On Radial Two-Species Onsager Vortices Near the Critical Temperature

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ABSTRACT. We compare two mean field equations describing hydrodynamic turbulence in equilibrium, which are derived under a deterministic vs. stochastic assumption on the variable vortex intensity distribution. Mathematically, such equations correspond to non-local Liouville type problems, and the critical temperature corresponds to the optimal Moser-Trudinger constant. We consider the radial case and we assume that the inverse temperature is near its critical value. Under these assumptions we show that, unlike previously existing results, the qualitative properties of the solution set in the deterministic case is more similar to the single vortex intensity case than the stochastic case. Some new variational interpretations of the value-explicit values of the critical temperature are also provided.

1. INTRODUCTION

In recent years, in the context of the statistical mechanics of two-dimensional point vortices, as introduced in the pioneering work of Onsager [26], several non-local elliptic equations, including an exponential type nonlinearity, have been derived in order to take into account variable vortex intensities. (See, e.g., [7, 39] and the references therein.)

In particular, assuming that the distribution of the (normalized) vortex intensity α is determined by a probability distribution $\mathcal{P}(d\alpha)$, $\alpha \in [-1, 1]$, the following multi-species Boltzmann-Poisson type problem was derived in [35] (see

also [11, 37]):

$$(1.1) \quad \begin{cases} -\Delta v = \lambda \int_{[-1,1]} \alpha \frac{e^{\alpha v}}{\int_{\Omega} e^{\alpha v} dx} \mathcal{P}(d\alpha) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, v denotes the stream function of the turbulent Euler flow, and $\lambda > 0$ is a physical constant related to the inverse temperature. It should be mentioned that problems of the form (1.1) also appear in the description of equilibrium states for Brownian gases and chemotaxis systems with different types of particle species, subject to the conserved mass constraint for each species (see [8, 9, 14, 18] and the references therein), as well as in affine geometry (see [16]). Here, we are particularly interested in the “two-species, positive case,” where $\mathcal{P}$ is given by

$$(1.2) \quad \mathcal{P}(d\alpha) = \tau \delta_1(d\alpha) + (1 - \tau) \delta_{\gamma}(d\alpha),$$

for some $\tau, \gamma \in (0, 1)$. Such an assumption corresponds to the physically relevant case where the point vortices have intensity 1 or γ , in proportion τ and $1 - \tau$, respectively, and are all equally oriented. For such a choice of $\mathcal{P}$, problem (1.1) reduces to the problem

$$(1.3) \quad \begin{cases} -\Delta v = \lambda \left(\tau \frac{e^v}{\int_{\Omega} e^v dx} + (1 - \tau) \gamma \frac{e^{\gamma v}}{\int_{\Omega} e^{\gamma v} dx} \right) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, the following problem, formally similar to (1.1), was derived in [24] in order to describe stationary turbulent flows under a “stochastic” assumption on the vorticity distribution:

$$(1.4) \quad \begin{cases} -\Delta v = \lambda \frac{\int_{[-1,1]} \alpha e^{\alpha v} \mathcal{P}(d\alpha)}{\iint_{\Omega \times [-1,1]} e^{\alpha v} dx \mathcal{P}(d\alpha)} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, it is assumed that the vortex intensities $\alpha \in [-1, 1]$ of each point vortex are independent identically distributed random variables with probability distribution $\mathcal{P}(d\alpha)$. In the “two-species case” where $\mathcal{P}$ is given by (1.2), problem (1.4) takes the form

$$(1.5) \quad \begin{cases} -\Delta v = \lambda \frac{\tau e^v + (1 - \tau) \gamma e^{\gamma v}}{\int_{\Omega} \{\tau e^v + (1 - \tau) e^{\gamma v}\} dx} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that we can also view Problem (1.4) as the stationary state of a multi-species Boltzmann-Poisson evolution system, where the masses of the individual population species are allowed to vary, provided the total population mass is conserved (see [18]).

In the special case $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$, equation (1.1) and equation (1.4) both reduce to the well-understood single-species Boltzmann-Poisson system, which is also known in the mathematical literature as the “standard” mean field equation

$$(1.6) \quad \begin{cases} -\Delta v = \lambda \frac{e^v}{\int_{\Omega} e^v dx} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

which has been extensively analyzed in view of its applications in biology, differential geometry, and physics. In the context of turbulence, problem (1.6) was rigorously derived and analyzed in [6]. (See [17, 20, 21, 39] and the references therein for results concerning problem (1.6).)

In recent years, a considerable effort has been devoted to compare qualitative properties of the solutions to problem (1.3) and problem (1.5) (see [16, 22, 25, 30–32, 34]). In [22, 33, 34] it is shown that problem (1.5) may be viewed as a perturbation of (1.6), and in particular it shares analogous mass quantization properties and an optimal Moser-Trudinger constant (which physically corresponds to the critical temperature) as (1.6), independently of $\mathcal{P}$ (see also [39]). On the other hand, in [13, 16, 25, 30] it is shown that the blow-up masses and the optimal Moser-Trudinger constant for (1.3) significantly depend on $\mathcal{P}$, and that (1.3) is not in general a perturbation of (1.6), with the exception of the case where the probability measure $\mathcal{P}(d\alpha)$ is sufficiently “close” to the Dirac mass $\delta_1(d\alpha)$ (see [32]). Sign-changing cases were considered in [15, 17, 27, 28, 31].

Our aim in this note is to exhibit a first situation where the deterministic problem (1.3) behaves more similarly to the “standard” problem (1.6) than the stochastic problem (1.5). More precisely, we shall show that when Ω is a disc and $\mathcal{P}$ is of the form (1.2), the critical temperature for (1.3) determines a threshold for the existence of solutions, exactly as it happens for (1.6), independently of τ, γ . On the other hand, the bifurcation diagram of the solution set near the critical

temperature for (1.5) significantly depends on τ, γ , and does not necessarily determine a threshold for the existence of solutions. We expect similar properties to hold true for probability measures $\mathcal{P}$ more general than (1.2). However, the corresponding computations become significantly more complicated in the three mass case, as may be seen in [40], in a different context, so that a different approach appears to be necessary.

This note is organized as follows. In the following section, we state our main results, namely Theorem 2.3 and Theorem 2.4 below. Several preliminary properties necessary to the proofs are derived in Section 3. The proof of Theorem 2.3 is provided in Section 4, and the proof of Theorem 2.4 is provided in Section 5. Finally, some remarks on the explicit value of the critical temperature for the deterministic problem (1.3), namely the constant $\bar{\lambda}_{\tau, \gamma}$ defined in (2.4) below, are provided in Appendix A.

