Asymptotics for the Spectrum of the Dirichlet Laplacian on Horn-shaped Regions

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ABSTRACT. Let Ω be an open bounded set in $\mathbb{R}^{m-1}$ with a piecewise smooth boundary, and star-shaped with respect to $(0, \dots, 0) \in \mathbb{R}^{m-1}$. We extend a result of G.V. Rozenbljum for the spectrum of the Dirichlet Laplace operator for $\{(x, y) \in \mathbb{R}^m : y \in (1+x)^{-\alpha}\Omega, x > 0\}$ in $\mathbb{R}^m$, where $\alpha > 0$. For $2m-1 > \alpha > (1-m + [(m-1)(9m-17)]^{1/2})/2$ we obtain two-term asymptotics and a remainder estimate for the Dirichlet counting function. For $\alpha \geq 2m-1$ or $m > 2$ and $(1-m + [(m-1)(9m-17)]^{1/2})/2 \geq \alpha > 0$ we recover Rozenbljum's result for the leading term of the Dirichlet counting function, together with an improved remainder estimate.

1. INTRODUCTION

Let D be an open set in euclidean space $\mathbb{R}^m$ with boundary ∂D , and let $-\Delta_D$ be the Dirichlet laplacian for D . A theorem of H. Weyl [24] asserts that, if D is bounded in $\mathbb{R}^m$ ($m = 2, 3, \dots$) and if ∂D is piecewise smooth, then the spectrum of $-\Delta_D$ is discrete $\lambda_1 \leq \lambda_2 \leq \dots$. Furthermore, if N_D is the counting function defined by

$$(1.1) \quad N_D(\lambda) = \#\{j \in \mathbb{Z}^+ : \lambda_j < \lambda\},$$

then

$$(1.2) \quad \lim_{\lambda \rightarrow \infty} \frac{N_D(\lambda)}{W_D(\lambda)} = 1,$$

where W_D is the Weyl function defined by

$$(1.3) \quad W_D(\lambda) = (4\pi)^{-m/2} \left(\Gamma\left(\frac{m}{2} + 1\right) \right)^{-1} |D|_m \lambda^{m/2},$$

and $|D|_m$ is the m -dimensional Lebesgue measure of D .

Weyl's theorem (1.2) has been improved in various directions. First of all, it was shown in [17], [18] that (1.2) holds under the minimal hypothesis D open and $|D|_m < \infty$ (see also [20, Theorem 10.6]). Secondly, if D is bounded and ∂D is piecewise smooth, then (1.2) can be improved to

$$(1.4) \quad N_D(\lambda) = W_D(\lambda) + O(\lambda^{(m-1)/2}), \quad \lambda \rightarrow \infty,$$

([14], [19]). If in addition the measure of periodic geodesics in D is equal to zero, then (1.4) can be improved to

$$(1.5) \quad N_D(\lambda) = W_D(\lambda) - 4^{-1}(4\pi)^{-(m-1)/2} \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{-1} A(\partial D) \lambda^{(m-1)/2} + o(\lambda^{(m-1)/2}), \quad \lambda \rightarrow \infty,$$

where $A(\partial D)$ is the area of ∂D ([8], [23]).

In this paper we consider unbounded open sets in $\mathbb{R}^m$ with possibly infinite volume. A necessary and sufficient condition in the infinite volume case for $-\Delta_D$ to have discrete spectrum has been obtained in [11]. Though various bounds for N_D have been obtained in this case [16], [17], [18], [5], we do not know of any formula which could possibly replace (1.2). In fact, some of the examples in [10] illustrate the very complicated oscillatory asymptotic behaviour that may occur, even for regions in $\mathbb{R}^2$. Here we concentrate on a wide class of unbounded horn-shaped regions in $\mathbb{R}^m$, where detailed asymptotic analysis of $N_D(\lambda)$ for $\lambda \rightarrow \infty$ remains possible.

