Gevrey Hypo-Ellipticity for Sums of Squares of Vector Fields in $\mathbb{R}^2$ with Quasi-Homogeneous Polynomial Vanishing

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ABSTRACT. Analytic and Gevrey hypo-ellipticity are studied for operators of the form

$$P(x, y, D_x, D_y) = D_x^2 + \sum_{j=1}^N (p_j(x, y) D_y)^2,$$

in $\mathbb{R}^2$. We assume that the vector fields D_x and $p_j(x, y) D_y$ satisfy Hörmander's condition, that is, that they as well as their Poisson brackets generate a two-dimensional vector space. It is also assumed that the polynomials p_j are quasi-homogeneous of degree m_j , that is, that $p_j(\lambda x, \lambda^\theta y) = \lambda^{m_j} p_j(x, y)$, for every positive number λ . We prove that if the associated Poisson-Trèves stratification is not symplectic, then P is Gevrey s hypo-elliptic for an s which can be explicitly computed. On the other hand, if the stratification is symplectic, then P is analytic hypo-elliptic.

1. INTRODUCTION AND STATEMENT OF THEOREM 1.3

In $\mathbb{R}^2$, we consider a sum of squares of two vector fields, X and Y . Assume that X and Y satisfy Hörmander's condition, that is, that at any point (x, y) of an open neighborhood of the origin, the brackets of X and Y up to a certain order generate a two-dimensional vector space.

It is then evident that one of the two vector fields can be reduced to D_x via a change of variable and that, moreover, the change of variables is real analytic if the coefficients of X and Y are real analytic functions.

The general form of the vector fields X and Y is then

$$X = D_x, \quad Y = \alpha(x, y)D_x + \beta(x, y)D_y.$$

Without loss of generality, we may restrict ourselves to the case when x derivatives do not appear in the second vector field, that is, when

$$X = D_x, \quad Y = \beta(x, y)D_y.$$

Thus, here and elsewhere replacing D_x by ∂_x , D_y by ∂_y , and so on, we let

$$P = X^2 + Y^2 = \partial_x^2 + (\beta(x, y) \partial_y)^2$$

satisfy Hörmander's condition, for example, to the order m .

The characteristic variety of P , $\text{Char}(P)$, is the set where the symbol of P vanishes in the cotangent bundle of $\mathbb{R}^2$ (or rather, of Ω , Ω being a given neighborhood of the origin) minus the zero section:

$$\text{Char}(P) = \{(x, y, \xi, \eta) \mid (\xi, \eta) \neq (0, 0), \xi = 0, \beta(x, y) = 0\}.$$

Thus, $\text{Char}(P)$ can be identified with a subset of the (x, y) -plane.

Moreover, we must examine the iterated brackets of X and Y . Consider the first bracket, $[X, Y]$. Its symbol is $\partial_x \beta(x, y) \eta = \beta_x \eta$. Since $\eta \neq 0$, there are only two possibilities: at a point where β vanishes, $\partial_x \beta$ either does or does not vanish.

In the second case, we see at once that β vanishes on a real analytic manifold of co-dimension 1 in the (x, y) -plane (or rather in Ω) and there is no loss of generality in identifying the base point in question with the origin.

In this case, $\text{Char}(P) \cap \Omega$ is a symplectic submanifold of the co-tangent bundle $T^*\Omega$, and the operator P is then known to be analytic hypoelliptic (ahe). This result is due to Tartakoff, [12], [13], and Trèves, [14].

In the first case, we then have $\beta(0, 0) = \partial_x \beta(0, 0) = 0$. Consider the brackets of order 3: $[X, [X, Y]]$ and $[Y, [X, Y]]$. Their symbols are $\partial_x^2 \beta(x, y) \eta$ and

$$\{\beta(x, y) \eta, \partial_x \beta(x, y) \eta\} = \beta(x, y) \partial_{xy}^2 \beta(x, y) \eta - \partial_y \beta(x, y) \partial_x \beta(x, y) \eta.$$

Computing the latter quantities at the origin, we see that the second vanishes, so that the only non-trivial bracket of order 3 at the origin is $\text{ad}_X^2 Y$, whose symbol is $H_X^2 Y = \partial_x^2 Y(x, y, \eta) = \partial_x^2 \beta(x, y) \eta$.

In general, we have the following lemma.

Lemma 1.1. *Let X and Y be two vector fields defined as before. Consider (the symbol of) a commutator of order k , $k \geq 3$,*

$$X_I = \{X_{i_1}, \{X_{i_2}, \{\dots \{X_{i_{k-1}}, X_{i_k}\} \dots\}\}\},$$

where $i_j \in \{1, 2\}$ and $I = (i_1, \dots, i_k)$. If the k -element set

$$\{i_1, i_2, \dots, i_k\} \neq \{1, 1, \dots, 1, 2\},$$

then X_I is a polynomial in the derivatives $\partial_x^j \beta(x, y)$, for $j = 0, 1, \dots, k-2$, that is,

$$(1.1) \quad X_I(x, y, \eta) = \sum_{j=0}^{k-2} \gamma_j^{(k)}(x, y) (\partial_x^j \beta(x, y))^{m_j} \eta,$$

where the m_j are non-negative integers and the $\gamma_j^{(k)}$ are analytic functions depending on the derivatives of β with respect to x and y . Here, we denoted X by X_1 and Y by X_2 .

Proof. For $k = 3$, the assertion is easily verified. Assume $k > 3$, and proceed by induction. Consider a commutator X_I of the form specified above. Suppose that $I' = (i_2, \dots, i_k) \neq (1, \dots, 2)$. If this is not the case, the assertion is even easier to prove. By induction,

$$\{X_{i_2}, \{X_{i_3}, \{\dots \{X_{i_{k-1}}, X_{i_k}\}\}\}\} = \sum_{j=0}^{k-3} \gamma_j^{(k-1)} (\partial_x^j \beta)^{m_j} \eta.$$

If $i_1 = 1$, then

$$X_I = \partial_x \left(\sum_{j=0}^{k-3} \gamma_j^{(k-1)} (\partial_x^j \beta)^{m_j} \eta \right),$$

and this is easily recognized to have the same form as $X_{I'}$. Suppose now that $i_1 = 2$. Then,

$$X_I = \left\{ \beta \eta, \sum_{j=0}^{k-3} \gamma_j^{(k-1)} (\partial_x^j \beta)^{m_j} \eta \right\}.$$

The right-hand side of the above relation is

$$\beta \partial_y \left(\sum_{j=0}^{k-3} \gamma_j^{(k-1)} (\partial_x^j \beta)^{m_j} \eta \right) - \partial_y \beta \left(\sum_{j=0}^{k-3} \gamma_j^{(k-1)} (\partial_x^j \beta)^{m_j} \eta \right),$$

which has the desired form. □

A straightforward consequence of the above lemma is the following result.