2. STATEMENT OF THE MAIN RESULTS

We first recall the variational formulations of problem (1.1) and problem (1.4), given by the functionals

$$(2.1) \quad J_{\lambda, \mathcal{P}}^{(d)}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \lambda \int_{[-1,1]} \left[\ln \int_{\Omega} e^{\alpha v} dx \right] \mathcal{P}(d\alpha), \quad v \in H_0^1(\Omega),$$

$$(2.2) \quad J_{\lambda, \mathcal{P}}^{(s)}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \lambda \ln \left[\iint_{\Omega \times [-1,1]} e^{\alpha v} dx \mathcal{P}(d\alpha) \right], \quad v \in H_0^1(\Omega),$$

respectively. For $\mathcal{P}(d\alpha) = \delta_1(d\alpha)$, (2.1) and (2.2) reduce to the variational functional for (1.6), given by

$$I_{\lambda}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \lambda \ln \int_{\Omega} e^v dx, \quad v \in H_0^1(\Omega).$$

In view of the Moser-Trudinger inequality [23, 41]

$$\int_{\Omega} e^{4\pi v^2 / \int_{\Omega} |\nabla v|^2 dx} \leq C_{MT} |\Omega| \quad \text{for all } v \in H_0^1(\Omega) \setminus \{0\},$$

where the constant $C_{MT} > 0$ does not depend on the domain Ω and where the constant 4π is best possible, using the inequality $ab \leq (a^2 + b^2)/2$ we derive

$$v \leq \frac{4\pi v^2}{\int_{\Omega} |\nabla v|^2 dx} + \frac{1}{16\pi} \int_{\Omega} |\nabla v|^2 dx.$$

It follows that

$$\int_{\Omega} e^v dx \leq C_{MT} |\Omega| e^{(1/16\pi) \int_{\Omega} |\nabla v|^2 dx} \quad \text{for all } v \in H_0^1(\Omega).$$

In particular, it follows that

$$(2.3) \quad \inf_{H_0^1(\Omega)} I_\lambda > -\infty \quad \text{if and only if } \lambda \leq 8\pi,$$

and that for all $\lambda < 8\pi$ a solution for (1.6) may be obtained by direct minimization of I_λ . The limit value $\lambda = 8\pi$ corresponds to the “critical temperature” for the single species case $\mathcal{P}(\mathrm{d}\alpha) = \delta_1(\mathrm{d}\alpha)$. Such a solution is also unique [2, 38]. On the other hand, if Ω is a disc, then a Pohozaev identity argument, as may be found, for example, in [3] (or, alternatively, the explicit form of the solutions; see [1]) implies that problem (1.6) does not admit solutions when $\lambda \geq 8\pi$. Such existence/non-existence properties were employed in an essential way in [19] to establish the biologically relevant finite-time blow-up of solutions to the corresponding evolution problem for (1.6).

Motivated by the recent results in [18], our aim in this note is to investigate similar properties for (1.3) and (1.5). In order to state our results precisely, we first recall the corresponding extensions of the “optimal Moser-Trudinger inequality” (2.3) to the functionals $J_{\lambda,\mathcal{P}}^{(d)}$ and $J_{\lambda,\mathcal{P}}^{(s)}$ as in [30, 33, 39].

Let $\bar{\lambda}_{\mathcal{P}}^{(d)}$ be defined by

$$\bar{\lambda}_{\mathcal{P}}^{(d)} := 8\pi \inf \left\{ \frac{\mathcal{P}(K_{\pm})}{\left(\int_{K_{\pm}} \alpha \mathcal{P}(\mathrm{d}\alpha) \right)^2} : K_{\pm} \subset I_{\pm}, \mathcal{P}(K_{\pm}) > 0 \right\},$$

where $I_+ := [0, 1]$, $I_- := [-1, 0]$ and where $K_{\pm}$ are Borel sets. We say that $\mathcal{P}$ is discrete if $\mathcal{P} = \sum_{i=1}^m \tau_i \delta_{y_i}$ for some $m \in \mathbb{N}$, $\tau_i \in (0, 1]$, $\sum_{i=1}^m \tau_i = 1$, $y_i \in [-1, 1]$. The following optimal conditions for boundedness from below of $J_{\lambda,\mathcal{P}}^{(d)}$ and $J_{\lambda,\mathcal{P}}^{(s)}$ hold true.

Proposition 2.1 ([30, 33, 36]). *It holds that*

$$(i) \quad \inf_{H_0^1(\Omega)} J_{\lambda,\mathcal{P}}^{(d)} > -\infty \text{ if } \lambda < \bar{\lambda}_{\mathcal{P}}^{(d)} \text{ and } \inf_{H_0^1(\Omega)} J_{\lambda,\mathcal{P}}^{(d)} = -\infty \text{ if } \lambda > \bar{\lambda}_{\mathcal{P}}^{(d)}.$$

Moreover, if $\mathcal{P}$ is discrete, then $\inf_{H_0^1(\Omega)} J_{\lambda,\mathcal{P}}^{(d)}|_{\lambda=\bar{\lambda}_{\mathcal{P}}^{(d)}} > -\infty$.

$$(ii) \quad \text{It holds that } \inf_{H_0^1(\Omega)} J_{\lambda,\mathcal{P}}^{(s)} > -\infty \text{ if and only if } \lambda \leq 8\pi.$$

In the special case where $\mathcal{P}$ is of the two-species form (1.2), the functional $J_{\lambda,\mathcal{P}}^{(d)}$ takes the form

$$J_{\lambda}^{(d)}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \mathrm{d}x - \lambda \tau \ln \int_{\Omega} e^v \mathrm{d}x - \lambda(1-\tau) \ln \int_{\Omega} e^{\gamma v} \mathrm{d}x, \quad v \in H_0^1(\Omega),$$

corresponding to the variational functional for (1.3). In view of Proposition 2.1 (i), the resulting Moser-Trudinger inequality for $J_{\lambda}^{(d)}$ is given by

$$\inf_{H_0^1(\Omega)} J_{\lambda}^{(d)} > -\infty \quad \text{if and only if } \lambda \leq \bar{\lambda}_{\tau,\gamma}$$

where $\bar{\lambda}_{\tau, \gamma} = \bar{\lambda}_P^{(d)}|_{P=\tau\delta_1+(1-\tau)\delta_\gamma}$ is given by

$$(2.4) \quad \begin{aligned} \bar{\lambda}_{\tau, \gamma} &= 8\pi \min \left\{ \frac{1}{\{\tau + (1-\tau)\gamma\}^2}, \frac{1}{\tau} \right\} \\ &= \begin{cases} \frac{8\pi}{\tau}, & \text{if } 0 < \gamma \leq \frac{\sqrt{\tau}}{1+\sqrt{\tau}}, \\ \frac{8\pi}{\{\tau + (1-\tau)\gamma\}^2}, & \text{if } \frac{\sqrt{\tau}}{1+\sqrt{\tau}} < \gamma < 1. \end{cases} \end{aligned}$$

Remark 2.2. Roughly speaking, the case $\bar{\lambda}_{\tau, \gamma} = 8\pi/\tau$ corresponds to the case where the term $e^{\gamma v}$ is a perturbation of the term e^v , whereas the case $\bar{\lambda}_{\tau, \gamma} = 8\pi/\{\tau + (1-\tau)\gamma\}^2$ corresponds to the situation where both terms e^v , $e^{\gamma v}$ influence the Moser-Trudinger constant. When $\gamma \geq \frac{1}{2}$, straightforward considerations involving “Liouville bubbles” imply that both nonlinearities should concentrate along a blowing-up sequence, so that the second case is natural. On the other hand, the threshold value satisfies $\sqrt{\tau}/(1+\sqrt{\tau}) < \frac{1}{2}$ for all $\tau \in (0, 1)$. So, in the interval $(\sqrt{\tau}/(1+\sqrt{\tau}), \frac{1}{2})$ the functional is also influenced by the lower-order term, although it does not blow up along the sequence of minimizers. See Appendix A for some considerations which motivate the role of τ in this context.

