ISOPERIMETRIC INEQUALITIES FOR EIGENVALUES OF TRIANGLES

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Abstract. Isoperimetric lower bounds for the first eigenvalue for the Dirichlet Laplacian on arbitrary triangles are proved using various symmetrization techniques. It is also shown that among triangles, the equilateral triangle maximizes the spectral gap and (under additional assumption) the ratio of the first two eigenvalues.

1. Introduction

The purpose of this paper is to prove new isoperimetric–type bounds for the eigenvalues of the Dirichlet Laplacian on arbitrary triangles. The problem of finding good bounds for eigenvalues of various domains has been of interest for many years. See for example [1, 9, 15, 16] for general bounds, [2] for result about polygons, and [10, 11, 18] for bounds for triangles. Comprehensive overview of this subject along with methods used to tackle it can be found in the book [12].

We will use "eigenvalue of the domain D" to refer to an eigenvalue of the Dirichlet Laplacian on the domain D. The eigenvalues for a domain form a nondecreasing sequence $\{\lambda_i, j \geq 1\}$ with $\lambda_1 < \lambda_2$. Throughout the paper we use A for the area of a domain, L for its perimeter, R for its inradius (the supremum of the radii of disks contained in the domain) and d for its diameter.

In [15, Section 7.4], Pólya and Szegö conjectured that with fixed area, among all polygons with n sides the regular one minimizes the first eigenvalue. The conjecture remains open except for the following cases:

$$\lambda_1 A|_{Triangle} \geq \lambda_1 A|_{Equilateral}, \tag{1.1}$$

$$\lambda_1 A|_{Quadrilateral} \geq \lambda_1 A|_{Square}, \tag{1.2}$$

$$\lambda_1 A \geq \lambda_1 A|_{Ball}. \tag{1.3}$$

The notation $|_{set}$ is used to indicate the set for the quantity to the left of it.

There are also upper bounds where the ball is extremal.

$$\lambda_1 R^2 \leq \lambda_1 R^2|_{Ball}, \tag{1.4}$$

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{Ball}, \tag{1.5}$$

$$(\lambda_2 - \lambda_1) R^2 \leq (\lambda_2 - \lambda_1) R^2|_{Ball}. \tag{1.6}$$

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The first inequality follows trivially from the domain monotonicity. The second and the third are known as the Payne-Pólya-Weinberger conjecture proved by Ashbaugh and Benguria [1]. The difference of the eigenvalues in the last bound is called the spectral gap. It can be regarded as a measure of the speed of convergence to equilibrium (see [3, 19]).

A ball provides bounds for general domains, we want to show that an equilateral triangle has this property among triangles. We already have the lower bound (1.1) as an analog of (1.3) and

$$\lambda_1 R^2|_{Triangle} \leq \lambda_1 R^2|_{Equilateral}, \tag{1.7}$$

proved in [18] (compare with (1.4)). The first result of this paper is the following

Theorem 1.1. *For arbitrary triangle*

$$(\lambda_2 - \lambda_1) R^2|_{Triangle} \leq (\lambda_2 - \lambda_1) R^2|_{Equilateral}. \tag{1.8}$$

If a triangle is acute we also have

$$\left.\frac{\lambda_2}{\lambda_1}\right|_{Triangle} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{Equilateral}. \tag{1.9}$$

We believe that the second result should hold for all triangles. In fact, it is likely that similar results hold for general polygons with regular polygon as extremal domain.

The second goal of this paper is to establish sharper lower bounds for the first eigenvalues of arbitrary triangles. In addition to (1.1) the following bounds are known

$$\lambda_1 \geq \frac{\pi^2}{4}\left(R^{-2} + d^{-2}\right), \tag{1.10}$$

$$\lambda_1 \geq \pi^2 \left(\frac{4}{d^2} + \frac{d^2}{4A^2}\right). \tag{1.11}$$

The first bound is due to Protter [16] and the second result was proved recently by Freitas [10]. We obtained new lower bounds which (together with (1.1)) provide the best results.

Theorem 1.2. *For an arbitrary triangle with the shortest altitude h,*

$$\lambda_1|_{Triangle} \geq \pi^2 \left(\frac{4}{d^2 + h^2} + \frac{d^2 + h^2}{4A^2}\right). \tag{1.12}$$

From now on $I(A, \gamma)$ will always denote an isosceles triangle with area A and angle γ between its sides of equal length. This angle will be called the vertex angle. The side opposite to that angle will be called the base and the other two the arms.

We also obtained a sharp bound based on circular sectors.

Theorem 1.3. *Let γ be the smallest angle of a triangle. Let also $S(A, \gamma)$ be a circular sector with area A and angle γ. Then*

$$\lambda_1|_{Triangle} \geq \lambda_1|_{I(A,\gamma)} \geq \lambda_1|_{S(A,\gamma)}. \tag{1.13}$$

The function

$$f(\gamma) = \lambda_1|_{I(A,\gamma)} \tag{1.14}$$

is decreasing for $\gamma \in (0, \pi/3)$ and increasing for $\gamma \in (\pi/3, \pi)$.

This result is a generalization of the Pólya's isoperimetric bound. The proof of (1.1) follows by first getting isosceles minimizer. The first bound can be used to get numerical bound in terms of zeros of Bessel functions.

The proofs of (1.11) and (1.10) are not based on symmetrization arguments. In contrast, the proofs of our lower bounds rely on certain symmetrization techniques similar to those used in the proof of Pólya's isoperimetric inequality. Using the same techniques one can also prove Freitas's bound (1.11).

The rest of the paper is organized as follows. Section 2 contains the definitions and explanations of various forms of symmetrization techniques which are used in Section 3 to prove the lower bounds. Section 4 contains the proofs of the upper bounds. Variational methods are used to obtain complicated polynomial bounds for the eigenvalues. These bounds are then simplified to the final results by proving certain polynomial inequalities. The algorithm for solving such polynomial inequalities is given in Section 5.

2. Symmetrization techniques

In this section we present various geometric transformations that decrease the first eigenvalue. Here we just remark that the important common property of those transformations is that they are contractions on $W^{1,2}(\mathbb{R}^d) \cap C_c(\mathbb{R}^d)$ and isometries on $L^2(\mathbb{R}^d)$. Those properties, together with the minimax formula for the first Dirichlet eigenvalue, give $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$. The most general reference for those results is [12].

