INCOMPRESSIBLE NAVIER-STOKES-FOURIER LIMIT FROM THE BOLTZMANN EQUATION: CLASSICAL SOLUTIONS

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ABSTRACT. We first construct the global existence of the classical solutions near the absolute equilibrium to the Boltzmann equation with general collision kernels under the Navier-Stokes scaling in whole space. In particular, a global energy estimate uniform in the Knudsen number is derived. Then using the local conservation laws of these sequence of solutions to the Boltzmann equation, by taking the zero Knudsen number limit, we obtain the global classical solution to the incompressible Navier-Stokes-Fourier equation with small initial data.

1. Introduction

1.1. The Boltzmann equation: basic properties. In kinetic theory, one of the most fundamental equation is the so-called Boltzmann equation which models a rarefied gas by describing the evolution of the distribution of molecules in phase-space. For a complete introduction to the Boltzmann equation, see [16] and [17]. The Boltzmann equation can be written in the following form:

\[
\begin{align*}
\frac{\partial}{\partial t} f + v \cdot \nabla_x f &= Q(f, f), \\
f|_{t=0} &= f_0 \geq 0.
\end{align*}
\]

Here, the non-negative measurable function \( f \equiv f(t, x, v) \) denotes the number density of the gas molecules at position \( x \in \mathbb{R}^3 \) and time \( t \geq 0 \). Furthermore, \( Q(f, f) \) is the collision operator which describes the binary elastic collision between particles and is given in the classical \( \sigma \)-representation by

\[
Q(g, f) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v, \sigma) \{ g'_* f' - g_* f \} \, d\sigma \, dv_* ,
\]

which is well-defined for suitable non-negative functions \( f \) and \( g \) specified later. In above expression, \( f'_* = f(t, x, v'_*) \), \( f' = f(t, x, v') \), \( f_* = f(t, x, v_*) \), \( f = f(t, x, v) \), and for \( \sigma \in S^2 \),

\[
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma ,
\]

which give the relations between the post and pre collisional velocities that follow from the conservation laws of momentum and kinetic energy. More specifically, the conservations of momentum and energy for particle pairs with the same mass during the elastic collisions are expressed as

\[
v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.
\]

On the physical grounds, the non-negative (a.e. finite) weight function \( B(v - v_*, \sigma) \) appeared in (1.2), called cross-section, is assumed to depend only on \( |v - v_*| \) (modulus of the relative velocity) and the scalar product \( \frac{v - v_*}{|v - v_*|} \cdot \sigma \) (cosine of the deviation angle). For a given interaction model, the cross section can be computed in a semiexplicit way by solving scattering problem.

We assume that the cross section takes the form

\[
B(v - v_*, \cos \theta) = |v - v_*|^2 b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma , \quad 0 \leq \theta \leq \frac{\pi}{2},
\]

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where \( \gamma > -3 \) is the index of the kinetic factor. Usually for \( \gamma \geq 0 \), we call the collision kernel as hard potential, in particular, for \( \gamma = 0 \), we call it Maxwell collision kernel, and for \(-3 < \gamma < 0\), we call it soft potential. Note that generally \( 0 \leq \theta \leq \pi \), but we assume the cross section is supported in the set \( 0 \leq \theta \leq \frac{\pi}{2} \), i.e. \( (v - v_*, \sigma) \geq 0 \). If not, we can reduce to this case by replacing \( B \) by its symmetrized version

\[
\overline{B}(v - v_*, \sigma) = [B(v - v_*, \sigma) + B(v - v_*, -\sigma)]1_{(v-v_*,\sigma)>0}.
\]

For the angular factor \( b(\cos \theta) \), we consider two cases:

- The non-cutoff case, \( b(\cos \theta) \) behaves like
  \[
b(\cos \theta) \sim K\theta^{-2-2s}, \quad \text{when } \theta \to 0^+,
\]
  for some constants \( K > 0 \) and \( 0 < s < 1 \).
- The Grad angular cutoff case, \( b(\cos \theta) \) satisfies
  \[
  \int_0^{\frac{\pi}{2}} b(\cos \theta) \sin \theta \, d\theta < \infty.
  \]

Next, we list some basic properties of the Boltzmann equation. The detailed proof of these properties can be found in the classic books [16, 17] and lecture notes [24]. We denote by \( \langle \xi \rangle \) the integral over \( \mathbb{R}^3 \) with respect to the standard measure \( dv \), i.e.

\[
\langle \xi \rangle = \int_{\mathbb{R}^3} \xi(v) \, dv.
\]

Formally, if \( f \) solves the Boltzmann equation (1.1), then it satisfies local conservation laws of mass, momentum and energy:

\[
\begin{aligned}
\partial_t \langle f \rangle + \nabla_x \cdot \langle vf \rangle &= 0, \\
\partial_t \langle vf \rangle + \nabla_x \cdot \langle v \otimes vf \rangle &= 0, \\
\partial_t \langle |v|^2 f \rangle + \nabla_x \cdot \langle \frac{1}{2} |v|^2 f \rangle &= 0.
\end{aligned}
\]

Integrating these over space yields the global conservation laws of mass, momentum and energy:

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \langle (1, v, \frac{|v|^2}{2})^\top f \rangle \, dv = 0,
\]

where \( W^\top \) denotes the transpose of the vector \( W \).

It was Ludwig Boltzmann who observed the dissipation law

\[
-\langle \log f \mathcal{Q}(f, f) \rangle = \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times S^2} \log \left( \frac{f_{\nu'} f}{f_f} \right) (f_{\nu'} f - f_f) \, d\sigma dv, \quad \mathcal{Q}(f, f) \geq 0,
\]

for every \( f \) for which the integrals make sense. Boltzmann then found the celebrated \( H \)-Theorem, i.e. the following three statements are equivalent:

1. \( \mathcal{Q}(f, f) = 0 \);
2. \( \langle \log f \mathcal{Q}(f, f) \rangle = 0 \);
3. \( f = \exp (\alpha + \beta \cdot v + \gamma \frac{1}{2} |v|^2) \) for some \( (\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \).

The equilibria characterized in (3) above will have finite mass, momentum and energy density when \( \gamma < 1 \). In that case \( f \) can be written as the classical (local) Maxwellian, i.e. \( f = \mu(\rho, u, \theta), \)

\[
\mu(\rho, u, \theta) = \frac{\rho}{(2\pi\theta)^{3/2}} \exp \left( -\frac{1}{2\theta} |v - u|^2 \right),
\]

and where the density \( \rho \geq 0 \), the velocity \( u \in \mathbb{R}^3 \) and the temperature \( \theta > 0 \) are determined by the relations

\[
\rho = \langle \mu(\rho, u, \theta) \rangle, \quad \rho u = \langle v \mu(\rho, u, \theta) \rangle, \quad \frac{1}{2} \rho |u|^2 + \frac{3}{2} \rho \theta = \langle \frac{1}{2} |v|^2 \mu(\rho, u, \theta) \rangle.
\]
When \((\rho, u, \theta)\) are independent of \((x, t)\), we call the corresponding Maxwellian as global Maxwellian. In particular, we denote \(\mu = \mu(1, 0, 1)\):

\[
\mu = (2\pi)^{-\frac{3}{2}} \exp \left( -\frac{|u|^2}{2} \right).
\]

The main concern of this paper is to study the incompressible Navier-Stokes limit from the Boltzmann equation. After suitable adimensionalization (for the detailed process, see [7, 8]), in the incompressible Navier-Stokes scaling, the Boltzmann equation (1.1) can be rescaled as:

\[
\begin{aligned}
\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon &= \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon), \\
 f_\varepsilon|_{t=0} &= f_{\varepsilon,0},
\end{aligned}
\]

where \(\varepsilon\) denotes the Knudsen number, which is the ratio of the mean free path and the macroscopic length scale. Physically, the smaller the Knudsen number \(\varepsilon\) is, the behavior of the dilute gas is more like a continuum fluid. For the more detailed introduction of nondimensionalization of the Boltzmann equation and the scalings (compressible, incompressible Euler and Navier-Stokes), see [7, 22, 24]. In particular, the incompressible Navier-Stokes equations can be derived from the Boltzmann equation in the scaling indicated in (1.14) and fluctuating around the global Maxwellian \(\mu\) with size \(\varepsilon\). More specifically, by setting \(f_\varepsilon(t, x, v) = \mu + \varepsilon \sqrt{\mu} g_\varepsilon(t, x, v)\), and

\[
\Gamma(g, h) = \mu^{-1/2} Q(\sqrt{\mu}g, \sqrt{\mu}h),
\]

the linearized Boltzmann operator \(L\) taking the form

\[
Lg = -\Gamma(\sqrt{\mu}g) - \Gamma(g, \sqrt{\mu}),
\]

the original problem (1.14) is reduced to the Cauchy problem for the fluctuation \(g_\varepsilon\)

\[
\begin{aligned}
\partial_t g_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon^2} Lg_\varepsilon &= \frac{1}{\varepsilon} \Gamma(g_\varepsilon, g_\varepsilon), \\
g_\varepsilon|_{t=0} &= g_{\varepsilon,0},
\end{aligned}
\]

where \(g_{\varepsilon,0}\) is given by \(f_{\varepsilon,0}(x, v) = \mu + \varepsilon \sqrt{\mu} g_{\varepsilon,0}(x, v) \geq 0\).

It is well known that the linearized Boltzmann operator \(L\) is self-adjoint under the standard \(L^2\)-inner product, i.e. \((Lg, f)_{L^2_\mu} = (g, Lf)_{L^2_\mu}\), where \(L^2_\mu(\mathbb{R}^3)\) and \((\cdot, \cdot)_{L^2_\mu}\) is the standard \(L^2\) inner product on \(\mathbb{R}^3\). Moreover, the null space \(N\) of \(L\) is a 5-dimensional linear space and is spanned by the set of collision invariants:

\[
N = \text{Span}\{\sqrt{\mu}, \sqrt{v}, |v|^2, \sqrt{\mu}|v|^2\} = \text{Span}\{\varphi_k(v), k = 1, \cdots, 5\},
\]

that is, \((Lg, g)_{L^2_\mu} = 0\) if and only if \(g \in N\). We also let \(N^\perp\) denote the orthogonal space of \(N\) with respect to the inner product \((\cdot, \cdot)_{L^2_\mu}\). Furthermore, the quadratic term \(\Gamma(f, g) \in N^\perp\).

The proof of all the above properties can be found in [16, 17, 24].

Now we define the fluid variables with respect to the fluctuation \(g_\varepsilon\) as follows:

\[
\begin{aligned}
\rho_\varepsilon &= (g_\varepsilon, \sqrt{\mu})_{L^2_\mu}, & u_\varepsilon &= (g_\varepsilon, \sqrt{v} \mu)_{L^2_\mu}, & \theta_\varepsilon &= (g_\varepsilon, (|v|^2 - 1) \sqrt{\mu})_{L^2_\mu}.
\end{aligned}
\]

Note that \((\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)\) are the linearized fluctuations of mass, momentum and the thermal energy around the global equilibrium \(\mu\). This matches with the relations (1.13).

Taking inner products with the Boltzmann equation (1.15) in \(L^2_\mu\) by \(\sqrt{\mu}, \sqrt{v} \mu\) and \((|v|^2 - 1) \sqrt{\mu}\) respectively gives the local conservation laws around the global Maxwellian \(\mu\).

\[
\begin{aligned}
\partial_t \rho_\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon &= 0, \\
\partial_t u_\varepsilon + \frac{1}{\varepsilon} \nabla_x (\rho_\varepsilon + \theta_\varepsilon) + \nabla_x (\sqrt{\mu} \hat{A}, \frac{1}{\varepsilon} Lg_\varepsilon)_{L^2(\mathbb{R}^3)} &= 0, \\
\partial_t \theta_\varepsilon + \frac{2}{\varepsilon} \nabla_x \cdot u_\varepsilon + \frac{2}{\varepsilon} \nabla_x (\sqrt{\mu} \hat{B}, \frac{1}{\varepsilon} Lg_\varepsilon)_{L^2(\mathbb{R}^3)} &= 0.
\end{aligned}
\]

Here \(\hat{A}\) are 3 \(3\) matrices and \(B, \hat{B}\) are vectors. All of their components are elements in \(N^\perp\), and \(\hat{A}, \hat{B}\) are the unique \(L\)-inverse of \(A, B\) respectively in \(N^\perp\). More specifically,

\[
A_{ij} = v_i v_j - \frac{|v|^2}{3}, \quad B_i = v_i (\frac{|v|^2}{2} - \frac{3}{2}), \quad \text{and} \quad L\hat{A}_{ij} = A_{ij}, \quad L\hat{B}_i = B_i.
\]
For more information about $\hat{A}$ and $\hat{B}$, see [24], for example.

Formally, it can be shown that as the Knudsen number $\varepsilon \to 0$, $(P u, 3\theta - 2\rho_0) \to (u, \theta)$, and $(u, \theta)$ is a solution of the incompressible Navier-Stokes equations. Here $P$ is the usual Leray projection in the whole space $\mathbb{R}^3$. The detailed derivations will be given in section 4. The main goal of the current paper is justifying this convergence rigorously in the context of classical solutions. In the next subsection, we first review the well-posedness of the Boltzmann equation.