Let $m = 2, 3, \dots$ and let Ω be an open bounded set in $\mathbb{R}^{m-1}$, star-shaped with respect to $(0, \dots, 0)$. Let $f : [0, \infty) \rightarrow [0, 1]$ be monotonically decreasing to zero with $f(0) = 1$. Let

$$(1.6) \quad D(\Omega, f) = \left\{ (x, y) \in \mathbb{R}^m : \frac{y}{f(x)} \in \Omega, x > 0 \right\}.$$

$D(\Omega, f)$ is an example of a horn-shaped region in $\mathbb{R}^m$. It is well-known that the spectrum of $-\Delta_{D(\Omega, f)}$ is discrete. Estimates for the corresponding counting function have been obtained in [4, Theorem 8]. However, in order to analyse the asymptotic behaviour of $N_{D(\Omega, f)}$ and to exclude possible oscillatory behaviour as observed in [10], it is necessary to restrict the class of functions for f even further. In fact, the case where $f = f_\alpha$, $\alpha > 0$, and

$$(1.7) \quad f_\alpha(x) = (1+x)^{-1/\alpha}, \quad x > 0,$$

subdivides already into many subcases. G.V. Rozenbljum ([17, p. 246], [18]) proved the following. Let $|\Omega|_{m-1} < \infty$ and $\alpha > m-1$. Then for $\lambda \rightarrow \infty$,

$$(1.8) \quad N_{D(\Omega, f_\alpha)}(\lambda) = \zeta(\alpha, \Omega) \lambda^{(1+\alpha)/2} + o(\lambda^{(1+\alpha)/2}),$$

where the spectral zeta function $\zeta(\alpha, \Omega)$ is given by

$$(1.9) \quad \zeta(\alpha, \Omega) = (4\pi)^{-1/2} \frac{\Gamma((\alpha+2)/2)}{\Gamma((\alpha+3)/2)} \sum_{j=1}^{\infty} \mu_j^{-\alpha/2},$$

and $\mu_1 < \mu_2 \leq \dots$ is the spectrum of the $(m-1)$ -dimensional Dirichlet laplacian $-\Delta_{\Omega}$ for Ω .

The special case where $\Omega = \Omega_1$, the unit disc in $\mathbb{R}^{m-1}$, has been investigated by many authors ([3], [4], [9], [13], [21], [22] and the references therein). In particular the case where $m = 2$ and $\Omega = (-1, 1)$ has received a great deal of attention. In this case $\mu_j = (\pi j/2)^2$, $j \in \mathbb{Z}^+$, and the spectral zeta function in (1.9) reduces to a constant multiple of the Riemann zeta function. One easily recovers [21, Theorem 1.3], [3, Examples 1 and 2], and the results of [9, Section 3].

The higher dimensional situation where $\Omega = \Omega_1$, $m = 2, 3, \dots$, and $f = f_{\alpha}$ has been investigated in [13] under the hypothesis that the measure of periodic geodesics on $D(\Omega_1, f_{\alpha})$ is equal to zero (billiard conditions). The author obtained the structure of the asymptotic behaviour for the corresponding counting function, including some results for the finite volume case. However, some of the coefficients in the asymptotic formulae remained undetermined. Moreover, by fixing the geometry of the cross-section $\Omega = \Omega_1$, it is not possible to give a geometric interpretation of the coefficients. See also the remark on [21, line 4, p. 86].

Estimates for the counting function for elliptic operators of even order with variable coefficients stabilizing at infinity in a regular cusp were obtained by Ivrii in [8, Chapter 12.1]. Some of these results were obtained under the hypothesis that the influence of the periodic bi-characteristics is small (billiard conditions).

In this paper we avoid making the hard-checkable billiard conditions by using the variational method. We obtain upper and lower bounds for the spectral counting function for the Dirichlet laplacian on $D(\Omega, f)$ given by (1.6), where f is an arbitrary to zero decreasing function, and where Ω is an open, bounded and star-shaped set in $\mathbb{R}^{m-1}$ satisfying mild regularity conditions (Theorem 3.1 and Theorem 5.1). In the special case where $f = f_{\alpha}$, given by (1.7), we refine Rozenbljum's result (1.8), (1.9) by obtaining an explicit expression for the second term in (1.8) (Theorem 1.1). It turns out that if $2m-1 > \alpha > m-1$, then this second term is a multiple of $|\Omega|_{m-1} \lambda^{m/2}$. This provides the geometric interpretation, which was lacking in the earlier results. In addition we obtain for $2m-1 > \alpha > m-1$ non-trivial remainder estimates. For the finite volume case $m-1 > \alpha$ we show that the leading term is given by the corresponding Weyl function, whereas the second term is a multiple of $\lambda^{(1+\alpha)/2}$. The coefficient involves a regularized zeta function.