In particular, for all $\lambda < \bar{\lambda}_{\tau, \gamma}$ there exists a solution to (1.3) corresponding to the minimum of $J_\lambda^{(d)}$. Our first result states that if Ω is the unit disc, similarly as for the “standard” case $\tau = 1$, the optimal Moser-Trudinger constant $\bar{\lambda}_{\tau, \gamma}$ provides a *sharp threshold* for the existence of solutions to (1.3), as follows.

Theorem 2.3. *Suppose $\Omega = B_1(0)$. For any $\tau, \gamma \in (0, 1)$, a solution to (1.3) exists if and only if $\lambda < \bar{\lambda}_{\tau, \gamma}$.*

On the other hand, the variational functional for (1.5) is given by

$$J_\lambda^{(s)}(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \lambda \ln \left(\tau \int_\Omega e^v dx + (1-\tau) \int_\Omega e^{\gamma v} dx \right), \quad v \in H_0^1(\Omega),$$

corresponding to the functional $J_{\lambda, P}^{(s)}$ with P satisfying the two-species form (1.2). In view of Proposition 2.1 (ii) (see also [39]), we have that the optimal Moser-Trudinger constant for $J_\lambda^{(s)}$ is independent of τ, γ :

$$\inf_{H_0^1(\Omega)} J_\lambda^{(s)} > -\infty \quad \text{if and only if} \quad \lambda \leq 8\pi.$$

Our next result shows that, on the contrary, the solution set of (1.5) for λ near 8π significantly depends on the parameters τ, γ . Indeed, for fixed τ, γ , let

$$(2.5) \quad \lambda_{\tau, \gamma}^* := \sup\{\lambda > 0 : (1.5) \text{ admits a solution}\}.$$

By direct minimization of $J_\lambda^{(s)}$ and the Moser-Trudinger inequality in [33], we have $\lambda_{\tau, \gamma}^* \geq 8\pi$. If Ω is the unit disc, for small values of τ , the *strict* inequality holds true.

Theorem 2.4. *Suppose $\Omega = B_1(0)$. For every fixed $\gamma \in (0, 1)$ there exists $\underline{\tau}_\gamma \in (0, 1)$ such that $\lambda_{\tau, \gamma}^* > 8\pi$ for all $\tau \in (0, \underline{\tau}_\gamma)$.*

In particular, Theorem 2.3 and Theorem 2.4 imply that the bifurcation diagram for the deterministic problem (1.3) is qualitatively more similar to the bifurcation diagram for (1.6) than the corresponding diagram for (1.5). This appears to be the first situation where (1.3) is more similar to (1.6) than (1.5). A more precise description of $\underline{\tau}_\gamma \in (0, 1)$ requires different arguments and will be addressed in a forthcoming paper.

Henceforth, we assume $\Omega = B_1(0)$.

Notation. We omit the integration variable when it is clear from the context. For $\sigma > 0$ we denote by B_σ the disc centered at 0 with radius σ . We denote by $C > 0$ a general constant whose actual value is allowed to vary from line to line.

3. PRELIMINARIES

We begin by establishing relations between the non-local mean field problems (1.3) and (1.5) defined on B_1 , namely, problem

$$(3.1) \quad \begin{cases} -\Delta v = \lambda \left(\tau \frac{e^v}{\int_{B_1} e^v} + (1 - \tau) \gamma \frac{e^{\gamma v}}{\int_{B_1} e^{\gamma v}} \right) & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1, \end{cases}$$

and problem

$$(3.2) \quad \begin{cases} -\Delta v = \lambda \frac{\tau e^v + (1 - \tau) \gamma e^{\gamma v}}{\int_{B_1} \{\tau e^v + (1 - \tau) e^{\gamma v}\}} & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1, \end{cases}$$

and the following local problem defined on whole space:

$$(3.3) \quad \begin{cases} -\Delta z = e^z + e^{\gamma z} & \text{in } \mathbb{R}^2, \\ z(0) = \max_{\mathbb{R}^2} z = \alpha \in \mathbb{R}, \\ \int_{\mathbb{R}^2} \{e^z + e^{\gamma z}\} = m(\alpha) < +\infty, \end{cases}$$

which is a special case of the cosmic string problem considered in [10, 29]. We note that solutions to (3.1) and (3.2) are radially symmetric [12].

Indeed, our aim in this section is to establish the following properties.

Proposition 3.1 (Reduction of the deterministic problem to (3.3)). *Let v be a solution to problem (3.1). Then, the function z defined by*

$$(3.4) \quad z(y) = v\left(\frac{y}{\sigma}\right) + \frac{1}{1-y} \ln \left[\frac{\tau}{(1-\tau)y} \frac{\int_{B_1} e^{yv}}{\int_{B_1} e^v} \right], \quad y \in B_\sigma$$

where σ is defined by

$$\sigma^{2(1-y)} = \frac{\lambda^{1-y}(1-\tau)y}{\tau^y} \frac{\left(\int_{B_1} e^v\right)^y}{\int_{B_1} e^{yv}}$$

is a solution to the problem

$$\begin{cases} -\Delta z = e^z + e^{yz} & \text{in } B_\sigma, \\ \int_{B_\sigma} e^z = \lambda\tau, \\ \int_{B_\sigma} e^{yz} = \lambda(1-\tau)y, \end{cases}$$

satisfying

$$z(0) = \max_{\mathbb{R}^2} z = v(0) + \frac{1}{1-y} \ln \left[\frac{\tau}{(1-\tau)y} \frac{\int_{B_1} e^{yv}}{\int_{B_1} e^v} \right].$$

Similarly, we also have the following result.

Proposition 3.2 (Reduction of the stochastic problem to (3.3)). *Let v be a solution to problem (3.2). Then, the function z defined by*

$$(3.5) \quad z(y) = v\left(\frac{y}{\sigma}\right) + \frac{1}{1-y} \ln \left[\frac{\tau}{(1-\tau)y} \right], \quad y \in B_\sigma$$

where σ is defined by

$$\sigma^{2(1-y)} = \frac{\lambda^{1-y}(1-\tau)y}{\tau^y} \left(\frac{1}{\int_{B_1} \{\tau e^v + (1-\tau)e^{yv}\}} \right)^{1-y}$$

is a solution to the problem

$$\begin{cases} -\Delta z = e^z + e^{yz}, & \text{in } B_\sigma, \\ z = \frac{1}{1-y} \ln \frac{\tau}{(1-\tau)y}, & \text{on } \partial B_\sigma, \\ \int_{B_\sigma} \left\{ e^z + \frac{e^{yz}}{y} \right\} = \lambda, \end{cases}$$

satisfying

$$z(0) = \max_{\mathbb{R}^2} z = v(0) + \frac{1}{1-y} \ln \left[\frac{\tau}{(1-\tau)y} \right].$$

Once we are reduced to problem (3.3), we may use the following known results.