We begin with the well known Steiner symmetrization (see for example [15, Note A] or [12, Chapter 2]). In short we center cross-sections perpendicular to a fixed line around this line to obtain a new domain with a line of symmetry. This transformation preserves the area of a domain, the perimeter decreases and the inradius increases. But the most interesting property from our point of view is that the first eigenvalue of the Dirichlet Laplacian on Ω is bigger than that on Ω^* (see [1, 12, 15] for applications).

The second type of symmetrization is the continuous Steiner symmetrization introduced by Pólya and Szegö in [15, Note B]. Different versions of it have been studied by Solynin [20] and Brock [4, 5]. The difference in case of convex domains is only in the way of defining "continuity parameters". The version studied in [4, 5] is more general and it works even for not connected domains. We refer the reader to these three papers and to [12] for the properties and strict definition of this transformation.

Continuous Steiner symmetrization is a partial Steiner symmetrization with cross-sections shifted by a fixed fraction of their total translation. Figure 1a shows the action of continuous Steiner symmetrization on triangles. We can see that $\Omega^0 = \Omega$ and $\Omega^1 = \Omega^*$. The first eigenvalue of the domain Ω^α is decreasing when α is increasing (see [4, 5, 12]) just like in the case of the classical Steiner symmetrization.

The last technique we want to introduce is called polarization. It was used in [9] to prove general inequalities for capacities and eigenvalues. It has also been useful in studying other types of symmetrization (see [6, 20]) and heat kernels for certain operators (see [8]). Let l be an arbitrary line, H_1 and H_2 two halfspaces with boundary l. For $x \in H_2$ let $\bar{x}$ denote the reflection of x with respect to l. Polarization of a domain Ω is defined pointwise by the following transformation

Definition 2.1. *The polarization of a set Ω with respect to l is a set Ω^P with the following properties.*

1. *If $x \in \Omega \cap H_1$ then $x \in \Omega^P$.*
2. *If $x \in \Omega \cap H_2$ then $\bar{x} \in \Omega^P$.*
3. *If both x and $\bar{x}$ are in Ω, then both x and $\bar{x}$ are in Ω^P,*

In essence, we reflect the set Ω with respect to l, the intersection of Ω with its reflection is in the polarized domain Ω^P and all other points of H_1 that are in either Ω or its reflection also belong to Ω^P. Figure 1b shows a triangle, its reflection with respect to l and the polarized domain (outlined polygon).

For the eigenvalue monotonicity and other properties of polarization we refer the reader to [6, 8] and to references in these papers.

3. THE PROOFS OF THE LOWER BOUNDS

Proof of Theorem 1.2. First, we use Steiner symmetrization with respect to the line perpendicular to the longest side. Next, we perform one more Steiner symmetrization but with respect to the longest side. This gives a rhombus with diagonals of length $|CD| = h$ and $|AB| = d$ (see Figure 2).

If we apply the Steiner symmetrization one more time but with respect to the altitude DE of the rhombus, we obtain the rectangle with base AC and height DE. Using Pythagorean theorem we find that $|AC| = \sqrt{d^2 + h^2}/2$. Since the area A of the triangle remains constant under symmetrization, we also have $|DE| = A/|AC|$. These, together with the explicit formula for eigenvalue of a rectangle, give the proof of Theorem 1.2. ☐

A longer version of this paper including numerical comparisons between all bounds is available at *http://arxiv.org/abs/0707.3631*. Here we just compare the above result to (1.11) and (1.1).

Both bounds (1.11) and Theorem 1.2 are of the form

$$f(x) = \pi^2 \left(\frac{4}{x} + \frac{x}{4A^2} \right). \tag{3.1}$$

We want to find values of x required for such a bound to be better than (1.1)

$$\frac{4\sqrt{3}}{3A} \geq \frac{4}{x} + \frac{x}{4A^2}. \tag{3.2}$$

Put $x = 4yA$. Then

$$\frac{4\sqrt{3}}{3} \geq \frac{1}{y} + \frac{y}{1}. \tag{3.3}$$

One can check that the equality holds if $y = \sqrt{3}$ or $y = \sqrt{3}/3$. Hence (1.1) is better than a bound of the type (3.1) if $x \in \left(4A\sqrt{3}/3, 4A\sqrt{3}\right)$. We also observe that right hand side is increasing for $x \geq 4A\sqrt{3}$.

In (1.11) we have $x = d^2$. Hence this bound is better than Pólya's bound if

$$d^2 \notin \left(4A\sqrt{3}/3, 4A\sqrt{3}\right). \tag{3.4}$$

Or equivalently

$$d \notin \left(2h\sqrt{3}/3, 2h\sqrt{3}\right). \tag{3.5}$$

Observe that d cannot be smaller than $2h\sqrt{3}/3$ and it is equal to this quantity only for an equilateral triangle. Hence Freitas's bound is better than Pólya's bound if

$$d > 2h\sqrt{3}. \tag{3.6}$$

In Theorem 1.2 we have $x = d^2 + h^2$ which is a bigger than in Freitas's bound. Therefore this bound is better for every triangle such that the above condition is true.

Let us note that Freitas's bound can also be proved using polarization even though the original proof was unrelated to any kind of symmetrization. The proof would be similar to the proof of the following lemma.

Lemma 3.1. *Let T be a triangle with the smallest angle γ. Let T' be a triangle with same area A and same smallest angle γ but with a smaller diameter. Then*

$$\lambda_1|_T \geq \lambda_1|_{T'}. \tag{3.7}$$

The first inequality in Theorem 1.3 follows from this lemma.

Proof. We have to apply a suitable sequence of polarizations to obtain the result. First, let T'_ε be a triangle similar to T' but with area $A + \varepsilon$. We will "symmetrize" T into a set contained in T'_ε. Then

$$\lambda_1|_T \geq \lambda_1|_{T'_\varepsilon}. \tag{3.8}$$

But when $\varepsilon \to 0$, the eigenvalue of T'_ε converges to the eigenvalue of T' and this ends the proof. Indeed, the triangles are similar hence we get the convergence due to scaling property of the eigenvalues.

