions for a hyperbolic 3-manifold and its hyperbolic volume. Their invariants are defined by a sequence of $\text{SL}_n(\mathbb{C})$-representations induced from the holonomy representation corresponding to the complete hyperbolic structure. The sequences of the Ray–Singer and the Reidemeister torsions have exponential growth and the logarithms have the order of $n^2$. They showed that the leading coefficient of the logarithm converges to the product of the hyperbolic volume and $-1/(4\pi)$.

We can also consider the similar sequences given by the Reidemeister torsions for non–hyperbolic 3-manifolds, especially for Seifert fibered spaces. We can use the simplicial volume instead of the hyperbolic volume in the studies of Müller and Menal-Ferrer & Porti. This is due to the fact that the simplicial volume of a hyperbolic manifold coincides with the product of the hyperbolic volume with the volume of an ideal tetrahedron. The simplicial volume of a Seifert fibered space is equal to zero. It is expected that the growth of the logarithm of the higher dimensional Reidemeister torsion for a Seifert fibered space has the growth with order less than $n^2$ and the leading coefficient converges to some geometric quantity of the Seifert fibered space. Actually we will see that the order of growth is the same as $n$ and the limits of the leading coefficients form a finite set, in which the maximum is given by $-\chi \log 2$ where $\chi$ is the Euler characteristic of the base orbifold of a Seifert fibered space.

To observe the asymptotic behaviors for Seifert fibered spaces, we focus on the decomposition of a Seifert fibered space. We prepare a surgery formula of asymptotic behaviors for gluing solid tori to a compact orientable 3-manifold with torus boundary. This is due to the fact that every Seifert fibered space admits the canonical decomposition into the trivial $S^1$-bundle over a compact surface and several solid tori (for details, see Subsection 4.1).

We must also draw attention to the choice of homomorphisms from fundamental groups into $\text{SL}_2(\mathbb{C})$ since Seifert fibered spaces do not admit complete hyperbolic structures. Concerning this problem, we focus on the acyclicity properties for the induced twisted chain complexes defined by $\text{SL}_n(\mathbb{C})$-representations for hyperbolic 3-manifolds. It was shown in [Rag65], [MFP12] that for a hyperbolic 3-manifold, the $\text{SL}_2n(\mathbb{C})$-representations induced from the holonomy representation define acyclic chain complexes, i.e., all of those homology groups vanish. We will restrict our attention to $\text{SL}_2(\mathbb{C})$-representations such that all the induced $\text{SL}_2n(\mathbb{C})$-representations define the acyclic twisted chain complexes. The conditions which our $\text{SL}_2(\mathbb{C})$-representations should satisfy will be referred to as acyclicity conditions (see Definition 3.5 for the acyclicity conditions).

The following is a surgery formula (Theorem 3.7) for the limits of sequences given by the logarithms of the higher dimensional Reidemeister torsions.
Surgery formula for the asymptotics (Theorem 3.7). Let $M(\frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n})$ be a compact orientable 3-manifold (possibly with boundary) obtained by gluing solid tori $S_1, \ldots, S_m$ with the slopes $\frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}$ to a compact orientable manifold $M$ with torus boundary.

We denote by $\rho$ an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$ which can be extended to a homomorphism of $\pi_1(M(\frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}))$ and by $\rho_{2N}$ the $2N$-dimensional representation induced from $\rho$. Then the asymptotics of the Reidemeister torsion $\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}); \rho_{2N})$ is expressed as follows:

\begin{align*}
(i) \lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}); \rho)|}{(2N)^2} = \lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2}, \\
(ii) \lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_n}{p_n}); \rho)|}{2N} = \lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N} - \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right) \log 2
\end{align*}

where $2\lambda_j$ is the order of $\rho(\ell_j)$ and $\ell_j$ is the homotopy class of the core of $S_j$.

Applying the above surgery formula to the decomposition of Seifert fibered spaces, we can show the asymptotic behaviors for Seifert fibered spaces. Moreover the limit of the leading coefficient is determined by each component in the representation spaces. By the invariance of the Reidemeister torsion under the conjugation of representations, we can assign each component of the character variety to the limit of the leading coefficient. Namely, we can define a locally constant function on the character variety for a Seifert fibered space. We will discuss on which components our locally constant function takes the maximum and minimum in the $\text{SU}(2)$-character varieties of Seifert fibered homology spheres. We can find the top dimensional components which give the maximum in the $\text{SU}(2)$-character variety. In particular, for Seifert fibered homology spheres given by sequences of prime integers, Theorem 4.14 shows that all top-dimensional components give the maximum and some 0-dimensional components give the minimum.

Organization. We review the definition of the Reidemeister torsion and the construction of the higher dimensional ones in Section 2. Section 3 is devoted to prepare our surgery formula under the acyclicity conditions which are deduced from the observation in Subsection 3.1. The examples of the surgery formula for integral surgeries along torus knots are exhibited in Subsection 3.3. We discuss the asymptotic behaviors of the sequences given by the logarithm of the higher dimensional Reidemeister torsion for Seifert fibered spaces in Section 4. We review on Seifert fibered spaces and prepare notations in Subsection 4.1. Subsection 4.2 gives a general formula of the asymptotic behavior for a Seifert fibered space. Furthermore we observe the relation between the limits of the leading coefficients and the components of the $\text{SU}(2)$-character varieties for Seifert fibered homology spheres in Subsection 4.3. The last Subsection 4.4 gives explicit examples for the limits of the leading coefficients and the $\text{SU}(2)$-character varieties.

2. Preliminaries

Although one can find the similar preliminaries in [Yam13a], we give a review on the Reidemeister torsion, needed in our observation, to make this article self-contained.

2.1. Reidemeister torsion.

Torsion for acyclic chain complexes. Torsion is an invariant defined for based chain complexes. We denote by $(C_*, \mathcal{C}) = (\cup_j \mathcal{C}_j)$ a based chain complex:

$$C_* : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$
where each chain module $C_i$ is a vector space over a field $F$ and equipped with a basis $\mathbf{e}^i$. We are mainly interested in an acyclic chain complex $C_\ast$, which means that the homology of the complex is trivial, i.e., $H_i(C_\ast) = 0$. The chain complex $C_\ast$ also has a basis determined by the boundary operators $\partial_i$, which arises from the following decomposition of chain modules.

We suppose that a based chain complex $(C_\ast, \mathbf{e}^\ast)$ is acyclic. For each boundary operator $\partial_i$, we denote by $\ker \partial_i \subset C_i$ by $Z_i$ and the image of $\partial_i$ by $B_i \subset C_{i-1}$. The chain module $C_i$ is expressed as the direct sum of $Z_i$ and a lift of $B_i$, denoted by $\tilde{B}_i$. Moreover we can rewrite the kernel $Z_i$ as the image of boundary operator $\partial_{i+1}$:

$$C_i = Z_i \oplus \tilde{B}_i = \partial_{i+1}\tilde{B}_{i+1} \oplus \tilde{B}_i$$

where $Z_i = B_{i+1}$ is written as $\partial_{i+1}\tilde{B}_{i+1}$.

We denote by $\tilde{\mathbf{b}}^i$ a basis of $\tilde{B}_i$. Then the set $\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i$ forms a new basis of the vector space $C_i$. We define the torsion of $(C_\ast, \mathbf{e}^\ast)$ as the following alternating product of determinants of base change matrices:

$$\text{Tor}(C_\ast, \mathbf{e}^\ast) = \prod_{i \geq 0} \left[ \partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i \right]^{[-1]} \in F^* = F \setminus \{0\}$$

where $[\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i]$ denotes the determinant of the base change matrix from the given basis $\mathbf{e}^i$ to the new one $\partial_{i+1}(\tilde{\mathbf{b}}^{i+1}) \cup \tilde{\mathbf{b}}^i$.

Note that the right hand side is independent of the choice of bases $\tilde{\mathbf{b}}^i$. The alternating product in (1) is determined by the based chain complex $(C_\ast, \mathbf{e}^\ast)$.

**Reidemeister torsion for CW–complexes.** We apply the torsion (1) of a based chain complex to the twisted chain complex given by a CW–complex and a homomorphism from its fundamental group to some linear group. Let $W$ denote a finite CW–complex and $(V, \rho)$ a representation of $\pi_1(W)$, which means $V$ is a vector space over $F$ and $\rho$ is a homomorphism from $\pi_1(W)$ into $\text{GL}(V)$. We will call $\rho$ a $\text{GL}(V)$-representation of $\pi_1(W)$ simply.

**Definition 2.1.** We define the twisted chain complex $C_\ast(W; V, \rho)$ which consists of the twisted chain module as:

$$C_i(W; V, \rho) := V \otimes_{\mathbb{Z}[\pi_1(W)]} C_i(\tilde{W}; \mathbb{Z})$$

where $\tilde{W}$ is the universal cover of $W$ and $C_i(\tilde{W}; \mathbb{Z})$ is the left $\mathbb{Z}[\pi_1(W)]$-module in which the action of $\pi_1(W)$ is given by the covering transformation. In taking the tensor product, we regard $V$ as a right $\mathbb{Z}[\pi_1(W)]$-module under the homomorphism $\rho^{-1}$. We identify a chain $v \otimes c$ with $\rho(y)^{-1}(v) \otimes c$ in $C_i(W; V, \rho)$.

We call $C_\ast(W; V, \rho)$ the twisted chain complex with the coefficient $V$ under $\rho$. Choosing a basis of the vector space $V$, we give a basis of the twisted chain complex $C_\ast(W; V, \rho)$. To be more precise, let $\{e^i_1, \ldots, e^i_{d_i}\}$ be the set of $i$-dimensional cells of $W$ and $\{v_1, \ldots, v_d\}$ a basis of $V$ where $d = \dim_F V$. Choosing a lift $\tilde{e}^i_j$ of each cell and taking tensor product with the basis of $V$, we have the following basis of $\tilde{C}_\ast(W; V, \rho)$:

$$\mathbf{e}^\ast(W; V) = \{v_1 \otimes \tilde{e}^1_1, \ldots, v_d \otimes \tilde{e}^1_1, \ldots, v_1 \otimes e^d_{m_1}, \ldots, v_d \otimes e^d_{m_d}\}.$$
based chain complex, we define the Reidemeister torsion for $W$ and an acyclic representation $(V, \rho)$ as the torsion of $C_* (W; V_\rho)$, i.e.,

\[
\text{Tor}(W; V_\rho) = \text{Tor}(C_* (W; V_\rho), \bar{e}' (W; V)) \in \mathbb{F}^*
\]

up to a factor in $\{ \pm \det(\rho(\gamma)) | \gamma \in \pi_1(W) \}$ since we have many choices of lifts $\bar{e}'_j$ and orders and orientations of cells $e'_j$. We call Tor$(W; V_\rho)$ the Reidemeister torsion of $W$ and a $\text{GL}(V)$-representation $\rho$.