Corollary 1.2. *Under the same hypotheses as in Lemma 1.1, if*

$$(\text{ad}_{X_1}^j X_2)(0, 0, \eta) = 0,$$

for $j = 0, \dots, k-2$, then all the brackets of length k , $X_I(0, 0, \eta)$ are also 0 provided that at least two indices i_j are equal to 2.

It is then evident that it suffices to consider the commutators of the form

$$(1.2) \quad \text{ad}_X^j Y.$$

Their symbols are, respectively,

$$(1.3) \quad H_X^j(x, y, \partial_x, \partial_y)Y(x, y, \eta) = \partial_x^j Y(x, y, \eta),$$

for $j = 0, 1, 2, \dots$.

If Hörmander's condition is verified to order m , then we have from (1.3) that

$$(1.4) \quad \begin{aligned} \partial_x^j \beta(0, 0) &= 0, \quad \text{for } j = 0, 1, \dots, m-1, \\ \partial_x^m \beta(0, 0) &\neq 0. \end{aligned}$$

We may apply Weierstraß theorem to β and obtain that there is a factorization of β of the form

$$(1.5) \quad \beta(x, y) = e(x, y) \left(x^m + \sum_{j=1}^m x^{m-j} a_j(y) \right) = e(x, y) p(x, y),$$

where e denotes a real analytic function defined in a neighborhood of the origin and such that $e(0, 0) \neq 0$; the a_j are also real analytic functions such that we have $a_j(0) = 0$. It is also clear that, possibly shrinking the open neighborhood of the origin Ω , we may assume that $e(x, y) \neq 0$ for every $(x, y) \in \Omega$.

Returning to the analysis of the characteristic variety of P , the only way for $\text{Char}(P)$ to be a symplectic submanifold of $T^*\Omega$ is for the analytic set $p^{-1}(0)$ to be a 1-dimensional submanifold of Ω , that is, that

$$(1.6) \quad p(x, y) = e(x, y)(x - \lambda(y))^m$$

for a certain real analytic function λ vanishing when $y = 0$ and a non-vanishing real analytic function $e(x, y)$ defined near the origin.

This case has been studied by Okaji, [11], in 1985, and reproved in a different way by Cordaro and Hanges, [7], in 2009.

We are interested in the case when $p^{-1}(0)$ is not a real analytic manifold—that is, since all the roots coincide at the origin, when $p^{-1}(0)$ is a union of real analytic curves issuing from the origin.

In this case, the Poisson stratification associated with P is composed of a number of symplectic strata lying at certain depths, strictly less than m , and by a non-symplectic stratum

$$(1.7) \quad \{(0, 0; 0, \eta)\}$$

at depth m of co-dimension 3 in $T^*\Omega$. In this sense, the operator P can be viewed as a generalization of the Métivier operator

$$D_x^2 + (x^2 + y^2)D_y^2.$$

In this paper, we study the operator P when it exhibits a non-symplectic stratum of the form (1.7), and the function ep , which we denote henceforth by just p , is a quasi-homogeneous polynomial, that is, a polynomial in x and y such that

$$p(\lambda x, \lambda^\theta y) = \lambda^m p(x, y),$$

for a suitable positive number θ and for any positive λ . We remark that in our setting, the pair (m, θ) is uniquely determined for polynomials depending on both x and y .

Note that when $\theta = 1$ we get the usual homogeneous case. In the theorem below, we actually allow the presence of an arbitrary number of vector fields, and we shall see that the resulting Gevrey regularity will depend also on more than one vector field. We prove the following result.

Theorem 1.3. *Let P denote the operator*

$$(1.8) \quad P(x, y, D_x, D_y) = D_x^2 + \sum_{j=1}^N (p_j(x, y) D_y)^2,$$

where the polynomials p_j satisfy

$$(1.9) \quad p_j(\lambda x, \lambda^\theta y) = \lambda^{m_j} p_j(x, y), \quad \lambda > 0,$$

that is, they are homogeneous polynomials in $\mathbb{R}^2$. It is useful to make the following convention: if there are polynomials depending only on x as well as polynomials depending only on y , we choose θ in such a way that the homogeneous degree of the minimum degree of the y -polynomials be equal to the least degree of the x -polynomials (see Remark 1.6 for a famous example).

We denote by m_j , $j = 1, \dots, N$, their homogeneity degrees. We may always assume that

$$(1.10) \quad m_1 \leq m_2 \leq \dots \leq m_N.$$

Suppose, moreover, that for a number r , $1 \leq r \leq N$, we have

$$(1.11) \quad p_r(1, 0) \neq 0, \quad p_j(1, 0) = 0, \quad \text{for } j < r.$$

Write

$$p_j(x, y) = \sum_{k=0}^{m_j} \alpha_{jk} x^k y^{q_{jk}},$$

where the q_{jk} are non-negative integers, $q_{jm_j} = 0$, and otherwise $q_{jk} \geq 1$.
Then, P is (C^∞ and) G^s hypo-elliptic for

$$(1.12) \quad s \geq \frac{1}{1 - \lambda},$$

where

$$(1.13) \quad \lambda = \frac{\theta}{m_r + 1} \max_{1 \leq j \leq r} \max_{0 \leq k < m_j, \alpha_{jk} \neq 0} \frac{m_r - k}{m_j - k},$$

or, because of the homogeneity of the polynomials p_j ,

$$(1.14) \quad \lambda = \frac{1}{m_r + 1} \max_{1 \leq j \leq r} \max_{0 \leq k < m_j, \alpha_{jk} \neq 0} \frac{m_r - k}{q_{jk}}.$$

A couple of remarks are in order. First, we observe that $\lambda < 1$. In fact, let us first suppose that $j < r$, and that $m_r - m_j > 0$. Then, we have to show that

$$1 + \frac{m_r - m_j}{m_j - k} < \frac{m_r + 1}{\theta} = \frac{m_r + 1}{m_j - k} q_{jk}.$$

This means that

$$\frac{m_r - m_j}{m_j - k} < \frac{1}{m_j - k} [(m_r + 1)q_{jk} - m_j + k],$$

which gives for the numerators

$$m_r < (m_r + 1)q_{jk} + k,$$

and this is true. Let us consider now the case $j < r$, but $m_j = m_r$. Again, we have to show that

$$1 > \frac{\theta}{m_r + 1} \frac{m_r - k}{m_j - k} = \frac{\theta}{m_r + 1} = \frac{m_r - k}{q_{jk}(m_r + 1)},$$

and the last inequality is clear. Almost the same argument takes care of the case left: $j = r$. This completes the proof that $\lambda < 1$.