Ideally, we would like to just define a sequence polarizations that folds T inside T'_ε. Unfortunately, it is virtually impossible to choose a correct line even for the first polarization. Each polarization must be with respect to a line cutting the longest and the shortest sides, but its exact position is not clear. The first line should be close to the vertex and the following lines should move away from the vertex. To give a precise position of each line we need to consider a temporary reversed sequence of transformations. The first line in this reversed sequence (or the last line for polarization) is easy to define. Having this line we can get another, and so on. The precise construction is split into 4 steps.

Step 1: Definition of elementary transformation.

We start with two triangles $T = ABC$ and $T'_\varepsilon = DBE$ such that E is on the interval BC and A is on the interval DB. We also assume that the angle with vertex B is the smallest in the triangle T and that the area of T'_ε equals the area of T plus ε. The triangles are shown on Figure 3a. Let l_1 be the bisector of the angle between the intervals DE and AC. If we reflect T with respect to this bisector we obtain another triangle $A'B'C'$ shown on Figure 3b. Let G be the intersection of $C'B'$ and AC.

We define the elementary transformation of T to be the polygon $ABEC'G$. This construction is valid due to two conditions. The first, the area of DBE is bigger than the area of ABC. The second, the angle $\angle ACB$ is smaller than the angle $\angle EDB$. Those two conditions guarantee that the bisector cuts the sides AC and BC and not the side AB and validates the construction of the elementary transformation.

The same transformation can be applied to more general domains. For example any domain contained in an infinite cone ACB with vertex at C. We will need this observation later on.

Notice also that the elementary transformation is equivalent to polarization with respect to the bisector for any domain containing ABC as long as: the intersection of this domain with the bisector is the same as the intersection of ABC with the bisector, the new part (triangle $C'GF$ in the case above) does not intersect the domain. In particular we could take a domain consisting of ABC and any subset of a half-plane disjoint with ABC and with boundary containing AB unless this domain intersects the bisector or $C'GF$. We also do not need $C'GF$ to be a triangle, it is enough that this set does not intersect the domain.

Step 2: Sequence of elementary transformations.

Now we want to describe a sequence of elementary transformations that changes $T = ABC$ into a subset of $T'_\varepsilon = DBE$.

The first transformation is described in *Step 1.* It is clear that the area of ADF is bigger than the area of $C'GF$. We have

$$\angle GC'F = \angle ACB < \angle ACB + \angle ABC = \angle DAF. \tag{3.9}$$

Therefore the second elementary transformation can be applied to triangles $C'GF$ and ADF (see Figure 4a). Here we completely disregard the presence of the quadrilateral $AFEB$.

This elementary transformation introduces a new triangular piece that may intersect the triangle ABC near vertex A (see Figure 4b). The intersection means that the elementary transformation is not equivalent to the polarization of $ABEC'G$ (we did not care about the quadrilateral $AFEB$ while making a transformation). This intersection will not occur if the triangle ADF is large enough to fit the triangle $C'GF$ inside. It would follow that the second elementary transformation is also equivalent to polarization. In such a case the construction of the sequence of transformations would be finished.

If the intersection occurs, we apply another elementary transformation to the triangle introduced during the previous transformation and to the difference between the triangles considered for the previous transformation (triangle AGH). Here we again disregard the presence of other part of the domain. The third elementary transformation is shown on Figure 4c. To validate this step we need to check that the angle $\angle AGH$ is bigger than the angle $\angle ACB$. Indeed, it is equal to the angle $\angle GC'F (= \angle ACB)$ plus the angle $\angle C'FG$.

If the newly introduced triangular piece does not intersect the already obtained domain (as on Figure 4c), the construction in finished. If it does, we apply another elementary transformation (or as many elementary transformations as needed) to fully "fold" the triangle ABC inside of the triangle DBE. The final subset of DBE that has the same area as ABC will have a spiral-like pattern of triangular pieces introduced by consecutive elementary transformations. First we need to show that this process is finite. Later we need to modify the procedure to get a sequence of valid polarizations. In particular we need to avoid self-intersecting that happens at every stage.

Step 3: Finiteness of the construction.

To estimate the area of the subset of ADF that is covered using three consecutive elementary transformations, we shift the triangular pieces obtained each time so that the angle equal to the angle ACB has a vertex at D, A and F respectively (after the second, third and fourth elementary transformation). The triangle $A'B'C'$ is shifted so that vertex C' moves to D, G moves to F' and the angle $\angle FDF'$ is equal to the angle $\angle ACB$. We do the same with other triangles. See Figure 5 for the picture of the shifted pieces. This decreases the covered area, but does not change the angles between any of the lines.

We have

$$\angle A'F'D' + \angle AF'A' = \angle F'DF + \angle F'FD. \tag{3.10}$$

This implies that the angle $\angle A'F'D'$ equals the angle $\angle AFD$. Similarly we show that other angles of the triangles AFD and $A'F'D'$ are also equal. This means that three elementary transformations reduce the uncovered part to a triangle similar to AFD. The size of this triangle is not bigger than the size of $A'D'F'$. Therefore the area of the uncovered part is shrinking at a constant rate. Due to the difference in the area of T'_ε and T the procedure must end as soon as the area of the uncovered part is smaller than ε.

Step 4: Sequence of polarizations.

The presence of intersections in the second and all consecutive elementary transformations stops us from using the sequence of elementary transformations as a sequence of polarizations. To fix this problem we consider the same sequence of transformations, but in the reverse order. Each elementary transformation was applied to a triangle that is a part of the reflection of the triangle ABC. Therefore we can treat those elementary transformations as a transformations on ABC.

As we remarked in *Step 1* such transformations are equivalent to polarizations. We use the reflection lines used by the elementary transformations to produce polarizations. Figures 6a and 6b show the domain after the first two polarizations or the last two elementary transformations (outlined sets). The construction from *Step 2* ensures that the triangular pieces introduced at every step have no intersection with the reflection lines. We also avoid self-intersecting since the triangular pieces are introduced in the reverse order. This means that we have a sequence of valid polarizations.