Theorem 1.1. Suppose that Ω in $\mathbb{R}^{m-1}$ is an open, bounded set and star-shaped with respect to 0. For $m = 3, 4, \dots$ we also assume that $\partial\Omega$ is piecewise smooth. Let

$$(1.10) \quad D(\Omega, f_\alpha) = \{(x, y) \in \mathbb{R}^{m-1} : (1+x)^{1/\alpha} y \in \Omega, x > 0\}.$$

Then for $\lambda \rightarrow \infty$ and

$$(i) \quad \boxed{\alpha \geq 2m-1}$$

$$(1.11) \quad N_{D(\Omega, f_\alpha)}(\lambda) = \zeta(\alpha, \Omega) \lambda^{(1+\alpha)/2} + O(\lambda^{(1+\alpha)/4} \log \log \lambda),$$

$$(ii) \quad \boxed{2m-1 > \alpha > 2m-2}$$

$$(1.12) \quad N_{D(\Omega, f_\alpha)}(\lambda) = \zeta(\alpha, \Omega) \lambda^{(1+\alpha)/2} - \\ - \frac{\alpha}{\alpha+1-m} (4\pi)^{-m/2} \left(\Gamma\left(\frac{m}{2}+1\right) \right)^{-1} |\Omega|_{m-1} \lambda^{m/2} + O(\lambda^{(1+\alpha)/4} \log \log \lambda),$$

$$(iii) \quad \boxed{2m-2 \geq \alpha > m-1}$$

$$(1.13) \quad N_{D(\Omega, f_\alpha)}(\lambda) = \zeta(\alpha, \Omega) \lambda^{(1+\alpha)/2} - \\ - \frac{\alpha}{\alpha+1-m} (4\pi)^{-m/2} \left(\Gamma\left(\frac{m}{2}+1\right) \right)^{-1} |\Omega|_{m-1} \lambda^{m/2} + O(\lambda^{(2m-1)/4} \log \lambda),$$

$$(iv) \quad \boxed{\alpha = m-1}$$

$$(1.14) \quad N_{D(\Omega, f_{m-1})}(\lambda) \\ = \frac{m-1}{2} \left(\Gamma\left(\frac{m}{2}+1\right) \right)^{-1} \lambda^{m/2} \\ \cdot \left((4\pi)^{-m/2} |\Omega|_{m-1} \left[\log \lambda - (-1)^m \log 4 - 2 \sum_{k=1}^m \frac{(-1)^{k+m}}{k} - \log \mu_1 \right] \right. \\ \left. + (4\pi)^{-1/2} \Gamma\left(\frac{m+1}{2}\right) \int_{[\mu_1, \infty)} \mu^{-(m+1)/2} (N_\Omega(\mu) - W_\Omega(\mu)) d\mu \right) \\ + O(\lambda^{(3m-2)/6} \log \lambda),$$

$$(v) \quad \boxed{m-1 > \alpha > (1-m + [(m-1)(9m-17)]^{1/2})/2}$$

$$(1.15) \quad N_{D(\Omega, f_\alpha)}(\lambda) = W_{D(\Omega, f_\alpha)}(\lambda) + \eta(\alpha, \Omega) \lambda^{(1+\alpha)/2} \\ + O(\lambda^{(m\alpha+2m^2-4m+2)/(2\alpha+4m-4)} \log \lambda),$$

$$(vi) \quad (1 - m + [(m - 1)(9m - 17)]^{1/2})/2 \geq \alpha > 0$$

$$(1.16) \quad N_{D(\Omega, f_\alpha)}(\lambda) = W_{D(\Omega, f_\alpha)}(\lambda) + O(\lambda^{(m\alpha + 2m^2 - 4m + 2)/(2\alpha + 4m - 4)} \log \lambda),$$

where the spectral zeta function $\zeta(\alpha, \Omega)$ in (1.11)-(1.13) is given by (1.9), the regularized spectral zeta function $\eta(\alpha, \Omega)$ in (1.15) is given by

$$(1.17) \quad \eta(\alpha, \Omega) = (4\pi)^{-1/2} \frac{\alpha \Gamma((\alpha + 2)/2)}{2\Gamma((\alpha + 3)/2)} \int_{[0, \infty)} \mu^{-(2+\alpha)/2} \{N_\Omega(\mu) - W_\Omega(\mu)\} d\mu,$$

and the Weyl function $W_{D(\Omega, f_\alpha)}$ in (1.15) and (1.16) is given by (1.3), with

$$(1.18) \quad |D(\Omega, f_\alpha)|_m = \frac{\alpha}{m - 1 - \alpha} |\Omega|_{m-1}.$$

Remark 1.2.