As a second remark, we want to compute λ when $r = 1$, that is, when the Lie algebra is generated using the vector fields ∂_x and $p_1 \partial_y$. In this case, $m_1 \leq m_j$ for every j , so that the maximum in (1.13) is 1. Hence,

$$\lambda = \frac{\theta}{m_1 + 1} \quad \text{and} \quad \frac{1}{1 - \lambda} = \frac{m_1 + 1}{m_1 + 1 - \theta}.$$

As a third remark, in the case that every non-pure (in x) monomial is present in each of the $p_1, \dots, p_r$, the value of λ has the simpler form

$$\lambda = \frac{\theta}{m_r + 1}(m_r - m_1 + 1),$$

and the Gevrey class is given by

$$\frac{1}{1 - \bar{\lambda}} = \frac{m_r + 1}{m_r + 1 - \theta(m_r - m_1 + 1)};$$

that is, the maximum value arises from the term $x^{m_1-1}y^{1/\theta}\partial_y$ (and *not* the term yielding subellipticity!).

Example 1.4. Let $N = 2$, and consider the two polynomials (m denotes here a positive integer satisfying some condition that we specify later)

$$\begin{aligned} p_1(x, y) &= x^{m-1}y + y^2, \\ p_2(x, y) &= x^{2m} + y^{2m/(m-1)}, \end{aligned}$$

where we assume that $m > 1$ and that the fraction $2m/(m-1)$ is a positive integer, which implies $m = 2$ or $m = 3$. It is obvious that, replacing the $2m$ above with a higher multiple of m , we may construct other examples. Here, we have $m_1 = 2m - 2$, $m_2 = 2m$, $\theta = m - 1$, $r = 2$. Then, Theorem 1.3 gives us

$$\lambda = \frac{m + 1}{2m + 1}.$$

We will denote by Z_j the operators which are maximally bounded in the *a priori* estimate

$$Z_0 = \partial_x, \quad Z_j = p_j(x, y) \partial_y, \quad Z_{N+1} = \Lambda^{1/m_r+1} \sim \langle D_y \rangle^{1/m_r+1}$$

(we use Λ and $\partial/\partial y$ interchangeably and without further comment, since we need fractional powers of $\partial/\partial y$ or $(1 + D_y^2)^{1/2}$, which only makes rigorous sense in the pseudo-differential context, and proper support is not a concern here):

$$(1.15) \quad \sum_{j=0}^{N+1} \|Z_j v\|_0^2 \lesssim |(Pv, v)_0| + \|v\|_0^2, \quad v \in C_0^\infty,$$

which by itself yields G^s hypoellipticity for $s \geq m_r + 1$, but we show much more.

Remark 1.5. In [9] Matsuzawa studied the Gevrey regularity of some classes of Grušin operators. In particular, he obtained that a generalization of the Métivier operator

$$D_x^2 + (x^{2\ell} + y^{2k})D_y^2$$

is Gevrey s hypo-elliptic for

$$s \geq \frac{k(\ell + 1)}{k(\ell + 1) - \ell}.$$

Let us see that we recover the same result. In this case, the choice of θ is made according to the convention in the statement of Theorem 1.3: since we have here only two polynomials, they will end up with the same homogeneity degree, so that we may call $p_1(x) = x^\ell$ and $p_2(y) = y^k$. Now, necessarily $m_1 = \ell$. Choose then θ in such a way that the homogeneity degree of p_2 , $2k\theta$, coincide with ℓ . Thus, $\theta = \ell/k$.

Now we have two polynomials with $m_1 = m_2$, so that we may choose either $r = 1$ or $r = 2$. In the first case, we obtain that

$$\lambda = \frac{\ell}{k(\ell + 1)},$$

which gives the result mentioned above. In the second case, that is, when $r = 2$, we observe that the factor $\theta/(\ell + 1)$ is fixed, while the fraction that we maximize contains the homogeneity degrees of p_1 and p_2 , which are equal. Hence, the max is just 1, and we have the same result as above.

Finally, we note that both Métivier operator and its generalization present a slight difference when compared to ours. Let us discuss this in the case of Matsuzawa [9] operator. In our case, the third vector field should be $y^k D_y$, which, once squared, gives $y^{2k} D_y^2 + k y^{2k-1} D_y$. In other words, a first-order term appears. This first-order term is actually a real vector field vanishing of order $2k - 1$ on $y = 0$. Even though in general the problem of treating lower-order terms might be highly non-trivial and, to our knowledge, still uncharted territory (see, e.g., the papers [3] and [4] for the case of complex vector fields), in this case the lower-order term appearing in $(Pu, u)_0$ is easily estimated by the left-hand side of the estimate (1.15), and thus will not disturb our proof.

Remark 1.6. Our result is generically optimal, as the operator (operator of Métivier) shows:

$$P(x, y, D_x, D_y) = D_x^2 + (x^2 + y^2) D_y^2.$$

It is known, after Métivier, [10], that P is Gevrey 2 hypo-elliptic and not better. Using (1.13), we also obtain $s = 2$ in our general case.

At the end of this long introduction, we would like to point out that although we completely understand that the homogeneity assumption (1.9) certainly limits the generality of the statement of Theorem 1.3, the general case in two variables is still an open problem, and the present result in a sense deals with the “principal homogeneous part” of the general case. For the time being, it seems very difficult

to use the subelliptic inequality (in any form) to prove Theorem 1.3 in the general case. For a study of a non-homogeneous class, we refer to the paper [5].

It should also be noted that the methods of [1], involving a study of the subelliptic inequality for the FBI transform of the problem, could in all likelihood be modified with additional work to prove the same result as Theorem 1.3. In this paper, however, we preferred to give a more direct analysis, showing that the Gevrey regularity can be computed by elementary methods involving the subelliptic estimate, and with the observation that, even for the local result, cutoff functions play no role.

2. PROOF OF THEOREM 1.3

The proof will proceed by estimating high derivatives in $\partial/\partial x$ and $\partial/\partial y$ applied to the solution in L^2 norms: $\|D_x^a D_y^b u\| \leq CC^{a+b}(a+b)!^s$, and since no localization is required (in the sense that away from the origin the operator has known and stronger regularity), we will use the estimate as if the solution already had small support.

2.1. High derivatives. Derivatives in x and those in y will behave differently, and generally x -derivatives will be much better behaved, since when we need a Z vector field we merely use a power of ∂_x ; certain things, however, are common to both x - and y -derivatives. Here, we derive a somewhat better estimate that will be exploited below (differently) for x - and for y -derivatives.