The last polarization (the first elementary transformation) fits the whole domain inside $T'_\varepsilon = DBE$ (see Figure 6c). If the sequence is longer than on the example, we proceed in the same manner, building the spiral-like structure starting from the smallest inner triangle.

This proves that for arbitrary $\varepsilon > 0$ there exists a finite sequence of polarizations, which transforms T into a subset of T'_ε.

This sequence of polarizations can be constructed even for domains consisting of triangle ABC and any set contained in the half-plane with boundary AB and disjoint with ABC.

□

Remark 3.2. *The same proof as for Lemma 3.1 works for any domain contained in an infinite cone ACB and containing T.*

Lemma 3.3. *Let Ω be a kite, that is a quadrilateral symmetric with respect to the longer diagonal and with perpendicular diagonals. Assume also that the length of the longer diagonal is smaller than the length of one of the sides. Given the fixed area and the smallest angle, the first eigenvalue is decreasing with the diameter. In particular, an isosceles triangle has a maximal first eigenvalue among kites with fixed area and the smallest angle, the kite consisting of two isosceles triangles has minimal first eigenvalue.*

Proof. We can split an isosceles triangle ABC into two right triangles and use Lemma 3.1 and the remark above to symmetrize any one of those right triangles (for example DBC on Figure 7a). Furthermore, we can perform all but the last polarization on both right triangles (Figure 7b shows the domain at this stage of construction). Finally we can perform the last polarizations to get the quadrilateral with vertex E. The last transformations are indeed polarizations since the bisectors are cutting the line BD outside of interval DE.

□

The proof of the second inequality in Theorem 1.3 requires a repeated use of the above results. As before we take a slightly larger sector $S(A+\varepsilon,\gamma)$ instead of $S(A,\gamma)$. We need to transform the isosceles triangle $I(A,\gamma)$ into a subset of $S(A+\varepsilon,\gamma)$. Using Lemma 3.3 we can symmetrize the triangle ABC into a quadrilateral (see Figure 7c) with longer diagonal equal to the radius of the circular sector $S(A+\varepsilon,\gamma)$.

Now we divide both halves of the quadrilateral into N (to be chosen later) parts by dividing the angle with vertex B into equal parts (see Figure 7d).

We work on each half separately. Consider a triangle formed by all but the inner most part. We use Remark 3.2 to symmetrize this triangle into a "more isosceles" triangle with one vertex on an intersection of a circular part and the innermost line. This makes the part outside of the circular sector smaller. Now we repeat this using decreased number of parts. The picture after two steps is shown on Figure 8. By choosing a large enough N and performing all steps we can fit the whole half of the quadrilateral inside of the circular sector.

This ends the proof of the second inequality in the Theorem 1.3. The last thing to prove is the monotonicity property for the isosceles triangles. Here we need to apply the continuous symmetrization to obtain the result. Suppose that we start with the isosceles triangle $I(A,\gamma)$ with the vertex angle smaller than $\pi/3$. We want to show that if we increase the angle to $\pi/3 \geq \gamma' > \gamma$ while the area remains fixed, then the first eigenvalue decreases. First, we apply the continuous Steiner symmetrization with respect to the line perpendicular to one of the arms. The length of the base increases, and its maximal length is obtained when we reach the full Steiner symmetrization. Compare with Figure 1a where the shortest side of Ω^α increases with α.

If this maximum is smaller than the base of $I(A,\gamma')$, we use the Steiner symmetrization to get an isosceles triangle with the angle γ'' between γ and γ' and the base equal to this maximum. We can repeat the above procedure on the triangle $I(A,\gamma'')$.

On the other hand, if the maximum is bigger than the base of $I(A,\gamma')$, then we can stop the continuous Steiner symmetrization at the time when the length of the enlarged base of $I(A,\gamma)$ is equal to the base of $I(A,\gamma')$ (on Figure 1a we choose α such that the shortest side is equal to the base of $I(A,\gamma')$. Now, the Steiner symmetrization with respect to the base gives an isosceles triangle with the base equal to the base of $I(A,\gamma')$. This implies that it must be equal to $I(A,\gamma')$.

Suppose that the vertex angle $\gamma > \pi/3$. The same procedure as above shortens the base in this case. The same argument applies, but with maximum replaced with minimum. This ends the proof of the Theorem 1.3.

4. The proofs of the upper bounds

In this section we prove Theorem 1.1. We start with the minimax formula for the second eigenvalue.

$$\lambda_2|_D = \inf_{f_1,f_2} \sup_{\alpha\in\mathbb{R}} \frac{\int_D |\nabla(f_1+\alpha f_2)|^2}{\int_D |f_1+\alpha f_2|^2}, \tag{4.1}$$

where $f_1, f_2 \in H_0^1(D)$ are linearly independent. This formula is a special case of the general minimax formula for an arbitrary eigenvalue (see e.g. [7]). As in [18], we will use known eigenfunctions for equilateral or right triangles to obtain test functions for arbitrary triangles.

Consider the equilateral triangle T_e with vertices $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$. The complete set of eigenfunctions is well known. For the exact formulas for these eigenfunctions we refer the reader to [13, 14]. In particular, [13] gives simple formulas that we will use in this paper. Let

$$z = \frac{\pi}{3}(2x-1), \tag{4.2}$$

$$t = \pi\left(1 - \frac{2}{\sqrt{3}}y\right). \tag{4.3}$$

The first eigenfunction is given by

$$\varphi_1(x,y) = (\cos(3z) - \cos(t))\sin(t). \tag{4.4}$$

The second eigenvalue has multiplicity two. We will follow the notation from [13] to name these eigenfunctions. The two eigenfunctions (symmetric and antisymmetric) belonging to the second eigenvalue are

$$\varphi_{S21}(x) = \cos(4z)\sin(2t) + \cos(5z)\sin(t) - \cos(z)\sin(3t), \tag{4.5}$$

$$\varphi_{A21}(x) = \sin(4z)\sin(2t) + \sin(5z)\sin(t) - \sin(z)\sin(3t). \tag{4.6}$$

Let T be an arbitrary triangle. We can assume that one side of this triangle is equal to the segment from $(0,0)$ to $(1,0)$, and that the last vertex (u,v) is in the upper halfspace. Then, there exists a unique linear transformation L from T onto T_e. We will compose L with the eigenfunctions of T_e to obtain suitable test functions for T.