Lemma 3.3 ([42]). *For any $a, b > 0$ and for any $\alpha \in \mathbb{R}$, the initial value problem*

$$(3.6) \quad \begin{cases} -\left(\eta'' + \frac{1}{r}\eta'\right) = ae^\eta + be^{y\eta}, & r > 0, \\ \eta(0) = \alpha, \\ \eta'(0) = 0 \end{cases}$$

admits a unique globally defined solution for any $\alpha \in \mathbb{R}$. Moreover, there exists $\beta > 2/y$ such that

$$\lim_{r \rightarrow +\infty} \frac{\eta(r)}{\ln r} = -\beta.$$

Proof. In order to obtain the local existence of solutions to (3.6), we rewrite the problem in the form $\eta' = F(r, \eta)$, $\eta(0) = \alpha$, where

$$F(r, \eta) = \begin{cases} -\frac{1}{r} \int_0^r s f(\eta(s)) \, ds, & \text{if } r \geq 0, \\ 0 & \text{if } r < 0, \end{cases}$$

and $f(t) = ae^t + be^{yt}$. It is readily checked that F is continuous and locally Lipschitz continuous with respect to η . Hence, standard ODE theory yields the asserted local existence and uniqueness. Now, the global existence and the asymptotic behavior follow by Lemma 5.1 and Lemma 5.2 in [42] (see also [4]). $\square$

The sharp range of admissible values of the constant β in Lemma 3.3 was determined in [29].

Lemma 3.4 ([10, 29]). *For every $\alpha \in \mathbb{R}$ there exists a unique solution to (3.3). Such a solution is radially symmetric with respect to 0 and strictly decreasing with*

respect to $r = |x|$. Moreover, $m(\alpha)$ is strictly decreasing with respect to $\alpha \in \mathbb{R}$ and it satisfies

$$\lim_{\alpha \rightarrow +\infty} m(\alpha) = 8\pi \max \left\{ 1, \frac{1-\gamma}{\gamma} \right\} = \begin{cases} 8\pi & \text{if } \frac{1}{2} \leq \gamma < 1, \\ 8\pi \frac{1-\gamma}{\gamma} & \text{if } 0 < \gamma < \frac{1}{2}, \end{cases}$$

$$\lim_{\alpha \rightarrow -\infty} m(\alpha) = \frac{8\pi}{\gamma}.$$

We set

$$(3.7) \quad m_1(\alpha) := \int_{\mathbb{R}^2} e^z, \quad m_\gamma(\alpha) := \int_{\mathbb{R}^2} e^{\gamma z},$$

where z is a solution to (3.3). We also need an “energy identity” from [10].

Lemma 3.5 ([10]). *The following relation holds true:*

$$(m_1(\alpha) + m_\gamma(\alpha))^2 = 8\pi \left(m_1(\alpha) + \frac{m_\gamma(\alpha)}{\gamma} \right) \quad \text{for any } \alpha \in \mathbb{R}.$$

Remark 3.6. From Lemma 3.4 and Lemma 3.5 it follows in particular that

$$(3.8) \quad m_1(\alpha) < 8\pi \quad \text{for all } \alpha \in \mathbb{R}.$$

In order to establish Proposition 3.1 and Proposition 3.2, we begin by observing the following general fact.

Lemma 3.7. *Let v be a solution to the problem*

$$\begin{cases} -\Delta v = ae^v + be^{\gamma v} & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1, \end{cases}$$

where $a, b > 0$. Let $z(\gamma) := v\left(\frac{\gamma}{\sigma}\right) + \ln\left(\frac{a}{\sigma^2}\right)$, where $\gamma \in B_\sigma$, with $\sigma > 0$ defined by

$$(3.9) \quad \sigma^{2(1-\gamma)} = \frac{b}{a\gamma}.$$

Then, z is a solution to the problem

$$(3.10) \quad \begin{cases} -\Delta z = e^z + e^{\gamma z} & \text{in } B_\sigma, \\ z = \frac{1}{1-\gamma} \ln \frac{a}{b} & \text{on } \partial B_\sigma, \\ \int_{B_\sigma} e^{z(\gamma)} d\gamma = \int_{B_1} ae^{v(x)} dx, \\ \int_{B_\sigma} e^{\gamma z(\gamma)} d\gamma = \int_{B_1} be^{\gamma v(x)} dx. \end{cases}$$

Proof. The function $w := v + \ln a$ satisfies

$$\begin{cases} -\Delta w = e^w + \frac{b}{a^\gamma} e^{\gamma w} & \text{in } B_1, \\ w = \ln a & \text{on } \partial B_1, \\ \int_{B_1} e^w = \int_{B_1} a e^v \\ \int_{B_1} \frac{b}{a^\gamma} e^{\gamma w} = \int_{B_1} b e^{\gamma v}. \end{cases}$$

For $\sigma > 0$ we rescale

$$z(\gamma) := w\left(\frac{\gamma}{\sigma}\right) + 2 \ln \frac{1}{\sigma} = v\left(\frac{\gamma}{\sigma}\right) + \ln\left(\frac{a}{\sigma^2}\right), \quad \gamma \in B_\sigma.$$

Then, z satisfies

$$\begin{cases} -\Delta z = e^z + \sigma^{-2(1-\gamma)} \frac{b}{a^\gamma} e^{\gamma z} & \text{in } B_\sigma, \\ z = \ln \frac{a}{\sigma^2} & \text{on } \partial B_\sigma, \\ \int_{B_\sigma} e^{z(\gamma)} d\gamma = \int_{B_1} e^{w(x)} dx, \\ \int_{B_\sigma} \sigma^{-2(1-\gamma)} e^{\gamma z(\gamma)} d\gamma = \int_{B_1} e^{\gamma w(x)} dx. \end{cases}$$

We observe that (3.9) implies that

$$(3.11) \quad \frac{a}{\sigma^2} = \left(\frac{a}{b}\right)^{1/(1-\gamma)}.$$

We deduce that if σ is defined by (3.9), then z satisfies (3.10), as asserted. $\square$

Proof of Proposition 3.1. We apply Lemma 3.7 with

$$a = \frac{\lambda \tau}{\int_{B_1} e^v}, \quad b = \frac{\lambda(1-\tau)\gamma}{\int_{B_1} e^{\gamma v}}.$$

Choosing σ according to (3.9), that is,

$$\sigma^{2(1-\gamma)} = \frac{b}{a^\gamma} = \frac{\lambda^{1-\gamma}(1-\tau)\gamma}{\tau^\gamma} \frac{\left(\int_{B_1} e^v\right)^\gamma}{\int_{B_1} e^{\gamma v}},$$

we obtain, recalling (3.11), that

$$\frac{a}{\sigma^2} = \left[\frac{\tau}{(1-\tau)\gamma} \frac{\int_{B_1} e^{\gamma v}}{\int_{B_1} e^v} \right]^{1/(1-\gamma)}.$$

Thus, the function z defined by (3.4) satisfies the problem

$$\begin{cases} -\Delta z = e^z + e^{\gamma z} & \text{in } B_\sigma, \\ z = \frac{1}{1-\gamma} \ln \left[\frac{\tau}{(1-\tau)\gamma} \frac{\int_{B_1} e^{\gamma v}}{\int_{B_1} e^v} \right] & \text{on } \partial B_\sigma, \\ \int_{B_\sigma} e^z = \int_{B_1} a e^v = \lambda \tau, \\ \int_{B_\sigma} e^{\gamma z} = \int_{B_1} b e^{\gamma v} = \lambda(1-\tau)\gamma, \end{cases}$$

and the asserted reduction is established. □

Similarly, we establish the reduction of problem (3.2).