We have, for any operator A (though in practice A will stand for either ∂_x^r or ∂_y^t),

$$\begin{aligned} \|ZAu\|^2 &\lesssim |(PAu, Au)| + \|Au\|^2 \\ &\lesssim |(APu, Au)| + |([P, A]u, Au)| + \|Au\|^2 + E_r, \end{aligned}$$

where the second term on the right will be harmless (by subellipticity). Here, Z denotes any of the vector fields defining the operator P , and the symbol $\lesssim A$ means $\leq cA$ where the value of c may change from instance to instance.

Actually, it will be useful and simplify the proof if we include not only ZAu but also AZu on the left side of the estimate. To do so requires adding the first bracket to the right, but we will need to expand and treat them anyhow. That is,

$$\begin{aligned} \|ZAu\|^2 + \|AZu\|^2 &\lesssim |(PAu, Au)| + \|[Z, A]u\|^2 + \|Au\|^2 \\ &\lesssim |(APu, Au)| + \|[Z, A]u\|^2 + \|Au\|^2 + E_r, \end{aligned}$$

where for E_r , since $P = \sum Z_j^2$, we have (with the usual meaning of “s.c.” = “small constant” and “l.c.” = “large constant”),

$$\begin{aligned} E_r &\lesssim |(Z[Z, A]u, Au)| + |([Z, A]Zu, Au)| + \|[Z, A]u\|^2 \\ &\leq \text{s.c.} \|Z^*Au\| + |([Z, A]Zu, Au)| + \text{l.c.} \|[Z, A]u\|^2. \end{aligned}$$

We also note that, up to a zero order term, $Z^* = -Z$, and the zero order term is absorbed by the L^2 norm of Au while the small constant permits the first term on the right to be absorbed on the left.

It will not be necessary to carry along the term Pu , since in the analytic category this may always be taken to be zero, and in the Gevrey case it is known; also, its derivatives are known and excellently behaved—our task is to show that derivatives of u itself are as well behaved.

Thus, the above estimate becomes

$$\|ZAu\|^2 + \|AZu\|^2 \lesssim \|[Z, A]u\|^2 + |([Z, A]Zu, Au)| + \|Au\|^2.$$

2.2. Derivatives in x Derivatives in x always behave better than those in y , since $\partial_x = Z_0$ is optimally controlled by the estimate. Using prime (') to denote an x derivative (of $p(x, y)$), we have

$$\begin{aligned} [Z_j, \partial_x^r] &= \underline{r} p'_j \partial_y \partial_x^{r-1} + \dots \\ &= \underline{r} Z p'_j \partial_y \partial_x^{r-2} + \dots \rightsquigarrow C^{r/2} r!^{1/2} Z \partial_y^{r/2} + \dots \end{aligned}$$

In fact, from the formula

$$(2.1) \quad [Z^j, \tilde{Z}] = \sum_{k=1}^j \binom{j}{k} \text{ad}_Z^k(\tilde{Z}) Z^{j-k},$$

we deduce

$$[Z_j, \partial_x^r] = - \sum_{k=1}^r \binom{r}{k} \text{ad}_{\partial_x}^k(Z_j) \partial_x^{r-k} = - \sum_{k=1}^r \binom{r}{k} p^{(k)} \partial_y \partial_x^{r-k},$$

so that the term with $k = 1$ is the term shown in the expression above (with factor r .)

Rewriting the estimate just before this section for the case $A = \partial_x^r$, and recalling that p_j is a polynomial of fixed, bounded degree N , the sums should not be frightening: thus, modulo the term that would contain Pu ,

$$\|Z \partial_x^r u\|^2 + \|\partial_x^r Z u\|^2 \lesssim \|[Z, \partial_x^r]u\|^2 + |([Z, \partial_x^r]Zu, \partial_x^r u)| + \|\partial_x^r u\|^2$$

or

$$\begin{aligned} \|Z \partial_x^r u\|^2 + \|\partial_x^r Z u\|^2 &\lesssim \sum_{k=1}^{m_N} \binom{r}{k}^2 \|p^{(k)} \partial_y \partial_x^{r-k} u\|^2 \\ &\quad + \sum_{k=1}^{m_N} \binom{r}{k} |(p^{(k)} \partial_y \partial_x^{r-k} Z u, \partial_x^r u)| + \|\partial_x^r u\|^2, \end{aligned}$$

or again, bounding $p^{(k)} = p_j^{(k)}$ in norm by C , since $j \leq m_N$ (the largest of the degrees) is universally bounded by the problem,

$$\begin{aligned} \|Z \partial_x^r u\|^2 + \|\partial_x^r Z u\|^2 &\lesssim \|\partial_x^r u\|^2 + \sum_{k=1}^{m_N} (r^k \|\partial_y \partial_x^{r-k} u\|)^2 \\ &\quad + \sum_{k=1}^{m_N} r^k |(p^{(k)} \partial_y \partial_x^{r-k} Z u, \partial_x^r u)|. \end{aligned}$$

Modulo benign errors, we will move one ∂_x from the right-hand side of the last term to the left and both an r and the ∂_y to the right, obtaining a bound of

$$\begin{aligned} C \left(\sum_1^N \|r^{k-1} \partial_x^{r-(k-1)} Z u\| \right) \times \|r \partial_y \partial_x^{r-1} u\| \\ \leq \text{s.c.} \left(\sum_1^N C \|r^{k-1} \partial_x^{r-(k-1)} Z u\|^2 \right) + \text{l.c.} \|r \partial_y \partial_x^{r-2} Z u\|^2 \end{aligned}$$

as the principal term, that is, where p is not further differentiated.

This is good—the principal term in the parentheses on the right is absorbed on the left, and the others are successively lower order, while the last term on the right shows the prototypical transfer of *two* powers of ∂_x , one going to a factor of r and the other becoming a ∂_y . Iterated, this will lead to $r!$ from the parenthetical term and to the term $r!! \|\partial_y^{r/2} Z u\|$, which illustrates the need to consider pure ∂_y derivatives *in any case* of half the order. We have met this often before. One moral of this story is that one cannot hope to bound pure ∂_x derivatives without considering the ∂_y derivatives—they are closely linked.