Using formula (4.1) we obtain an upper bound

$$\lambda_2|_T \le \sup_{\alpha\in\mathbb{R}} \frac{\int_T |\nabla(g_\alpha \circ L)|^2}{\int_T |g_\alpha \circ L|^2}, \tag{4.7}$$

where g_α is a linear combination of two known eigenfunctions.

As the first two test functions we can take

$$g_\alpha^1(x,y) = \varphi_{S21}(x,y) + \alpha\varphi_1(x,y), \tag{4.8}$$

$$g_\alpha^2(x,y) = \varphi_{A21}(x,y) + \alpha\varphi_1(x,y). \tag{4.9}$$

If the triangle T is almost equilateral, we can expect that its eigenvalues and eigenfunctions are similar to the eigenvalues and eigenfunctions of T_e. Also, the linear transformation L should not perturb the bound (4.7) significantly. Therefore, we can expect that for almost equilateral triangles this should be a good upper bound for $\lambda_2|_T$.

Notice that the linear combinations in (4.8) and (4.9) consist of two orthogonal functions. Hence the second norm of this combination is just the sum of the second norms. Therefore

$$\frac{\int_T |\nabla(g_\alpha \circ L)|^2}{\int_T |g_\alpha \circ L|^2} = \frac{a\alpha^2 + b\alpha + c}{e\alpha^2 + f}, \tag{4.10}$$

where a,c,e,f are strictly positive and all coefficients depend on u and v. This rational function has a limit a/e, as $\alpha \to \pm\infty$. By taking its derivative we can also find the critical points

$$\alpha_\pm = \frac{1}{bd}\left(af - ce \pm \frac{1}{2}\sqrt{4b^2ef + (2af - 2ce)^2}\right). \tag{4.11}$$

If we evaluate the function at those points and simplify, we get

$$C_\pm = \frac{1}{2ef}\left(af + ce \pm \frac{1}{2}\sqrt{4b^2ef + (2af - 2ce)^2}\right). \tag{4.12}$$

The expression under the root is clearly nonnegative. It is zero if $b = 0$ and $af = ce$. In such case $C_- = C_+ = a/e$, hence the maximum is a/e. If the expression under the root is positive, then we have two distinct critical points, and $C_+ > C_-$. This means that this rational function has an absolute maximum C_+ and absolute minimum C_-.

This leads to a new bound for the eigenvalue

$$\lambda_2|_T \le C_+ = \frac{1}{2ef}\left(af + ce + \frac{1}{2}\sqrt{4b^2ef + (2af - 2ce)^2}\right). \tag{4.13}$$

To finish the proof of Theorem 1.1 we need to use one of the lower bounds for the first eigenvalue. Due to the simple form of the bound, we will use Freitas's result (1.11). To prove the first bound in Theorem 1.1 we need to show that

$$\left(C_+ - \pi^2\left(\frac{4}{d^2} + \frac{d^2}{4A^2}\right)\right)R^2 \le \frac{16\pi^2}{27}, \tag{4.14}$$

The number on the right side of the inequality is the exact value obtained for the equilateral triangle.

The second bound will be proved if we can show that

$$\frac{\sqrt{3}AC_+}{4\pi^2} \le \frac{7}{3}. \tag{4.15}$$

Notice that this time we use Pólya's isoperimetric bound (1.1).

Let us begin with the proof of (4.14). We first use g_α^1 as a test function. The expressions (4.13) and (1.11) can be written in terms of the vertices of the triangle T. But we assumed that the vertices are $(0,0)$, $(1,0)$ and (u,v). We denote the lengths of the sides of the triangle by 1, $M = \sqrt{u^2+v^2}$, $N = \sqrt{(1-u)^2+v^2}$ and we can assume that $N \ge M \ge 1$. Then the bound (4.14) (in terms of the lengths of the sides) is equivalent to

$$\begin{aligned} 0 \ge -1612800N^2(1+M+N)^2\pi^2 + 27\Big\{&-413343N^2V \\ &+11200(9(M^2-1)^2 + 2(M^2+1)N^2 + 20N^4)\pi^2 \\ &+N^2\sqrt{655128046899V^2 - 74071065600VW\pi^2 + 8028160000W^2\pi^4}\Big\}, \end{aligned} \tag{4.16}$$

where $V = M^2 + N^2 - 2$ and $W = M^2 + N^2 + 1$.

The expression is quite complicated since the integrals of the function g^1_α are very cumbersome to calculate. This task could be accomplished by hand since this function is just a sum of the products of the trigonometric functions. However, symbolic calculations in Mathematica may be used to quickly simplify the expression. A longer version of this paper including Mathematica scripts is available at *http://arxiv.org/abs/0707.3631* .

One can check numerically where this inequality is true and find out that one test function will not be enough. If we put $U = N - M$ then the set of all possible triangles can be characterized by $U \in [0, 1)$ and $M \geq 1$. We divide all triangles into subregions of $U \in [0, 1)$ and $M \geq 1$.

We now prove inequality (4.16) on $U \in [0, 0.03]$ and $M \in [1.03, 1.39]$.

The inequality (4.16) can be written as

$$P(N, M) + Q(N, M)\sqrt{R(N, M)} \leq 0, \tag{4.17}$$

where P, Q and R are polynomials in N and M. It will be proved, if we can show that

$$P(N, M) \leq 0, \tag{4.18}$$

$$Q^2(N, M)R(N, M) - P^2(M, N) \leq 0. \tag{4.19}$$

This is a system of polynomial inequalities. Unfortunately the degrees of those polynomials are 4 and 8 in each variable. Therefore there is almost no hope to solve this system using any conventional method.

Instead, we developed an algorithm for proving such polynomial inequalities on rectangles. The next section contains a detailed description of this algorithm and the proof of its correctness. It turned out that in our case this method gives the proof of the inequality for any test function we need.

To finish the proof of the first part of Theorem 1.1 we need to define the other test functions and rectangles for each corresponding inequality.