- (i) We note the very complicated coefficient of $\lambda^{m/2}$ in borderline case $\alpha = m - 1$ in (1.14). The role of the principal eigenvalue μ_1 of $-\Delta_\Omega$ in that coefficient is not special: the coefficient remains unchanged if we replace μ_1 by any $\varepsilon \in (0, \mu_1]$.
- (ii) We note that for $m = 2$, case (vi) becomes vacuous and the remainder in (1.15) is $o(\lambda^{(1+\alpha)/2})$. This implies two-term asymptotics for all finite volume cases $1 > \alpha$. In fact, case (vi) only occurs because the remainder estimate in (1.15) dominates the second term $\lambda^{(1+\alpha)/2}$ for $m > 2$.
- (iii) Assuming billiard conditions, it is possible, by using results of [8, Chapter 12.1], to improve the remainder estimates for $m - 1 > \alpha$. For $m - 1 > \alpha > m - 2$ one recovers two-term asymptotics with the second term as in (1.15), whereas for $m - 2 > \alpha > 0$ one recovers (1.5).

Theorem 1.1 implies the following improvement of [4, Corollary 3].

Corollary 1.2. *Let $D_\alpha \subset \mathbb{R}^2$ be given by*

$$(1.19) \quad D_\alpha = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \cdot |x_2|^\alpha < 1\}.$$

Then for $\lambda \rightarrow \infty$,

$$(1.20) \quad N_{D_\alpha}(\lambda) = \beta(\alpha) \lambda^{(1+\alpha)/2} + \beta\left(\frac{1}{\alpha}\right) \lambda^{(1+\alpha)/(2\alpha)} + O(\lambda^{(1+\alpha)/4} \log \lambda),$$

$$1 < \alpha < 2,$$

$$(1.21) \quad N_{D_\alpha}(\lambda) = \beta(\alpha) \lambda^{(1+\alpha)/2} + O(\lambda^{(1+\alpha)/4} \log \lambda), \quad \alpha \geq 2,$$

$$(1.22) \quad N_{D_1}(\lambda) = \pi^{-1} \lambda \log \lambda + 2\pi^{-1} (\gamma - \log \pi) \lambda + O(\lambda^{2/3} \log \lambda),$$

where

$$(1.23) \quad \beta(\alpha) = 2^\alpha \pi^{-\alpha-1/2} \zeta(\alpha) \frac{\Gamma((\alpha+2)/2)}{\Gamma((\alpha+3)/2)},$$

and $\zeta(\alpha)$ is the Riemann zeta function defined on $\mathbb{C} \setminus \{1\}$. The constant γ in (1.22) is Euler's constant.

We note that (1.20) gives two-term asymptotics for $1 < \alpha < 2$, whereas [4, Corollary 3] gave two-term asymptotics only for $1 < \alpha < 2^{1/2}$. It is possible to extend Corollary 1.2 to the situation in $\mathbb{R}^m$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m$, and

$$(1.24) \quad D_\alpha = \{(x_1, \dots, x_m) \in \mathbb{R}^m : |x_1|^{\alpha_1} \cdots |x_m|^{\alpha_m} < 1\}.$$

We will not present the details, except for the following special case.

Theorem 1.3. Let $m = 3, 4, \dots$, $\alpha = (1, 1, \dots, 1)$ and let D_α be given by (1.24). Then for $\lambda \rightarrow \infty$ and $m = 3, 4, \dots$,

$$(1.25) \quad N_{D_\alpha}(\lambda) = a_m \lambda^{m/2} (\log \lambda)^{m-1} + b_m \lambda^{m/2} (\log \lambda)^{m-2} + O(\lambda^{m/2} (\log \lambda)^{m-3}),$$