Finally, the terms we have ignored in the immediately preceding analysis, where p receives additional derivatives and powers of ∂_x or the ∂_y disappear, are of lower order and pose no complications. In fact, these appear in the errors (shifting the index k and writing Z_0 for ∂_x)

$$\sum_{k=1}^N r^{k-1} |(p^{(k+1)} \partial_x^{r-k} Z u, r \partial_x^r u)| \sim C \sum_{k=0}^{N-1} r^k \|\partial_x^{r-1-k} Z u\| \|r \partial_x^{r-1} Z_0 u\|$$

and

$$\sum_{k=1}^N |r^{k-1} (p^{(k+1)} \partial_x^{r-k} Z u, r \partial_y \partial_x^{r-1} u)| \sim C \sum_{k=0}^{N-1} r^k \|\partial_x^{r-1-k} Z u\| \|r \partial_y \partial_x^{r-2} Z_0 u\|,$$

which are the same form as the terms just treated with an extra harmless derivative on p and not requiring the *weighted* Schwarz inequality.

2.3. Derivatives in the radial direction (powers of ∂_ρ). For the derivatives in $\mathcal{Y}$, the situation is less straightforward, since $\partial_{\mathcal{Y}}$ is not an “elliptic” vector field (i.e., the *a priori* estimate bounds x -derivatives maximally but not $\mathcal{Y}$ -derivatives, only $p(x)D_{\mathcal{Y}}$ derivatives), but we will make essential use of a “radial” vector field

$$\partial_\rho = x \partial_x + \theta \mathcal{Y} \partial_{\mathcal{Y}},$$

where θ denotes the same constant as in (1.9). Note that ∂_ρ has the easily verified property that

$$[\partial_\rho, \partial_x] = \partial_x \quad \text{and} \quad [\partial_\rho, Z_j] = (m_j - \theta)Z_j, \quad j = 1, \dots, N,$$

in view of the quasi-homogeneity of the polynomials $p_j(x, \mathcal{Y})$.

These bracket relations for ∂_ρ lead almost at once to the analytic growth of derivatives in ∂_ρ of u by repeating the argument of the previous section and using the above relations at each step.

In the end, we obtain the following result.

Proposition 2.1. *There exists a constant C such that for all t ,*

$$(2.2) \quad \|\partial_\rho^t u\| \leq C^{t+1} t!,$$

if we assume this is true of Pu .

2.4. Derivatives in $\mathcal{Y}$.

2.4.1. Derivatives in $\mathcal{Y}$ in some model cases. To motivate the reader for the general case, we sketch the computation here of $\mathcal{Y}$ -derivatives for some model cases. The polynomials p_j are given by

$$\begin{aligned} p_1 &= x\mathcal{Y}^2 + \mathcal{Y}^4, & (m_1 = 2), \\ p_2 &= \mathcal{Y}^4, & (m_2 = 2), \\ p_3 &= x^6 + x\mathcal{Y}^{10}, & (m_3 = m_r = 6), \\ p_4 &= x^8 + x^6\mathcal{Y}^4, & (m_4 = 8), \\ p_5 &= x\mathcal{Y}^{14} + \mathcal{Y}^{16}, & (m_5 = 8), \\ p_6 &= x^8\mathcal{Y}^2, & (m_6 = 9), \end{aligned}$$

which should cover the cases we will encounter. Note that in the above example, $\theta = \frac{1}{2}$.

In all cases, we will write the operator $P = \partial_x^2 + \sum_j (p_j \partial_{\mathcal{Y}})^2 = Z_0^2 + \sum_j Z_j^2$.

The *a priori* estimate may be written, since we have assumed all along that the Hörmander condition is satisfied, here leading to a subelliptic estimate with

$$\varepsilon = \frac{1}{7},$$

$$\sum_1^6 \|Z_j \partial_y^t u\|^2 + \|\partial_y^{t+1/7} u\|^2 \leq |(P \partial_y^t u, \partial_y^t u)| + \|\partial_y^t u\|^2,$$

with Z_j corresponding to $p_j(x, y)$.

We want to expand the left-hand side of the estimate to include terms roughly interpolated between $\|\partial_y^{t+1/7} u\|^2$ and $\|Z_3 \partial_y^t u\|^2$, namely the whole string

$$\sum_1^6 \|Z_j \partial_y^t u\|^2 + \|\partial_y^{t+1/7} u\|^2 + \|x \partial_y^{t+2/7} u\|^2 + \cdots + \|x^6 \partial_y^{t+7/7} u\|^2,$$

since this last term on the left is *nearly* equal to $\|Z_3 \partial_y^t u\|^2$, which is already estimated. In fact, the error committed is $\|x y^{10} \partial_y \partial_y^t u\|^2$, which, by homogeneity, may be estimated:

$$\|x y^{10} \partial_y \partial_y^t u\|^2 \lesssim \text{s.c.} \|x^6 \partial_y \partial_y^t u\|^2 + \text{l.c.} \|y^{12} \partial_y \partial_y^t u\|^2,$$

since the first term on the right will be absorbable on the left provided we include the last term (or some bound for it) on the right.

The bound we shall use for this last term involves the radial derivative introduced above, since radial derivatives of the solution *grow analytically*. Using the notation involved above in estimating x -derivatives, we will write

$$\|y^{12} \partial_y \partial_y^t u\| \sim \|\partial_\rho^{12} \partial_y^{t-11} u\|,$$

and then, abusively, continue this list of similarities:

$$\|y^{12} \partial_y \partial_y^t u\| \sim \|\partial_\rho^{12} \partial_y^{t-11} u\| \sim t^{12} \|W_0 \partial_y^{t-11-1/7} u\|.$$

We use the notation

$$W_j = x^j \partial_y^{(j+1)/7},$$

but the “abusive” part of this string is that once we have grouped $y \partial_y$ as ∂_ρ , we have “removed it from the norm as a power of t .” Of course what really happens is that we leave the ∂_ρ^{12} in place, but since we will need to account for a gain in ∂_y derivatives, we should leave *inside the norm* the expression $\partial_\rho^{12}/t^{12}$ and balance it with t^{12} outside. Then, in view of the excellent commutation properties of ∂_ρ with everything in sight, and knowing that

- (a) the powers of ∂_ρ here are bounded by the degree of p in y , and
- (b) eventually the norm of $(\partial_\rho^N/t^N)u$ will be bounded by C_u^N ,

we may suppress $\partial_\rho^{12}/t^{12}$ and, if desired, bring a constant to the power 12 out of the norm, which is absorbed by $\sim$.