1.2. Well-posedness. Since the Boltzmann equation is the cornerstone of kinetic theory, its mathematical theory of well-posedness has been drawn lots of attentions for several decades. In the context of weak solutions, the first result on global solutions (renormalized solutions) to its Cauchy problem with large initial datum was obtained by DiPerna-Lions [19] under Grad’s cutoff assumption. Here the initial datum is “large” means that it only satisfies some finite physical bounds: density, momentum, energy and entropy, and there is no smallness assumption needed. The Grad’s cutoff assumption exclude the long range interaction, in particular the Coulomb force. The global existence of renormalized solutions for long range interaction kernels were obtained later by Alexandre and Villani in [5]. Compared to the cutoff case [19], there exists a defect measure in the definition of renormalized solutions in [5]. As for the initial boundary problem, the first complete answer was given by Mischler [46] while Boltzmann equation was endowed with Maxwell reflection boundary condition for cutoff case.

On the other research direction, Ukai obtained the first global in time close-to-equilibrium classical solution result in [49] for hard potential cut-off collision kernels. After almost 30 years, using the nonlinear energy method, Guo [28, 29] proved the same type result for soft potential case for both periodic domain and whole space. Without Grad’s angular cutoff kernel assumption, Gressman-Strain [26] and Alexandre-Morimoto-Ukai-Xu-Yang [3] proved the existence and regularity of global classical solution near the equilibrium for whole space (also for torus domain with minor modifications). In recent years, there has been significant progress on the strong solutions of the Boltzmann equations in convex bounded domain endowed with different boundary conditions, for example, specular reflection and complete diffusive boundary conditions, started by Guo [31]. For more developments in this direction, see [35, 35, 13, 12].

1.3. Hydrodynamic limits. One of the most important features of the Boltzmann equations (or more generally, kinetic equations) is its connection to the fluid equations. The so-called fluid regimes of the Boltzmann equation are those of asymptotic dynamics of the scaled Boltzmann equations when the Knudsen number $\varepsilon$ is very small. Justifying these limiting processes rigorously has been an active research field from late 70’s. Among many results obtained, the main contributions are the incompressible Navier-Stokes and Euler limits. There are two types of results in this field:

1. First obtaining the solutions of the scaled Boltzmann equation uniform in the Knudsen number $\varepsilon$, then extracting a convergent (at least weakly) subsequence converging to the solutions of the fluid equations as $\varepsilon \to 0$.
2. First obtaining the solutions for the limiting fluid equations, then constructing a sequence of special solutions (around the Maxwellian) of the scaled Boltzmann equations for small Knudsen number $\varepsilon$.

The key difference between the results of type (1) and (2) are: in type (1), the solutions of the fluid equations are not known a priori, and are completely obtained from taking limits from the Boltzmann equation. In short, it is “from kinetic to fluid”; In type (2), the solutions of the fluid equations are known first. In short, it is “from fluid to kinetic”.

The most successful program in type (1) is the so-called BGL program. As mentioned above, the DiPerna-Lions’s renormalized solutions for cutoff kernel [19] (also the non-cutoff kernels in [5]) are the only solutions known to exist globally without any restriction on
the size of the initial data so far. From late 80’s, Bardos-Golse-Levermore initialized the program (BGL program in brief) to justify Leray’s solutions to the incompressible Navier-Stokes equations from DiPerna-Lions’ renormalized solutions [7], [8]. They proved the first convergence result with 5 additional technical assumptions. After 10 years effects by Bardos, Golse, Levermore, Lions and Saint-Raymond, see for example [9, 42, 43, 21], the first complete convergence result without any additional compactness assumption was proved by Golse and Saint-Raymond in [23] for cutoff Maxwell collision kernel, and in [25] for hard cutoff potentials. Later on, it was extended by Levermore-Masmoudi [41] to include soft potentials. Recently Arsenio got the similar results for non-cutoff case [6]. Furthermore, by Jiang, Levermore, Masmoudi and Saint-Raymond, these results were extended to bounded domain where the Boltzmann equation was endowed with the Maxwell reflection boundary condition [45, 37, 38], based on the solutions obtained by Mischler [46].

The BGL program says that, given any $L^2$-bounded functions $(\rho_0, u_0, \theta_0)$, and for any physically bounded initial data (as required in DiPerna-Lions solutions) $F_{\varepsilon,0} = \mu + \varepsilon \sqrt{\mu} g_{\varepsilon,0}$, such that suitable moments of the fluctuation $g_{\varepsilon,0}$, say, $(P(g_{\varepsilon,0}, v\sqrt{\mu})_{L^2(R^3)}, (g_{\varepsilon,0}, (|v|^2 - \frac{5}{3} - 1)\sqrt{\mu})_{L^2(R^3)})$ converges in the sense of distributions to $(u_0, \theta_0)$, the corresponding DiPerna-Lions solutions are $F_{\varepsilon}(t, x, v)$. Then the fluctuations $g_{\varepsilon}$ (defined by $F_{\varepsilon} = \mu + \varepsilon \sqrt{\mu} g_{\varepsilon}$) has weak compactness, such that the corresponding moments of $g_{\varepsilon}$ converge weakly in $L^1$ to $(u, \theta)$ which is a Leray solution of the incompressible Navier-Stokes equation whose viscosity and heat conductivity coefficients are determined by microscopic information, with initial data $(u_0, \theta_0)$. Under some situations, for example the well-prepared initial data or in bounded domain with suitable boundary condition, the convergence could be strong $L^1$.

We emphasize that the BGL program indeed gave a new proof of Leray’s solutions to the incompressible Navier-Stokes equation, in particular the energy inequality which can be derived from the entropy inequality of the Boltzmann equation. Any a priori information of the Navier-Stokes equation is not needed, and completely derived from the microscopic Boltzmann equation. In this sense, BGL program is spiritually a part of Hilbert’s 6th problem: derive and justify the macroscopic fluid equations from the microscopic kinetic equations (see [48]).

Another direction in type (1) is in the context of classical solutions. The first work in this type is Bardos-Ukai [10]. They started from the scaled Boltzmann equation (1.15) for cut-off hard potentials, and proved the global existence of classical solutions $g_{\varepsilon}$ uniformly in $0 < \varepsilon < 1$. The key feature of Bardos-Ukai’s work is that they only need the smallness of the initial data, and did not assume the smallness of the Knudsen number $\varepsilon$. After having the uniform in $\varepsilon$ solutions $g_{\varepsilon}$, taking limits can provide a classical solution of the incompressible Navier-Stokes equations with small initial data. Bardos-Ukai’s approach heavily depends on the sharp estimate especially the spectral analysis on the linearized Boltzmann operator $L$, and the semigroup method (the semigroup generated by the scaled linear operator $\varepsilon^{-2}L + \varepsilon^{-1}v \cdot \nabla_x$). It seems that it is hardly extended to soft potential cutoff, and even harder for the non-cutoff cases, since it is well-known that the operator $L$ has continuous spectrum in those cases. Recently, on the torus, semigroup approach has been employed by Briant [11] and Briant, Merino-Aceituno and Mouhot [14] to prove incompressible Navier-Stokes limit by employing the functional analysis breakthrough of Guadaani-Mischler-Mouhot [27]. Again, their results are for cut-off kernels with hard potentials.

Most of the type (2) results are based on the Hilbert expansion and obtained in the context of classical solutions. It was started from Nishida and Caflisch’s work on the compressible Euler limit [47, 15, 40]. Their approach was revisited by Guo, Jang and Jiang, combining with nonlinear energy method to apply to the acoustic limit [32, 33, 36]. After then this process was used for the incompressible limits, for examples, [18] and [30]. In [18], De Masi-Esposito-Lebowitz considered Navier-Stokes limit in dimension 2. More recently, using the nonlinear energy method, in [30] Guo justified the Navier-Stokes limit (and beyond, i.e. higher order terms in Hilbert expansion). This result was extended in [39] to more general initial data
which allow the fast acoustic waves. These results basically say that, given the initial data which is needed in the classical solutions of the Navier-Stokes equation, it can be constructed the solutions of the Boltzmann equation of the form $F_\varepsilon = \mu + \varepsilon \sqrt{\mu}(g_1 + \varepsilon g_2 + \cdots + \varepsilon^n g_n)$, where $g_1, g_2, \cdots$ can be determined by the Hilbert expansion, and $g_\varepsilon$ is the error term. In particular, the first order fluctuation $g_1 = \rho_1 + u_1 \cdot v + \theta_1 \left( \frac{|v|^2}{2} - \frac{3}{2} \right)$, where $(\rho_1, u_1, \theta_1)$ is the solutions to the incompressible Navier-Stokes equations.

In the present paper, we prove a result of type (1) for classical solutions. Comparing to the similar results [10, 11, 14] in this direction as mentioned above, we consider much larger class of collision kernels for both cut-off and non-cutoff cases. We use a nonlinear energy method, in particular the nice properties of the non-isotropic norm defined in (1.20), which was recently developed in the series of works [1, 2, 3] and equivalently in [26]. In particular, using this non-isotropic norm, we obtain a new estimate on the triple-nonlinear term (Lemma 2.2). Then we proved in Theorem 1.1 the uniform in $\varepsilon$ global existence of the Boltzmann equation with or without cut-off assumption and established the global energy estimates. Then taking limit as $\varepsilon \to 0$, proved the incompressible Navier-Stokes limit in Theorem 1.2. Our result in fact give a microscopic proof of the small initial data classical solution to the incompressible Navier-Stokes equation.

1.4. Notations. Before we state our main theorems, we introduce some notations.

Let us recall that the non-isotropic norm introduced in the series work of Alexandre-Morimoto-Ukai-Xu-Yang. Here for simplicity we use [AMUXY] to denote the references [1, 2, 3, 4].

\[
\|g\|^2 = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B(v - v_\ast, \sigma) \mu_\ast^2 (g' - g)^2 d\sigma dv_x dv
\]

(1.20)

\[
\quad + \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B(v - v_\ast, \sigma) g_\ast^2 (\mu' - \mu)^2 d\sigma dv_x dv.
\]

See also Gressman-Strain [26] for another equivalent definition of this norm.

Using this non-isotropic norm, we define that for $N \in \mathbb{N},$

\[
\|f\|_{X_N(\mathbb{R}_+^6)}^2 = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3} \|\partial_\alpha f\|^2 dx.
\]

Let us recall the macro-micro decomposition of solutions

\[
g = P g + (I - P) g = g_1 + g_2,
\]

where $P$ is the orthogonal projection to $\mathcal{N}$. More precisely,

\[
P g = \rho + v \cdot u + \theta \left( \frac{|v|^2}{2} - \frac{3}{2} \right),
\]

(1.22)

where $(\rho, u, \theta)$ can be computed as in (1.17), i.e.

\[
\rho = (g, \sqrt{\mu})_{L^2_x}, \quad u = (g, v \sqrt{\mu})_{L^2_x}, \quad \theta = (g, (\frac{|v|^2}{2} - 1) \sqrt{\mu})_{L^2_x}.
\]

We denote $g_1 = Pg$ the macroscopic (or fluid) projection of $g(t, x, v)$. In this paper, we often use the following notation:

\[
P g = \{a(t, x) + v \cdot b(t, x) + |v|^2 c(t, x)\} \sqrt{\mu} = \sum_{k=1}^{5} \eta_k(t, x) \varphi_k(v),
\]

(1.23)

where $\varphi_k(v)$'s are defined in (1.16). We also use the notation

\[
\mathcal{A}(g) = (a, b, c) = (\eta_1, \cdots, \eta_5).
\]

Comparing (1.22) and (1.23), we have the linear relations (which are not used in the rest of the paper):

\[
a = \rho - \frac{3}{2} \theta, \quad b = u, \quad c = \theta \frac{\theta}{2}.
\]

(1.25)
Furthermore, \( g_2 = (I - P)g \) is called the kinetic part of \( g \).

Notice that the standard Sobolev norm, denoted by \( H^N \) satisfies
\[
\|g\|_{H^N(R_3;L^2(R_3^3))}^2 \sim \|A(g)\|_{H^N(R_3^3)}^2 + \|g_2\|_{H^N(R_3;L^2(R_3^3))}^2,
\]
where the notation \( \sim \) denotes the norms of both sides are equivalent.

We introduce the following temporal energy functional and dissipation rate functional respectively
\[
\mathcal{E}_N^2(g) = \|g\|_{H^N(R_3;L^2(R_3^3))}^2 = \|g_1\|_{H^N(R_3;L^2(R_3^3))}^2 + \|g_2\|_{H^N(R_3;L^2(R_3^3))}^2
\]
\[
\mathcal{C}_N(g) = \|
abla x A(g)\|_{H^{N-1}(R_3^3)} \sim \|
abla x g_1\|_{H^{N-1}(R_3;L^2(R_3^3))},
\]
\[
\mathcal{D}_N(g) = \|g_2\|_{X^N(R_0^3)}.
\]

Remark that
\[
\mathcal{C}_N \leq \mathcal{E}_N.
\]

We also define the following weighted Sobolev spaces: let \( \langle v \rangle = (1 + |v|^2)^{\frac{1}{2}} \),
\[
L^2_v(R^3) = \{ g \in \mathcal{S}'(R^3) : \|g\|_{L^2_v(R^3)} = \|\langle v \rangle g\|_{L^2(R^3)} < +\infty \}.
\]

1.5. **Main Theorems.** The first theorem is about the global existence of the Boltzmann equation uniform with respect to the Knudsen number \( \varepsilon \). The second is on the incompressible Navier-Stokes limit as \( \varepsilon \to 0 \), taken from the solutions \( g_\varepsilon \) of the Boltzmann equation (1.15) which are constructed in the first theorem.