**Remark 2.2.** We mention some well-definedness of the torsion (2):

- The acyclicity of $C_* (W; V_\rho)$ implies that the Euler characteristic of $W$ is zero.
- Then the torsion (2) of $C_* (W; V_\rho)$ is independent of the choice of a basis in $V$.
- If we choose an $\text{SL}(V)$-representation $\rho$ with an even dimensional $V$, then the Reidemeister torsion Tor$(W; V_\rho)$ has no indeterminacy.
- The Reidemeister torsion has an invariance under the conjugation of representations.

We refer to Milnor’s survey [Mil66] and Turaev’s book [Tur01] for more details on the Reidemeister torsion. The following lemma for torsion (1) will be needed to derive our surgery formula:

**Lemma 2.3** (Lemma 1.3 in [Kit94]). *Let a short exact sequence of based chain complexes:*

\[
0 \rightarrow (C'_*, \bar{e}') \rightarrow (C_*, \bar{e}') \rightarrow (C''_*, \bar{e}''_*) \rightarrow 0
\]

*satisfy that $[\bar{e}' \cup \bar{e}''_i / \bar{e}'_i] = 1$ for all $i$. Suppose that any two of the complexes are acyclic. Then the third one is also acyclic and the torsion of the three complexes are well-defined. Furthermore we have the next equality:*

\[
\text{Tor}(C_*, \bar{e}') = (-1)^{\sum \beta_i \beta_i''} \text{Tor}(C'_*, \bar{e}') \text{Tor}(C''_*, \bar{e}'')
\]

*where $\beta_i = \dim \partial C_{i+1}$ and $\beta_i'' = \dim \partial C_i''$. *

This lemma is called Multiplicativity Lemma.

### 2.2. Higher dimensional Reidemeister torsion for $\text{SL}_2(\mathbb{C})$-representations.

We will consider a sequence of the Reidemeister torsions for a finite CW-complex $W$. This sequence of invariants corresponds to the sequence of the $\text{SL}_n(\mathbb{C})$-representations induced by an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$. Let $\rho$ be a homomorphism from $\pi_1(W)$ to $\text{SL}_2(\mathbb{C})$. Then the pair $(C_2^*, \rho)$ is an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(W)$ by the standard action of $\text{SL}_2(\mathbb{C})$ to $C_2^*$. It is known that the pair of the symmetric product $\text{Sym}^{m-1}(C_2^*)$ and the induced action by $\text{SL}_2(\mathbb{C})$ gives an $n$-dimensional irreducible representation of $\text{SL}_2(\mathbb{C})$. The symmetric product $\text{Sym}^{m-1}(C_2^*)$ can be identified with the vector space $V_n$ of homogeneous polynomials on $C_2^*$ with degree $n - 1$, i.e.,

\[
V_n = \text{span}_C(\zeta_1^{n-1}, \zeta_1^{n-2}, \zeta_2^{n-1}, \ldots, \zeta_1 \zeta_2^{n-2}, \zeta_2^{n-1})
\]

and the action of $A \in \text{SL}_2(\mathbb{C})$ is expressed as

\[
A \cdot p(z_1, z_2) = p(A^{-1}(z_1, z_2)) \quad \text{for} \quad p(z_1, z_2) \in V_n.
\]

We write $(V_n, \sigma_n)$ for the representation given by the action (3) of $\text{SL}_2(\mathbb{C})$ where $\sigma_n$ denotes the homomorphism from $\text{SL}_2(\mathbb{C})$ into $\text{GL}(V_n)$. In fact the image of $\sigma_n$ is contained in $\text{GL}_n(\mathbb{C})$. We will see this after the definition of the higher dimensional Reidemeister torsion.

We write $\rho_\sigma$ for the composition $\sigma \circ \rho$. Then we can take $V_n$ as a coefficient of a twisted chain complex for $W$ since the vector space $V_n$ is a right $\mathbb{Z}[\pi_1(W)]$-module of $\pi_1(W)$. We denote by $C_* (W; V_n)$ this twisted chain complex of $W$ defined by $(V_n, \rho_\sigma)$. We will drop the subscript $\rho_\sigma$ in the coefficient for simplicity when no confusion can arise.
Definition 2.4. When the twisted chain complex $C_*(W; V_n)$ is acyclic, we define the higher dimensional Reidemeister torsion for $W$ and $\rho_n$ as $\text{Tor}(W; V_n)$. We will use $\text{Tor}(W; \rho_n)$ for the higher dimensional Reidemeister torsion since the coefficient $V_n$ of $C_*(W; V_n)$ is determined by $\rho$ and $n$.

We obtain the sequence of the Reidemeister torsions $\text{Tor}(W; \rho_n)$ when every $C_*(W; V_n)$ is acyclic. We will observe the asymptotic behaviors of these sequences for Seifert fibered spaces in the subsequent sections.

We also review the eigenvalues of the image $\sigma_\rho(A)$ for $A \in \text{SL}_2(\mathbb{C})$. Let $a^{\pm 1}$ be the eigenvalues of $A$. By direct calculation, we can see that the eigenvalues of $\sigma_\rho(A) \in \text{SL}_n(\mathbb{C})$ are given by $a^{-n+1}, a^{-n+3}, \ldots, a^{n-1}$, i.e., the weight space of $\sigma_n$ is $\{-n+1, -n+3, \ldots, n-1\}$ and the multiplicity of each weight is 1.

Remark 2.5. If $n > 1$ is even (resp. odd), the eigenvalues of $\sigma_n(A)$ are the odd (resp. even) powers of the eigenvalues of $A$ by odd (resp. even) integers from 1 (resp. 0) to $n-1$. This implies that $\det \sigma_n(A) = 1$ for any $A \in \text{SL}_2(\mathbb{C})$ and $n \geq 1$.

2.3. Review of a standard surgery formula. Our surgery formula is based on the standard application of Lemma 2.3 (Multiplicativity Lemma) in Section 2.1. We review this standard argument in [MFP14, Lemma 5.7] and [Kit96, Proposition 2.1] to adapt the notation to our situation.

We will consider a connected compact orientable 3-manifold $M$ with torus boundary. For the $j$-th boundary component $T_j$, we denote a pair of a meridian and longitude by $(q_j, h_j)$, i.e., $\pi_1(T_j) = \langle q_j, h_j | [q_j, h_j] = 1 \rangle$. By Dehn filling with slopes $\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m$, we obtain a compact 3-manifold $M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m)$ (possibly with boundary). Here a slope $\alpha_j/\beta_j$ is the unoriented isotopy class of the essential simple loop $\alpha_j q_j + \beta_j h_j$ on the $j$-th boundary component $T_j$, i.e.,

$$M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m) = M \cup (\bigcup_{j=1}^m D_j^2 \times S^1_j) \quad \text{where} \quad \partial D_j^2 \times \{\ast\} \sim \alpha_j q_j + \beta_j h_j \quad (\forall j).$$

Our purpose is to express the higher dimensional Reidemeister torsion of resulting manifolds $M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m)$ by those of $M$ and solid tori $S_j = D_j^2 \times S^1_j$ ($j = 1, \ldots, m$). We start with a homomorphism $\rho$ from $\pi_1(M)$ to $\text{SL}_2(\mathbb{C})$. When $\rho$ satisfies that the equations $\rho(q_j)^n \rho(h_j)^{\beta_j} = I$ extends to a homomorphism of the fundamental group of the resulting manifold and also defines the higher dimensional representations $\rho_n$ of $\pi_1(M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m))$.

Then we can consider the short exact sequence with the coefficient $V_n$:

$$0 \to \oplus_{j=1}^m C_*(T_j; V_n) \to C_*(M; V_n) \oplus \oplus_{j=1}^m C_*(S_j; V_n) \to C_*(M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m); V_n) \to 0.$$

If the left and middle parts in the short exact sequence (4) are acyclic, then the higher dimensional Reidemeister torsion of $M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m)$ is expressed as

$$\text{Tor}(M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m); \rho_n) = \pm \text{Tor}(M; \rho_n) \cdot \prod_{j=1}^m \text{Tor}(S_j; \rho_n) \cdot \prod_{j=1}^m \text{Tor}(T_j; \rho_n)^{-1}$$

from Lemma 2.3 (Multiplicativity Lemma). Here we adopt the CW-structure of the resulting manifold $M(\alpha_1/\beta_1, \ldots, \alpha_m/\beta_m)$ given by those of $M$, $S_j$ and $T_j$.

We have seen that if $n$ is odd, then the image $\sigma_\rho(A)$ always has the eigenvalue 1 for any $A \in \text{SL}_2(\mathbb{C})$. This implies that the twisted chain complex $C_*(S_j; V_n)$ is never acyclic when $n = 2N - 1$ (this will be seen in the following Subsection 3.1). Hence we will focus on even dimensional representations $\rho_{2\mathbb{N}}$ to apply Multiplicativity Lemma for acyclic Reidemeister torsions.
3. Surgery formula for the asymptotics of higher dimensional Reidemeister torsion

We first observe a situation that a sequence of surgery formulas works well. In Subsection 3.1, we will give a sufficient condition for the twisted chain complexes for $T^2$ and $S_j$ to be acyclic for all $2N$. We have to deal with our surgery formula under the resulting conditions in Lemmas 3.2 and 3.3. From Eq. (5) we will derive a surgery formula for the asymptotic behavior of the sequence obtained by the higher dimensional Reidemeister torsions of $M(\frac{m}{n}, \ldots, \frac{m}{n})$ in Subsection 3.2. The last Subsection 3.3 gives the example of our surgery formula in the case that $M$ is a torus knot exterior.