The above estimates then become, if we examine only first brackets,

$$\begin{aligned} & \sum_1^6 \|Z_j \partial_y^t u\| + \sum_0^6 \|W_k \partial_y^t u\| \\ & \lesssim |(\partial_y^t P u, \partial_y^t u)|^{1/2} + t^{12} \|W_0 \partial_y^{t-11-1/7} u\| + \sum_1^6 \|[Z_j, \partial_y^t] u\| + \|\partial_y^t u\|. \end{aligned}$$

To give a feel for the effect of this “error,” if we drop the power of t on the left-hand side by $11\frac{1}{7}$ and pick up a factor of $(C)t^{12}$ on the right-hand side, ignoring the other commutators that will enter in moving P past ∂_y^t , there is a “gain” of $78/7$ derivatives at the expense of 12 powers of t , that is, a gain of $78/84$ powers of ∂_y *per* power of t , which by itself would lead to solutions belonging to the Gevrey class G^s with $s = 84/78 = 14/13 \sim 1.077$.

Returning to the last estimate above, we begin with p_1 , as follows.

$$\begin{aligned} [Y_1, \partial_y^t] &= [p_1 \partial_y, \partial_y^t] = [p_1, \partial_y^t] \partial_y = [x y^2, \partial_y^t] \partial_y + [y^4, \partial_y^t] \partial_y \\ &\sim \sum_1^2 \binom{t}{\ell} x y^{2-\ell} \partial_y^{t+1-\ell} + \sum_{\ell=1}^4 \binom{t}{\ell} y^{4-\ell} \partial_y^{t+1-\ell} \\ &\sim \sum_1^2 \binom{t}{\ell} x \partial_\rho^{2-\ell} \partial_y^{t+1-\ell} + \sum_{\ell=1}^4 \binom{t}{\ell} \partial_\rho^{4-\ell} \partial_y^{t+1-\ell} \end{aligned}$$

(writing $y \partial_y \equiv \partial_\rho \bmod x \partial_x$ which is elliptically controlled),

$$\sim \sum_1^2 \binom{t}{\ell} W_1 \partial_\rho^{2-\ell} \partial_y^{t+1-(2-\ell)-\ell-2/7} + \sum_{\ell=1}^4 \binom{t}{\ell} W_0 \partial_\rho^{4-\ell} \partial_y^{t+1-(4-\ell)-\ell-1/7}.$$

Now, as always, the principal term ($\ell = 1$) will guide the result, and we will think of $\binom{t}{\ell}$ as t^ℓ (modulo C^ℓ , and recall that all values of ℓ are bounded by the maximum power of y in the p_j , a nice bounded number depending only on the operator). Writing only the terms with $\ell = 1$, and dropping the summations, we have

$$\|[Z_1, \partial_y^t] u\| \lesssim t \|W_1 \partial_\rho \partial_y^{t-1-2/7} u\| + t \|W_0 \partial_\rho^3 \partial_y^{t-3-1/7} u\|.$$

Thus, *if we only needed to consider the bracket with p_1* , we would have

$$\begin{aligned} & \sum_1^6 \|Z_j \partial_y^t u\| + \sum_0^6 \|W_k \partial_y^t u\| \\ & \lesssim |(\partial_y^t P u, \partial_y^t u)|^{1/2} + t^{12} \|W_0 \partial_y^{t-11-1/7} u\| \\ & \quad + t^2 \|W_1 \partial_y^{t-1-2/7} u\| + t^4 \|W_0 \partial_y^{t-3-1/7} u\| + \|\partial_y^t u\|, \end{aligned}$$

replacing ∂_ρ by powers of t since we have seen that radial derivatives play out analytically.

The third and fourth terms on the right come from p_1 and will iterate to gains of $\frac{9}{7}$ derivative with penalty of t^2 and of $22/7$ derivatives with a penalty of t^4 , respectively. That is, $G^{14/9}$ and $G^{28/22=14/11}$, of which we would have to take the weaker class, $G^{14/9}$, to control the regularity.

Note that the second term on the right we have seen would lead to $G^{14/13}$, a class we shall see again.

We must naturally do the same thing with the other fields.

Some shorthand will help. Let us describe the path p_1 takes as

$$[Z_1, \partial_y^t] \rightsquigarrow t^2 W \partial_y^{t-9/7}, t^4 W \partial_y^{t-22/7} \rightsquigarrow t^{14/9}, t^{28/22} \rightsquigarrow G^{14/9}, G^{28/22=14/11},$$

where W denotes a generic W_j , since the remaining term above that we analyzed first led to $G^{84/78=13/12}$, which is harmless.

How about p_2 ? For p_2 , the first term produces no bracket with any of the $p_j \partial_y$, and the second has just been treated.

The term with p_3 is the one that yielded the subelliptic estimate but no brackets here, while the last term could have been treated by interpolation, but more directly, brackets

$$\begin{aligned} [Z_3, \partial_y^t] &\sim t W_1 y^9 \partial_y^{t-2/7} \sim t W_1 \partial_\rho^9 \partial_y^{t-9-2/7} \\ &\sim t^{10} W \partial_y^{t-9-2/7=65/7} \rightsquigarrow G^{70/65} = G^{14/13}. \end{aligned}$$

Entirely similarly,

$$[Z_4, \partial_y^t] \sim t W_6 y^3 \partial_y^{t-1} \sim t W_6 \partial_\rho^3 \partial_y^{t-3-1} \sim t^4 W_6 \partial_y^{t-4} \rightsquigarrow G^1.$$

For p_5 , we have

$$\begin{aligned} [Z_5, \partial_y^t] &\sim t \{x y^{13} + y^{15}\} \partial_y^t \sim t^{14} W_1 \partial_y^{t-13-2/7}, t^{16} W_0 \partial_y^{t-15-1/7} \\ &\rightsquigarrow G^{98/93}, G^{112/106}. \end{aligned}$$

Finally, for p_6 ,

$$[Z_6, \partial_y^t] \sim t x^8 y \partial_y^t \sim t x^2 W_6 \partial_\rho \partial_y^{t-1} \sim t^2 x^2 W_6 \partial_y^{t-2},$$

and even discarding the last x^2 , pulling it out of the norm with a constant, we have $G^{2/2} = G^1$.

The behavior of an entirely general term will be as follows. The degree of subellipticity will depend on the lowest-degree term with a pure power of x —we call that degree m_r , and that goes with the field that generates the Lie algebra. Other fields will contribute terms upon bracketing with the Z_j , that is, arising from brackets $[p_j \partial_y, \partial_y^t] = [p_j, \partial_y^t] \partial_y$. A typical term in p_j is of the form

$x^\ell y^{q_{j\ell}}$, whose bracket with ∂_y^t will play out as follows (as illustrated by the above examples):

$$\begin{aligned} [x^\ell y^{q_{j\ell}}, \partial_y^t] \partial_y &\rightsquigarrow \underline{t} x^\ell y^{q_{j\ell}-1} \partial_y^t \rightsquigarrow t x^\ell \partial_\rho^{q_{j\ell}-1} \partial_y^{t-(q_{j\ell}-1)} \\ &\rightsquigarrow t t^{q_{j\ell}-1} W_\ell \partial_y^{t-(q_{j\ell}-1)-(\ell+1)/(m_r+1)}, \end{aligned}$$

provided $\ell \leq m_r$ so that the x can be effectively combined with powers of ∂_y . (When this is not the case, we will be able to just “discard” some of the x .)