Proof of Proposition 3.2. We apply Lemma 3.7 with

$$a = \frac{\lambda \tau}{\int_{B_1} \{\tau e^v + (1-\tau)e^{\gamma v}\}}, \quad b = \frac{\lambda(1-\tau)\gamma}{\int_{B_1} \{\tau e^v + (1-\tau)e^{\gamma v}\}}.$$

Choosing $\sigma > 0$ according to (3.9), that is,

$$\sigma^{2(1-\gamma)} = \frac{b}{a^\gamma} = \frac{\lambda^{1-\gamma}(1-\tau)\tau^{-\gamma}\gamma}{\left(\int_{B_1} \{\tau e^v + (1-\tau)e^{\gamma v}\}\right)^{1-\gamma}},$$

we find, recalling (3.11), that

$$\frac{a}{\sigma^2} = \left[\frac{\tau}{(1-\tau)\gamma} \right]^{1/(1-\gamma)}.$$

Moreover,

$$\begin{aligned} \int_{B_\sigma} e^z &= \int_{B_1} a e^v = \frac{\lambda \tau \int_{B_1} e^v}{\int_{B_1} \{\tau e^v + (1-\tau)e^{\gamma v}\}}, \\ \int_{B_\sigma} e^{\gamma z} &= \int_{B_1} b e^{\gamma v} = \frac{\lambda(1-\tau)\gamma \int_{B_1} e^{\gamma v}}{\int_{B_1} \{\tau e^v + (1-\tau)e^{\gamma v}\}}. \end{aligned}$$

Then, the function z defined by (3.5) is a solution to the problem

$$\begin{cases} -\Delta z = e^z + e^{yz}, & \text{in } B_\sigma, \\ z = \frac{1}{1-y} \ln \frac{\tau}{(1-\tau)y}, & \text{on } \partial B_\sigma, \\ \int_{B_\sigma} e^z + \frac{e^{yz}}{y} = \lambda, \end{cases}$$

as asserted. □

4. PROOF OF THEOREM 2.3

In order to prove Theorem 2.3 we use the following Pohozaev identity. The radial symmetry is used in an essential way in the following arguments.

Lemma 4.1 (Pohozaev identity). *The following identity holds true:*

$$-\frac{1}{2} \int_{\partial B_1} |\nabla v|^2 d\sigma = -2\lambda + 2\pi\lambda \left(\frac{\tau}{\int_{B_1} e^v} + \frac{1-\tau}{\int_{B_1} e^{yv}} \right).$$

Proof. Multiplying (1.3) by $x \cdot \nabla v$ and integrating, we have

$$\int_{B_1} (x \cdot \nabla v)(-\Delta v) dx = -\frac{1}{2} \int_{\partial B_1} |\nabla v|^2 d\sigma.$$

Similarly,

$$\begin{aligned} & \int_{B_1} (x \cdot \nabla v) \lambda \left(\tau \frac{e^v}{\int_{B_1} e^v} + (1-\tau) y \frac{e^{yv}}{\int_{B_1} e^{yv}} \right) \\ &= \lambda \int_{B_1} x \cdot \left(\tau \frac{\nabla e^v}{\int_{B_1} e^v} + (1-\tau) \frac{\nabla e^{yv}}{\int_{B_1} e^{yv}} \right) \\ &= \lambda \int_{\partial B_1} \left(\frac{\tau}{\int_{B_1} e^v} + \frac{1-\tau}{\int_{B_1} e^{yv}} \right) - 2\lambda \\ &= -2\lambda + 2\pi\lambda \left(\frac{\tau}{\int_{B_1} e^v} + \frac{1-\tau}{\int_{B_1} e^{yv}} \right). \end{aligned}$$

□

Proof of Theorem 2.3. We prove separately the two cases, corresponding to the two possible values of the Moser-Trudinger constant $\bar{\lambda}_{\tau, \gamma}$ defined in (2.4). Throughout this proof, for the sake of simplicity, we denote $\bar{\lambda} = \bar{\lambda}_{\tau, \gamma}$.

Case 1: $\bar{\lambda} = 8\pi / \{\tau + (1 - \tau)\gamma\}^2$. Recalling that $\Omega = B_1$, and using the Schwarz inequality, we have

$$\begin{aligned} \int_{\partial B_1} |\nabla v|^2 d\sigma &= \int_{\partial B_1} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma \geq \frac{1}{|\partial B_1|} \left(\int_{\partial B_1} \left(-\frac{\partial v}{\partial \nu} \right) d\sigma \right)^2 \\ &= \frac{1}{2\pi} \left(\int_{B_1} (-\Delta v) dx \right)^2 = \frac{\lambda^2}{2\pi} \{\tau + (1 - \tau)\gamma\}^2. \end{aligned}$$

In view of Lemma 4.1 we deduce that

$$\begin{aligned} \lambda &\leq \frac{8\pi}{\{\tau + (1 - \tau)\gamma\}^2} \left\{ 1 - \pi \left(\frac{\tau}{\int_{B_1} e^v} + \frac{1 - \tau}{\int_{B_1} e^{\gamma v}} \right) \right\} \\ &< \frac{8\pi}{\{\tau + (1 - \tau)\gamma\}^2} = \bar{\lambda}. \end{aligned}$$

Case 2: $\bar{\lambda} = 8\pi / \tau$. Let z be defined as in Proposition 3.1. By Lemma 3.3 and Lemma 3.4, z coincides in B_σ with the unique solution z_α to (3.3) with $\alpha = \alpha^*$ given by

$$\alpha^* = v(0) + \frac{1}{1 - \gamma} \ln \frac{\tau \int_{B_1} e^{\gamma v}}{(1 - \tau)\gamma \int_{B_1} e^v}.$$

Consequently, (3.8) assures

$$\int_{B_\sigma} e^z = \lambda\tau < \int_{\mathbb{R}^2} e^{z_{\alpha^*}} = m_1(\alpha^*) < 8\pi.$$

In particular, we derive from the above that

$$\lambda < \frac{8\pi}{\tau} = \bar{\lambda}.$$

Now, Theorem 2.3 is completely established. $\square$

5. PROOF OF THEOREM 2.4

Our first aim in this section is to establish the following “converse” for Proposition 3.2. It is convenient to define

$$(5.1) \quad \beta_{\tau, \gamma} := \frac{1}{1 - \gamma} \ln \frac{\tau}{(1 - \tau)\gamma}.$$