3.1. Acyclicity conditions for the boundary and solid tori. The purpose of this subsection is to give the following sufficient condition for acyclicity of the twisted chain complexes $C_r(T^2; V_{2N})$ and $C_r(S_j; V_{2N})$.

**Proposition 3.1.** If $\rho$ is an $SL_2(\mathbb{C})$-representation of $\pi_1(M(\frac{m}{n}, \ldots, \frac{m}{n}))$ such that

- either $\rho(q_j)$ or $\rho(h_j)$ has an even order and;
- the order of $\rho(\ell_j)$ for the generator $\ell_j$ of $\pi_1(S_j)$ is also even,

then $C_r(T^2; V_{2N})$ and $C_r(S_j; V_{2N})$ are acyclic for all $2N$.

This follows from Lemmas 3.2 and 3.3. First, we recall the twisted chain complexes of $T^2$ and $D^2 \times S^1$. The torsion of $D^2 \times S^1$ coincides with that of core $[0] \times S^1$ since they are simple homotopy equivalent. We consider the twisted chain complexes of $S^1$ instead of $D^2 \times S^1$. Under the cell decomposition

$$S^1 = e^0 \cup e^1 \quad \text{and} \quad T^2 = e^0 \cup e^1_1 \cup e^1_2 \cup e^2,$$

the twisted chain complexes with the coefficient $V_n$ of $S^1$ and $T^2$ are described as follows:

$$C_r(S^1; V_n): V_n \xrightarrow{L \cdot \ell} C_0(S^1; V_n) = V_n \rightarrow 0,$$

$$C_r(T^2; V_n): V_n \xrightarrow{Q \cdot h} C_0(T^2; V_n) = V_n \rightarrow 0,$$

where $L$, $Q$ and $H$ denote $SL_n(\mathbb{C})$-matrices corresponding to the simple closed loops $\ell = e^0 \cup e^1$ in $S^1$, $q = e^0 \cup e^1_1$ and $h = e^0 \cup e^1_2$ in $T^2$.

The twisted homology group $H_1(S^1; \hat{V}_n)$ is the eigenspace of $L$ for the eigenvalue 1. As mentioned in Remark 2.5, if $n$ is odd, then the $SL_n(\mathbb{C})$-matrix $L$ always has the eigenvalue 1. Hence $C_r(S^1; V_{2N-1})$ cannot be acyclic. Here and subsequently, we focus only on the twisted chain complexes given by the even dimensional vector spaces $V_{2N}$.

We describe the acyclicity conditions for these twisted chain complexes $C_r(S^1; V_{2N})$ and $C_r(T^2; V_{2N})$ by the $SL_2(\mathbb{C})$-matrices corresponding to generators of $\pi_1(S^1)$ and $\pi_1(T^2)$. Any $SL_2(\mathbb{C})$-representation sends $\pi_1(S^1)$ and $\pi_1(T^2)$ to abelian subgroups in $SL_2(\mathbb{C})$. It is known that every abelian subgroup in $SL_2(\mathbb{C})$ is moved by conjugation into either the maximal abelian subgroups $\text{Hyp}$ or $\text{Para}$:

$$\text{Hyp} := \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C} \setminus \{0\} \right\}, \quad \text{Para} := \left\{ \begin{pmatrix} \pm 1 & w \\ 0 & \pm 1 \end{pmatrix} \mid w \in \mathbb{C} \right\}.$$

Since the conjugation of representations induces an isomorphism between twisted homology groups, we can assume that $SL_2(\mathbb{C})$-representations send $\pi_1(S^1)$ and $\pi_1(T^2)$ into $\text{Hyp}$ or $\text{Para}$.
Lemma 3.2. Let $\rho$ be an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(S^1) = \langle \ell \rangle$. The composition $\rho_2N = \sigma_{2N} \circ \rho$ is acyclic for all $N \geq 1$ if and only if $\rho(\ell)$ is neither of odd order nor parabolic with trace 2.

Proof. The dimensions of $H_0(S^1; V_{2N})$ and $H_1(S^1; V_{2N})$ are the same since the Euler characteristic of $S^1$ is zero. The homology group $H_1(S^1; V_{2N})$ is the eigenspace of $L = \rho_2N(\ell)$ for the eigenvalue 1. The acyclicity of $\rho_2N$ is equivalent to that the $\text{SL}_{2N}(\mathbb{C})$-matrix $L$ does not have the eigenvalue 1.

If $\rho(\ell) \in \text{Hyp}$ is not of odd order, then the eigenvalues of $L$ forms the set $\{e^{2\pi i(2k-1)} | k = 1, \ldots, N\}$ where $e^1_{2\pi}$ are the eigenvalues of $\rho(\ell)$. Since $e^1_{2\pi}$ is not of odd order, the $\text{SL}_{2N}(\mathbb{C})$-element $L$ does not have the eigenvalue 1 for all $N$. Thus $\rho_2N$ is acyclic for all $N$. If $\rho(\ell) \in \text{Para}$ has trace $-2$, then any eigenvalue of $L$ equals $-1$ for all $N$. The $\text{SL}_{2N}(\mathbb{C})$-representation $\rho_2N$ is also acyclic for all $N$.

Conversely suppose that $\rho(\ell)$ has order $2k_\ell - 1$. Then the set of eigenvalues of $L$ contains 1 when $N \geq k_\ell$. The twisted homology group $H_1(S^1; V_{2N})$ is non–trivial for $N \geq k_\ell$. Suppose that $\rho(\ell) \in \text{Para}$ has trace 2. Then any eigenvalue of $L$ equals $1$ for all $N$. The twisted homology group $H_1(S^1; V_{2N})$ is non–trivial for all $N$. □

Lemma 3.3. Let $\rho$ be an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(T^2) = \langle q, h \mid [q, h] = 1 \rangle$. The composition $\rho_2N = \sigma_{2N} \circ \rho$ is acyclic for all $N \geq 1$ if and only if either $\rho(q)$ or $\rho(h)$ is neither of odd order nor parabolic with trace 2.

Proof. The homology group $H_2(T^2; V_{2N})$ is generated by the common eigenvectors of $Q = \rho_2N(q)$ and $H = \rho_2N(h)$ for the eigenvalue 1. By the Euler characteristic of $T^2$, which equals to zero, and Poincaré duality, we can see that the twisted homology group $H_2(T^2; V_{2N})$ vanishes if and only if $H_2(T^2; V_{2N}) = \{0\}$. The acyclicity of $\rho_2N$ is equivalent to that we have no common eigenvectors of $Q$ and $H$ for the eigenvalue 1.

In part, we can assume that $\rho(q)$ is neither of odd order nor parabolic with trace 2. Then $Q$ does not have the eigenvalue 1 for all $N$ as in the proof of Lemma 3.2. We have no common eigenvectors of $Q$ and $H$ for the eigenvalue 1. Hence $\rho_2N$ is acyclic for all $N$.

Conversely suppose that $\rho(q)$ and $\rho(h)$ have orders $2k_q - 1$ and $2k_h - 1$. Then we have a common eigenvector of $Q$ and $H$ for the eigenvalue 1 at the weight $(2k_q - 1)(2k_h - 1)$ when $N$ is sufficiently large. Thus $\rho_2N$ is not acyclic for sufficiently large $N$. Suppose that $\rho(q)$ and $\rho(h)$ are parabolic with trace 2. Then $Q$ and $H$ are always upper triangular matrix whose all diagonal entries are 1. We have a common eigenvector of $Q$ and $H$ for the eigenvalue 1. Hence $\rho_2N$ is not acyclic for all $N$. □

Remark 3.4. One can show that the $\text{SL}_{2N-1}(\mathbb{C})$-representation $\rho_2N-1$ of $\pi_1(T^2)$ is not acyclic for all $N \geq 1$ by the similar argument in Lemma 3.3.

3.2. Surgery formula for the asymptotic behaviors. We show a surgery formula for the asymptotic behavior of the higher dimensional Reidemeister torsions of a compact 3-manifold $M(\frac{\alpha_1}{p_1}, \ldots, \frac{\alpha_m}{p_m}) = M \cup (\cup_{j=1}^n D_j \times S^1_j)$. To apply Lemma 2.3 (Multiplicativity Lemma) for acyclic chain complexes, we assume that the following acyclicity conditions for the twisted chain complexes $\cup_{j=1}^n D_j \times S^1_j$ with the presentations of the fundamental groups:

\[ \pi_1(T^2_j) = \langle q_j, h_j \mid [q_j, h_j] = 1 \rangle, \quad \pi_1(S_j) = \langle \ell_j \rangle. \]

Definition 3.5 (Acyclicity conditions). Let $\rho$ be a representation of $\pi_1(M)$ into $\text{SL}_2(\mathbb{C})$ such that $\rho(q_j^m h_j^m) = 1$ for all $j = 1, \ldots, m$. We use the same symbol $\rho$ for the induced homomorphism of $\pi_1(M(\frac{\alpha_1}{p_1}, \ldots, \frac{\alpha_m}{p_m}))$ and assume that for all $j = 1, \ldots, m$
(i) either $\rho(q_j)$ or $\rho(h_j)$ has an even order and;
(ii) the order of $\rho(\ell_j)$ is also even.

We will call the above conditions (i) and (ii) the acyclicity conditions.

**Remark 3.6.** The acyclicity conditions guarantee that all twisted chain complexes of $T_j^2$ and $S_j$ are acyclic. Our acyclicity conditions are more restricted as compared with the conditions in Lemmas 3.3 and 3.2. However, as seen in Section 4, it is reasonable to assume our conditions for the case that the resulting manifold $M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m})$ is a Seifert fibered space.

We assume that $\rho$ satisfies the acyclicity conditions and each $\text{SL}_{2N}(\mathbb{C})$-representation $\rho_{2N}$ of $\pi_1(M)$ is acyclic. Then the surgery formula (5) turns out to be

\begin{equation}
\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N}) = \text{Tor}(M; \rho_{2N}) - \prod_{j=1}^m \text{Tor}(S_j; \rho_{2N}) - \prod_{j=1}^m \text{Tor}(T_j^2; \rho_{2N})^{-1}.
\end{equation}

Note that every integer $\beta_j$ in Lemma 2.3 is even from the acyclicity of $C_j(T_j^2; V_{2N})$.