**Theorem 1.1.** Assume that the collision kernel \( B(\cdot, \cdot) \) satisfies (1.5). It also satisfies either the non-cutoff case (1.7) with \( 0 < s < 1, \gamma > \max\{-3, -\frac{3}{2} - 2s\} \), or the cutoff case (1.8) with \( \gamma > -3 \). For \( N \geq 3 \) and \( 0 < \varepsilon < 1 \), there exists a \( \delta_0 > 0 \), independent of \( \varepsilon \), such that if \( \|g_{\varepsilon,0}\|_{H^N(R_3^3;L^2(R_3^3))} \leq \delta_0 \), then the Cauchy problem (1.15) admits a global solution
\[
g_\varepsilon \in L^\infty([0, +\infty); H^N(R_3^3;L^2(R_3^3)))
\]
with the global energy estimate:
\[
\sup_{t \geq 0} \mathcal{E}_N^2(t) + c_0 \int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}_N^2(t) \, dt + c_0 \int_0^\infty \mathcal{C}_N^2(t) \, dt \leq \mathcal{E}_N^2(0),
\]
where \( c_0 > 0 \) is independent of \( \varepsilon \).

The next theorem is about the limit to the incompressible Navier-Stokes-Fourier equation:
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u, \\
\nabla \cdot u &= 0, \\
\partial_t \theta + u \cdot \nabla \theta &= \kappa \Delta \theta,
\end{aligned}
\]
where the viscosity and heat conductivity are given by
\[
\nu = \frac{1}{15}(\sqrt{\mu} A_{ij}, \mu \hat{A}_{ij})_{L^2(R_3^3)}, \quad \kappa = \frac{2}{15}(\sqrt{\mu} B_i, \sqrt{\mu} \hat{B}_i)_{L^2(R_3^3)}
\]
respectively, recalling the notations defined in (1.19). For the derivation of (1.30), i.e. the relation of \( \mu, \kappa \) with \( A, \hat{A}, B \) and \( \hat{B} \), see [7].

**Theorem 1.2.** Let the collision kernel satisfy the same assumptions as in Theorem 1.1. Let \( 0 < \varepsilon < 1, N \geq 3 \) and \( \delta_0 > 0 \) be as in Theorem 1.1. For any \( (\rho_0, u_0, \theta_0) \in H^N(R_3^3) \) with \( \|\rho_0, u_0, \theta_0\|_{H^N(R_3^3)} < \frac{\delta_0}{2} \), and \( \tilde{g}_{\varepsilon,0} \in N^\perp \) with \( \|\tilde{g}_{\varepsilon,0}\|_{H^N(R_3^3;L^2(R_3^3))} < \frac{\delta_0}{2} \), let
\[
g_{\varepsilon,0}(x,v) = \{\rho_0(x) + u_0(x) \cdot v + \theta_0(x)(\nu \frac{|v|^2}{2} - \frac{3}{2})\} \sqrt{\mu} + \tilde{g}_{\varepsilon,0}(x,v).
\]
Let $g_\varepsilon$ be the family of solutions to the Boltzmann equation (1.15) constructed in Theorem 1.1. Then,

$$g_\varepsilon \to u \cdot v + \theta \left( \frac{|v|^2}{2} - \frac{5}{2} \right) \quad \text{as} \quad \varepsilon \to 0,$$

where the convergence is weak-* for $t$, strongly in $H^{N-\eta}(\mathbb{R}^3)$ for any $\eta > 0$, and weakly in $L^2(\mathbb{R}^3)$, and $(u, \theta) \in C([0, \infty); H^{N-\eta}(\mathbb{R}^3)) \cap L^\infty([0, \infty); H^N(\mathbb{R}^3))$ is the solution of the incompressible Navier-Stokes-Fourier equation (1.29) with initial data:

$$\text{Preparation. 2.1.}$$

$$\eta > 0 \text{ for any } (1.34)$$

$$\text{Assume that this manuscript will focus on the more difficult non-cutoff case.}$$

$$\text{There, Let}$$

$$\text{Preparation.}$$

$$\text{Section 3, the uniform energy estimate and the global existence is established. In the final section, the incompressible Navier-Stokes limit is proved.}$$

$$\text{2. Construction of Local Solutions}$$

$$\text{2.1. Preparation. For convenience, we collect some known results about the collision operators. In the rest of the paper, we use the notation } a \lesssim b \text{ which means that there exists a generic constant (independent of } \varepsilon) C \text{ such that } a \leq Cb.$$}

**Proposition 2.1.** The following estimates hold:

- For any $\gamma > -3, 0 < s < 1$, there exists two generic constants $C_1, C_2 > 0$ such that

$$C_1 \left\{ \|g\|^2_{H^{s/2}(\mathbb{R}^3)} + \|g\|^2_{L^2 s+\gamma/2(\mathbb{R}^3)} \right\} \leq \|g\|^2 \leq C_2 \|g\|^2_{H^{s+\gamma/2}(\mathbb{R}^3)},$$

and

$$C_1 \|\langle \mathcal{P}g\rangle\|^2 \leq (\mathcal{L}g, g)_{L^2(\mathbb{R}^3)} \leq C_2 \|g\|^2.$$}

- For any $0 < s < 1$ and $\gamma > \max\{-3, -3/2 - 2s\}$, there exists $C > 0$ such that

$$\left\| (\Gamma f, g, h)_{L^2(\mathbb{R}^3)} \right\| \leq C \|f\|_{L^2(\mathbb{R}^3)} \|g\| \|h\|,$$

and

$$\left\| (\Gamma f, g, h)_{L^2(\mathbb{R}^3)} \right\| \leq C \left\{ \|f\|_{L^2 s+\gamma/2(\mathbb{R}^3)} \|g\| + \|g\|_{L^2 s+\gamma/2(\mathbb{R}^3)} \|f\| \right\}$$

$$+ \min\{\|f\|_{L^2 s+\gamma/2(\mathbb{R}^3)} \|g\|_{L^2(\mathbb{R}^3)}, \|g\|_{L^2 s+\gamma/2(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)}\} \|h\|.$$}

- For the cutoff case (1.8), we have that (2.1) holds true with $s = 0$ and the trilinear upper bounded estimate (2.3) holds true for $\gamma > -3$.

The estimates (2.1), (2.2) and (2.4) were proved in [3], and (2.3) was proved in [4]. For the cutoff case, the trilinear upper bounded estimate is just Theorem 3 of [28]. The rest of this manuscript will focus on the more difficult non-cutoff case.

Next, we prepare some lemmas about the upper bound estimate. The first is the following Gagliardo-Nirenberg type inequality which was proved in [3] (See Lemma 6.1 there.)

**Lemma 2.1.** Assume that $N \geq 3$ and let $\alpha \in \mathbb{N}^3, |\alpha| \leq N$. Then

$$\| \partial_x^\alpha A^2 \|_{L^2(\mathbb{R}^3)} \lesssim \| \nabla_x A \|_{H^{N-1}(\mathbb{R}^3)} \| A \|_{H^N(\mathbb{R}^3)}.$$
The notations in above inequality (2.5) should be understood in the following sense:
\begin{equation}
\| \partial_x^\alpha (\xi \eta) \|_{L^2(\mathbb{R}^3)} \lesssim \| \nabla_x \xi \|_{H^{N-1}(\mathbb{R}^3)} \| \eta \|_{H^N(\mathbb{R}^3)},
\end{equation}
if \( \xi(x) \) and \( \eta(x) \) are components of \( \mathcal{A}(g) \).

The next is an estimate on the nonlinear collision operator \( \Gamma \) in terms of temporal energy functional and dissipation rate.

**Lemma 2.2.** Under the assumption of Theorem 1.1, we have, for any \( N \geq 3 \),
\begin{equation}
(\Gamma(g, g), h)_{H^N(\mathbb{R}^3; L^2(\mathbb{R}^3))} \lesssim \mathcal{E}(g)\{ \mathcal{C}(g) + \mathcal{D}(g) \} \mathcal{D}(h).
\end{equation}

**Proof.** In the following, we fix an index \( |\alpha| \leq N \), choose any indices \( \alpha_1 \) and \( \alpha_2 \) such that \( \alpha_1 + \alpha_2 = \alpha \), and fix any \( \varphi_k, \varphi_m \) in \( \mathcal{N} \). Note that \( (\partial_x^\alpha \Gamma(g, g), \partial_x^\alpha h_1)_{L^2(\mathbb{R}^3)} = 0 \), we have
\begin{equation}
(\partial_x^\alpha \Gamma(g, g), \partial_x^\alpha h)_{L^2(\mathbb{R}^3)} = (\partial_x^\alpha \Gamma(g, g), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} = J^{11} + J^{12} + J^{21} + J^{22},
\end{equation}
where
\begin{equation}
J^{ij} = (\partial_x^\alpha \Gamma(g_i, g_j), \partial_x^\alpha h)_{L^2(\mathbb{R}^3)}.
\end{equation}

**Estimation of** \( J^{11} \)
\begin{equation}
J^{11} = (\partial_x^\alpha \Gamma(g_1, g_1), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} = (\partial_x^\alpha (\eta_k \eta_m) \Gamma(\varphi_k, \varphi_m), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)}.
\end{equation}
Here we use the notation introduced in (1.23). Thus,
\begin{equation}
J^{11} = \int_{\mathbb{R}^3} \partial_x^\alpha (\eta_k \eta_m) (\Gamma(\varphi_k, \varphi_m), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} dx.
\end{equation}
Then (2.3) yields
\begin{equation}
\left| (\Gamma(\varphi_k, \varphi_m), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} \right| \lesssim \| \partial_x^\alpha h_2 \|,
\end{equation}
Now (2.5) implies for \( |\alpha| \leq N \) and \( N \geq 3 \),
\begin{align*}
|J^{11}| & \lesssim \| \partial_x^\alpha A(\xi) \|_{L^2(\mathbb{R}^3)} \| h_2 \|_{H^N(\mathbb{R}^3)} \\
& \lesssim \| A(\xi) \|_{H^N(\mathbb{R}^3)} \| \nabla_x A(\xi) \|_{H^{N-1}(\mathbb{R}^3)} \| h_2 \|_{H^N(\mathbb{R}^3)} \\
& \lesssim \mathcal{E}(g)\{ \mathcal{C}(g) + \mathcal{D}(g) \} \mathcal{D}(h).
\end{align*}

**Estimation of** \( J^{12} \): Notice
\begin{equation}
J^{12} = \int_{\mathbb{R}^3} (\partial_x^\alpha \eta_k) (\Gamma(\varphi_k, \partial_x^\alpha g_2), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} dx.
\end{equation}
Again, (2.3), yielding
\begin{equation}
|J^{12}| \leq \int_{\mathbb{R}^3} |\partial_x^\alpha A(\xi)| \| \partial_x^\alpha g_2 \| \| \partial_x^\alpha h_2 \| dx,
\end{equation}
the Sobolev embedding \( H^{3/2} \hookrightarrow L^\infty \) (since \( N/2 + 3/2 \leq N \)) gives
\begin{equation}
|J^{12}| \lesssim \| A(\xi) \|_{H^N(\mathbb{R}^3)} \| g_2 \|_{H^N(\mathbb{R}^3)} \| h_2 \|_{H^N(\mathbb{R}^3)} \lesssim \mathcal{E}(g)\{ \mathcal{C}(g) + \mathcal{D}(g) \} \mathcal{D}(h).
\end{equation}

**Estimation of** \( J^{21} \):
\begin{equation}
J^{21} = \int_{\mathbb{R}^3} (\partial_x^\alpha \eta_k) (\Gamma(\partial_x^\alpha g_2, \varphi_k), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} dx.
\end{equation}
If \( \alpha_2 \neq 0 \), (2.3) and Sobolev inequality yields
\begin{align*}
|J^{21}| & \lesssim \| \nabla_x A(\xi) \|_{H^{N-1}(\mathbb{R}^3)} \| g_2 \|_{H^N(\mathbb{R}^3)} \| h_2 \|_{H^N(\mathbb{R}^3)} \\
& \lesssim \mathcal{E}(g)\{ \mathcal{C}(g) + \mathcal{D}(g) \} \mathcal{D}(h).
\end{align*}
If \( \alpha_2 = 0 \), (2.1) and (2.4) yields
\begin{align*}
|\Gamma(\partial_x^\alpha g_2, \varphi_k), \partial_x^\alpha h_2)_{L^2(\mathbb{R}^3)} | & \lesssim \| \partial_x^\alpha g_2 \|_{L^{2+\gamma}(\mathbb{R}^3)} \| \partial_x^\alpha h_2 \| \\
& \lesssim \| \partial_x^\alpha g_2 \| \| \partial_x^\alpha h_2 \|,
\end{align*}
thus
\[ |J^{21}| \lesssim \|A(g)\|_{H^2(\mathbb{R}^3)}\|g_2\|_{X^N(\mathbb{R}^6)}\|h_2\|_{X^N(\mathbb{R}^6)} \lesssim \mathcal{E}_N(g)D_N(g)D_N(h). \]

**Estimation of \( J^{22} \):**

\[ J^{22} = \int_{\mathbb{R}_3^3} (\Gamma(\partial_x^2 g_2, \partial_x^2 g_2), \partial_x^2 h_2)_{L^2(\mathbb{R}_3^3)} dx, \]

(2.4) yields
\[ |J^{22}| \lesssim \|g_2\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}\|g_2\|_{X^N(\mathbb{R}^6)}\|h_2\|_{X^N(\mathbb{R}^6)} \lesssim \mathcal{E}_N(g)D_N(g)D_N(h). \]

Now, combining the above estimates yields the estimate (3.3) and this completes the proof of the Lemma 2.2.

2.2. **Linear Problem.** We consider the following linear Cauchy problem

\[ \begin{aligned}
& \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2} L g = \frac{1}{\varepsilon} \Gamma(f, f), \\
& g|_{t=0} = g_0,
\end{aligned} \]

where \( f \) is a given function. We study the existence of solution in the function space \( H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3)) \).