where

$$(1.26) \quad a_m = \frac{2^{(m-1)(m-2)/2}}{\pi^{m/2} \Gamma(1 + m/2)},$$

and where the constants b_m are recursively defined by

$$(1.27) \quad b_m = 2^{(m^2-3m+4)/2} \frac{m-1}{\Gamma(m/2+1)} \pi^{-m/2} \cdot \left\{ (-1)^{m+1} \log 2 + \sum_{k=1}^m \frac{(-1)^{k+m+1}}{k} \right\} \\ + 2^{m-2} \frac{m-1}{m-2} \frac{\Gamma((m+1)/2)}{\Gamma(m/2+1)} \pi^{-1/2} b_{m-1},$$

$$(1.28) \quad b_2 = \frac{2}{\pi} (\gamma - \log \pi).$$

The proof of Theorem 1.1 is organized as follows. For $\alpha < m-1$ (the finite volume case) we let $L > 0$ be large and partition $D(\Omega, f_\alpha)$ into a “thick end” D_L^1 , and into a “horn-shaped end” D_L^2 :

$$(1.29) \quad D_L^1 = D(\Omega, f_\alpha) \cap \{x < L\}, \quad D_L^2 = D(\Omega, f_\alpha) \cap \{x > L\}.$$

At the “thick end” it is advantageous to use a Whitney partition of D_L^1 into cubes, while at the horn-shaped end it is advantageous to use a partition of D_L^2 into cuboids of constant width. Since $\partial\Omega$ is piecewise smooth, the area of D_L^1 is at most $O(L)$, for large L . Hence by the Courant-Hilbert estimates ([6, VI.4]),

$$(1.30) \quad N_{D_L^1}(\lambda) = W_{D_L^1}(\lambda) + O(L\lambda^{(m-1)/2} \log \lambda).$$

To estimate $N_{D_L^2}$ we use [4, Theorem 8] with $f_\alpha^L(x) = f_\alpha(x+L)$. We then choose L to be λ -dependent, in order to minimize the sum of the various remainders. Since L is λ -dependent one has to replace (1.30) by a uniform estimate. This will be done in Section 2 where we also obtain uniform bounds for open sets D in $\mathbb{R}^m$ which satisfy

$$(1.31) \quad \omega(\varepsilon) \leq C\varepsilon^{m-d}, \quad 0 < \varepsilon < \varepsilon_0,$$

for some $C < \infty$, $d \in [m-1, m)$, $\varepsilon_0 > 0$, and where ω is defined by

$$(1.32) \quad \omega(\varepsilon) = |\{x \in \mathbb{R}^m : \text{dist}(x, \partial D) < \varepsilon\}|_m.$$

In Section 3 we combine the estimates of Section 2 and the estimates for horn-shaped regions to obtain an estimate for $N_{D(\Omega, f)}(\lambda)$ (Theorem 3.1). Theorem 1.1 (v) and (vi) follow then by substituting $f = f_\alpha$ and by minimizing the remainder with respect to L . The proof of Theorem 1.1 (iv) ($\alpha = m-1$) also relies upon Theorem 3.1 and will be given in Section 4.

For $\alpha > m-1$ it turns out that the partition of D_L^1 into Whitney cubes does not pay off. However, it does pay off to partition $D(\Omega, f_\alpha)$ into a large λ -dependent number of horn-shaped pieces. The remainder term is minimized by choosing the width in each piece appropriately (Section 5).

The proof of Corollary 1.2 follows the details of the proof of [4, Corollary 3] with estimates (40), (43) being replaced by (1.14), (1.15) for $m = 2$ respectively. We omit the details.

The proof of Theorem 1.3 exploits the recursive structure of D_α : the cross-section at x_1 is a region of the form D_α in $\mathbb{R}^{m-1}$ where $\alpha = (1, \dots, 1)$, and $\lambda_k(x) = x^{2/(m-1)}\lambda_k(1)$, where $\lambda_k(1)$ are the eigenvalues corresponding to the region D_α in $\mathbb{R}^{m-1}$. The two-dimensional result (1.22) and minor modifications of the estimates of [4, Theorem 8] complete the proof. We omit the details.

2. COURANT AND HILBERT REMAINDER ESTIMATES

In this section we prove some uniform bounds on the remainder term for the spectral counting function for open sets with finite volume in $\mathbb{R}^m$. We carefully keep track of all the geometric constants in these bounds, refining some of the results in [6, Chapter VI.4] and [12].