Then the asymptotics of $\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N})|$ for $2N \to \infty$ is determined by that of $\log |\text{Tor}(M; \rho_{2N})|$ as follows:

**Theorem 3.7.** Let $\rho$ be an $\text{SL}_{2N}(\mathbb{C})$-representation of $\pi_1(M)$ satisfying the equation that $\rho(q_j^2 h_j^2) = I$ and the acyclicity conditions in Definition 3.5. Suppose that $\rho_{2N}$ of $\pi_1(M)$ is acyclic for all $N$. Then the asymptotics of the sequence $\{\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N})|\}_{N=1,\ldots}$ is expressed as follows:

\begin{align*}
(\text{i}) \quad & \lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N})|}{(2N)^2} = \lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2}, \\
(\text{ii}) \quad & \lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N})|}{2N} = \lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N} - \left(\sum_{j=1}^m \frac{1}{d_j}\right) \log 2
\end{align*}

where $2A_j$ is the order of $\rho(\ell_j)$ and $\ell_j$ is the homotopy class of $[0] \times S^1_j \subset S_j$.

**Proof.** By Eq. (7), the logarithm $\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N})|$ is expressed as

\begin{equation}
\log |\text{Tor}(M(\frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}); \rho_{2N})| = \log |\text{Tor}(M; \rho_{2N})| + \sum_{j=1}^m \log |\text{Tor}(S_j; \rho_{2N})| - \sum_{j=1}^m \log |\text{Tor}(T_j^2; \rho_{2N})|.
\end{equation}

Applying the following Propositions 3.8 and 3.9, we obtain Theorem 3.7.

**Proposition 3.8.** Let $\rho$ be an $\text{SL}_{2N}(\mathbb{C})$-representation of $\pi_1(T^2) = \langle q, h | [q, h] = 1 \rangle$ such that either $\rho(q)$ or $\rho(h)$ has an even order. Then $\text{Tor}(T^2; \rho_{2N}) = 1$ for all $N \geq 1$.

**Proof.** We can assume that $\rho(h)$ has an even order without loss of generality. The twisted chain complex $C_j(T^2; V_{2N})$ is described as the short exact sequence (6). The matrix $H - I$ in the boundary operators are non-singular for any $2N$. By definition, it follows that $\text{Tor}(T_j^2; \rho_{2N}) = 1$ (for the details, see [Miil66, KL99, Tur01]).

**Proposition 3.9.** Let $\rho$ be an $\text{SL}_{2N}(\mathbb{C})$-representation of $\pi_1(S^1) = \langle \ell \rangle$ such that $\rho(\ell)$ has an even order. Then we have the following limits of $\log |\text{Tor}(S^1; \rho_{2N})|$

\begin{align*}
(\text{i}) \quad & \lim_{N \to \infty} \frac{\log |\text{Tor}(S^1; \rho_{2N})|}{(2N)^2} = 0,
\end{align*}
the Reidemeister torsion as $\text{Tor}(S_{\ell})$.

Proof. We begin with computing the Reidemeister torsion $\text{Tor}(S^1; \rho_{2N})$. We can express the Reidemeister torsion as $\text{Tor}(S^1; \rho_{2N}) = \det(\rho_{2N}(\ell) - I)^{-1}$. The set of the eigenvalues of $\rho_{2N}(\ell)$ is obtained from the eigenvalues $e^{\pm \eta \sqrt{\ell}/\lambda}$ of $\rho(\ell)$, where $\eta$ is odd and $\eta$ and $\lambda$ are coprime. It turns into the set written as $\{e^{\pm \eta \sqrt{\ell}/\lambda} | k = 1, \ldots, N\}$. Thus the Reidemeister torsion $\text{Tor}(S^1; \rho_{2N})$ turns out

$$\text{Tor}(S^1; \rho_{2N}) = \prod_{k=1}^{N} [(e^{\eta \sqrt{\ell}/\lambda} - 1)(e^{-\eta \sqrt{\ell}/\lambda} - 1)]^{-1} = \prod_{k=1}^{N} \left(2 \sin \frac{(2k-1)\eta}{2\lambda}\right)^{-2}.$$ 

The logarithm of $|\text{Tor}(S^1; \rho_{2N})|$ is expressed as

$$\log |\text{Tor}(S^1; \rho_{2N})| = 2N \log 2 + 2\sum_{k=1}^{N} \log \left|\sin \frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1}.$$ 

We can now proceed to compute the limits in the list.

(i). From the following inequality

$$\left|\sin \frac{\pi}{2\lambda}\right| \leq \left|\sin \frac{\pi(2k-1)\eta}{2\lambda}\right| \leq 1$$

it follows that

$$N \log \left|\sin \frac{\pi}{2\lambda}\right| \geq \sum_{k=1}^{N} \log \left|\sin \frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1} \geq 0.$$ 

By the inequality (9) and explicit form (8) of $\log |\text{Tor}(S^1; \rho_{2N})|$, we can assert

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(S^1; \rho_{2N})|}{(2N)^2} = 0.$$ 

(ii). We can express the second limit as

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(S^1; \rho_{2N})|}{2N} = \log 2 + \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left|\sin \frac{\pi(2k-1)\eta}{2\lambda}\right|^{-1}.$$ 

The second term in the right hand side of (10) can be rewritten as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left|\sin \frac{\eta(2k-1)\eta}{2\lambda}\right|^{-1} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left|\sin \left(\frac{\eta k - 1}{\lambda}\right)^{-1}\right|^{-1}.$$ 

The sequence $\{\log |\sin (\eta k - 1)/\lambda)|^{-1}\}_{k=1,2,\ldots}$ has the minimum period $\lambda$ since $\eta$ and $\lambda$ are coprime. By Lemma 3.11, we can rewrite as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \log \left|\sin \left(\frac{\eta k - 1}{\lambda}\right)^{-1}\right|^{-1} = \frac{\lambda}{\lambda} \sum_{k=1}^{\lambda} \log \left|\sin \left(\frac{\eta k - 1}{\lambda}\right)^{-1}\right|^{-1}.$$
The right hand side of (10) turns into
\[
\log 2^{-1} + \frac{1}{\lambda} \sum_{k=1}^{\lceil \frac{n}{\lambda} \rceil} \log \left| \sin \left( \frac{\pi n}{2\lambda} + \frac{\pi (k-1)\eta}{\lambda} \right) \right|^{-1} = \frac{1}{\lambda} \log \left| 2 \sin \left( \frac{\pi n}{2\lambda} \right) \right|^{-1} = \frac{1}{\lambda} \log \left[ 2 \sin \left( \frac{\pi}{2} \right) \right]^{-1}
\]
by \(|2 \sin(n\theta)| = \prod_{k=1}^{\lceil n/\lambda \rceil} |2 \sin(\theta + k\pi/\lambda)|\). Therefore we obtain the limit
\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(S^1; \rho_{2N})|}{2N} = -\frac{\log 2}{\lambda}
\]
since \(\eta\) is odd. \(\square\)

Remark 3.10. We obtain \(|2 \sin(n\theta)| = \prod_{k=1}^{\lceil n/\lambda \rceil} |2 \sin(\theta + k\pi/\lambda)|\) from substituting \(z = e^{-2\pi \sqrt{-1}n/\lambda}\) to \(|z^n - 1| = \prod_{k=1}^{\lceil n/\lambda \rceil} |z - e^{2\pi \sqrt{-1}k/n}|\).

Lemma 3.11. Let \([a_k] a_k \in \mathbb{R}]_{k \geq 1}\) be a sequence such that \(a_k \geq 0\) and \(a_k + N_0 = a_k\). Then we have the following limit:
\[
\lim_{N \to \infty} \frac{a_1 + \cdots + a_N}{N} = \frac{a_1 + \cdots + a_{N_0}}{N_0}.
\]
Proof. It follows that \(\left[ \frac{N}{N_0} \right] \sum_{k=1}^{N_0} a_k \leq \sum_{k=1}^{N} a_k \leq \left( \left[ \frac{N}{N_0} \right] + 1 \right) \sum_{k=1}^{N_0} a_k\) where \([x]\) denotes the maximal integer less than or equal to \(x\). Note that \(\frac{N}{N_0} - 1 < \left[ \frac{N}{N_0} \right] \leq \frac{N}{N_0}\). \(\square\)

Remark 3.12. In Lemma 3.11, it is not required that the period \(N_0\) is minimum. However we have the same average for any period \(N_0\).

3.3. Example for Dehn fillings of torus knot exteriors. We give examples of Theorem 3.7 for integral surgeries along torus knots. Let \(M\) be the \((p, q)\)-torus knot exterior which is obtained by removing an open tubular neighbourhood of the knot from \(S^3\). After gluing a solid torus along the slope \(1/n\) \((n \in \mathbb{Z})\) on \(\partial M\), we have an integral homology sphere \(M(p/q)\). The resulting manifold \(M(p/q)\) is a Brieskorn homology sphere of the index \((p, q, pqn \pm 1)\). Here the sign in \(pqn \pm 1\) depends on the orientation of the preferred longitude on \(\partial M\).

The \((p, q)\)-torus knot group admits the following presentation:
\[
\pi_1(M) = \langle x, y | x^p = y^q \rangle.
\]
In this presentation, we can express a pair of meridian \(m\) and longitude \(\ell\) as
\[
m = x^u y^v, \quad \ell = m^p x^p
\]
where \(u\) and \(v\) are integers satisfying that \(pv - qu = 1\). Then the Brieskorn homology sphere \(M(p/q)\) has the index \((p, q, pqn + 1)\). We consider irreducible \(SL_2(\mathbb{C})\)-representations \(\rho\) of \(\pi_1(M)\) such that \(\rho(m) = I\), i.e., they extend to irreducible \(SL_2(\mathbb{C})\)-representations of \(\pi_1(M(p/q)) = \langle x, y | x^p = y^q, m^p = 1 \rangle\). Here irreducible means that there are no common non-trivial eigenvectors among all elements in \(\rho(\pi_1(M))\). Under the assumption of irreducibility for \(\rho\), the central element \(x^p(= y^q)\) must be sent to \pm I\). The requirement that \(\rho(m^p) = I\) turns into \(\rho(m)^{pq+1} = \pm I\). Hence we have the constrains on the order of \(\rho(x), \rho(y)\) and \(\rho(m)\) for every irreducible \(SL_2(\mathbb{C})\)-representation of \(\pi_1(M(p/q))\).