Proposition 5.1. Let z be a solution to (3.3) with $\alpha > \beta_{\tau,y}$. Let $\sigma > 0$ be defined by

$$(5.2) \quad z|_{\partial B_\sigma} = \beta_{\tau,y},$$

and let $a > 0$ be defined by

$$(5.3) \quad \frac{a}{\sigma^2} = e^{\beta_{\tau,y}} = \left[\frac{\tau}{(1-\tau)y} \right]^{1/(1-y)}.$$

Then, the function v defined by

$$(5.4) \quad v(x) = z(\sigma x) + \ln \frac{\sigma^2}{a} = z(\sigma x) - \beta_{\tau,y}$$

is a solution to problem (3.2) with

$$\lambda = \int_{B_\sigma} \left(e^z + \frac{e^{yz}}{y} \right).$$

Proof. The function v defined in (5.4) satisfies the equation

$$(5.5) \quad -\Delta v = ae^v + \sigma^{2(1-y)} a^y e^{yv} = \sigma^2 \left[\frac{a}{\sigma^2} e^v + \left(\frac{a}{\sigma^2} \right)^y e^{yv} \right].$$

Hence, we obtain from (3.2) the necessary condition

$$\frac{a}{\sigma^2 \lambda \tau} = \left(\frac{a}{\sigma^2} \right)^y \frac{1}{\lambda(1-\tau)y} = \frac{1}{\sigma^2 \int_{B_1} \{ \tau e^v + (1-\tau) e^{yv} \}},$$

from which (5.3) follows. On the other hand, (5.2) is a consequence of the Dirichlet boundary condition for v .

Finally, we compute, for v given by (5.4),

$$\begin{aligned} \int_{B_1} e^v dx &= \frac{\sigma^2}{a} \int_{B_1} e^{z(\sigma x)} dx = \frac{e^{-\beta_{\tau,y}}}{\sigma^2} \int_{B_\sigma} e^z, \\ \int_{B_1} e^{yv} &= \left(\frac{\sigma^2}{a} \right)^y \int_{B_1} e^{yz(\sigma x)} dx = \frac{e^{-y\beta_{\tau,y}}}{\sigma^2} \int_{B_\sigma} e^{yz}. \end{aligned}$$

In view of the elementary identity

$$(5.6) \quad \tau e^{-\beta_{\tau,y}} = \left[\frac{(1-\tau)y}{\tau^y} \right]^{1/(1-y)} = (1-\tau)y e^{-y\beta_{\tau,y}},$$

we may write

$$(5.7) \quad \int_{B_1} \{\tau e^v + (1 - \tau)e^{yv}\} = \sigma^{-2} \left[\frac{(1 - \tau)y}{\tau y} \right]^{1/(1-y)} \int_{B_\sigma} \left(e^z + \frac{e^{yz}}{y} \right).$$

Inserting (5.6)–(5.7) into (5.5), we deduce that

$$\begin{aligned} -\Delta v &= \sigma^2 [e^{\beta_{\tau,y}} e^v + e^{y\beta_{\tau,y}} e^{yv}] = \sigma^2 \left[\frac{\tau y}{(1 - \tau)y} \right]^{1/(1-y)} \{\tau e^v + (1 - \tau)y e^{yv}\} \\ &= \frac{\tau e^v + (1 - \tau)y e^{yv}}{\int_{B_1} \{\tau e^v + (1 - \tau)e^{yv}\}} \int_{B_\sigma} \left(e^z + \frac{e^{yz}}{y} \right). \end{aligned}$$

We deduce that v defined by (5.4) with (5.2)–(5.3) is a solution to problem (3.2) with

$$\lambda = \int_{B_\sigma} \left(e^z + \frac{e^{yz}}{y} \right),$$

as asserted. □

In view of Proposition 5.1 it is natural to consider, for fixed $\tau, y \in (0, 1)$ and $\alpha > \beta_{\tau,y}$, where $\beta_{\tau,y}$ is defined in (5.1), solution pairs $(z, \sigma) = (z_y(\alpha), \sigma_{\tau,y}(\alpha))$ where $z_y(\alpha)$ is a solution to problem (3.3) and $\sigma_{\tau,y}(\alpha)$ is uniquely defined by (5.2). Correspondingly, we obtain $\Lambda_{\tau,y}(\alpha)$ defined by

$$(5.8) \quad \Lambda_{\tau,y}(\alpha) = \int_{B_{\sigma_{\tau,y}(\alpha)}} \left\{ e^{z_y(\alpha)} + \frac{e^{yz_y(\alpha)}}{y} \right\}.$$

Let

$$\begin{aligned} \mathcal{I}_{\tau,y} &:= \{\lambda > 0 : \text{there exists a solution to (3.2)}\}, \\ \mathcal{I}'_{\tau,y} &:= \{\lambda > 0 : \lambda = \Lambda_{\tau,y}(\alpha) \text{ for some } \alpha > \beta_{\tau,y}\}. \end{aligned}$$

In view of Proposition 3.2 and Proposition 5.1, $\mathcal{I}_{\tau,y} = \mathcal{I}'_{\tau,y}$. Therefore, we are reduced to the study of $\mathcal{I}'_{\tau,y}$. Setting $\Lambda_{\tau,y}^* := \sup \mathcal{I}'_{\tau,y}$, we have $\lambda_{\tau,y}^* = \Lambda_{\tau,y}^* \geq 8\pi$, where $\lambda_{\tau,y}^*$ is defined in (2.5).

Proposition 5.2. *Let $\tau, y \in (0, 1)$ be fixed. The following properties hold true:*

- (i) *Quantities $\sigma_{\tau,y}(\alpha)$ and $\Lambda_{\tau,y}(\alpha)$ are continuous functions of $\alpha \in [\beta_{\tau,y}, +\infty)$, provided that we set $\sigma_{\tau,y}(\beta_{\tau,y}) = \Lambda_{\tau,y}(\beta_{\tau,y}) = 0$.*
- (ii) *It holds that $\lim_{\alpha \rightarrow +\infty} \sigma_{\tau,y}(\alpha) = 0$.*
- (iii) *It holds that $\lim_{\alpha \rightarrow +\infty} \Lambda_{\tau,y}(\alpha) = 8\pi$.*

Proof. Throughout this proof, for simplicity we omit the subscripts τ, γ .

Proof of (i). Given that $\alpha \in (\beta_{\tau, \gamma}, +\infty)$, $\sigma(\alpha)$ and $\Lambda(\alpha)$ are uniquely determined by $\beta_{\tau, \gamma}$ and $z(\alpha)$, the unique solution to (3.3), only. Moreover, it is also the unique solution to (3.6) for $a = b = 1$. Therefore, the desired continuity follows from the standard ODE theory, since $\beta_{\tau, \gamma}$ is independent of α .