Let D be an open set in $\mathbb{R}^m$ with finite volume $|D|_m$. For $\varepsilon > 0$ we define

$$(2.1) \quad \mu(\varepsilon) = |\{x \in D : \text{dist}(x, \partial D) < \varepsilon\}|_m.$$

Theorem 2.1. *Suppose there exist constants $d \in [m-1, m)$ and $C < \infty$ such that*

$$(2.2) \quad \mu(\varepsilon) \leq C\varepsilon^{m-d}, \quad \varepsilon > 0.$$

Then for $d = m-1$ and

$$(2.3) \quad \lambda \geq 4|D|_m^{-2/m},$$

we have

$$(2.4) \quad N_D(\lambda) \geq W_D(\lambda) - 3C\lambda^{(m-1)/2} \log\{\lambda(2|D|_m)^{2/m}\},$$

and for $d \in (m-1, m)$ and $\lambda > 0$ we have

$$(2.5) \quad N_D(\lambda) \geq W_D(\lambda) - 5C\lambda^{d/2}\{(m-d)(d-m+1)\}^{-1},$$

where the Weyl counting function W_D is defined by (1.3).

Proof. Let $a > 0$ be given by

$$(2.6) \quad a^m = 2|D|_m,$$

and form the Whitney decomposition of D into hypercubes of size $\{a2^{-p} : p = 0, 1, 2, \dots\}$. Let n_p denote the number of cubes of size $a2^{-p}$ contained in D . Given $p \in \mathbb{N}$, the hypercubes of size $\{a2^{-\ell} : \ell = 0, 1, \dots, p-1\}$ cover the set $\{x \in D : \text{dist}(x, \partial D) > a2^{-(p-1)}\sqrt{m}\}$. Hence the hypercubes of size $a2^{-p}$ are contained in the set $\{x \in D : \text{dist}(x, \partial D) \leq a2^{1-p}\sqrt{m}\}$. Hence

$$(2.7) \quad n_p \leq a^{-m}2^{mp}\mu(a2^{1-p}\sqrt{m}), \quad p = 1, 2, \dots$$

We note that $n_0 = 0$ by (2.6). □

Let N_a denote the spectral counting function for the Dirichlet laplacian on a hypercube of size a in $\mathbb{R}^m$. Then

$$(2.8) \quad N_a(\lambda) = \#\{(k_1, \dots, k_m) \in \mathbb{N}^m : \pi^2\{k_1^2 + \dots + k_m^2\} < \lambda a^2\}.$$

Suppose

$$(2.9) \quad \lambda a^2 > m\pi^2.$$

Counting the lattice points in (2.8) gives, by Gauss' estimate,

$$\begin{aligned}
 (2.10) \quad N_a(\lambda) &\geq 2^{-m} \left| \left\{ x \in \mathbb{R}^m : |x| < \left(\frac{\lambda a^2}{\pi^2} \right)^{1/2} - m^{1/2} \right\} \right|_m \\
 &= \pi^{m/2} 2^{-m} \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} \left(\frac{a \lambda^{1/2}}{\pi} - m^{1/2} \right)^m \\
 &\geq (4\pi)^{-m/2} a^m \lambda^{m/2} \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} \left(1 - \frac{\pi m^{3/2}}{a \lambda^{1/2}} \right).
 \end{aligned}$$

Note that the right hand side of (2.10) is negative for $0 < \lambda a^2 < m\pi^2$, and so (2.10) holds for all $\lambda > 0$, $a > 0$, $m = 1, 2, \dots$

By Dirichlet bracketing (see e.g. [15, XIII.15]),

$$\begin{aligned}
 (2.11) \quad N_D(\lambda) &\geq \sum_{p=1}^{\infty} n_p N_{a2^{-p}}(\lambda) \\
 &\geq \sum_{\{p \in \mathbb{N} : 2^p < a \lambda^{1/2}\}} n_p N_{a2^{-p}}(\lambda) \\
 &\geq \sum_{\{p \in \mathbb{N} : 2^p < a \lambda^{1/2}\}} n_p \frac{a^m \lambda^{m/2} 2^{-mp}}{(4\pi)^{m/2} \Gamma(1 + m/2)} \left(1 - \frac{\pi m^{3/2} 2^p}{a \lambda^{1/2}} \right) \\
 &\geq W_D(\lambda) - A_1 - A_2,
 \end{aligned}$$

where

$$(2.12) \quad A_1 = \sum_{\{p \in \mathbb{N} : 2^p \geq a \lambda^{1/2}\}} n_p \frac{a^m \lambda^{m/2} 2^{-mp}}{(4\pi)^{m/2} \Gamma(1 + m/2)},$$

$$(2.13) \quad A_2 = \pi m^{3/2} \sum_{\{p \in \mathbb{N} : 2^p < a \lambda^{1/2}\}} n_p \frac{a^{m-1} \lambda^{(m-1)/2} 2^{-(m-1)p}}{(4\pi)^{m/2} \Gamma(1 + m/2)}.$$