We already have two test functions (4.8) and (4.9) that came from the equilateral triangle. We can also use the first two eigenfunctions of the half of the equilateral triangle. That is, of the right triangle with angle $\pi/6$. Its eigenfunctions are certain antisymmetric eigenfunctions of the equilateral triangle. Hence, we get the third test function

$$g^3_\alpha(x, y) = \varphi_{A31}(x, y) + \alpha\varphi_{A21}(x, y), \tag{4.20}$$

where (using notation (4.2) and (4.3))

$$\varphi_{A31}(x, y) = \sin 5z \sin 3t - \sin 2z \sin 4t - \sin 7z \sin t. \tag{4.21}$$

In this case we also need to get a new linear transformation L' that transforms the given triangle into the right triangle.

Eigenfunctions are also known for right isosceles triangle. Its eigenfunctions are equal to eigenfunctions of the square with diagonal nodal lines. We get

$$g^4_\alpha(x, y) = \phi_2(x, y) + \alpha\phi_1(x, y), \tag{4.22}$$

where

$$\phi_1(x, y) = \sin 2\pi x \sin \pi y + \sin \pi x \sin 2\pi y, \tag{4.23}$$

$$\phi_2(x, y) = \sin 3\pi x \sin \pi y - \sin \pi x \sin 3\pi y. \tag{4.24}$$

We can also mix the eigenfunctions from the different triangles provided we use appropriate linear transformation for each of them. In this manner we obtain the last (fifth) case needed to prove the theorem by taking $g^1_\alpha(x,y)$ and $\frac{1}{2}\phi_2(x,y)$.

Each of the five cases requires a rectangle on which we can prove the bound. We take

(1) $U \in [0,0.03]$, $M \in [1.03,1.39]$,
(2) $U \in [0,0.2]$, $M \in [1,1.03]$,
(3) $U \in [0,1)$, $M \in [1.39,\infty)$,
(4) $U \in [0.2,1)$, $M \in [1,1.39]$,
(5) $U \in [0.03,0.2]$, $M \in [1.03,1.39]$.

Notice that these rectangles exactly cover the infinite strip $[0,1)\times[1,\infty)$. Since the sides of the triangle are $N \geq M \geq 1$ and $U = N - M$, the strip includes all possible combinations of lengths of the sides.

The proof in *cases (3)-(5)* is exactly the same as in *case (1)*. Additional step has to be performed in *case (2)*. Consider a triangle T' similar to T with sides 1, $N' = M/N$, $M' = 1/N$. We have $1 \geq N' \geq M'$. Consider inequality (4.14) for T'. Note that the diameter of T' used in (1.11) is now 1 (N in other cases). Just like before we get an expression similar to (4.16) but involving N' and M'. After a change of variable $M' \to N' - M' + 1$ and $N' \to 1 - U'$ we can apply our algorithm with a rectangle given in *case (2)*. This proves the bound (4.14) for T', and hence for T since the bound is invariant under scaling.

We need to check that the transformation

$$\frac{M}{M+U} = 1 - U', \tag{4.25}$$

$$\frac{1}{M+U} = 2 - U' - M', \tag{4.26}$$

changes the rectangle $U \in [0,0.2)$, $M \in [1,1.03)$ into a subset of the same rectangle in primed variables. From the first equation we get

$$0 \leq U' = 1 - \frac{M}{M+U} \leq 1 - \frac{1}{1.23} < 0.19. \tag{4.27}$$

From the second equation

$$1 \leq M' = \frac{M-1}{M+U} + 1 \leq \frac{M-1}{M} + 1 = 2 - \frac{1}{M} \leq 2 - \frac{1}{1.03} < 1.0292. \tag{4.28}$$

This ends the proof of the first part of Theorem 1.1. The second part requires an additional argument. We still need to use test functions $g^1_\alpha(x,y)$ through $g^5_\alpha(x,y)$ on the following rectangles

(1) $U \in [0,0.09]$, $M \in [1,1.37]$;
(2) not needed;
(3) $U \in [0,0.42]$, $M \in [1.37,2.05]$;
(4) $U \in [0.09,0.2]$, $M \in [1,1.37]$;
(5) $U \in [0.2,0.42]$, $M \in [1,1.37]$.

Notice that in each case $U \leq 0.42$. Since the triangles are acute we have $N^2 \leq M^2 + 1$, hence $U \leq \sqrt{2} - 1$. Also, all these rectangles cover only the cases with $M \leq 2.05$.

Therefore we need a circular sector type–bound similar to the one in [18]. We use the first part of Theorem 1.3 to obtain isosceles triangle $I(A,\gamma)$ and smallest sector containing this isosceles triangle for the upper bound for the first eigenvalue.

The upper bound for the second eigenvalue is based on the biggest circular sector contained in the given triangle. Suppose that we have an acute triangle with area A and smallest angle γ. Let $N \geq M \geq 2.05$ be the two longest sides. Since the triangle is acute, the largest sector contained in the triangle has the radius equal to the longest altitude H. We get

$$\frac{\lambda_2}{\lambda_1}\bigg|_{Triangle} \leq \frac{\lambda_2|_{S(\gamma H^2/2,\gamma)}}{\lambda_1|_{S(\gamma MN/2,\gamma)}}. \tag{4.29}$$

The eigenvalues of the circular sectors are given in terms of the zeros $j_{v,n}$ of the Bessel function of index v. Here n indicates n-th smallest zero. We have

$$\lambda_1|_{S(\gamma R^2/2,\gamma)} = R^{-2} j^2_{\pi/\gamma,1}, \tag{4.30}$$

$$\lambda_2|_{S(\gamma R^2/2,\gamma)} \leq R^{-2} j^2_{\pi/\gamma,2}. \tag{4.31}$$

Notice that we only have the inequality for the second eigenvalue due to the presence of the antisymmetric eigenfunction of the sector.

We can use the bounds for the zeros of the Bessel functions proved in [17]. That is,

$$v - \frac{a_k}{\sqrt[3]{2}}\sqrt[3]{v} < j_{v,k} < v - \frac{a_k}{\sqrt[3]{2}}\sqrt[3]{v} + \frac{3}{20}a_k^2\frac{\sqrt[3]{2}}{\sqrt[3]{v}}, \tag{4.32}$$

where a_k are zeros of the Airy function with $a_1 \approx -2.3381$ and $a_2 \approx -4.0879$.