Proof of (ii). Let $\alpha_k \rightarrow +\infty$. Let (z_k, σ_k) be the solution pair to (3.3) and (5.2) with $\alpha = \alpha_k$ and Λ_k defined by (5.8). We claim that $\lim_{k \rightarrow +\infty} \sigma_k = 0$. Indeed, suppose there exists $\sigma_0 > 0$ and a subsequence, still denoted σ_k , such that $\lim_{k \rightarrow +\infty} \sigma_k = \sigma_0 > 0$. We use an argument from [5]. Let $\underline{z}_k$ be defined as the unique solution to the problem

$$\begin{cases} -\Delta \underline{z}_k = e^{z_k} + e^{\gamma z_k} & \text{in } B_{\sigma_0/2}, \\ \underline{z}_k = z_k|_{\partial B_{\sigma_k}} = \beta_{\tau, \gamma} & \text{on } \partial B_{\sigma_0/2}. \end{cases}$$

Then, for k sufficiently large, we have $z_k \geq \underline{z}_k \geq \beta_{\tau, \gamma}$ in $B_{\sigma_0/2}$ in view of the maximum principle. On the other hand, setting $V_k(x) := 1 + e^{-(1-\gamma)z_k}$ we may write

$$\begin{cases} -\Delta z_k = e^{z_k} + e^{\gamma z_k} = V_k e^{z_k} & \text{in } B_{\sigma_k}, \\ \int_{B_{\sigma_k}} V_k e^{z_k} \leq \int_{B_{\sigma_k}} \left\{ e^{z_k} + \frac{e^{\gamma z_k}}{\gamma} \right\}, \end{cases}$$

with

$$\begin{aligned} \|V_k\|_{L^\infty(B_{\sigma_0})} &\leq 1 + e^{-(1-\gamma)\beta_{\tau, \gamma}}, \\ z_k(0) &= \max_{B_{\sigma_0}} z_k \rightarrow +\infty, \\ \int_{B_{\sigma_k}} e^{z_k} &\leq m_1(\alpha_k) \leq 8\pi, \\ \int_{B_{\sigma_k}} e^{\gamma z_k} &\leq m_\gamma(\alpha_k) \leq \frac{8\pi}{\gamma}, \end{aligned}$$

where $m_1(\alpha_k)$, $m_\gamma(\alpha_k)$ are the masses defined in (3.7). In view of an adaptation of the Brezis–Merle alternative, as may be found, for example, in [22, 25], we conclude there exist $n \geq 4\pi$ and $s \geq 0$, $s \in L^1(B_{\sigma_0})$ such that, up to subsequences,

$$e^{z_k} + e^{\gamma z_k} = V_k e^{z_k} \xrightarrow{*} n\delta_0(dx) + s(x) dx.$$

We deduce that

$$\underline{z}_k(x) \geq \frac{n}{2\pi} G_{B_{\sigma_0/2}}(x, 0) + \beta_{\tau, \gamma}$$

where $G_{B_{\sigma_0/2}}(\cdot, \cdot)$ denotes Green's function on $B_{\sigma_0/2}$. Since $n/(2\pi) \geq 2$, Fatou's lemma implies that

$$\int_{B_{\sigma_0/2}} e^{z_k} \geq \int_{B_{\sigma_0/2}} e^{\underline{z}_k} \rightarrow +\infty \quad \text{as } k \rightarrow +\infty,$$

a contradiction.

Proof of (iii). In view of Proposition 5.1, setting

$$\lambda_k = \int_{B_{\sigma_k}} \left\{ e^{z_k} + \frac{e^{y z_k}}{y} \right\} = \Lambda_k,$$

$$v_k(x) = z_k(\sigma_k x) - \beta_{\tau, y},$$

we obtain a solution v_k to problem (3.2) with $\lambda = \lambda_k$. Moreover, we have

$$v_k(0) = \alpha_k - \beta_{\tau, y} \rightarrow +\infty.$$

Therefore, (λ_k, v_k) is a blowing-up solution sequence for (3.2) with a unique blow-up point located at 0. Now, in view of the mass quantization results as established in [31, 34], we conclude that $\lim_{k \rightarrow +\infty} \lambda_k = \lim_{k \rightarrow +\infty} \Lambda_k = 8\pi$. $\square$

Now, we are ready to prove our main result for (3.2), using Proposition 5.2 and Lemma 3.4.

Proof of Theorem 2.4. Fix $y \in (0, 1)$, and observe that $\beta_{\tau, y}$ as defined in (5.1) is monotonically increasing with respect to $\tau \in (0, 1)$ and $\lim_{\tau \rightarrow 0^+} \beta_{\tau, y} = -\infty$. Hence, for any $\tau_1, \tau_2 \in (0, 1)$ with $\tau_1 < \tau_2$ and for any fixed $\alpha \in (\beta_{\tau_2, y}, +\infty)$, we have $\sigma_{\tau_1, y}(\alpha) > \sigma_{\tau_2, y}(\alpha)$, and consequently $\Lambda_{\tau_1, y}(\alpha) > \Lambda_{\tau_2, y}(\alpha) > 0$. Moreover, $\lim_{\tau \rightarrow 0^+} \sigma_{\tau, y}(\alpha) = +\infty$.

Let $\alpha_0 \in \mathbb{R}$ and $z_0 = z_0(x)$ be uniquely defined by the problem

$$\begin{cases} -\Delta z_0 = e^{z_0} + e^{y z_0}, & \text{in } \mathbb{R}^2, \\ z_0(0) = \max_{\mathbb{R}^2} z_0 = \alpha_0, \\ \int_{\mathbb{R}^2} \{e^{z_0} + e^{y z_0}\} = \frac{8\pi}{2} \left(1 + \frac{1}{y}\right) > 8\pi. \end{cases}$$

We deduce that $\lim_{\tau \rightarrow 0^+} \Lambda_{\tau, y}(\alpha_0) \geq 8\pi(1 + y^{-1})/2$. In particular, there exists $0 < \tau_0 = \tau_0(y) \ll 1$ such that, for all $\tau \in (0, \tau_0)$, we have $\Lambda_{\tau, y}(\alpha_0) > 8\pi$. $\square$

APPENDIX A. REMARKS ON $\tilde{\lambda}_{\tau, y}$

Our aim in this appendix is to show that the values of $\tilde{\lambda}_{\tau, y}$ given in (2.4) may be interpreted in terms of Liouville bubbles. Indeed, for $\delta > 0$ let

$$U_\delta(x) := \ln \frac{8\delta^2}{(\delta^2 + |x|^2)^2}, \quad x \in \mathbb{R}^2$$

be the family of “Liouville bubbles,” uniquely determined as solutions to the problem

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^U < +\infty. \end{cases}$$

Let PU_δ be the projection of U_δ on $H_0^1(B_1)$, namely,

$$PU_\delta(x) = \ln \frac{(\delta^2 + 1)^2}{(\delta^2 + |x|^2)^2}.$$

Proposition A.1. *For every $\gamma \in (0, 1)$ there exists some $t_\gamma > 0$ such that for all $\lambda > \bar{\lambda}_{\tau, \gamma}$, it holds that $J_\lambda(t_\gamma PU_\delta) \rightarrow -\infty$ as $\delta \rightarrow 0^+$.*

The following asymptotic behaviors are readily verified.

Lemma A.2. *The following expansions hold, as $\delta \rightarrow 0^+$:*

$$\begin{aligned} \int_{B_1} |\nabla PU_\delta(x)|^2 dx &= 16\pi \ln \frac{1}{\delta^2} + O(1), \\ \ln \int_{B_1} e^{aPU_\delta(x)} dx &= \begin{cases} O(1), & \text{if } 0 < a < \frac{1}{2}, \\ \ln \ln \frac{1}{\delta^2} + O(1), & \text{if } a = \frac{1}{2}, \\ (2a - 1) \ln \frac{1}{\delta^2} + O(1), & \text{if } a > \frac{1}{2}, \end{cases} \end{aligned}$$

for any $a > 0$.