**Proposition 2.3.** Under the assumption of Theorem 1.1, let \( g_0 \in H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3)) \) with \( N \geq 3 \) and for some \( T > 0 \), \( f \) satisfies
\[ \sup_{0 \leq t \leq T} \mathcal{E}_N^2(f(t)) + \int_0^T D_N^2(f(t)) dt < +\infty. \]

Then the Cauchy problem (2.10) admits an unique solution
\[ g \in L^\infty([0, T]; H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))). \]

**Proof.** We prove the existence of solution to the Cauchy problem (2.10) by the Hahn-Banach theorem. We define the linear operator \( T \) by We rewrite (2.10) into the following form
\[ \mathcal{T} g \equiv \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2} L g. \]

Then we rewrite (2.10) into the following form
\[ \mathcal{T} g = \frac{1}{\varepsilon} \Gamma(f, f), \quad g(0) = g_0. \]

For \( h \in C^\infty([0, T]; S(\mathbb{R}_x^6, \mathbb{R})) \) with \( h(T) = 0 \), we define \( \mathcal{T}_N^* \) through
\[ \left( g, \mathcal{T}_N^* h \right)_{L^2([0, T]; H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3)))} = \left( \mathcal{T} g, h \right)_{L^2([0, T]; H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3)))}, \]
so that \( \mathcal{T}_N^* \) is the adjoint of the operator \( \mathcal{T} \) in the Hilbert space \( L^2([0, T]; H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))) \).

Set
\[ \mathcal{W} = \left\{ w = \mathcal{T}_N h; \ h \in C^\infty([0, T]; S(\mathbb{R}_x^6, \mathbb{R})) \text{ with } h(T) = 0 \right\}, \]
which is a linear subspace of \( L^2([0, T]; H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))) \). And we also have
\[ \mathcal{T}_N^*(h) = -\partial_t h - \frac{1}{\varepsilon}(v \cdot \nabla_x) h + \frac{1}{\varepsilon^2} L h. \]

Then
\[ \left( h, \mathcal{T}_N^* h \right)_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} = -\frac{1}{2} \frac{d}{dt} \|h(t)\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}^2 - \frac{1}{\varepsilon} (v \cdot \nabla_x h, h)_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} + \frac{1}{\varepsilon^2} \left( \mathcal{L}(h), h \right)_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}. \]
Note that the second term above vanishes and the estimate (2.2), we have
\[
\int_t^T (h, T_N h)_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+))} dt \geq \frac{1}{2} \|h(t)\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+))}^2 + \frac{c}{\varepsilon^2} \int_t^T D^2_N(h(s)) ds.
\]
Thus, for all \(0 < t < T\),
\[
\|h(t)\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+))}^2 + \frac{c}{\varepsilon^2} \int_t^T D^2_N(h(s)) ds
\]
\[
\leq \|T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} \|h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))}.
\]
This implies that
\[
\|h\|_{L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} + \frac{c}{\varepsilon^2} \int_0^T D^2_N(h(s)) ds
\]
\[
\leq \||T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+))}\%)^2)\}
\[
\leq \sqrt{T} \|T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} \|h\|_{L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))}.
\]
This immediately gives
\[
\|h\|_{L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} \leq \sqrt{T} \|T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))}.
\]
Furthermore, (2.13) can be written as: let \(y = \|h\|_{L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))}^2\),
\[
\frac{c}{\varepsilon^2} \int_0^T D^2_N(h(s)) ds \leq \sqrt{T} \|T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} y - y^2
\]
\[
\leq \frac{T}{4} \|T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))}^2.
\]
Here we consider the right-hand side of the first line as a quadratic function of \(y\), then optimize it. Thus,
\[
\frac{1}{\varepsilon} \left( \int_0^T D^2_N(h(s)) ds \right)^{1/2} \leq \frac{C \sqrt{T}}{\varepsilon} \|T_N h\|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))}.
\]
Next, we define a functional \(G\) on \(\mathbb{W}\) as follows
\[
G(w) = \frac{1}{\varepsilon} (\Gamma(f, f, h)_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} + (g_0, h(0))_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+))}.
\]
Then, using (1.27) and (2.7)
\[
|G(w)| \leq \frac{1}{\varepsilon} \left( \int_0^T \{\mathcal{E}_N(f) + \mathcal{E}_N(f) \mathcal{D}_N(f)\} \mathcal{D}_N(h) dt
\]
\[
+ \|g_0\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} \|h(0)\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))
\]
\[
\leq \frac{1}{\varepsilon} \sup_{0 < t < T} \mathcal{E}_N(f) \sup_{0 < t < T} \mathcal{E}_N(f) + \left( \int_0^T \mathcal{D}_N^2(f) dt \right)^{1/2} \left( \int_0^T \mathcal{D}_N^2(h) dt \right)^{1/2}
\]
\[
+ \|g_0\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} \|h\|_{L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))},
\]
finally, (2.14) and (2.15) imply
\[
|G(w)| \leq C(f, g_0) |T_N h|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))} \leq C||w||_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))},
\]
where
\[
C(f, g_0) = \sup_{0 < t < T} \mathcal{E}_N(f) \sup_{0 < t < T} \mathcal{E}_N(f) + \left( \int_0^T \mathcal{D}_N^2(f) dt \right)^{1/2} + \sqrt{T} \|g_0\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+))}.
\]
Thus, \(G\) is a continuous linear functional on \((\mathbb{W}; \| \cdot \|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))})\). So by the Hahn-Banach Theorem, \(G\) can be extended from \(\mathbb{W}\) to \(L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3_+)))\). From
the Riesz representation theorem, there exists \( g \in L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3))) \) such that for any \( w \in \mathbb{W} \),
\[
G(w) = (g, w)_{L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)))}.
\]
For any \( h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}_x^6)) \) with \( h(T) = 0 \), we have
\[
\left( g, T_N^+ h \right)_{L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)))} = \frac{1}{\varepsilon} \left( \Gamma(f, f), h \right)_{L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)))} + (g_0, h(0))_{H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3))},
\]
and by the definition of the operator \( T_N \),
\[
\left( g, T_N^+ h \right)_{L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)))} = \left( T g, h \right)_{L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)))} = \left( g, \tilde{h} \right)_{L^2([0, T]; L^2(\mathbb{R}_x^6, \mathbb{R}_v^3))} + \left( g_0, \tilde{h}(0) \right)_{L^2(\mathbb{R}_x^6, \mathbb{R}_v^3)},
\]
where
\[
\tilde{h} = \Lambda^{2N} h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}_x^6)) \text{ with } \tilde{h}(T) = 0.
\]
Here \( \Lambda = (1 - \Delta_x)^{1/2} \), and \( \Lambda^{2N} \) is used for changing the inner product from \( H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)) \) to \( L^2(\mathbb{R}_x^2, L^2(\mathbb{R}_v^3)) \). Since \( \Lambda^{2N} \) is an isomorphism on \( \{ h : h \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}_x^6)) \text{ with } h(T) = 0 \} \), then \( g \in L^2([0, T]; H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3))) \) is a solution of the Cauchy problem (2.11).

2.3. Local existence for nonlinear problem. We consider now the following iterative scheme
\[
(2.17) \quad \begin{cases}
\partial_t g^{n+1} + \frac{1}{\varepsilon} v \cdot \nabla_x g^{n+1} + \frac{1}{\varepsilon^2} \mathcal{L} g^{n+1} = \frac{1}{\varepsilon} \Gamma(g^n, g^n), \\
g^{n+1}|_{t=0} = g_0,
\end{cases}
\]
with \( g^0 \equiv 0 \).

**Proposition 2.4.** There exists \( 0 < \delta_0 \leq 1, 0 < T \leq 1 \), such that for any \( 0 < \varepsilon \leq 1, g_0 \in H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3)), N \geq 3 \) with
\[
\|g_0\|_{H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3))} \leq \delta_0,
\]
the iteration problem (2.17) admits a sequence of solutions \( \{g^n\}_{n \geq 1} \) satisfying
\[
\sup_{t \in [0, T]} \mathcal{E}_N^2(g^n) + \frac{1}{\varepsilon^2} \int_0^T \mathcal{D}_N^2(g^n) \, dt \leq 4 \|g_0\|_{H^N(\mathbb{R}_x^2; L^2(\mathbb{R}_v^3))}^2.
\]

**Proof.** Notice that for the linear Cauchy problem (2.17), for given \( g^n \) satisfying (2.19), the existence of \( g^{n+1} \) is assured by the Proposition 2.3. So that it is enough to prove (2.19) by induction, using (2.2) and (2.7), there exists \( C > 0 \) such that
\[
\frac{d}{dt} \mathcal{E}_N^2(g^{n+1}) + \frac{1}{\varepsilon^2} \mathcal{D}_N^2(g^{n+1}) \leq \frac{C}{\varepsilon} \left\{ \mathcal{E}_N^2(g^n) + \mathcal{E}_N(g^n) \mathcal{D}_N(g^n) \right\} \mathcal{D}_N(g^{n+1})
\]
\[
\leq \frac{1}{2\varepsilon^2} \mathcal{D}_N^2(g^{n+1}) + 2C^2 \mathcal{E}_N^2(g^n) \{ \mathcal{E}_N^2(g^n) + \mathcal{D}_N^2(g^n) \}.
\]
Thus, we get
\[
\frac{d}{dt} \mathcal{E}_N^2(g^{n+1}) + \frac{1}{\varepsilon^2} \mathcal{D}_N^2(g^{n+1})
\leq 4C^2 \mathcal{E}_N^2(g^n) \{ \mathcal{E}_N^2(g^n) + \mathcal{D}_N^2(g^n) \}.
\]
Integration on $[0, T]$ with $T \leq 1$,

$$
\sup_{t \in [0, T]} \mathcal{E}^2_N(g^{n+1}) + \frac{1}{\varepsilon} \int_0^T \mathcal{D}^2_N(g^{n+1}) \, dt \leq \mathcal{E}^2_N(g_0)
$$

$$
+ 4C^2 \sup_{t \in [0, T]} \mathcal{E}^2_N(g^n) \left\{ \sup_{t \in [0, T]} \mathcal{E}^2_N(g^n) + \int_0^T \mathcal{D}^2_N(g^n) \right\},
$$

we complete the proof of the Proposition if we chose $\delta_0$ such that

$$
1 + 64C^2\delta_0^2 \leq 4.
$$

Finally, from the uniform estimate (2.19), we can prove the convergence of $\{g^n\}$, thus the following local existence result through a standard argument as in [1].

**Theorem 2.5.** There exists $\delta_0 > 0, T > 0$, such that for any $0 < \varepsilon < 1, g_{\varepsilon,0} \in H^N(\mathbb{R}^3_\varepsilon; L^2(\mathbb{R}^3_\varepsilon))$, $N \geq 3$ with

$$
(2.20) \quad \|g_{\varepsilon,0}\|_{H^N(\mathbb{R}^3_\varepsilon; L^2(\mathbb{R}^3_\varepsilon))} \leq \delta_0,
$$

then the Cauchy problem (1.15) admits an unique solution $g_{\varepsilon} \in L^\infty([0, T]; H^N(\mathbb{R}^3_\varepsilon; L^2(\mathbb{R}^3_\varepsilon)))$ and it satisfies

$$
(2.21) \quad \sup_{t \in [0, T]} \mathcal{E}^2_N(g_{\varepsilon}) + \frac{1}{\varepsilon} \int_0^T \mathcal{D}^2_N(g_{\varepsilon}) \, dt \leq 4\|g_{\varepsilon,0}\|_{H^N(\mathbb{R}^3_\varepsilon; L^2(\mathbb{R}^3_\varepsilon))}^2.
$$

**Proof.** It is enough to prove that $\{g^n\}$ is a Cauchy sequence in $L^\infty([0, T]; L^2(\mathbb{R}^6))$. Set $w^n = g^{n+1} - g^n$ and deduce from (2.17),

$$
\begin{cases}
\partial_t w^n + \frac{1}{\varepsilon} v \cdot \nabla_x w^n + \frac{1}{\varepsilon} \mathcal{L} w^n = \frac{1}{\varepsilon} \left[ \Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}) \right], \\
w^n|_{t=0} = 0.
\end{cases}
$$

Since, for any $h \in L^2$

$$
\left( \Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}), \mathcal{P} h \right)_{L^2(\mathbb{R}^3_\varepsilon)} = 0,
$$

$$
\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}) = \Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}),
$$

then

$$
\left( \Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}), w^n \right)_{L^2(\mathbb{R}^6)} = \left( \Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), w^n \right)_{L^2(\mathbb{R}^6)},
$$

using (2.3) and $H^N(\mathbb{R}^3_\varepsilon) \hookrightarrow L^\infty(\mathbb{R}^3_\varepsilon)$ for $N \geq 2$,

$$
\left| \left( \frac{1}{\varepsilon} \left[ \Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}) \right], w^n \right)_{L^2(\mathbb{R}^6)} \right|
$$

$$
\leq \frac{C}{\varepsilon} \mathcal{E}_N(g^n)(\|w^{n-1}\|_{\chi^0} + \mathcal{D}_0(w^{n-1})) \mathcal{D}_0(w^n)
$$

$$
+ \frac{C}{\varepsilon} \mathcal{E}_0(w^{n-1})(\|g^{n-1}\|_{\chi^N} + \mathcal{D}_N(g^{n-1})) \mathcal{D}_0(w^n)
$$

$$
\leq C_\delta \mathcal{E}_N^2(g^n)(\|w^{n-1}\|_{\chi^0}^2 + \mathcal{D}_0^2(w^{n-1})) + \frac{\delta}{\varepsilon^2} \mathcal{D}_0^2(w^n)
$$

$$
+ C_\delta \mathcal{E}_0^2(w^{n-1})(\|w^{n-1}\|_{\chi_N}^2 + \mathcal{D}_N^2(g^{n-1})).
$$
Thus, fix a small $\delta > 0$, we get
\[
\frac{d}{dt}\|w^n\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\varepsilon^2}D_0^2(w^n) \leq C\delta \mathcal{E}_N'(g^n)(\|w^{n-1}\|_{\mathcal{A}_0}^2 + D_0^2(w^{n-1})) + C\delta \mathcal{E}_N'(g^{n-1})(\|g_1^{n-1}\|_{\mathcal{A}_N}^2 + D_N^2(g^{n-1}))\].