Then, the bracket and its sequellae lead to the paradigm

$$Z_j \partial_y^t \rightsquigarrow \underline{t}^{q_{j\ell}} Z \partial_y^{t-(q_{j\ell}-1)-(\ell+1)/(m_r+1)},$$

which, as we have seen, leads to the Gevrey class $G^{q_{j\ell}/((q_{j\ell}-1)+(\ell+1)/(m_r+1))}$.

The homogeneity of p_j may be written $m_j = \ell + \theta q_{j\ell}$ or

$$q_{j\ell} = \frac{m_j - \ell}{\theta},$$

where, in our particular example, $\theta = \frac{1}{2}$.

Clearly, each term in the p_j will contribute a (generally unique) class, and the *largest* of these classes will be the one to govern the regularity of the operator P . This Gevrey class may always be written as $G^{1/(1-\lambda)}$, with $\lambda < 1$, and a brief calculation gives this maximum value of λ to be

$$\lambda = \frac{\theta}{m_r + 1} \max_{1 \leq j \leq r} \max_{0 \leq k < m_j, \alpha_{jk} \neq 0} \frac{m_r - k}{m_j - k},$$

which is the Gevrey index of the theorem.

Finally, in the case when the degree is larger than m_r , there are two choices: either the degree of x in a given a term is larger than m_r or it is not. If it is, then merely discarding the remaining x so that a W_{m_r} is generated, together with some y , which we convert to ∂_ρ 's as above, we find a smaller Gevrey exponent. When such a term has no more than m_r powers of x , using homogeneity again we may estimate such a term by

$$\text{s.c.} |x^{m_r}| + \text{l.c.} |y^{(m_j-m_r)/\theta}|,$$

both of which are at least as good as the terms arising when we handled p_{m_r} .

2.4.2. Derivatives in y in general. For derivatives in y in general, we argue in the same way as in Subsection 2.1. That is, we assumed that $Pu = 0$, and consider the estimate, now with $A = \partial_y^t$,

$$\|ZAu\|^2 + \|AZu\|^2 \lesssim \|[Z, A]u\|^2 + |([Z, A]Zu, Au)| + \|Au\|^2.$$

Now, however, we will need the subellipticity as well, and we note that the only Z that will give non-zero brackets is $Z_j = p_j \partial_y$. However, now Z^* differs from Z by p_y (the last term on the right below), which must be more closely followed than was the case with pure ∂_x derivatives. We have, following the generic argument above,

$$\begin{aligned} & \|Z \partial_y^t u\|^2 + \|\partial_y^t Z u\|^2 + \|\partial_y^{t+1/(m_r+1)} u\|^2 \\ & \lesssim \|[p \partial_y, \partial_y^t] u\|^2 + |([p \partial_y, \partial_y^t] \circ p \partial_y u, \partial_y^t u)| + \|\partial_y^t u\|^2 + \|p_y \partial_y^t u\|^2. \end{aligned}$$

By formula (2.1), with $\partial_y^k p_j(x, y)$ just written $p_y^{(k)}$,

$$(2.3) \quad [p \partial_y, \partial_y^t] = - \sum_{k=1}^t \binom{t}{k} p_y^{(k)} \partial_y^{t-k+1},$$

so that, using Cauchy-Schwarz on the inner product above to obtain the last term below,

$$\begin{aligned} & \sum_j \|Z_j \partial_y^t u\|^2 + \sum_j \|\partial_y^t Z_j u\|^2 + \|\partial_y^{t+1/(m_r+1)} u\|^2 \lesssim \|\partial_y^t u\|^2 + \|p_y \partial_y^t u\|^2 \\ & + \left(\sum_{k=1}^t \binom{t}{k} \|p_y^{(k)} \partial_y^{t-k+1} u\| \right)^2 + \sum_{k=1}^t \binom{t}{k} \|\partial_y^{t-k+1} \circ p \partial_y u\| \times \|p_y^{(k)} \partial_y^t u\|. \end{aligned}$$

In this last term, we will bound the binomial coefficient just by t^k , and then apportion the powers of t between the two terms in the product depending upon the value of k (a fixed apportionment will not work).

Remarkably, it turns out that each of the (bounded number of) error contributions leads to the *same* Gevrey index as the leading terms from the previous line.

To give an example of this, recall the field corresponding to $p_3 = x^6 + x y^{10}$ that by itself led to the class $G^{14/13}$ if one in effect ignored higher commutators, that is, ignored the last line above.

But now let us look at the last line, and in particular the last term. *Keeping this as a product*, we may write a careful bound for this term as

$$\begin{aligned} & \sum_{k=1}^t \binom{t}{k} \|\partial_y^{t-k+1} \circ p \partial_y u\| \times \|p_y^{(k)} \partial_y^t u\| \\ & \lesssim \sum_1^{10} \bar{C} t^k \|\partial_y^{t-k+1} Z_3 u\| \times t^{10-k} \|W_1 \partial_y^{t-(10-k)-2/7} u\|, \end{aligned}$$

since $p_y^{(k)} \sim x y^{10-k}$, and if we combine each y with a ∂_y to obtain ∂_ρ modulo (for the minute) $x \partial_x$, and ignore the fact that all the y are to the left of all D_y .

Then, we have taken the powers of ∂_ρ out of the norm, and, recalling that radial derivatives ultimately grow analytically, we have replaced each ∂_ρ by a power of t . Recall that $W_1 = \varkappa \Lambda^{2/7}$. Note that $p_3^{(k)} = 0$ if $k > 10$. Of course, to be honest, the factor t^{10-k} should not be there, and the ∂_ρ^{10-k} left inside the norm.