In view of Lemma A.2 we may write

$$J_\lambda(tPU_\delta) = \begin{cases} \{8\pi t^2 - 2\lambda[\tau + (1 - \tau)\gamma]t + \lambda\} \ln \frac{1}{\delta^2} + O(1), & \text{if } t\gamma > \frac{1}{2}, \\ (8\pi t^2 - 2\lambda\tau t + \lambda\tau) \ln \frac{1}{\delta^2} + O(1), & \text{if } t\gamma < \frac{1}{2}. \end{cases}$$

For later convenience, we note that

$$(A.1) \quad 0 < \frac{\tau}{1 + \tau} < \frac{\sqrt{\tau}}{1 + \sqrt{\tau}} < \frac{1}{2}, \quad \text{for all } 0 < \tau < 1.$$

Lemma A.3. *Suppose that $\gamma > \tau/(1 + \tau)$. Then, there exists a solution $t = t_\gamma$ to the system*

$$(A.2) \quad \begin{cases} t\gamma > \frac{1}{2}, \\ 8\pi t^2 - 2\lambda(\tau + (1 - \tau)\gamma)t + \lambda < 0, \end{cases}$$

provided that

$$(A.3) \quad \lambda > \frac{8\pi}{(\tau + (1 - \tau)\gamma)^2}.$$

Proof. By elementary considerations, a necessary condition for a solution t to the second inequality in (A.2) to exist is that

$$\lambda^2(\tau + (1 - \tau)y)^2 - 8\pi\lambda > 0,$$

and moreover $t_- < t < t_+$ where

$$t_{\pm} = \frac{\lambda(\tau + (1 - \tau)y) \pm \sqrt{\lambda^2(\tau + (1 - \tau)y)^2 - 8\pi\lambda}}{8\pi}.$$

Hence, we readily derive (A.3), so, assuming that (A.3) holds true, we are left to check compatibility with the first inequality in (A.2). To this end, we observe that in view of (A.3) we have

$$t_+ = \frac{\lambda(\tau + (1 - \tau)y) + \sqrt{\lambda^2(\tau + (1 - \tau)y)^2 - 8\pi\lambda}}{8\pi} > \frac{1}{\tau + (1 - \tau)y}.$$

In particular, we have that a sufficient condition for $t_+y > \frac{1}{2}$ is given by

$$\frac{y}{\tau + (1 - \tau)y} > \frac{1}{2}.$$

The latter inequality is equivalent to $y > \tau/(1 + \tau)$, which holds true by assumption. We conclude there exists a solution t_y to (A.2) satisfying $0 < t_+ - t_y \ll 1$. $\square$

Lemma A.4. *If $0 < y < \frac{1}{2}$, there exists a solution t_y to the system*

$$(A.4) \quad \begin{cases} \frac{1}{2} < t < \frac{1}{2y}, \\ 8\pi t^2 - \lambda\tau(2t - 1) < 0, \end{cases}$$

provided $\lambda > 8\pi/\tau$.

Proof. By elementary considerations, solutions to the second inequality in (A.4) exist if and only if $(\lambda\tau)^2 - 8\pi\lambda\tau > 0$, which is equivalent to $\lambda > 8\pi/\tau$. Moreover, solutions satisfy $t_- < t < t_+$ where

$$t_{\pm} = \frac{\lambda\tau \pm \sqrt{(\lambda\tau)^2 - 8\pi\lambda\tau}}{8\pi}.$$

Thus, in the inequality $\frac{1}{2} < t_- < 1/(2y)$ we can find a sufficient condition for a solution to (A.4) to exist. It is readily checked that the inequality $t_- > \frac{1}{2}$ is always satisfied, since we may write

$$t_- = \frac{(\lambda\tau)^2 - [(\lambda\tau)^2 - 8\pi\lambda\tau]}{8\pi(\lambda\tau + \sqrt{(\lambda\tau)^2 - 8\pi\lambda\tau})} > \frac{1}{2}.$$

On the other hand, the inequality $t_- < 1/(2\gamma)$ is equivalent to

$$\sqrt{(\lambda\tau)^2 - 8\pi\lambda\tau} > \lambda\tau - \frac{4\pi}{\gamma}.$$

Therefore, if $\lambda\tau - 4\pi/\gamma < 0$, the second inequality holds true. Assuming that $\lambda\tau - 4\pi/\gamma \geq 0$, that is, $\lambda > 4\pi/(\tau\gamma)$, we obtain from above the inequality

$$\gamma^2[(\lambda\tau)^2 - 8\pi\lambda\tau] > (\lambda\tau\gamma - 4\pi)^2.$$

We deduce the equivalent condition $\lambda\tau\gamma(1 - \gamma) > 2\pi$. Since $\lambda > 4\pi/(\tau\gamma)$, a sufficient condition for the above is $4\pi(1 - \gamma) > 2\pi$, equivalently $\gamma < \frac{1}{2}$. $\square$

Now we can prove Proposition A.1.

Proof of Proposition A.1. We first assume that $\gamma \geq \sqrt{\tau}/(1 + \sqrt{\tau})$ and therefore $\bar{\lambda}_{\tau,\gamma} = 8\pi/(\lambda\tau\gamma(1 - \gamma))^2$. In view of (A.1), we have $\sqrt{\tau}/(1 + \sqrt{\tau}) > \tau/(1 + \tau)$, so in particular $\gamma > \tau/(1 + \tau)$. We deduce from Lemma A.3 that there exists a solution t_γ to system (A.2). Now, Lemma A.2 implies that

$$\begin{aligned} J_\lambda(t_\gamma PU_\delta) &= \{8\pi t_\gamma^2 - \lambda[\tau(2t_\gamma - 1) + (1 - \tau)(2t_\gamma\gamma - 1)]\} \ln \frac{1}{\delta^2} + O(1) \\ &= \{8\pi t_\gamma^2 - 2\lambda(\tau + (1 - \tau)\gamma)t_\gamma + \lambda\} \ln \frac{1}{\delta^2} + O(1) \rightarrow -\infty \end{aligned}$$

as $\delta \rightarrow 0^+$. Hence, the desired asymptotics is established in this case.

Now, we assume $\gamma < \sqrt{\tau}/(1 + \sqrt{\tau})$. Then, $\bar{\lambda}_{\tau,\gamma} = 8\pi/\tau$. Since

$$\frac{\sqrt{\tau}}{1 + \sqrt{\tau}} < \frac{1}{2},$$

we may apply Lemma A.4 to conclude that there exists a solution t_γ to system (A.4). Lemma A.2 yields

$$J_\lambda(t_\gamma PU_\delta) = \{8\pi t_\gamma^2 - \lambda\tau(2t_\gamma - 1)\} \ln \frac{1}{\delta^2} + O(1) \rightarrow -\infty \quad \text{as } \delta \rightarrow 0^+.$$

Now, Proposition A.1 is completely established. $\square$

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