Using the last two facts we obtain

$$\begin{aligned} \frac{\lambda_2}{\lambda_1}\bigg|_{Triangle} &\leq \frac{NM}{H^2}\left(\frac{x^3 - \frac{a_2}{\sqrt[3]{2}}x + \frac{3}{20}a_2^2\frac{\sqrt[3]{2}}{x}}{x^3 - \frac{a_1}{\sqrt[3]{2}}x}\right)^2 \\ &= \frac{NM}{H^2}\left(\frac{1 - \frac{a_2}{\sqrt[3]{2}}x^{-2} + \frac{3}{20}a_2^2\frac{\sqrt[3]{2}}{x^4}}{1 - \frac{a_1}{\sqrt[3]{2}}x^{-2}}\right)^2, \end{aligned} \tag{4.33}$$

where $x = (\pi/\gamma)^{1/3}$. Note that for acute triangles we have $N^2 \leq M^2 + 1$. We need to consider two cases based on the length of H. First assume that $H^2 \geq M^2 - 1/16$. This is equivalent to the condition that the altitude H divides the shortest side (with the length 1) into two parts with one of them not longer than 1/4. Using the inequality $N^2 \leq M^2 + 1$, we obtain

$$\frac{NM}{H^2} \leq \frac{M\sqrt{M^2+1}}{M^2 - \frac{1}{16}} = \sqrt{1 + M^{-2}}\left(1 - \frac{1}{16M^2}\right)^{-1} =: z_1(M). \tag{4.34}$$

If $M^2 - 1/4 \leq H^2 \leq M^2 - 1/16$ then the altitude H divides the shortest side into two parts, one of length δ satisfying $1/4 < \delta < 1/2$ and the other of length $1 - \delta$. Then

$$N^2 = H^2 + (1-\delta)^2 = M^2 - \delta^2 + (1-\delta)^2 = M^2 + 1 - 2\delta \leq M^2 + \frac{1}{2}. \tag{4.35}$$

This gives

$$\frac{NM}{H^2} \leq \frac{M\sqrt{M^2 + \frac{1}{2}}}{M^2 - \frac{1}{4}} = \sqrt{1 + \frac{1}{2M^2}}\left(1 - \frac{1}{4M^2}\right)^{-1} =: z_2(M). \tag{4.36}$$

Let $z(M) = \max\{z_1(M), z_2(M)\}$. Note that $z(M)$ is decreasing since both $z_1(M)$ and $z_2(M)$ are decreasing. Making the substitution $y = x^{-2}$ we get

$$\begin{aligned} \frac{\lambda_2}{\lambda_1}\bigg|_{Triangle} &\le z(M)\left(\frac{1 - \frac{a_2}{\sqrt[3]{2}}y + \frac{3\sqrt[3]{2}}{20}a_2^2y^2}{1 - \frac{a_1}{\sqrt[3]{2}}y}\right)^2 \\ &= z(M)\left(c_1 + c_2y + \frac{c_3}{1+c_4y}\right)^2, \end{aligned} \tag{4.37}$$

where the constants c_i satisfy

$$c_1 = \frac{(10a_1 - 3a_2)a_2}{10a_1^2} \approx 0.83133 > 0, \tag{4.38}$$

$$c_2 = -\frac{3a_2^2}{10\sqrt[3]{2}a_1} \approx 1.70183 > 0, \tag{4.39}$$

$$c_3 = \frac{10a_1^2 - 10a_1a_2 + 3a_2^2}{10a_1^2} \approx 0.16867 > 0, \tag{4.40}$$

$$c_4 = -\frac{a_1}{\sqrt[3]{2}} \approx 1.85575 > 0. \tag{4.41}$$

We want to show that the right hand side of (4.37) is decreasing with M (note that y also depends on M since it depends on γ). First we can show that this expression is increasing with y for a fixed M. Indeed, the derivative with respect to y is $c_2 - \frac{c_3}{(1+c_4y)^2}$ and it is positive for $y > 0$ due to the condition $c_2 > c_3$. This means that the expression is also increasing in γ since $y = (\pi/\gamma)^{-2/3}$. If we fix M, then from all triangles with sides 1, $M \ge 1$, $N \ge M$, the isosceles triangle ($M = N$) has the biggest angle γ. In this case $\cos\gamma = 1 - \frac{1}{2M^2}$. As a result we get an upper bound for the ratio of the first two eigenvalues in terms of M. That is, we have

$$\frac{\lambda_2}{\lambda_1}\bigg|_{Triangle} \le z(M)\left(c_1 + c_2y_M + \frac{c_3}{1+c_4y_M}\right)^2, \tag{4.42}$$

with $y_M = (\arccos(1 - (2M^2)^{-1})/\pi)^{2/3}$. But y_M and $z(M)$ are decreasing with M, hence the right hand side is decreasing with M. To finish the proof we just need to check that this is smaller than $7/3$, as required in (4.15), for $M = 2.05$ (we get $\approx 2.3285 < 7/3$). This ends the proof of Theorem 1.1.

5. An algorithm for polynomial inequalities

This section gives the algorithm for proving polynomial inequalities in two variables with arbitrary degrees. The domain we deal with is a rectangle. The proof of the correctness is also given.

Suppose that we have an inequality

$$P(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}x^iy^j \le 0, \tag{5.1}$$

for $x \in (0,a)$ and $y \in (0,b)$. Any other rectangle can be shifted to the origin, hence reducing the problem to this case.

The idea behind the algorithm is to use the following inequalities

$$c_{i,j}x^iy^j \le c_{i,j}\min\{ax^{i-1}y^j, bx^iy^{i-1}\}, \text{ if } c_{i,j} > 0, \tag{5.2}$$

$$c_{i,j}x^iy^j \le c_{i,j}\max\{a^{-1}x^{i+1}y^j, b^{-1}x^iy^{i+1}\}, \text{ if } c_{i,j} < 0. \tag{5.3}$$

We can use this simple observation to reduce the number of positive coefficients in $P(x,y)$. Clearly, if we apply any of the above inequalities finite number of times on any of the monomials in P(x,y), we obtain an upper bound for $P(x,y)$. If we can reduce the whole polynomial to 0, we proved inequality (5.1).