The conjugacy classes of irreducible \(SL_2(\mathbb{C})\)-representations of \(\pi_1(M(p/q))\) form a finite set. Each member of the finite set corresponds to a triple of integers. This was shown by D. Johnson [Joh] and he also gave the explicit form of the Reidemeister torsion for acyclic \(SL_2(\mathbb{C})\)-representations as follows.
Theorem 3.13 (Johnson [Joh]). The conjugacy classes of irreducible \( SL_2(\mathbb{C}) \)-representations \( \rho \) of \( \pi_1(M_{\frac{1}{n}}) \) are given by triples \( (a, b, c) \) such that

(i) \( 0 < a < p, 0 < b < q, a \equiv b \mod 2 \),
(ii) \( 0 < c < r = \lfloor pqn + 1 \rfloor, c \equiv na \mod 2 \),
(iii) \( tr(\rho) = 2 \cos \pi c / p \),
(iv) \( tr(\rho) = 2 \cos \pi b / q \),
(v) \( tr(\rho) = 2 \cos \pi c / r \).

The Reidemeister torsion is given by

\[
\text{Tor}(M_{\frac{1}{n}}; \rho) = \begin{cases} 
2^{-4} \sin^2 \frac{\pi a}{2p} \sin^2 \frac{\pi b}{2q} \sin^2 \frac{\pi (cq - r)}{2r} & a \equiv b \equiv 1, c \equiv n \mod 2 \\
\text{non-acyclic} & a \equiv b \equiv 0 \text{ or } c \not\equiv n \mod 2
\end{cases}
\]

for \( \rho \in (a, b, c) \).

In the remainder of this subsection, we denote by \( (a, b, c) \) the corresponding conjugacy class of irreducible \( SL_2(\mathbb{C}) \)-representations. We also refer to [Fre92] for the Reidemeister torsion of Brieskorn homology spheres.

Remark 3.14. The parameters \( a \) and \( b \) determine the image of the central element \( x^b(= y^b) \) by \(-I^b(= -I^b)\).

To apply Theorem 3.7, we need to find

- a condition on \( (a, b, c) \) for all \( \rho_{2N}|_{\pi_1(M)} \) to be acyclic and
- the orders of \( SL_2(\mathbb{C}) \)-elements in the acyclicity conditions (Definition 3.5).

The author has shown in [Yam13a, Proposition 3.1] that \( \rho_{2N}|_{\pi_1(M)} \) induces an acyclic \( SL_2(\mathbb{C}) \)-representation for all \( N \) if and only if the parameters \( a \) and \( b \) of \( \rho_{2N}|_{\pi_1(M)} \) satisfy that \( a \equiv b \equiv 1 \mod 2 \).

Since the surgery slope for \( M_{\frac{1}{n}} = M \cup D^2 \times S^1 \) is \( 1/n \), the homotopy class of the core \( [0] \times S^1 \) is given by \( \ell^\pm 1 \) in \( \pi_1(M_{\frac{1}{n}}) \). To check the acyclicity conditions, we only need to find the order of \( \rho(\ell) \). Let \( \rho \) be in the conjugacy class \((a, b, c)\) such that \( a \equiv b \equiv 1 \mod 2 \).

Then it follows from \( c \equiv na \mod 2 \) that

\[ \rho(\ell)'^{-(pqc - r)} = -I. \]

Hence the eigenvalues of \( \rho(\ell) \) are given by \( e^{\pm \eta \sqrt{-1}} \) for some odd integer \( \eta \), which shows that \( \rho(\ell) \) has an even order.

Let us apply Theorem 3.7 to the Brieskorn homology sphere \( M_{\frac{1}{n}} \). We obtain

\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M_{\frac{1}{n}}; \rho_{2N})|}{(2N)^2} = \lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{(2N)^2} - \frac{\log 2}{\ell'}
\]

where \( 2\ell' \) is the order of \( \rho(\ell) \). Note that it is seen from the g.c.d. \((p, r) = (q, r) = 1\) that \( r' = r/(c, r) \).

It has shown in [Yam13a, Theorem 4.2] that the right hand side in (11) vanishes. The higher dimensional Reidemeister torsion of the torus knot exterior \( M \) is expressed as, by [Yam13a, Proposition 4.1],

\[
\text{Tor}(M; \rho_{2N}) = \prod_{k=1}^{2N} 4^k \sin^2 \frac{\pi (2k-1)u}{2p} \sin^2 \frac{\pi (2k-1)v}{2q}
\]
By a similar argument to the proof of Proposition 3.9, we can see that the limit in the right hand of (12) turns out
\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M; \rho_{2N})|}{2N} = \left(1 - \frac{1}{p'} - \frac{1}{q'}\right) \log 2
\]
where \(p' = p/(p,a)\) and \(q' = q/(q,b)\).

**Theorem 3.15.** The growth of \(\log |\text{Tor}(M(\frac{1}{2}); \rho_{2N})|\) has the same order as \(2N\) for any acyclic irreducible representation \(\rho\) of \(\pi_1(M(\frac{1}{2}))\). Moreover if \(\rho\) is contained in the conjugacy class \((a,b,c)\), then the leading coefficient in \(2N\) converges as
\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{1}{2}); \rho_{2N})|}{2N} = \left(1 - \frac{1}{p'} - \frac{1}{q'} - \frac{1}{r'}\right) \log 2
\]
where \(p' = p/(a,p)\), \(q' = q/(b,q)\) and \(r' = r/(c,r)\).

In the case that \((a,p) = (b,q) = (c,r) = 1\), the leading coefficient converges to the maximum \((1 - 1/p - 1/q - 1/r) \log 2\).

For more details on the limits of \(\log |\text{Tor}(M; \rho_{2N})|/(2N)\), we refer to [Yam13b].

### 4. Asymptotics of the Higher Dimensional Reidemeister Torsion for Seifert fibered spaces

We will apply Theorem 3.7 to Seifert fibered spaces and study the asymptotic behaviors of their higher dimensional Reidemeister torsions. We will see the growth of the logarithm of the higher dimensional Reidemeister torsion has the same order as the dimension of representation. We also give explicit forms on the limits of the leading coefficients.

The limits of the leading coefficients are determined by each of components in the \(\text{SL}_2(\mathbb{C})\)-representation space of the fundamental group of a Seifert fibered space. Hence we obtain a locally constant function on the \(\text{SL}_2(\mathbb{C})\)-representation space. The Reidemeister torsion has the invariance under the conjugation of representations. We also obtain a locally constant function on the character variety of the fundamental group of a Seifert fibered space.

We will focus on the \(\text{SU}(2)\)-character varieties for Seifert fibered homology spheres and describe explicit values of the locally constant functions. Our calculation shows that these locally constant functions take the maximum values on the top dimensional components. The explicit maximums are given by \(-\chi \log 2\) where \(\chi\) is the Euler characteristic of the base orbifold of a Seifert fibered homology sphere.

We start with a brief review on Seifert fibered spaces in Subsection 4.1. Subsection 4.2 shows the application of Theorem 3.7 to Seifert fibered spaces. We will observe the relation between the limits of the leading coefficients and the components of the \(\text{SU}(2)\)-character variety for a Seifert fibered homology sphere in Subsection 4.3.

#### 4.1. Seifert fibered spaces.

A Seifert fibered space is referred as an \(S^1\)-fibration over a closed 2-orbifold. We consider the orientable Seifert fibered space given by the following Seifert index:

\[ \{b, (a, g); (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}. \]

where \(\alpha_j \geq 2\) \((j = 1, \ldots, m)\) and each pair of \(\alpha_j\) and \(\beta_j\) is coprime. For a Seifert index, we refer to [NJ81], [Orl72].

We can regard a Seifert fibered space as an \(S^1\)-bundle over a closed orientable surface \(\Sigma\) with \(m + 1\) exceptional fibers, where the genus of \(\Sigma\) is \(g\). From this viewpoint, we can decompose a Seifert fibered space into tubular neighbourhoods of exceptional fibers and
their complement. Set \( \Sigma = \Sigma \setminus \text{int}(D_0^2 \cup \ldots \cup D_m^2) \) where \( D_0^2, \ldots, D_m^2 \) are disjoint disks in \( \Sigma \). Let \( M \) be the trivial \( S^1 \)-bundle \( \Sigma \times S^1 \). We have the canonical decomposition of the Seifert fibered space as the following union of \( M \) and solid tori:

\[
M \cup (S_0 \cup \ldots \cup S_m) = M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right).
\]

The solid torus \( S_0 \) corresponds to the triviality obstruction \( b \) and the others \( S_j \) \((1 \leq j \leq m)\) correspond to the exceptional fibers with the index \((a_j, b_j)\). Then the fundamental group of \( M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right) \) admits the following presentation:

\[
\pi_1(M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right)) = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_m, h | [a_i, h] = [b_i, h] = [q_j, h] = 1, \quad q_j^a h^b = 1, q_1 \cdots q_m[a_1, b_1] \cdots [a_g, b_g] = h^p \rangle
\]

where \( a_i \) and \( b_j \) correspond to generators of \( \pi_1(\Sigma) \) and \( q_j \) is the corresponding to the circle \( \partial D_j^2 \subset \Sigma \) and \( h \) is the homotopy class of a regular fiber in \( M \). Note that the presentation of \( \pi_1(M) = \pi_1(\Sigma \times S^1) \) is written as

\[
\left\{ a_1, b_1, \ldots, a_g, b_g, q_0, q_1, \ldots, q_m, h \mid [a_i, h] = [b_i, h] = [q_j, h] = 1, \quad q_1 \cdots q_m[a_1, b_1] \cdots [a_g, b_g] = q_0 \right\}
\]

We also touch the first homology groups of Seifert fibered spaces since we will consider Seifert fibered homology spheres in Subsection 4.3. It is known that the first homology group of \( M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right) \) is expressed as \( H_1(M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right); \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus T \) where \( T \) is a finite abelian group with the order \( a_1 \cdots a_m | b + \sum_{j=1}^m b_j/\alpha_j | \) if \( b + \sum_{j=1}^m b_j/\alpha_j \) is not zero. In the case that \( b + \sum_{j=1}^m b_j/\alpha_j = 0 \), the homology group \( H_1(M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right); \mathbb{Z}) \) has the free rank of \( 2g + 1 \). Hence, for any Seifert fibered homology sphere \( M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right) \), the genus of the base orbifold is zero and we have the equation:

\[
\alpha_1 \cdots \alpha_m (b + \sum_{j=1}^m b_j/\alpha_j) = 1.
\]

In particular, this implies that \( \alpha_j \) are pairwise coprime.