Note that
\[
\|w_1^{n-1}\|_{\mathcal{A}_0}^2 \leq C\mathcal{E}_N'(g^{n-1}), \quad \|g^{n-1}_1\|_{\mathcal{A}_N}^2 \leq C\mathcal{E}_N'(g^{n-1}),
\]
we have proved
\[
\|w^n\|^2_{L^\infty([0,T];L^2(\mathbb{R}^d))} + \frac{1}{\varepsilon^2} \int_0^T D_0^2(w^n) \, dt \leq C \sup_{t \in [0,T]} \mathcal{E}_N'(g^n)(T \sup_{t \in [0,T]} \mathcal{E}_N'(w^{n-1}) + \int_0^T D_0^2(w^{n-1}) \, dt) + C \sup_{t \in [0,T]} \mathcal{E}_N'(w^{n-1})(T \sup_{t \in [0,T]} \mathcal{E}_N'(g^{n-1}) + \int_0^T D_N^2(g^{n-1}) \, dt).
\]

Using now (2.19) with $\delta_0 > 0$ small enough, we get that for any $0 < \varepsilon \leq 1$
\[
\sup_{t \in [0,T]} \mathcal{E}_N'(w^n) + \frac{1}{\varepsilon^2} \int_0^T D_0^2(w^n) \, dt \leq \frac{1}{2} \left( \sup_{t \in [0,T]} \mathcal{E}_N'(w^{n-1}) + \frac{1}{\varepsilon^2} \int_0^T D_0^2(w^{n-1}) \, dt \right).
\]

Thus we have proved that $\{g^n\}$ is a Cauchy sequence in $L^\infty([0,T];L^2(\mathbb{R}^{d+6}_x))$.

Combining with the estimate (2.19) and interpolation, $\{g^n\}$ is a Cauchy sequence in $L^\infty([0,T],H^{N-\eta}(\mathbb{R}^3_x,L^2(\mathbb{R}^3_v)))$ for any $\eta > 0$ and the limit is in $L^\infty([0,T],H^N(\mathbb{R}^3_x,L^2(\mathbb{R}^3_v)))$. Finally the estimate (2.21) follows from weak lower semicontinuity.

We prove now the uniqueness of local solutions, let $g_1^\varepsilon, g_2^\varepsilon \in 2$ local solutions of (1.15), then $\tilde{g}_\varepsilon = g_1^\varepsilon - g_2^\varepsilon \in L^\infty([0,T],H^N(\mathbb{R}^3_x,L^2(\mathbb{R}^3_v)))$ and it satisfy
\[
\begin{cases}
\partial_t \tilde{g}_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{g}_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{L} \tilde{g}_\varepsilon = \frac{1}{\varepsilon} \Gamma(g_1^\varepsilon, \tilde{g}_\varepsilon) + \frac{1}{\varepsilon} \Gamma(\tilde{g}_\varepsilon, g_2^\varepsilon), \\
\tilde{g}_\varepsilon|_{t=0} = 0.
\end{cases}
\]

Then similary procedure prove that $\tilde{g}_\varepsilon \equiv 0$ which imply the uniqueness of weak solutions. □

3. **Uniform estimate and global solutions**

Let $g_\varepsilon$ be a local solution of Cauchy problem (1.15), we use the continuation argument of local solutions to prove the existence of global solutions in the space $H^N(\mathbb{R}^3_x,L^2(\mathbb{R}^3_v))$, $N \geq 3$.

For this, we need to carry out the smallness assumption of initial data.

3.1. **Microscopic Energy Estimate.** We study firstly the estimate on the microscopic component $g_2$ in the function space $H^N(\mathbb{R}^3_x,L^2(\mathbb{R}^3_v))$. For notational simplification, we drop the sub-index $\varepsilon$ of $g$, and also drop $g$ in the notations $A, \mathcal{E}_N, \mathcal{C}_N, \mathcal{D}_N$. Actually, we shall establish

**Proposition 3.1.** Let $g \in L^\infty([0,T];H^N(\mathbb{R}^3_x,L^2(\mathbb{R}^3_v)))$ be a solution of the equation (1.15) constructed in Theorem 2.5, then there exists a constant $C$ independent of $\varepsilon$ such that the following estimate holds:

\[
(3.1) \quad \frac{d}{dt} \mathcal{E}_N + \frac{1}{\varepsilon^2} \mathcal{D}_N \leq C \left\{ \frac{1}{\varepsilon} \mathcal{E}_N \mathcal{D}_N + (\mathcal{E}_N \mathcal{C}_N)^2 \right\}.
\]

**Proof.** We apply $\partial^\alpha_x$ to (1.15) and take the $L^2(\mathbb{R}^d_{x,v})$ inner product with $\partial^\alpha_x g$. Since the inner product including $v \cdot \nabla_x g$ vanishes by integration by parts, we get

\[
(3.2) \quad \frac{1}{2} \frac{d}{dt} \mathcal{E}_N + \frac{1}{\varepsilon^2} \sum_{|\alpha| \leq N} (\mathcal{L} \partial^\alpha_x g, \partial^\alpha_x g)_{L^2(\mathbb{R}^d_{x,v})} = \frac{1}{\varepsilon} \sum_{|\alpha| \leq N} (\partial^\alpha_x \Gamma(g,g), \partial^\alpha_x g)_{L^2(\mathbb{R}^d_{x,v})}.
\]
Note that the above identity makes sense is guaranteed by (2.21). In view of (2.2), we have,
\[
\sum_{|\alpha| \leq N} (L \partial_x^\alpha g, \partial_x^\alpha g)_{L^2(\mathbb{R}^6_v)} \geq C_1 \|g_2\|_{L^N(\mathbb{R}^6_v)}^2 = C_1 D_2^2.
\]
Lemma 2.2 implies that for $|\alpha| \leq N$,
\[
\frac{1}{\varepsilon} \left| (\partial_x^\alpha \Gamma(g, g), \partial_x^\alpha g)_{L^2(\mathbb{R}^6_v)} \right| \leq \frac{C_2}{\varepsilon} E_N(C_N D_N + D_2^2) \leq \frac{C_2}{\varepsilon} E_N D_2^2 + \frac{1}{4\eta} (E_N C_N)^2 + \eta \frac{1}{\varepsilon} D_2^2.
\]
Taking $\eta = \frac{C_1}{2}$, then Proposition 3.1 can be concluded by plugging these two estimates into (3.2).

\[\square\]

### 3.2. Macroscopic energy estimates

We study now the energy estimate for the macroscopic part $P g$ where $g$ is a solution of the equation (1.15). First we decompose the equation (1.15) into microscopic and macroscopic parts, i.e. using the notations (1.21) and (1.23) to rewrite it into the following equation

\[
\partial_t \{a + b \cdot v + c|v|^2\} \mu^{1/2} + \frac{1}{\varepsilon} v \cdot \nabla_x \{a + b \cdot v + c|v|^2\} \mu^{1/2} = -\partial_t g_2 - \frac{1}{\varepsilon} v \cdot \nabla_x g_2 - \frac{1}{\varepsilon^2} \mathcal{L} g_2 + \frac{1}{\varepsilon} \Gamma(g_\varepsilon, g_\varepsilon),
\]

\[\text{(3.3)}\]

**Lemma 3.1.** Let $\alpha \in \mathbb{N}^3, |\alpha| \leq N - 1$. If $g$ is a solution of the Boltzmann equation (1.15), and $A = A(g) = (a, b, c)$ defined in (1.23), then
\[
\varepsilon \|\partial_x^\alpha A\|_{L^2(\mathbb{R}^3_v)} \lesssim C_N + D_N.
\]

**Proof.** Let $g$ be a solution of the scaled Boltzmann equation (1.15) which can be rewritten as
\[
\partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g_1 = -\frac{1}{\varepsilon} v \cdot \nabla_x g_2 - \frac{1}{\varepsilon} \mathcal{L} g + \Gamma(g, g).
\]

The equation (3.6) taking inner product with $\sqrt{\mu}, v\sqrt{\mu}$ and $(|v|^2 - 1)\sqrt{\mu}$ respectively yields the local conservation laws
\[
\begin{aligned}
\partial_t \rho + \frac{1}{\varepsilon} \nabla_x \cdot u &= \frac{1}{\varepsilon} (v \cdot \nabla_x g_2, \sqrt{\mu})_{L^2(\mathbb{R}^3_v)}, \\
\partial_t u + \frac{1}{\varepsilon} (\nabla_x \rho + \nabla_x \theta) &= -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, v\sqrt{\mu})_{L^2(\mathbb{R}^3_v)}, \\
\partial_t \theta + \frac{1}{\varepsilon} \nabla_x \cdot u &= -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, (|v|^2 - 1)\sqrt{\mu})_{L^2(\mathbb{R}^3_v)}.
\end{aligned}
\]

Using the relations between $(\rho, u, \theta)$ and $(a, b, c)$, (see (1.25)), the local conservation laws (3.7) read as
\[
\begin{aligned}
\partial_t a &= \frac{1}{\varepsilon} (v \cdot \nabla_x g_2, (5 - |v|^2)\sqrt{\mu})_{L^2(\mathbb{R}^3_v)}, \\
\partial_t b + \frac{1}{\varepsilon} (\nabla_x a + 5\nabla_x c) &= -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, v\sqrt{\mu})_{L^2(\mathbb{R}^3_v)}, \\
\partial_t c + \frac{1}{\varepsilon} \nabla_x \cdot b &= -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, (|v|^2 - 3)\sqrt{\mu})_{L^2(\mathbb{R}^3_v)},
\end{aligned}
\]

from which we can deduce that
\[
\|\partial_t \partial_x^\alpha A\|_{L^2(\mathbb{R}^3_v)} \lesssim \frac{1}{\varepsilon} \|

\text{(3.8)}
\]

Here we use the fact that for any $k, m \geq 0, |v|^k\sqrt{\mu} \leq C(v)^m$. This completes the proof of the lemma by using (2.1). In particular, from the first equation of (3.8), we have
\[
\varepsilon \|\partial_x^\alpha a\|_{L^2(\mathbb{R}^3_v)} \lesssim D_N.
\]

\[\square\]
Next, we put the so-called 13-moments

\begin{equation}
\{e_j^{13}\}_{j=1} = \left\{ \mu^{1/2}, v_i \mu^{1/2}, v_i v_j \mu^{1/2}, v_i |v|^2 \mu^{1/2} \right\}.
\end{equation}

This set of functions spans a 13-dimensional subspace of \( L^2(\mathbb{R}_+^3) \). The left-hand side of (3.4) can be rewritten in the expansion of these moments:

\begin{equation}
\frac{1}{\varepsilon} \nabla_x e \cdot |v|^2 + \sum_{j=1}^{3} (\partial_t + \frac{b_j}{\varepsilon} \partial_j) v_j^2 + \frac{1}{\varepsilon} \sum_{i \neq j}^{3} (\partial_i b_j + \partial_j b_i) v_i v_j + \sum_{j=1}^{3} (\partial_t b_i + \frac{1}{\varepsilon} \partial_j a) v_j + \partial_t a \right\} \mu^{1/2}
\end{equation}

Let \( \{e_k^j\}_{k=1}^{13} \) be a corresponding bi-orthogonal basis, i.e. a basis such that

\begin{equation}
(e_j^*, e_k^j)_{L^2(\mathbb{R}_+^3)} = \delta_{j,k}, \quad j, k = 1, \cdots, 13,
\end{equation}

hold. It is clear that \( e_k^j \) is given as a linear combination of the 13 moments in (3.10). For the precise construction of \( e_k^j \), see section 3.9 in the book [20]. On the other hand, the 13 moments in (3.10) are also linear combinations of \( e_k^j \). Note that from (3.11), the left-hand side of the equation (3.4) is a linear combination of 13-moments with the coefficients being the time and spacial derivatives of macroscopic variables \( (a,b,c) \). Now the key idea of Guo in [29] is to project the right-hand side of the equation (3.4) also onto 13-moments (3.10). This is equivalent to take inner product of the right-hand side of the equation (3.4) with linear combinations of \( e_k^j \). By doing so, similar to [29], we have that the macroscopic component \( g_1 = P g \sim A = (a,b,c) \), satisfies the following set of equations

\begin{equation}
\begin{cases}
v |v|^2 \mu^{1/2} : & \frac{1}{\varepsilon} \nabla_x e = -\partial_t r_c + \frac{1}{\varepsilon} m_c + \frac{1}{\varepsilon} l_c + \frac{1}{\varepsilon} h_c,

v_i^{1/2} : & \partial_t c + \frac{1}{\varepsilon} \partial_i b_i = -\partial_t r_i + \frac{1}{\varepsilon} m_i + \frac{1}{\varepsilon} l_i + \frac{1}{\varepsilon} h_i,

v_i v_j^{1/2} : & \frac{1}{\varepsilon} b_j + \frac{1}{\varepsilon} \partial_j b_i = -\partial_t r_{ij} + \frac{1}{\varepsilon} m_{ij} + \frac{1}{\varepsilon} l_{ij} + \frac{1}{\varepsilon} h_{ij}, \quad i \neq j,

v_i^{1/2} : & \partial_t b_i + \frac{1}{\varepsilon} \partial_i a = -\partial_t r_{bi} + \frac{1}{\varepsilon} m_{bi} + \frac{1}{\varepsilon} l_{bi} + \frac{1}{\varepsilon} h_{bi},

\mu^{1/2} : & \partial_t a = -\partial_t r_a + \frac{1}{\varepsilon} m_a + \frac{1}{\varepsilon} l_a + \frac{1}{\varepsilon} h_a.
\end{cases}
\end{equation}

In fact, one obtains each equation of the second column, if one multiplies (3.4) by such an \( e_j^* \) and integrating in \( \nu \), where \( r_c, \cdots, h_a \) are the inner products of the form

\begin{equation}
r = (g_2, e^*)_{L^2(\mathbb{R}_+^3)},
m = -(v \cdot \nabla_x g_2, e^*)_{L^2(\mathbb{R}_+^3)},
h = (\Gamma(g, g), e^*)_{L^2(\mathbb{R}_+^3)},
\end{equation}

\begin{equation}
l = -(\mathcal{L} g_2, e^*)_{L^2(\mathbb{R}_+^3)},
\end{equation}

in which \( e^* \) stands for the corresponding \( e_k^j \). The precise expressions of how the 13-moments in (3.10) are linearly combined by \( e_k^j \) are not important here. So we only express in a simple way as (3.13). Here and below we drop the index \( c, i, ij, b_i, a \) since the computations are similar.