We need to describe a sequence of reductions, that leads to 0 or to a polynomial with only positive coefficients. If the second case happens, the algorithm fails, but rectangles can be divided into smaller rectangles and algorithm can be reapplied on each piece. It is worth noting that for 8 out of the 9 polynomials we have in the previous section this method works on the whole rectangle (without sub–dividing). Only in the case of the third test function in the second part of Theorem 1.1 we need to split the given rectangle into halves.

The only way to reduce a positive coefficient is by lowering the power of one of the variables (using (5.2)). Similarly, to reduce a negative coefficient we have to increase one of the powers (using (5.3)). Each time, two of the coefficients combine giving a new, possibly negative coefficient. Write

$$P(x,y) = \sum_{i=0}^{n} x^iQ_i(y), \tag{5.4}$$

where

$$Q_i(y) = \sum_{j=0}^{m} c_{i,j}y^j. \tag{5.5}$$

To avoid ambiguity, we start with Q_n. Any negative coefficient in Q_n can be used only to combine with some positive coefficient with higher power of y. It is impossible to use them to interact with Q_i for $i < n$. Therefore we inductively (starting from $j = 0$) check if $c_{n,j} < 0$ and in case this is true we get an upper bound

$$c_{n,j}x^ny^j + c_{n,j+1}x^ny^{j+1} \le (c_{n,j}b^{-1} + c_{n,j+1})x^ny^{j+1}, \tag{5.6}$$

and we redefine $c_{n,j+1} = c_{n,j}b^{-1} + c_{n,j+1}$. If the last coefficient turns out to be negative, we can just change it to 0.

As a result we changed $Q_n(y)$ into a polynomial $Q'_n(y)$ with nonnegative coefficients (possibly all equal to 0). A positive coefficient can only be altered by lowering one of the powers. We could lower the power of y but, ultimately, if we want to obtain a constant as a final bound we have to also lower the power of x in all coefficients of $x^nQ'_n(y)$. Hence we get an upper bound

$$x^nQ'_n(y) + x^{n-1}Q_{n-1}(y) \le (aQ'_n(y) + Q_{n-1}(y))x^{n-1}, \tag{5.7}$$

and we redefine $Q_{n-1}(y) = aQ'_n(y) + Q_{n-1}(y)$. This approach guarantees that any negative coefficient of $Q_{n-1}(y)$ can be used to reduce as many positive coefficients from $Q_n(y)$ as possible.

Now we repeat the whole procedure for $Q_{n-1}(y)$ and for all others by induction. At the end we get $Q'_0(y)$ which has only nonnegative coefficients. If any of those coefficients is strictly positive, the algorithm failed. But $Q_0(y) = 0$ means that $P(x,y) \le 0$ on the rectangle $(0,a) \times (0,b)$.

The implementation of this algorithm is a part of the Mathematica script available at *http://arxiv.org/abs/0707.3631* .

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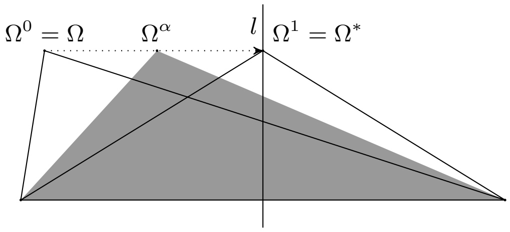
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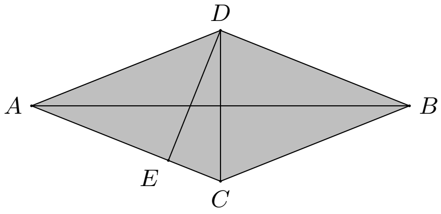
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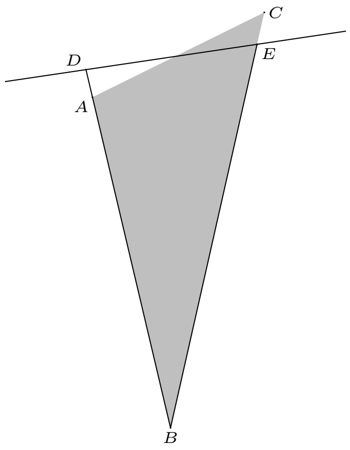
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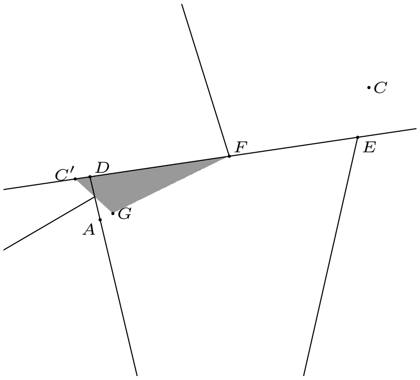
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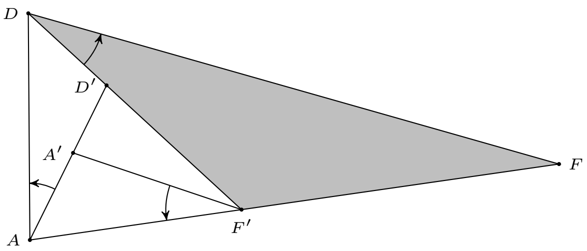
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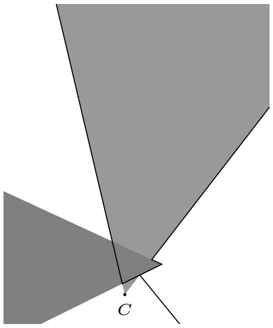
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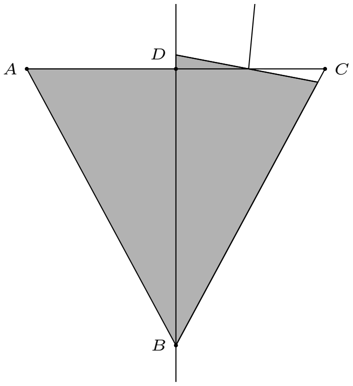
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