By (2.7), we obtain for A_1

$$(2.14) \quad A_1 \leq \sum_{\{p \in \mathbb{N} : 2^p \geq a \lambda^{1/2}\}} \frac{\lambda^{m/2} \mu(a 2^{1-p} \sqrt{m})}{(4\pi)^{m/2} \Gamma(1 + m/2)},$$

and by (2.2),

$$\begin{aligned}
 (2.15) \quad A_1 &\leq \sum_{\{p \in \mathbb{N} : 2^p \geq a \lambda^{1/2}\}} \frac{C \lambda^{m/2} a^{m-d} (4m)^{(m-d)/2}}{(4\pi)^{m/2} \Gamma(1 + m/2)} 2^{-p(m-d)} \\
 &\leq \frac{4C \lambda^{d/2} m^{1/2}}{(4\pi)^{m/2} \Gamma(1 + m/2) (\log 2) (m-d)}.
 \end{aligned}$$

To estimate A_2 suppose first that $d = m - 1$. By (2.2) and (2.7),

$$(2.16) \quad A_2 \leq \sum_{\{p \in \mathbb{N} : 2^p < a\lambda^{1/2}\}} \frac{2C\pi m^2 \lambda^{(m-1)/2}}{(4\pi)^{m/2} \Gamma(1 + m/2)}.$$

By (2.3) and (2.6),

$$(2.17) \quad \#\{p \in \mathbb{N} : 2^p < a\lambda^{1/2}\} \leq \frac{\log(a^2\lambda)}{\log 4}.$$

Hence

$$(2.18) \quad A_2 \leq \frac{2C\pi m^2}{(4\pi)^{m/2} \Gamma(1 + m/2) \log 4} \lambda^{(m-1)/2} \log\{(2|D|_m)^{2/m} \lambda\}.$$

Inequality (2.4) now follows from (2.11), (2.15), and (2.18). Suppose $m - 1 < d < m$. By (2.7) and (2.2),

$$(2.19) \quad \begin{aligned} A_2 &\leq \frac{C\pi m^{3/2} a^{m-d-1} (4m)^{(m-d)/2} \lambda^{(m-1)/2}}{(4\pi)^{m/2} \Gamma(1 + m/2)} \sum_{\{p \in \mathbb{N} : 2^p < a\lambda^{1/2}\}} 2^{(1-m+d)p} \\ &\leq \frac{2C\pi m^2}{(4\pi)^{m/2} \Gamma(1 + m/2)} \lambda^{d/2} (2^{1+d-m} - 1)^{-1} \\ &\leq \frac{2C\pi m^2}{(4\pi)^{m/2} \Gamma(1 + m/2) (\log 2) (1 + d - m)} \lambda^{d/2}. \end{aligned}$$

Inequality (2.5) follows from (2.11), (2.15), and (2.19), since for $m = 1, 2, \dots$,

$$(2.20) \quad 2\pi m^2 + 4m^{1/2} \leq 5(4\pi)^{m/2} \Gamma\left(1 + \frac{m}{2}\right) \log 2.$$

Theorem 2.2. *Suppose that there exist constants $d \in [m - 1, m)$ and $C < \infty$ such that*

$$(2.21) \quad \omega(\varepsilon) \leq C\varepsilon^{m-d}, \quad 0 < \varepsilon \leq |D|_m^{1/m}.$$

Then for $d = m - 1$ and

$$(2.22) \quad \lambda \geq 4|D|_m^{-2/m},$$

we have

$$(2.23) \quad N_D(\lambda) \leq W_D(\lambda) + 10^m C \lambda^{(m-1)/2} \log\{\lambda(2|D|_m)^{2/m}\},$$

and for $d \in (m - 1, m)$ and $\lambda > 0$,

$$(2.24) \quad N_D(\lambda) \leq W_D(\lambda) + 10^m C \lambda^{d/2} \{(m - d)(d - m + 1)\}^{-1}.$$

Remark 2.3.