The next lemma gives estimates on the various terms involved in the right-hand side of the macroscopic system (3.12). Moreover, in the left-hand side of the following estimates, \( r, m, l \) and \( h \) stand for one of the corresponding terms of the macroscopic system as explained above.

\textbf{Lemma 3.2.} Let \( r, m, l, h \) be the ones defined by (3.13) with \( e^* \) replaced by any linear combination of the basis functions \( e_j^* \). Let \( |\alpha| \leq N - 1 \). Then, one has

\begin{align}
\| \partial_\alpha^x \partial_\alpha^z r \|_{L^2(\mathbb{R}_+^3)} & \lesssim \min\{||g_2||_{H^N(\mathbb{R}_+^3; L^2(\mathbb{R}_+^3))}, D_N\}, \\
\| \partial_\alpha^z m \|_{L^2(\mathbb{R}_+^3)} & \lesssim \min\{||g_2||_{H^N(\mathbb{R}_+^3; L^2(\mathbb{R}_+^3))}, D_N\}, \\
\| \partial_\alpha^z l \|_{L^2(\mathbb{R}_+^3)} & \lesssim \min\{||g_2||_{H^{N-1}(\mathbb{R}_+^3; L^2(\mathbb{R}_+^3))}, D_{N-1}\}, \\
\| \partial_\alpha^z h \|_{L^2(\mathbb{R}_+^3)} & \lesssim E_{N-1}(C_{N-1} + D_{N-1}).
\end{align}
Proof. Since \( e^*_i \) can be expressed as a linear combination of basis functions \( \{ e_i \} \), we may compute \( r, m, h \) with \( e^* \) any linear combination \( e \) of \( \{ e_i \} \). Remark that \( e = \hat{e} \sqrt{\mu} \), then, for any \( \ell \in \mathbb{R} \),

\[
\| \partial_x \partial^\alpha_a r \|_{L^2(\mathbb{R}^2)} = \| (\partial_x \partial^\alpha_a g_2, e)_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \\
\lesssim \| \partial_x \partial^\alpha_a g_2 \|_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \lesssim \| g_2 \|_{H(N;L^2(\mathbb{R}^2))},
\]

\[
\| \partial^\alpha_a l \|_{L^2(\mathbb{R}^2)} = \| (\mathcal{L}(\partial^\alpha_a g_2, e)_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \\
= \| (\partial^\alpha_a g_2, \mathcal{L} e)_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \lesssim \| \partial^\alpha_a g_2 \|_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \lesssim \| g_2 \|_{H(N;L^2(\mathbb{R}^2))},
\]

\[
\| \partial^\alpha_a m \|_{L^2} = \| (\nabla_x \partial^\alpha_a g_2, v e)_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \\
\lesssim \| \nabla_x \partial^\alpha_a g_2 \|_{L^2(\mathbb{R}^2)} \|_{L^2(\mathbb{R}^2)} \lesssim \| g_2 \|_{H(N;L^2(\mathbb{R}^2))},
\]

chose \( \ell = 0 \) and \( \ell = s + \gamma/2 \), thus \( (2.1) \) imply \( (3.15), (3.16) \) and \( (3.17) \). The estimate \( (3.18) \) is a direct consequence of \( (2.7) \), since \( h \) is computed as follows.

\[
h = (\Gamma(g, g), e).
\]

\[\square\]

Using the previous three lemmas, we are now able to prove our first differential inequality, which estimates \( \alpha + 1 \) derivatives of the macroscopic part \( A \) in terms of the microscopic part \( g_2 \), for \( |\alpha| \leq N - 1 \). Below, on the left-hand side, \( r \) stands for the vector of all the previous \( r \).

**Lemma 3.3.** Let \( |\alpha| \leq N - 1 \), and let \( g \) be a solution of the scaled Boltzmann equation \( (1.15) \). Then there exists a positive constant \( C \) independent of \( \varepsilon \), such that the following estimate holds:

\[
(3.19) \quad \varepsilon \frac{d}{dt} \left\{ \left( \partial_x^\alpha r, \nabla_x \partial_x^\alpha (a, -b, c) \right)_{L^2(\mathbb{R}^2)} + \left( \partial_x^\alpha b, \nabla_x \partial_x^\alpha a \right)_{L^2(\mathbb{R}^2)} \right\} + C_N^2 \leq \tilde{C} \left\{ \frac{1}{\varepsilon^2} D_N^2 + E_N(C_N^2 + D_N^2) \right\}.
\]

**Proof.** Recall that

\[
C_N^2 = \| \nabla_x \partial_x^\alpha A \|_{L^2(\mathbb{R}^2)}^2 = \| \nabla_x \partial_x^\alpha a \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla_x \partial_x^\alpha b \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla_x \partial_x^\alpha c \|_{L^2(\mathbb{R}^2)}^2.
\]

(a) **Estimate of \( \nabla_x \partial_x^\alpha a \).** From the macroscopic equations \( (3.12) \),

\[
\| \nabla_x \partial_x^\alpha a \|_{L^2(\mathbb{R}^2)}^2 = \| \nabla_x \partial_x^\alpha a \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla_x \partial_x^\alpha b \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla_x \partial_x^\alpha c \|_{L^2(\mathbb{R}^2)}^2.
\]

Here,

\[
\varepsilon R_1 = -\varepsilon (\partial^\alpha_a \mathcal{L} b + \partial^\alpha_a \mathcal{L} r, \nabla_x \partial_x^\alpha a)_{L^2(\mathbb{R}^2)}
\]

\[
= -\varepsilon \frac{d}{dt} \left( \partial^\alpha_a (b + r), \nabla_x \partial_x^\alpha a \right)_{L^2(\mathbb{R}^2)} - \varepsilon (\nabla_x \partial_x^\alpha (b + r), \partial^\alpha_a a)_{L^2(\mathbb{R}^2)}.
\]

Note that the estimate \( (3.9) \) and \( (3.12) \), \( \varepsilon (\nabla_x \partial_x^\alpha b, \partial^\alpha_a a)_{L^2(\mathbb{R}^2)} \lesssim D_N^2 \), and

\[
\varepsilon (\nabla_x \partial_x^\alpha b, \partial^\alpha_a a)_{L^2(\mathbb{R}^2)} \lesssim \eta \| \nabla_x \partial_x^\alpha b \|_{L^2(\mathbb{R}^2)} + \frac{1}{4\eta} D_N^2.
\]

Furthermore, Lemma 3.2 implies that

\[
| (\partial_x^\alpha a, \nabla_x \partial_x^\alpha a)_{L^2(\mathbb{R}^2)} | \lesssim D_N \| \nabla_x A \|_{H^{-1}(\mathbb{R}^2)} \lesssim D_N C_N,
\]

\[
\frac{1}{\varepsilon} | (\partial_x^\alpha a, \nabla_x \partial_x^\alpha a)_{L^2(\mathbb{R}^2)} | \lesssim \frac{1}{\varepsilon} D_N C_N \leq \frac{1}{\varepsilon} D_N C_N,
\]

\[
| (\partial_x^\alpha h, \nabla_x \partial_x^\alpha a)_{L^2(\mathbb{R}^2)} | \lesssim E_{-1}(C_{N-1} + D_{N-1}) C_N.
\]
Hence, for some small $0 < \eta < 1$,
\[
\varepsilon \frac{d}{dt} (\partial^a_x (b + r), \nabla_x \partial^a_x a)_{L^2(\mathbb{R}^3)} + \| \nabla_x \partial^a_x a \|_{L^2(\mathbb{R}^3)}^2 \leq \eta \| \nabla_x \partial^a_x b \|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{4\eta} \mathcal{D}^2_N + \eta (C^2_N + \mathcal{D}^2_N)
\]
\[
+ \mathcal{D}_N \mathcal{C}_N + \frac{1}{\varepsilon} \mathcal{D}_N \mathcal{C}_N + \mathcal{E}_N(C_N + \mathcal{D}_N) \mathcal{C}_N
\]
\[
\leq \frac{1}{4\eta \varepsilon^2} \mathcal{D}^2_N + \eta C^2_N + \mathcal{E}_N(C^2_N + \mathcal{D}^2_N)
\]
Thus,
\[
(3.20) \quad \varepsilon \frac{d}{dt} (\partial^a_x (b + r), \nabla_x \partial^a_x a)_{L^2(\mathbb{R}^3)} + \| \nabla_x \partial^a_x a \|_{L^2(\mathbb{R}^3)}^2 \leq \eta C^2_N + \frac{C}{\eta} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2_N + \mathcal{E}_N(C^2_N + \mathcal{D}^2_N) \right\},
\]
where $C > 0$ independent of $\varepsilon$ and $\eta$.

(b) **Estimate of** $\nabla_x \partial^a_x b_i$. Recall $b = (b_1, b_2, b_3)$. From (3.12),
\[
\Delta_x \partial^a_x b_i + \partial^a_x \partial^a_x b_i = \partial^a_x \left[ \sum_{j \neq i} \partial_j (\partial_j b_i + \partial_i b_j) + \partial_i (2\partial_i b_i - \sum_{j \neq i} \partial_j b_j) \right]
\]
\[
= \partial^a_x \left[ \nabla_x (-\varepsilon r + m + \frac{1}{\varepsilon} l + h) - \partial_i (2\partial_i c - 2\partial_i c) \right],
\]
where $r, l, h$ stands for linear combinations of $r_i, l_i, h_i$ and $r_{ij}, l_{ij}, h_{ij}$ for $i, j = 1, 2, 3$ respectively. Then
\[
\| \nabla_x \partial^a_x b_i \|_{L^2(\mathbb{R}^3)}^2 + \| \partial_i \partial^a_x b_i \|_{L^2(\mathbb{R}^3)}^2 = -(\Delta_x \partial^a_x b_i + \partial^a_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)}
\]
\[
= \varepsilon R_2 + R_3 + R_4 + R_5,
\]
where
\[
\varepsilon R_2 = -\varepsilon (\nabla_x \partial^a_x \partial r, \partial^a_x b_i)_{L^2(\mathbb{R}^3)}
\]
\[
= -\varepsilon \frac{d}{dt} (\partial^a_x r, -\nabla_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)} - \varepsilon (\partial^a_x r, \partial_i \nabla_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)}
\]
\[
\lesssim -\varepsilon \frac{d}{dt} (\partial^a_x r, \nabla_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)} + \frac{1}{4\eta} \mathcal{D}^2_N + \eta \varepsilon^2 \| \partial_i \partial^a_x b_i \|_{L^2(\mathbb{R}^3)}^2,
\]
\[
R_3 = -(\partial^a_x m, \nabla_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{4\eta} \mathcal{D}^2_N + \eta \| \nabla_x \partial^a_x b_i \|_{L^2(\mathbb{R}^3)}^2,
\]
\[
R_4 = -\frac{1}{\varepsilon} (\partial^a_x l, \nabla_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)} \lesssim \frac{1}{\varepsilon} \mathcal{C}_N \mathcal{D}_N \leq \frac{1}{\varepsilon} \mathcal{C}_N \mathcal{D}_N,
\]
\[
R_5 = -(\partial^a_x h, \nabla_x \partial^a_x b_i)_{L^2(\mathbb{R}^3)} \lesssim \mathcal{E}_N(C^2_N + \mathcal{D}^2_N),
\]
where in above estimates we apply Lemma 3.2.