4.2. The asymptotic behavior of the higher dimensional Reidemeister torsions for Seifert fibered spaces. We apply Theorem 3.7 to a Seifert fibered space \( M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right) \) for \( SL_2(\mathbb{C}) \)-representations \( \rho \) which satisfy \( \rho(h) = -I \). This assumption means that the central element \( h \) is sent to the non-trivial central element \(-I\) of \( SL_2(\mathbb{C}) \). Moreover every \( \rho_{2N} \) sends \( h \) to \(-I \in SL_{2N}(\mathbb{C}) \). We begin with the acyclicity for \( M, \partial M \) and the solid tori \( S_j \) and confirm that the condition that \( \rho(h) = -I \) is only required in our situation.

**Proposition 4.1.** Let \( \rho \) be an \( SL_2(\mathbb{C}) \)-representation of \( \pi_1(M\left(\frac{1}{p_1}, \frac{a_1}{p_1}, \ldots, \frac{a_m}{p_m}\right)) \). Suppose that \( \rho \) sends \( h \) to \(-I \). Then the restriction of \( \rho \) gives an acyclic twisted chain complex \( C_*(M; V_{2N}) \) for any \( N \). Moreover \( \rho \) satisfies the acyclicity conditions in Definition 3.5, i.e., \( C_*(T_j^2; V_{2N}) \) and \( C_*(S_j; V_{2N}) \) are also acyclic for any \( N \).

**Proof.** It is shown from [Kit96, the proof of Proposition 3.1] that \( C_*(M; V_{2N}) \) is acyclic if \( \rho_{2N}(h) = -I \). It holds for all \( N \) that \( \rho_{2N}(h) = -I \) under the assumption that \( \rho(h) = -I \) since every weight of \( \sigma_{2N} \) is odd. We have also shown that such an \( SL_2(\mathbb{C}) \)-representation \( \rho \) satisfies the acyclicity condition (i) in Definition 3.5 for all boundary components \( T_j^2 \).

This is due to that \( \pi_1(T_j^2) \) has the presentation \( \langle q_j, h | [q_j, h] = 1 \rangle \) and \( \rho(h) = -I \) has the order of 2.
It remains to prove that the acyclicity for the twisted chain complexes of $S_j$ (the condition (ii) in Definition 3.5). This follows from the following Lemma 4.2.

**Lemma 4.2.** Let $\rho$ be an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(M_1, a_1, \ldots, a_m)$ such that $\rho(h) = -I$ and $\ell_j$ denote the homotopy class of the core of $S_j$ for $j = 0, \ldots, m$. Then $\rho(\ell_j)$ is of even order for all $j$.

**Proof.** Set $\alpha_0 = 1$ and $\beta_0 = -\beta$, which are the corresponding slope to the solid torus $S_0$. We can express each $\ell_j$ ($j = 0, 1, \ldots, m$) as $\ell_j = q_j^{\rho(h_j)}$ where integers $\mu_j$ and $v_j$ satisfy that $v_0 + b\mu_0 = -1$ and $\alpha_j v_j - \beta \mu_j = -1$ ($0 < \mu_j < \alpha_j$) for $j \geq 1$. We will show that every $\rho(\ell_j)^{\nu_j}$ turns into $-I$. For $j = 0$, the matrix $\rho(\ell_0)^{\mu_0}(= \rho(\ell_0))$ turns out

$$\rho(\ell_0)^{\nu_0} = \rho(q_1 \cdots q_m [a_1, b_1] \cdots [a_g, b_g])^{\nu_0} \rho(h_0)^{\nu_0} = \rho(h_0)^{\nu_0 + \nu_0} = -I.$$  

The relation $q_j^{\rho(h_j)} = 1$ shows that $\rho(\ell_j)^{\nu_j}$ turns into

$$\rho(q_j)^{\nu_j} \rho(h_j)^{\nu_j} = \rho(h_j)^{\nu_j} \rho(\ell_j)^{\nu_j} = -I.$$  

Hence, for all $j$, the eigenvalues of $\rho(\ell_j)$ are given by $e^{\pm \pi \eta_j \sqrt{-1/\alpha_j}}$ where some odd integer $\eta_j$, which implies that the order of $\rho(\ell_j)$ is even. \qed

**Remark 4.3.** For every $j = 0, 1, \ldots, m$, it holds that $\rho(\ell_j)^{\nu_j} = -I$ and $\rho(\ell_j)^{2\nu_j} = I$. The order of $\rho(\ell_j)$ must be less than or equal to $2\alpha_j$. However the order of $\rho(\ell_0)$ is always $2$.

We turn to the higher dimensional Reidemeister torsion of $M = \Sigma \times S^1$ for $\rho_{2N}$. We have the following explicit values under the assumption that $\rho(h) = -I$.

**Proposition 4.4.** (Proposition 3.1 in [Kit96]). The $2N$-dimensional Reidemeister torsion $\text{Tor}(M, \rho_{2N})$ is given by $2^{-2N(1-2g-m)}$, i.e., $\log \text{Tor}(M, \rho_{2N}) = -2N(1-2g-m) \log 2$.

Now we are in position to apply our surgery formula (Theorem 3.7) to a Seifert fibered space $M = (\Sigma_1, a_1, \ldots, a_m)$. By Proposition 4.4 and Theorem 3.7, we can derive the asymptotic behavior of the higher dimensional Reidemeister torsion for $M = (\Sigma_1, \alpha_1, \ldots, \alpha_m)$.

**Theorem 4.5.** Let $\rho$ be an $\text{SL}_2(\mathbb{C})$-representation of $M = (\Sigma_1, \alpha_1, \ldots, \alpha_m)$ such that $\rho(h) = -I$. Then we can express the asymptotics of the sequence $\log \text{Tor}(M, \rho_{2N})$ as follows:

(i) $\lim_{N \to \infty} \frac{\log \text{Tor}(M(\frac{1}{2}, \frac{a_1}{2}, \ldots, \frac{a_m}{2}); \rho_{2N})}{(2N)^2} = 0$.

(ii) $\lim_{N \to \infty} \frac{\log \text{Tor}(M(\frac{1}{2}, \frac{a_1}{2}, \ldots, \frac{a_m}{2}); \rho_{2N})}{2N} = -2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\alpha_j} \log 2$.

where $2\lambda_j$ is the order of $\rho(\ell_j)$.

In particular, if $\lambda_j$ is equal to $\alpha_j$ for all $j$, then we have

$$\lim_{N \to \infty} \frac{\log \text{Tor}(M(\frac{1}{2}, \frac{a_1}{2}, \ldots, \frac{a_m}{2}); \rho_{2N})}{2N} = -2 - 2g - \sum_{j=1}^{m} \frac{\alpha_j - 1}{\alpha_j} \log 2 = -\chi \log 2$$

where $\chi$ is the Euler characteristic of the base orbifold.
Proof. Applying Theorem 3.7 and Proposition 4.4, we obtain
\[
\lim_{N \to \infty} \log |\text{Tor}(M( \frac{1}{b}, \frac{a}{b}, \ldots, \frac{a}{b}); \rho_{2N})|/(2N)^2 = \lim_{N \to \infty} \log |\text{Tor}(M; \rho_{2N})|/(2N)^2 = 0.
\]

Also it follows that
\[
\lim_{N \to \infty} \log |\text{Tor}(M( \frac{1}{b}, \frac{a}{b}, \ldots, \frac{a}{b}); \rho_{2N})|/2N = \lim_{N \to \infty} \log |\text{Tor}(M; \rho_{2N})|/2N - \left( \sum_{j=0}^{m} \frac{1}{\lambda_j} \right) \log 2
\]
\[
= -(1 - 2g - m) - \left( 1 + \sum_{j=1}^{m} \frac{1}{\lambda_j} \right) \log 2
\]
\[
= -(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}) \log 2.
\]
\[\Box\]

Remark 4.6. It follows from the proof of Lemma 4.2 that \(\rho(\ell_j)^{2N_j} = I\) for all \(j\). Each \(\lambda_j\) in Theorem 4.5 is a divisor of the corresponding \(\alpha_j\).

Corollary 4.7. The product \(\chi \log 2\) is the maximum in the limits of the leading coefficients \((\text{ii). in Theorem 4.5}) for all SL\(_2(\mathbb{C})\)-representations sending \(h\) to \(-I\).

Proof. We can rewrite the second limit in Theorem 4.5 as
\[
\lim_{N \to \infty} \log |\text{Tor}(M( \frac{1}{b}, \frac{a}{b}, \ldots, \frac{a}{b}); \rho_{2N})|/2N = -(2 - 2g - \sum_{j=1}^{m} \frac{\lambda_j - 1}{\lambda_j}) \log 2
\]
\[
= \chi \log 2 - \log 2 \sum_{j=1}^{m} \frac{1}{\lambda_j - 1} / \alpha_j.
\]
Our claim follows from that each \(\lambda_j\) is a divisor of \(\alpha_j\) for \(j = 1, \ldots, m\). \[\Box\]

We also give the explicit form of the higher dimensional Reidemeister torsion for a Seifert fibered space. The following is the direct application of Lemma 2.3 (Multiplicativity Lemma of the Reidemeister torsion).

Proposition 4.8. Let \(\rho\) be an SL\(_2(\mathbb{C})\)-representation of \(\pi_1(M( \frac{1}{b}, \frac{a}{b}, \ldots, \frac{a}{b})\) such that \(\rho(h) = -I\). Then we can express \(\text{Tor}(M( \frac{1}{b}, \frac{a}{b}, \ldots, \frac{a}{b}); \rho_{2N})\) as
\[
\text{Tor}(M( \frac{1}{b}, \frac{a}{b}, \ldots, \frac{a}{b}); \rho_{2N}) = 2^{-2N(2g-m)} \prod_{j=1}^{m} \prod_{k=1}^{N} \left( 2 \sin \frac{\pi(2k-1)\eta_j}{2\alpha_j} \right)^2
\]
where \(e^{i\pi \eta_j}, \sqrt{\alpha_j}\) are the eigenvalues of \(\rho(\ell_j)\).