Now, in all there are 10 powers of t , and there has been a gain in derivatives of

$$k - 1 + 10 - k + \frac{2}{7} = 9 + \frac{2}{7}.$$

Of course, this gain is not equally apportioned between the two factors in the product, but *neither do the powers of t need to be*. The product is entirely at our disposal. With a suitable division of t^{10} between the two terms, *each* of them will “lead to” the same Gevrey class (after a Schwarz inequality), namely, $G^{14/13}$. In particular, the split, for a given value of k , will not be the one above into k and $10 - k$, but will be as follows:

$$\begin{aligned} & t^k \|\partial_y^{t-k+1} Z_3 u\| \times t^{10-k} \|W_2 \partial_y^{t-(10-k)-2/7} u\| \\ &= t^{(14/13)(k-1)} \|\partial_y^{t-k+1} Z_3 u\| \times t^{(14/13)(10-k+2/7)} \|W_1 \partial_y^{t-(10-k)-2/7} u\|, \end{aligned}$$

since the sum of the exponents is 10, and this clearly yields a factor $t^{14/13}$ per gain of one in the value of t .

To see that these are completely general results, write p ($= p_j$) as

$$(2.4) \quad p_j(x, y) = \sum_{\ell=0}^{m_j} a_{j,\ell} x^\ell y^{q_{j\ell}},$$

where, according to assumptions (1.9) and (1.11), $a_{j,m_j} = 0$ for $j = 1, \dots, r-1$, $a_{r,m_r} \neq 0$, and $q_{j\ell}$ is a non-negative integer such that

$$q_{j,m_j} = 0, \quad \ell + \theta q_{j\ell} = m_j, \quad \text{for } j = 1, \dots, N.$$

Hence, any derivative with respect to y of the p_j destroys, if present, the term x^{m_j} . In addition, once a derivative lands on p_j , we will need to treat each term individually, so we may generally drop summations for terms involving brackets. Also, the bounded degree of the p_j means that we need not be concerned with precise coefficients in obtaining estimates since everything will be either $\lesssim$ or $\sim$, where coefficients are implicit.

Thus,

$$\partial_y^k p_j(x, y) \sim \sum_{\ell} x^\ell y^{q_{j\ell}-k},$$

that is,

$$[p \partial_y, \partial_y^t] = - \sum_{k=1}^t \binom{t}{k} p_y^{(k)} \partial_y^{t-k+1} = \sum_{k,\ell} \binom{t}{k} x^\ell y^{q_{j\ell}-k},$$

so that

$$\begin{aligned}
& \|Z \partial_y^t u\|^2 + \|\partial_y^t Z u\|^2 + \|\partial_y^{t+1/(m_r+1)} u\|^2 \\
& \lesssim \|\partial_y^t u\|^2 + \|p_y \partial_y^t u\|^2 + \left(\sum_{k=1}^{m_N} \binom{t}{k} \|p_y^{(k)} \partial_y^{t-k+1} u\| \right)^2 \\
& \quad + \sum_{k=1}^{m_N} \binom{t}{k} \|\partial_y^{t-k+1} \circ p \partial_y u\| \times \|p_y^{(k)} \partial_y^t u\| \\
& \lesssim \|\partial_y^t u\|^2 + \|p_y \partial_y^t u\|^2 + \left(\sum_{k=1, \ell}^{m_N} t^k \|x^\ell y^{q_{j\ell}-k} \partial_y^{t-k+1} u\| \right)^2 \\
& \quad + \sum_{k=1}^{m_N} t^k \|\partial_y^{t-k+1} \circ p \partial_y u\| \times \|x^\ell y^{q_{j\ell}-k} \partial_y^t u\|,
\end{aligned}$$

with $\ell \leq m_j$, since in the next-to-last term the powers of x and y come from one (arbitrary) p_j , and all derivatives of order greater than N of the p_j are zero.

At this point, we treat powers of $y \partial_y$ in the norms in the last two terms as ∂_ρ (modulo the maximally estimated $\partial_x = Z_0$), and realize that a radial derivative will (eventually) be bounded analytically, as in the previous section:

$$\begin{aligned}
& \|\partial_y^t u\|^2 + \|p_y \partial_y^t u\|^2 + \left(\sum_{k=1}^{m_N} \sum_{\ell=0}^{m_j} t^k \|x^\ell y^{q_{j\ell}-k} \partial_y^{t-k+1} u\| \right)^2 \\
& \quad + \sum_{k=1}^{m_N} \sum_{\ell=0}^{m_j} t^k \|\partial_y^{t-k+1} \circ p \partial_y u\| \times \|x^\ell y^{q_{j\ell}-k} \partial_y^t u\| \\
& \lesssim \|\partial_y^t u\|^2 + \|p_y \partial_y^t u\|^2 + \left(\sum_{k=1}^{m_N} \sum_{\ell=0}^{m_j} t^{q_{j\ell}} \|W_\ell \partial_y^{t-(q_{j\ell}-1)-(\ell+1)/(m_r+1)} u\| \right)^2 \\
& \quad + \sum_{k=1}^{m_N} \sum_{\ell=0}^{m_j} t^{q_{j\ell}} \|\partial_y^{t-k+1} \circ p \partial_y u\| \times \|W_\ell \partial_y^{t-(q_{j\ell}-k)-(\ell+1)(m_r+1)} u\|,
\end{aligned}$$

with a special comment below when $m_j \geq \ell > m_r$, that is, when there are too many x to take advantage of the subellipticity array of the W_j . Actually, in this case, we will just “throw away” the extra x (pull them out of the norm, estimated by a constant) in a remarkable moment of wasteful largesse.

Now, we employ the same subtle division of $q_{j\ell}$ that we did of t^{10} above, into two parts, parts that are fitted to the gain of derivatives inside the norm but that add up to $q_{j\ell}$.

Recall in the example above that the parts were $\frac{14}{13}(k-1)$ and $\frac{14}{13}(10-k+\frac{2}{7})$. Here, the analogous parts will be $s(k-1)$ and $s((q_{j\ell}-k)+(\ell+1)/(m_r+1))$ for suitable s to make the sum equal to $q_{j\ell}$. As should be clear, this is the precise

division that will lead to the Gevrey class G^s upon iteration as it led to $G^{14/13}$ in the example.

To solve for s , we note that a glance at the previous line suggests that the price, in powers of t , for a gain of *one* power of ∂_y is

$$s = \frac{q_{j\ell}}{(q_{j\ell} - 1) + (\ell + 1)/(m_r + 1)},$$

and then the summands $s(k - 1)$ and $s((q_{j\ell} - k) + (\ell + 1)/(m_r + 1))$ obviously lead to the same class in each of the norms in the last line of the estimate above, if one makes this choice.

Of course, the maximum value of such s , over all j , will govern the Gevrey class.

Remark 2.2. Since, when $j \leq r$,

$$\frac{m_r - k}{m_j - k} \geq 1,$$

whereas, for $j \geq r$,

$$\frac{m_r - k}{m_j - k} \leq 1,$$

we conclude that the controlling Gevrey class is obtained by the first r vector fields, leading to the stated value of λ in Theorem 1.3.

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