Thus
\[
(3.21) \quad \varepsilon \frac{d}{dt} (\partial^a_x r, -\nabla_x \partial^a_x b)_{L^2(\mathbb{R}^3)} + \| \nabla_x \partial^a_x b \|_{L^2(\mathbb{R}^3)}^2 \leq \eta C^2_N + \frac{C}{\eta} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2_N + \mathcal{E}_N(C^2_N + \mathcal{D}^2_N) \right\},
\]
where $C > 0$ independent of $\varepsilon$ and $\eta$. 
(c) Estimate of $\nabla_x \partial_x^2 c$. From (3.12),
\[
\|\nabla_x \partial_x^2 c\|_{L^2(\mathbb{R}^3)}^2 = (\nabla_x \partial_x^2 c, \nabla_x \partial_x^2 c)_{L^2(\mathbb{R}^3)}
\]
\[
= (\partial_x^2 (-\varepsilon \partial_t r + m + \frac{1}{\varepsilon} l + h), \nabla_x \partial_x^2 c)_{L^2(\mathbb{R}^3)}
\]
\[
\lesssim \varepsilon R_6 + \eta C_N^2 + \frac{1}{\eta} D_N^2 + \frac{1}{\varepsilon^2} D_N^2 + \mathcal{E}_N (C_N^2 + D_N^2),
\]
where
\[
\varepsilon R_6 = -\varepsilon (\partial_x^2 \partial_t r, \nabla_x \partial_x^2 c)_{L^2(\mathbb{R}^3)} = -\varepsilon \frac{d}{dt} (\partial_x^2 r, \nabla_x \partial_x^2 c)_{L^2(\mathbb{R}^3)} - \varepsilon (\nabla_x \partial_x^2 r, \partial_t \partial_x^2 c)_{L^2(\mathbb{R}^3)}
\]
\[
\lesssim -\varepsilon \frac{d}{dt} (\partial_x^2 r, \nabla_x \partial_x^2 c)_{L^2(\mathbb{R}^3)} + \frac{1}{\eta^2} D_N^2 + \eta^2 \|\partial_t \partial_x^2 c\|_{L^1(\mathbb{R}^3)}^2.
\]
Thus
\[
(3.22)
\]
\[
\varepsilon \frac{d}{dt} (\partial_x^2 r, \nabla_x \partial_x^2 c)_{L^2(\mathbb{R}^3)} + \|\nabla_x \partial_x^2 c\|_{L^2(\mathbb{R}^3)}^2
\]
\[
\leq \eta C_N^2 + \frac{C}{\eta} \left\{ \frac{1}{\varepsilon^2} D_N^2 + \mathcal{E}_N (C_N^2 + D_N^2) \right\},
\]
where $C > 0$ independent of $\varepsilon$ and $\eta$.

By combining the above estimates (3.20), (3.21), (3.22) and taking $\eta > 0$ sufficiently small, then we get the estimate (3.19) uniformly for $0 < \varepsilon < 1$, thus complete the proof of Lemma 3.3.

Based on the microscopic estimate (3.1) and the macroscopic estimate (3.19), we can derive the uniform energy estimate. Take the following form of linear combination of (3.1) and (3.19),
\[
\frac{d}{dt} \left\{ \mathcal{E}_N + d_1 \varepsilon \sum_{|a| \leq N-1} \left( (\partial_x^2 r, \nabla_x \partial_x^2 (a, -b, c))_{L^2(\mathbb{R}^3)} + (\partial_x^2 b, \nabla_x \partial_x^2 a)_{L^2(\mathbb{R}^3)} \right) \right\} + \frac{1}{\varepsilon^2} D_N^2
\]
\[
+ d_1 C_N^2 \leq C \left( \frac{1}{\varepsilon} \mathcal{E}_N D_N^2 + (\mathcal{E}_N C_N)^2 \right) + \frac{d_1 \tilde{C}}{\varepsilon^2} D_N^2 + d_1 \tilde{C} \mathcal{E}_N (C_N^2 + D_N^2).
\]

Choose $d_1$ firstly such that $1 - d_1 \tilde{C} > 0$, we have for $0 < \varepsilon < 1$
\[
\frac{d}{dt} \left[ \mathcal{E}_N^2 + d_1 \varepsilon \sum_{|a| \leq N-1} \left( (\partial_x^2 r, \nabla_x \partial_x^2 (a, -b, c))_{L^2(\mathbb{R}^3)} + (\partial_x^2 b, \nabla_x \partial_x^2 a)_{L^2(\mathbb{R}^3)} \right) \right]
\]
\[
+(1 - d_1 \tilde{C}) \frac{1}{\varepsilon^2} D_N^2 + d_1 C_N^2 \leq (C + d_1 \tilde{C}) \frac{1}{\varepsilon} \mathcal{E}_N D_N^2 + (C \mathcal{E}_N + d_1 \tilde{C}) \mathcal{E}_N C_N^2.
\]

Set
\[
\mathcal{E}_N^2 = \left[ \mathcal{E}_N^2 + \frac{d_1 \varepsilon}{\varepsilon^2} \sum_{|a| \leq N-1} \left( (\nabla_x \partial_x^2 r, \partial_x^2 (a, -b, c))_{L^2(\mathbb{R}^3)} - (\partial_x^2 b, \nabla_x \partial_x^2 a)_{L^2(\mathbb{R}^3)} \right) \right].
\]

(3.15) implies that
\[
\left| \sum_{|a| \leq N-1} (\nabla_x \partial_x^2 r, \partial_x^2 (a, -b, c))_{L^2(\mathbb{R}^3)} + (\partial_x^2 b, \nabla_x \partial_x^2 a)_{L^2(\mathbb{R}^3)} \right| \lesssim \|g_2\|_{H^N(\mathbb{R}^3)}^2 + \|A\|_{H^N(\mathbb{R}^3)}^2 = \mathcal{E}_N^2,
\]
then we can choose $d_1 > 0$ even smaller such that, for any $0 < \varepsilon < 1$
\[
(3.23)
\]
\[
c_1 \mathcal{E}_N \leq \mathcal{E}_N \leq c_2 \mathcal{E}_N
\]
for some positive constants $c_1$ and $c_2$. Thus we proved the following theorem:
Theorem 3.2. (Global Energy Estimate) For $N \geq 2$, if $g$ is a solution of the scaled Boltzmann equation (1.15), then there exists a constant $c_0 > 0$ independent of $\varepsilon$ such that if $E_N \leq 1$, then

\begin{equation}
\frac{d}{dt} E_N^2 + \frac{1}{\varepsilon^2} D_N^2 + C_N^2 \leq c_0 E_N \left\{ \frac{1}{\varepsilon^2} D_N^2 + C_N^2 \right\}
\end{equation}

holds as far as $g$ exists.

3.3. Proof of Theorem 1.1. Now, we are ready to prove Theorem 1.1 by the usual continuation arguments.

Proof. We choose the initial data $g_{0,\varepsilon}$ such that

\[ E_N(0) = \|g_{0,\varepsilon}\|_{H^N(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))} \leq M, \]

where $M$ is defined as

\begin{equation}
M = \min\{\delta_0, \frac{1}{c_2}, \frac{1}{4c_0c_2}\}.
\end{equation}

Recall that $c_0, c_1, c_2$ are constants in (3.24) and (3.23), and $\delta_0$ appears in Theorem 2.5. Note that $E_N(0) \leq M \leq \delta_0$, then from Theorem 2.5 there exists a solution $g \in L^\infty([0,T]; H^N(\mathbb{R}^3_x), L^2(\mathbb{R}^3_v))$ for some $T > 0$, and from the local estimate (2.21), we have $E_N(t) \leq 2M$ for $0 < t < T$. We define

\[ T^* = \sup \{ t \in \mathbb{R}^+ | E_N(t) \leq 2M \leq 4 \frac{M}{c_1} \} > 0. \]

Note that on $[0, T]$ for $0 < T < T^*$, $E_N(t) \leq c_2 E_N(t) \leq 2c_2 M < 1$. Then the global energy estimate (3.24) implies that

\begin{equation}
\frac{d}{dt} E_N^2 + (1 - 2c_0 M) \left\{ \frac{1}{\varepsilon^2} D_N^2 + C_N^2 \right\} \leq 0.
\end{equation}

From the choice of $M$, $1 - 2c_0 M > \frac{1}{2}$. Thus

\[ E_N^2(T) + \frac{1}{2} \int_0^T \left\{ \frac{1}{\varepsilon^2} D_N^2 + C_N^2 \right\} dt \leq E_N^2(0), \]

which implies $E_N(T) \leq \frac{c_2}{c_1} M$. Thus $T^* = \infty$, and we finish the proof of Theorem 1.1. \hfill \Box

4. LIMIT TO INCOMPRESSIBLE NAVIER-STOKES-FOURIER EQUATIONS

4.1. The limit from the global energy estimate. Based on Theorem 1.1, there exists a $\delta_0 > 0$, such that the Boltzmann equation (1.15) admits a global solution $g_\varepsilon$ with initial data $g_{\varepsilon,0}(x,v) = \{\rho_0(x) + u_0(x) \cdot v + \theta_0(x) (|v|^2 - \frac{3}{2}) \} \sqrt{\mu} + g_{\varepsilon,0}(x,v)$, where

\[ \| \{\rho_0, u_0, \theta_0\} \|_{H^N(\mathbb{R}^3_x)} \leq \delta_0, \]

and

\[ \tilde{g}_{\varepsilon,0} \in \mathcal{N}^\ell, \quad \text{with} \quad \| \tilde{g}_{\varepsilon,0} \|_{H^N(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))} \leq \delta_0. \]

Furthermore, the global energy estimate (1.28) holds, i.e. there exists a positive constant $C$ independent of $\varepsilon$, such that

\begin{equation}
\sup_{t \geq 0} E_N^2(t) = \sup_{t \geq 0} \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3_x} |\partial_x^\alpha g_\varepsilon(t)|^2 \, dv \, dx \leq C,
\end{equation}

and

\begin{equation}
\int_0^\infty D_N^2(t) \, dt = \sum_{|\alpha| \leq N} \int_0^\infty \int_{\mathbb{R}^3_x} \| \partial_x^\alpha \{I - P\} g_\varepsilon(t) \|^2 \, dv \, dx \, dt \leq C \varepsilon^2,
\end{equation}

and

\begin{equation}
\int_0^\infty C_N^2(t) \, dt = \sum_{|\alpha| \leq N} \int_0^\infty \int_{\mathbb{R}^6_{x,v}} |\partial_x^\alpha P g_\varepsilon(t)|^2 \, dv \, dx \, dt \leq C.
\end{equation}
From the energy bound (4.1), there exists a $g_0 \in L^\infty(\mathbb{R}^3; H^N(\mathbb{R}^3; L^2(\mathbb{R}^3)))$, such that
\begin{equation}
 g_\varepsilon \to g_0 \quad \text{as} \quad \varepsilon \to 0,
\end{equation}
where the convergence is weak-* for $t \geq 0$, strongly in $H^{N-\eta}(\mathbb{R}^3)$ for any $\eta > 0$, and weakly in $L^2(\mathbb{R}^3)$. 

From the energy dissipation bound (4.2) and the inequality (2.1), we have
\begin{equation}
 (\mathbf{I} - \mathbf{P})g_\varepsilon \to 0, \quad \text{in} \quad L^2([0, +\infty); H^N(\mathbb{R}^3; L^2(\mathbb{R}^3))) \quad \text{as} \quad \varepsilon \to 0.
\end{equation}
Combining the convergence (4.4) and (4.5), we have $(\mathbf{I} - \mathbf{P})g_0 = 0$. Thus, there exists $(\rho, u, \theta) \in L^\infty(\mathbb{R}^3)$, such that
\begin{equation}
 g_0(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \theta(t, x)(\frac{|v|^2}{2} - \frac{3}{2}).
\end{equation}

### 4.2. The limiting equations

Now we define the fluid variables as follows:
\[
\rho_\varepsilon = (g_\varepsilon, \sqrt{\mu}) L^2, \quad u_\varepsilon = (g_\varepsilon, v\sqrt{\mu}) L^2, \quad \theta_\varepsilon = (g_\varepsilon, (|v|^2 - 1)\sqrt{\mu}) L^2,
\]
where $L^2_\varepsilon$ denotes $L^2(\mathbb{R}^3)$. It follows from (4.4) that
\begin{equation}
 (\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon) \to (\rho, u, \theta) \quad \text{as} \quad \varepsilon \to 0,
\end{equation}
where the convergence is weak-* for $t \geq 0$, strongly in $H^{N-\eta}(\mathbb{R}^3)$ for any $\eta > 0$.