For the Reidemeister torsion of a Seifert fibered space \((g > 1)\) with more general irreducible SL\(_2(\mathbb{C})\)-representations, we refer to [Kit96].

Remark 4.9. We do not require the irreducibility of \(\rho_{2N} = \sigma_{2N} \circ \rho\). Our assumption that \(\rho(h) = -I\) guarantees the acyclicity of \(\rho_{2N}\) for all \(N\).
4.3. The leading coefficients and the SU(2)-character varieties for Seifert fibered homology spheres. We have shown the explicit lists of the leading coefficients in the higher dimensional Reidemeister torsions for Seifert fibered spaces $M(\frac{1}{b}, \frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m})$. The limit of the leading coefficient depends only on the orders of $\rho(\ell_j)$ for an $SU(2)$-representation $\rho$ of $\pi_1(M(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}))$. In the fundamental group $\pi_1(M(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}))$, we have the relations that $\ell_j = q_j^{\alpha_j} h^{\nu_j}$ and $q_j^{\alpha_j} h^{\beta_j} = 1$ where $\alpha_j, \beta_j, \nu_j = -1$. Under the assumption that $\rho(h) = -I$, the order of $\rho(\ell_j)$ is determined by the order $\rho(q_j)$, i.e., the eigenvalues of $\rho(q_j)$.

Here and subsequently, following the previous studies [FS90], [KK91], [BO90], we focus on Seifert fibered homology spheres and irreducible SU(2)-representations of their fundamental groups. It has shown by [FS90], [KK91], [BO90] that the set of conjugacy classes of irreducible SU(2)-representations for a Seifert fibered homology sphere can be regarded as the set of smooth manifolds with even dimensions. We deal with Seifert fibered homology spheres $M(\frac{1}{b}, \frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m})$ with $b = 0$ and denote it briefly by $M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m})$.

The set of conjugacy classes of irreducible SU(2)-representations of $\pi_1(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}))$ is denoted by

$$\mathcal{R}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m})) = \text{Hom}^\text{irr}(\pi_1(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m})), SU(2)/\text{conjugate}).$$

This set is called the SU(2)-character variety.

Each component in $\mathcal{R}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}))$ is determined by the set of eigenvalues for the SU(2)-elements corresponding to $q_1, \ldots, q_m$. By the relation that $q_j^{\alpha_j} h^{\beta_j} = 1$, the eigenvalues of $\rho(q_j)$ for an irreducible SU(2)-representation $\rho$ are given by $e^{\pi \ell_j(0, b_j)} (0 \leq \ell_j \leq m)$. We will use the $m$-tuple $(\xi_1, \ldots, \xi_m)$ to denote the corresponding component in $\mathcal{R}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}))$ (for details, we refer to [FS90], [KK91], [BO90]).

**Proposition 4.10** ([FS90], [KK91], [BO90]). Let $\rho$ be an irreducible SU(2)-representation of $M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m})$. Suppose that the conjugacy class of $\rho$ is contained in a component $(\xi_1, \ldots, \xi_m)$. Then the dimension of $(\xi_1, \ldots, \xi_m)$ is equal to $2(n - 3)$ where $n$ is the number of $\xi_j$ such that $\xi_j \neq 0, \alpha_j$, i.e., $\rho(q_j) \neq \pm I$, in $j = 1, \ldots, m$.

We will find components of $\mathcal{R}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}))$ containing the conjugacy class of an irreducible SU(2)-representation $\rho$ which makes the leading coefficient in the logarithm of $|\text{Tor}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}); \rho_{2\pi})|$ converge to $-\chi \log 2$.

**Proposition 4.11.** Let $\rho$ be an irreducible SU(2)-representation such that $\rho(h) = -I$. The leading coefficient of $\log |\text{Tor}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}); \rho_{2\pi})|$ converges to the maximum $-\chi \log 2$ if and only if the conjugacy class $[\rho]$ is contained in a $2(m - 3)$-dimensional component $(\xi_1, \ldots, \xi_m)$ of $\mathcal{R}(M(\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}))$ such that $\alpha_j$ and $\xi_j$ are coprime for all $j$.

Before proving Proposition 4.11, we observe a relation between the order of $\rho(\ell_j)$ and the integer $\xi_j$.

**Lemma 4.12.** Suppose that an irreducible SU(2)-representation $\rho$ satisfies that $\rho(h) = -I$ and its conjugacy class is contained in a component $(\xi_1, \ldots, \xi_m)$. Then the order of $\rho(\ell_j)$ is equal to $2\alpha_j$, i.e., $\lambda_j = \alpha_j$, if and only if $\xi_j$ and $\alpha_j$ are coprime.

**Proof.** We can express $\ell_j$ as $\ell_j = q_j^{\alpha_j} h^{\nu_j}$ where $\alpha_j, \beta_j, \nu_j = -1$. Since the eigenvalues of $\rho(q_j)$ are given by $e^{\pi \ell_j(0, b_j)}$, we can diagonalize $\rho(\ell_j)$ as

$$\rho(\ell_j) \sim \begin{pmatrix} e^{\pi \eta_j \sqrt{-1} \alpha_j} & 0 \\ 0 & e^{-\pi \eta_j \sqrt{-1} \alpha_j} \end{pmatrix} \quad (\eta_j = \mu \xi_j - \alpha_j \nu_j).$$
Suppose that $\xi_j$ and $\alpha_j$ are coprime. It follows from $(\alpha_j, \mu_j) = 1$ that the g.c.d. $(\alpha_j, \eta_j)$ coincides with $(\alpha_j, \xi_j) = 1$. By $p(\xi_j)^0 = -I$, we have that $e^{\log(\alpha_j)\xi_j} = -I$. This implies that $\alpha_j$ divides $\xi_j$. Together with the fact that $\xi_j$ is a divisor of $\alpha_j$, we have that $\xi_j = \alpha_j$. Conversely, when the order of $p(\xi_j)$ is $2\alpha_j$, the g.c.d. $(\alpha_j, \xi_j)$ must be 1. 

**Remark 4.13.** Under the assumption that $\rho(h) = -I$, if $\xi_j$ is equal to 0 or $\alpha_j$, then $\rho(\ell_j) = -I$. Hence $\lambda_j = 1$.

**Proof of Proposition 4.11.** According to Eq. (14), every $\lambda_j$ coincides with $\alpha_j$ if and only if the leading coefficient of $\log |\text{Tor}(M(\pi_1, \ldots, \pi_n); \rho_{2N})|$ converges to $-\chi \log 2$. From Lemma 4.12, we can rephrase $\lambda_j = \alpha_j$ for all $j$ since all integers $\xi_j$ satisfy that $(\alpha_j, \xi_j) = 1$. In particular, it is seen from Proposition 4.10 that the dimension of $(\xi_1, \ldots, \xi_m)$ is equal to $2(m-3)$. 

In special cases that every $\alpha_j$ is prime in the Seifert index of a Seifert homology sphere $M(\pi_1, \ldots, \pi_n)$, we obtain a simple correspondence between the limits of the sequences given by $\log |\text{Tor}(M(\pi_1, \ldots, \pi_n); \rho_{2N})|$ and the components of $\mathcal{R}(M(\pi_1, \ldots, \pi_n))$.

**Theorem 4.14.** We assume that there exists an irreducible SU(2)-representation $\rho$ of $\pi_1(M(\pi_1, \ldots, \pi_n))$ such that $\rho(h) = -I$. Suppose that every $\alpha_j$ is prime and $\alpha_1 < \cdots < \alpha_m$.

If the conjugacy class $[\rho]$ is contained in a component $(\xi_1, \ldots, \xi_m)$, then the leading coefficient of $\log |\text{Tor}(M(\pi_1, \ldots, \pi_n); \rho_{2N})|$ converges to

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\pi_1, \ldots, \pi_n); \rho_{2N})|}{2N} = -(2 - \sum_{\xi_j \neq \alpha_j} \frac{\alpha_j - 1}{\alpha_j}) \log 2. \quad (15)$$

If the set $\{[\rho] \in \mathcal{R}(M(\pi_1, \ldots, \pi_n)) \mid \rho(h) = -I\}$ has components of dimension 0 and $2(m-3)$, then the limit takes the maximum $-\chi \log 2$ on only all top-dimensional components and takes the minimum on some 0-dimensional components.

**Proof.** As seen in Remark 4.6, we have $\lambda_j = 1$ if $\xi_j = 0$ or $\alpha_j$. Substituting $\lambda_j = 1$ (and $g = 0$) into the equality above Eq. (14) for the corresponding index $j$, we have the limit (15). From Proposition 4.11 and our assumption, it follows that the limit takes the maximum $-\chi \log 2$ on all top-dimensional components.

It remains to prove that the limit takes the minimum on a 0-dimensional component. Since the limit is expressed as Eq. (15), we consider the minimum of $\sum_{\xi_j \neq 0, \alpha_j} (\alpha_j - 1)/\alpha_j$. Each 0-dimensional component is given by $(\xi_1, \ldots, \xi_m)$ for all $\xi_j = 0, \alpha_j$ except three $\xi_{j_1}$, $\xi_{j_2}$ and $\xi_{j_3}$. We need to consider two cases: (i) $\alpha_1 = 2$ and (ii) $\alpha_1 \geq 3$. In the case that $\alpha_1 = 2$, the integer $\xi_1$ must be 1 for all components $(\xi_1, \ldots, \xi_m)$ since we have $\rho(\alpha_1) = -I$ from the assumption that $\rho(h) = -I$. The sum $(\alpha_{j_1} - 1)/\alpha_{j_1} + (\alpha_{j_3} - 1)/\alpha_{j_3}$ turns into

$$\frac{1}{2} + \frac{\alpha_{j_1} - 1}{\alpha_{j_1}} + \frac{\alpha_{j_3} - 1}{\alpha_{j_3}} < \frac{5}{2},$$

On the other hand, it is easily seen that for higher dimensional components,

$$\frac{1}{2} + \frac{\alpha_{j_1} - 1}{\alpha_{j_1}} + \frac{\alpha_{j_2} - 1}{\alpha_{j_2}} + \cdots + \frac{\alpha_{j_4} - 1}{\alpha_{j_4}} + \cdots \geq \frac{1}{2} + \frac{3 - 1}{3} + \frac{5 - 1}{5} + \frac{7 - 1}{7} > \frac{5}{2}.$$ 

Hence the minimum lies in a 0-dimensional component.