Taking inner products with the Boltzmann equation (1.15) in $L^2_\varepsilon$ by $\sqrt{\mu}, v\sqrt{\mu}$ and $(|v|^2 - 1)\sqrt{\mu}$ respectively gives the local conservation laws:
\begin{equation}
\left\{
\begin{array}{ll}
\partial_t \rho_\varepsilon + \frac{1}{2} \nabla_x \cdot u_\varepsilon = 0, \\
\partial_t u_\varepsilon + \frac{1}{2} \nabla_x (\rho_\varepsilon + \theta_\varepsilon) + \nabla_x \cdot (\sqrt{\mu} \hat{A} + \frac{1}{2} \mathcal{L} g_\varepsilon) L^2(\mathbb{R}^3) = 0, \\
\partial_t \theta_\varepsilon + \frac{2}{3} \nabla_x \cdot u_\varepsilon + \frac{2}{3} \nabla_x \cdot (\sqrt{\mu} \hat{B} + \frac{1}{2} \mathcal{L} g_\varepsilon) L^2(\mathbb{R}^3) = 0,
\end{array}
\right.
\end{equation}
Here we use the notations defined in (1.17) and properties of $\hat{A}, \hat{B}, \sqrt{\mu}$ and $\sqrt{\mu}$. Self-adjointness of $\mathcal{L}$ to obtain the following well-known calculations:
\[
(v \cdot \nabla_x g_\varepsilon, \sqrt{\mu}) L^2(\mathbb{R}^3) = \nabla_x \cdot (\hat{A} g_\varepsilon \sqrt{\mu}) L^2(\mathbb{R}^3) + \nabla_x \left(\frac{|v|^2}{2} g_\varepsilon \sqrt{\mu}\right) L^2(\mathbb{R}^3)
= \nabla_x \cdot (\mathcal{L} \hat{A}, g_\varepsilon \sqrt{\mu}) L^2(\mathbb{R}^3) + \nabla_x (\rho_\varepsilon + \theta_\varepsilon),
\]
and
\[
(v \cdot \nabla_x g_\varepsilon, (|v|^2 - 1) \sqrt{\mu}) L^2(\mathbb{R}^3) = \frac{2}{3} \nabla_x \cdot (\hat{B} g_\varepsilon \sqrt{\mu}) L^2(\mathbb{R}^3) + \frac{2}{3} \nabla_x \cdot (v g_\varepsilon \sqrt{\mu}) L^2(\mathbb{R}^3)
= \frac{2}{3} \nabla_x \cdot (\mathcal{L} \hat{B}, g_\varepsilon \sqrt{\mu}) L^2(\mathbb{R}^3) + \frac{2}{3} \nabla_x \cdot u_\varepsilon.
\]

### Incompressibility and Boussinesq relation

From the first equation of (4.8) and the global energy bound (4.1), it is easy to deduce
\begin{equation}
 \nabla_x \cdot u_\varepsilon \to 0 \quad \text{in the sense of distributions} \quad \text{as} \quad \varepsilon \to 0.
\end{equation}

Combining with the convergence (4.7), we have
\begin{equation}
 \nabla_x \cdot u = 0.
\end{equation}

From the second equation of (4.8),
\begin{equation}
 \nabla_x (\rho_\varepsilon + \theta_\varepsilon) = -\varepsilon \partial_t u_\varepsilon + \nabla_x \cdot (\mathcal{L} \sqrt{\mu} \hat{A}, (\mathbf{I} - \mathbf{P})g_\varepsilon) L^2(\mathbb{R}^3).
\end{equation}
From the global energy dissipation (4.2), it follows that
\begin{equation}
 \nabla_x (\rho_\varepsilon + \theta_\varepsilon) \to 0 \quad \text{in the sense of distributions} \quad \text{as} \quad \varepsilon \to 0,
\end{equation}
which gives the Boussinesq relation
\begin{equation}
 \nabla_x (\rho + \theta) = 0.
\end{equation}
Convergence of $\frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon$: The third equation minus $\frac{2}{3}$ times the first equation in (4.8) gives
\begin{equation}
\partial_t (\frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon) + \frac{2}{5}\nabla \cdot (\sqrt{\mu_B} \cdot \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon)_{L^2(\mathbb{R}^3)} = 0.
\end{equation}
From the global energy estimate (4.1), we have that for almost every $t \in [0, \infty)$, \(\| (\frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon)(t) \|_{H^N(\mathbb{R}^3)} \leq C\). Then there exists a $\tilde{\theta} \in L^\infty([0, \infty); H^N(\mathbb{R}^3))$, so that
\begin{equation}
(\frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon)(t) \to \tilde{\theta}(t) \quad \text{in} \quad H^{N-\eta}(\mathbb{R}^3),
\end{equation}
for any $\eta > 0$ as $\varepsilon \to 0$. Furthermore, using the equation (4.13), we can show the equi-continuity in $t$. Indeed, for any $[t_1, t_2] \subset [0, \infty)$, any test function $\chi(x)$ and $|\alpha| \leq N - 1$,
\begin{equation}
\int_{\mathbb{R}^3} \left[ \partial_\alpha^\varepsilon \left( \frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right)(t_2) - \partial_\alpha^\varepsilon \left( \frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right)(t_1) \right] \chi(x) \, dx \leq \frac{1}{\varepsilon^2} \int_{t_1}^{t_2} \mathcal{D}_N^2(g_\varepsilon(t)) \, dt.
\end{equation}
Thus the energy dissipation estimate (4.2) implies the equi-continuity in $t$. From the Arzelà-Ascoli Theorem, $\tilde{\theta} \in C(\mathbb{R}; H^{N-1-\eta}(\mathbb{R}^3)) \cap L^\infty([0, \infty); H^{N-\eta}(\mathbb{R}^3))$, and
\begin{equation}
\frac{2}{3}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \to \tilde{\theta} \quad \text{in} \quad C([0, \infty); H^{N-1-\eta}(\mathbb{R}^3)) \cap L^\infty([0, \infty); H^{N-\eta}(\mathbb{R}^3)),
\end{equation}
as $\varepsilon \to 0$ for any $\eta > 0$. Note that $\tilde{\theta} = \frac{3}{5}\theta - \frac{2}{5}\rho$ and $\theta = (\frac{3}{5}\theta - \frac{2}{5}\rho) + \frac{2}{5}(\rho + \theta)$, and the relation (4.12), we get $\theta = \theta$ and $\rho + \theta = 0$. 

Convergence of $\mathcal{P}u_\varepsilon$: Taking the Leray projection operator $\mathcal{P}$ on the second equation of (4.8) gives
\begin{equation}
\partial_t \mathcal{P}u_\varepsilon + \mathcal{P} \nabla \cdot \left( \sqrt{\mu_A} \cdot \frac{1}{\varepsilon} \mathcal{L} g_\varepsilon \right)_{L^2(\mathbb{R}^3)} = 0.
\end{equation}
Similar arguments as above deduce that there exists a divergence free $\tilde{u} \in L^\infty([0, \infty); H^N(\mathbb{R}^3))$, such that
\begin{equation}
\mathcal{P}u_\varepsilon \to \tilde{u} \quad \text{in} \quad C([0, \infty); H^{N-1-\eta}(\mathbb{R}^3)) \cap L^\infty([0, \infty); H^{N-\eta}(\mathbb{R}^3)),
\end{equation}
as $\varepsilon \to 0$ for any $\eta > 0$. Note that $\tilde{u} = \mathcal{P}u$ and (4.10), we have $\tilde{u} = u$.

Follow the standard calculations, (for example, [7]), the local conservation laws can be rewritten as
\begin{equation}
\begin{aligned}
\partial_t \rho_\varepsilon + \frac{1}{\varepsilon} \nabla \cdot u_\varepsilon &= 0, \\
\partial_t u_\varepsilon + \frac{1}{\varepsilon} \nabla \rho_\varepsilon + \nabla \cdot (u_\varepsilon \otimes u_\varepsilon - |u_\varepsilon|^2 I) &= \nu \nabla \cdot \Sigma(u_\varepsilon) + \nabla \cdot R_{\varepsilon,u}, \\
\partial_t \theta_\varepsilon + \frac{2}{5} \nabla \cdot u_\varepsilon + \nabla \cdot (u_\varepsilon \theta_\varepsilon) &= \kappa \nabla \cdot \left( \nabla \theta_\varepsilon \right) + \nabla \cdot R_{\varepsilon,\theta},
\end{aligned}
\end{equation}
where $\Sigma(u) = \nabla u + \nabla u^T - \frac{2}{3} \nabla u I$, and $R_{\varepsilon,u}, R_{\varepsilon,\theta}$ have the form
\begin{equation}
\begin{aligned}
- \varepsilon (\zeta(v), \partial_t g_\varepsilon)_{L^2(\mathbb{R}^3)} + (\zeta(v), v \cdot \nabla \{(I - \mathcal{P}) g_\varepsilon\})_{L^2(\mathbb{R}^3)} + (\zeta(v), \Gamma((I - \mathcal{P}) g_\varepsilon, \{I - \mathcal{P}\} g_\varepsilon)_{L^2(\mathbb{R}^3)} \\
+ (\zeta(v), \Gamma((I - \mathcal{P}) g_\varepsilon, \mathcal{Q} g_\varepsilon)_{L^2(\mathbb{R}^3)} + (\zeta(v), \mathcal{Q}(I - \mathcal{P}) g_\varepsilon, \{I - \mathcal{P}\} g_\varepsilon)_{L^2(\mathbb{R}^3)}).
\end{aligned}
\end{equation}
For $R_{\varepsilon,u}$, take $\zeta(v) = \sqrt{\mu_A}$, while for $R_{\varepsilon,\theta}$, take $\zeta(v) = \sqrt{\mu_B}$.

The equations of $\theta$ and $u$: Decompose $u_\varepsilon = \mathcal{P}u_\varepsilon + \mathcal{Q}u_\varepsilon$, where $\mathcal{Q} = \nabla \Delta_\varepsilon^{-1} \nabla \cdot$ is a gradient. Denote $\tilde{\theta}_\varepsilon = \frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon$. Then from (4.19), the following equation is satisfied in the sense of distributions:
\begin{equation}
\partial_t \tilde{\theta}_\varepsilon + \frac{3}{5} \nabla \cdot (\mathcal{P} u_\varepsilon \tilde{\theta}_\varepsilon) - \frac{2}{5} \kappa \Delta_\varepsilon \tilde{\theta}_\varepsilon = \nabla \cdot R_{\varepsilon,\theta},
\end{equation}
where
\begin{equation}
R_{\varepsilon,\theta} = \frac{3}{5} R_{\varepsilon,\theta} - \frac{6}{25} \mathcal{P} u_\varepsilon (\rho_\varepsilon + \theta_\varepsilon) - \frac{6}{25} \mathcal{Q} u_\varepsilon (\rho_\varepsilon + \theta_\varepsilon) - \frac{3}{5} \mathcal{Q} u_\varepsilon \tilde{\theta}_\varepsilon + \frac{6}{25} \kappa \Delta_\varepsilon (\rho_\varepsilon + \theta_\varepsilon).
\end{equation}
For any $T > 0$, let $\phi(t, x)$ be a test function satisfying $\phi(t, x) \in C^1([0, T], C_\infty^c(\mathbb{R}^3))$ with $\phi(0, x) = 1$ and $\phi(t, x) = 0$ for $t \geq T'$, where $T' < T$. Noting (4.20), and using the global bounds (4.1), (4.2) and (4.3), it is easy to show that

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \nabla_x \cdot R_\varepsilon(t, x) \phi(t, x) \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \to 0,
\end{equation}

where $R_\varepsilon = R_{\varepsilon, u}$ or $R_{\varepsilon, \theta}$. For other terms in (4.21), noting that the convergence (4.9) and (4.11), together with (4.22), we have

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \nabla_x \cdot \tilde{R}_{\varepsilon, \theta}(t, x) \phi(t, x) \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{equation}

From the convergence (4.16) and (4.18), for $N > 1$, as $\varepsilon \to 0$,

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \partial_t \tilde{\theta}_\varepsilon(t, x) \phi(t, x) \, dx \, dt \to -\int_{\mathbb{R}^3} (3\theta_0 - 2\varepsilon \rho_0)(x) \, dx - \int_0^T \int_{\mathbb{R}^3} \theta(t, x) \partial_t \phi(t, x) \, dx \, dt,
\end{equation}

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \Delta_x \tilde{\theta}_\varepsilon(t, x) \, dx \, dt \to \int_0^T \int_{\mathbb{R}^3} \theta(t, x) \Delta_x \phi(t, x) \, dx \, dt,
\end{equation}

and

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \nabla_x \cdot (\mathcal{P} u_\varepsilon \tilde{\theta}_\varepsilon)(t, x) \phi(t, x) \, dx \, dt \to -\int_0^T \int_{\mathbb{R}^3} u(t, x) \theta(t, x) \cdot \nabla_x \phi(t, x) \, dx \, dt.
\end{equation}

Acting the Leray projection $\mathcal{P}$ on the second equation of (4.19), we have the following equation

\begin{equation}
\partial_t \mathcal{P} u_\varepsilon + \mathcal{P} \nabla_x \cdot (\mathcal{P} u_\varepsilon \otimes \mathcal{P} u_\varepsilon) - \nu \Delta_x \mathcal{P} u_\varepsilon = \mathcal{P} \nabla_x \cdot \tilde{R}_{\varepsilon, u}
\end{equation}

where

\begin{equation}
\tilde{R}_{\varepsilon, u} = R_{\varepsilon, u} - \mathcal{P} \cdot (\mathcal{P} u_\varepsilon \otimes \mathcal{P} u_\varepsilon - \mathcal{Q} u_\varepsilon \otimes \mathcal{Q} u_\varepsilon + \mathcal{Q} u_\varepsilon \otimes \mathcal{Q} u_\varepsilon).
\end{equation}

Similar as above we can take the vector-valued test function $\psi(t, x)$ with $\nabla_x \cdot \psi = 0$, and prove that as $\varepsilon \to 0$

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} (\partial_t \mathcal{P} u_\varepsilon + \mathcal{P} \nabla_x \cdot (\mathcal{P} u_\varepsilon \otimes \mathcal{P} u_\varepsilon) - \nu \Delta_x \mathcal{P} u_\varepsilon) \cdot \psi(t, x) \, dx \, dt
\end{equation}

\begin{equation}
\to -\int_{\mathbb{R}^3} \mathcal{P} u_0(x) \cdot \psi(0, x) \, dx - \int_0^T \int_{\mathbb{R}^3} (u \cdot \partial_t \psi + u \otimes u : \nabla_x \psi - \nu u \cdot \Delta_x \psi) \, dx \, dt,
\end{equation}

and

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \mathcal{P} \nabla_x \cdot \tilde{R}_{\varepsilon, u}(t, x) \phi(t, x) \, dx \, dt \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{equation}

Collecting all above convergence results, we have shown that $(u, \theta) \in C([0, \infty); H^{N-1}(\mathbb{R}^3)) \cap L^\infty([0, \infty); H^N(\mathbb{R}^3))$ satisfies the following incompressible Navier-Stokes equations

\begin{equation}
\begin{cases}
\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \\
\nabla_x \cdot u = 0,
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta,
\end{cases}
\end{equation}

with initial data:

\begin{equation}
u(0, x) = \mathcal{P} u_0(x), \quad \theta(0, x) = \frac{3}{3} \theta_0(x) - \frac{2}{5} \rho_0(x).
\end{equation}

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