In the other case that $\alpha_1 \geq 3$, it is clear that

$$\frac{\alpha_{j_1} - 1}{\alpha_{j_1}} + \frac{\alpha_{j_2} - 1}{\alpha_{j_2}} + \frac{\alpha_{j_3} - 1}{\alpha_{j_3}} < 3.$$
On the other hand, we can see that
\[
\frac{\alpha_1 - 1}{\alpha_1} + \frac{\alpha_2 - 1}{\alpha_2} + \frac{\alpha_3 - 1}{\alpha_3} + \frac{\alpha_i - 1}{\alpha_i} + \cdots \geq \frac{3 - 1}{3} + \frac{5 - 1}{5} + \frac{7 - 1}{7} + \frac{11 - 1}{11} > 3.
\]
The minimum of the limits lies on 0-dimensional components.

4.4. Examples for Seifert fibered homology spheres. We will see two examples of Theorem 4.14 and one more example. The last example shows that if some \(\alpha_j\) is not prime and \((\alpha_j, \xi_j) \neq 1\), then there exists a top-dimensional component where the leading coefficient of \(\log |\text{Tor}(M(\mu_1, \ldots, \mu_3); \rho_{2N})|\) does not converge to the maximum \(-\chi \log 2\).

4.4.1. \(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7})\). We can choose \(\beta_1 = 1\) and \(\beta_2 = \beta_3 = -1\) by the requirement that \(2 \cdot 3 \cdot 7(\beta_1/2 + \beta_2/3 + \beta_3/7) = 1\). The Brieskorn homology sphere \(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7})\) also corresponds to the surgery along \((2, 3)\)-torus knot with slope 1 in Subsection 3.3. From the presentation:
\[
\pi_1(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7})) = (q_1, q_2, q_3, h | [q_j, h] = 1, q_1^{\beta_j}h^\beta_j = 1, q_1q_2q_3 = 1),
\]
every irreducible SU(2)-representation sends \(h\) to \(-I\) and the SU(2)-character variety of \(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7})\) consists of \((\xi_1, \xi_2, \xi_3) = (1, 1, 3)\) and \((1, 1, 5)\). For details about the computation of SU(2)-character varieties, we refer to [FS90], [KK91] and [Sav99, Lecture 14].

Let \(\rho\) be an irreducible SU(2)-representation of \(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7})\). By the relations that \(\ell_j = q_j^\alpha h^\beta\) and \(\alpha_j \gamma_j - \beta_j \mu_j = -1\) \((0 < \mu_j < \alpha_j)\), we obtain that
\[
\rho(\ell_1) = \rho(q_1), \rho(\ell_2) = -\rho(q_2)^2, \rho(\ell_3) = -\rho(q_3)^6.
\]
From Eqs. \(\rho(q_1)^3 = -I, \rho(q_2)^3 = -I\) and \(\rho(q_3)^7 = -I\), the orders 2\(\lambda_j\) of \(\rho(\ell_j)\) are given by
\[
2\lambda_1 = 4, 2\lambda_2 = 6, 2\lambda_3 = 14.
\]
By Theorem 4.5, in the both cases of \([\rho] \in (1, 1, 3)\) and \([\rho] \in (1, 1, 5)\), we can see that
\[
\lim_{N \to \infty} \frac{\log |\text{Tor}(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7}); \rho_{2N})|}{2N} = \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7}\right) \log 2.
\]
The limit of the leading coefficient takes the maximum \(-\chi \log 2\) on all top-dimensional components (see also Theorem 3.15).

4.4.2. \(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})\). We can choose \(\beta_1 = 1, \beta_2 = \beta_3 = -2\) and \(\beta_4 = 4\). The subvariety \([\rho] \in \mathcal{R}(M(\frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})) | \rho(h) = -I\) consists of eight points and six 2-dimensional spheres. (For details, see [Sav99, Lecture 14])

Each 0-dimensional component corresponds to the parameter \((\xi_1, \xi_2, \xi_3, \xi_4)\) in:
\[
(16) \quad \left\{ (1, 0, 2, 2), (1, 0, 2, 4), (1, 0, 2, 6), (1, 0, 4, 4), \right\}
\cdot \left\{ (1, 2, 0, 2), (1, 2, 0, 4), (1, 2, 2, 0), (1, 2, 2, 0) \right\}
\]
and each 2-dimensional components are given by \((\xi_1, \xi_2, \xi_3, \xi_4)\) in
\[
\left\{ (1, 2, 2, 2), (1, 2, 2, 4), (1, 2, 2, 6), \right\}
\cdot \left\{ (1, 2, 4, 2), (1, 2, 4, 4), (1, 2, 4, 6) \right\}
\]
We can express \(\ell_j\) \((1 \leq j \leq 4)\) as
\[
\ell_1 = q_1, \ell_2 = q_2 h^{-1}, \ell_3 = q_3^2 h^{-1}, \ell_4 = q_4^2 h.
\]
Let $\rho$ be an irreducible SU(2)-representation of $\pi_1(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}))$ such that $\rho(h) = -I$. We have the following table between the 0-dimensional components and the orders of $\rho(\ell_j)$ for $j = 1, 2, 3, 4$:

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\lambda_j$; the half of the order of $\rho(\ell_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1, \xi_2, \xi_3, \xi_4$</td>
<td>$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 7$</td>
</tr>
<tr>
<td>$\xi_1, 0, \xi_3, \xi_4$</td>
<td>$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 7$</td>
</tr>
<tr>
<td>$\xi_1, \xi_2, 0, \xi_4$</td>
<td>$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$, $\lambda_4 = 7$</td>
</tr>
<tr>
<td>$\xi_1, \xi_2, \xi_3, 0$</td>
<td>$\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 1$.</td>
</tr>
</tbody>
</table>

By Theorem 4.5, for $[\rho] \in (\xi_1, \xi_2, \xi_3, \xi_4)$ ($\xi_2 = 0$), we obtain

$$\lim_{N \to \infty} \log \left| \frac{\text{Tor}(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}); \rho_{2N})}{2N} \right| = -(2 - \sum_{j \neq 2} \frac{\lambda_j - 1}{\lambda_j}) \log 2$$

$$= -\chi \log 2 - \frac{2}{3} \log 2,$$

for $[\rho] \in (\xi_1, \xi_2, \xi_3, \xi_4)$ ($\xi_3 = 0$),

$$\lim_{N \to \infty} \log \left| \frac{\text{Tor}(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}); \rho_{2N})}{2N} \right| = -(2 - \sum_{j \neq 3} \frac{\lambda_j - 1}{\lambda_j}) \log 2$$

$$= -\chi \log 2 - \frac{4}{5} \log 2,$$

and for $[\rho] \in (\xi_1, \xi_2, \xi_3, \xi_4)$ ($\xi_4 = 0$),

$$\lim_{N \to \infty} \log \left| \frac{\text{Tor}(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}); \rho_{2N})}{2N} \right| = -(2 - \sum_{j \neq 4} \frac{\lambda_j - 1}{\lambda_j}) \log 2$$

$$= -\chi \log 2 - \frac{6}{7} \log 2.$$

When the conjugacy class $[\rho]$ is contained in the 2-dimensional components in (16), it is seen that $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$ and $\lambda_4 = 7$. Hence we obtain the maximum in the above limits:

$$\lim_{N \to \infty} \log \left| \frac{\text{Tor}(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}); \rho_{2N})}{2N} \right| = -\chi \log 2.$$

The limit of $\log |\text{Tor}(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}); \rho_{2N})|/(2N)$ takes the minimum at the 0-dimensional components $(1, 2, 2, 0)$ and $(1, 2, 4, 0)$.

4.4.3. $M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$. We can choose $\beta_1 = 3$, $\beta_2 = -1$ and $\beta_3 = -3$. The subvariety

$$\{ [\rho] \in \mathcal{R}(M(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})) | \rho(h) = -I \}$$

consists of eight points which are given by $(\xi_1, \xi_2, \xi_3)$ in the following set:

$$\left\{ (1, 1, 1), (1, 3, 3), (1, 5, 5), (3, 1, 5), (3, 3, 1), (3, 3, 3), (3, 3, 5), (3, 5, 3) \right\}.$$

We separate the above subvariety into two subsets $X_1$ and $X_2$ as follows:

$$X_1 := \{(1, 1, 1), (1, 5, 5), (3, 1, 5), (3, 3, 5)\},$$
$$X_2 := \{(1, 3, 3), (3, 3, 1), (3, 3, 3), (3, 3, 5)\},$$

where every component is 0-dimensional. Each component in $X_1$ satisfies that $(\alpha_j, \xi_j) = 1$ for all $j$. On the other hand, each component in $X_2$ satisfies that $(\alpha_1, \xi_1) = (\alpha_3, \xi_3) = 1$ and $(\alpha_2, \xi_2) = 3$. 
In the case that the conjugacy class $[\rho]$ is contained in $X_1$, we can compute $\lambda_1 = 5$, $\lambda_2 = 6$ and $\lambda_3 = 7$. By Theorem 4.5, we obtain the following limit:

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(M(5, 3, 6 - 1, 7 - 3); \rho_{2N})|}{2N} = -\chi \log 2.$$ 

In the case of the conjugacy class $[\rho]$ is contained in $X_2$, the order of $\rho(\ell_2)$ is 4 since $\ell_2 = q_2^h - 1$ and $\rho(q_2^2) = -I$. Hence $\lambda_2$ equals to 2. Similar computations yield $\lambda_1 = 5$ and $\lambda_3 = 7$. Theorem 4.5 gives the following limit:

$$\lim_{N \to \infty} \frac{\log |\text{Tor}(M(5, 3, 6 - 1, 7 - 3); \rho_{2N})|}{2N} = -(2 - \frac{3}{3}) \log 2 = -\chi \log 2 - \frac{1}{3} \log 2.$$ 

Therefore we have the top-dimensional components where the limit of the leading coefficient in $\log |\text{Tor}(M(5, 3, 6 - 1, 7 - 3); \rho_{2N})|$ does not take the maximum.

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**References**


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