- (i) Note that hypothesis (2.21) implies that D is bounded.
- (ii) It is possible to prove upper bounds of the form (2.23), (2.24) assuming (2.2) instead of (2.21), and using [12, Lemma 3.3]. Alternatively, it is possible to obtain (2.23), (2.24) assuming (2.2) and a strong Hardy inequality. See [2] for details and [5, Theorem 5] for a related application.

Proof of Theorem 2.2. Let $a > 0$ be such that (2.6) holds, and form the Whitney partition of D into hypercubes of size $\{a2^{-p} : p = 0, 1, 2, \dots\}$. Let $p(\lambda) \in \mathbb{N}$ be the unique integer such that

$$(2.25) \quad 2^{p(\lambda)} \leq a\lambda^{1/2} < 2^{p(\lambda)+1}.$$

This is possible by (2.22) and (2.6). The cubes of size $\{a2^{-p} : p = 0, 1, 2, \dots, p(\lambda) - 1\}$ in the Whitney partition cover the set $\{x \in D : \text{dist}(x, \partial D) > a2^{1-p(\lambda)}m^{1/2}\}$. We consider the set $\{x \in D : \text{dist}(x, \partial D) \leq a2^{1-p(\lambda)}m^{1/2}\}$, and we denote by $n'_{p(\lambda)}$ the number of hypercubes of size $a2^{-p(\lambda)}$ that are needed to cover this set. As in the proof of Theorem 2.1 we have $n_0 = 0$,

$$(2.26) \quad n_p \leq a^{-m} 2^{mp} \mu(a2^{1-p}m^{1/2}), \quad p = 1, 2, \dots, p(\lambda) - 1,$$

and

$$(2.27) \quad n'_{p(\lambda)} \leq a^{-m} 2^{mp(\lambda)} \omega(a2^{1-p(\lambda)}m^{1/2}).$$

Since $\mu(\varepsilon) \leq \omega(\varepsilon)$ we have

$$(2.28) \quad n_p \leq a^{-m} 2^{mp} \omega(a2^{1-p}m^{1/2}), \quad p = 1, 2, \dots, p(\lambda).$$

Let $\tilde{N}_a(\lambda)$ denote the spectral counting function for the Neumann laplacian on a hypercube of size a in $\mathbb{R}^m$. Then

$$(2.29) \quad \tilde{N}_a(\lambda) = \#\{(k_1, k_2, \dots, k_m) \in (\mathbb{N} \cup \{0\})^m : \pi^2\{k_1^2 + \dots + k_m^2\} < \lambda a^2\}.$$

By (2.22) and (2.6),

$$(2.30) \quad \lambda^{1/2}a \geq 2^{1+1/m}.$$

Hence by (2.29) and (2.30),

$$\begin{aligned}
 (2.31) \quad \tilde{N}_a(\lambda) &\leq 2^{-m} \left| \left\{ x \in \mathbb{R}^m : |x| < \frac{\lambda^{1/2}a}{\pi} + m^{1/2} \right\} \right|_m \\
 &= (4\pi)^{-m/2} \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} a^m \lambda^{m/2} \left\{ 1 + \frac{\pi m^{1/2}}{\lambda^{1/2}a} \right\}^m \\
 &\leq (4\pi)^{-m/2} \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} a^m \lambda^{m/2} \left\{ 1 + \frac{\pi m^{3/2}}{\lambda^{1/2}a} \left(1 + \frac{\pi m^{1/2}}{2^{1+1/m}} \right)^m \right\}.
 \end{aligned}$$

By Neumann bracketing (see e.g. [15, XIII.15]) and by (2.31),

$$\begin{aligned}
 (2.32) \quad N_D(\lambda) &\leq \sum_{p=1}^{p(\lambda)-1} n_p \tilde{N}_{a2^{-p}}(\lambda) + n'_{p(\lambda)} \tilde{N}_{a2^{-p(\lambda)}}(\lambda) \\
 &\leq \sum_{p=1}^{p(\lambda)-1} n_p (4\pi)^{-m/2} \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} a^m 2^{-mp} \lambda^{m/2} \\
 &\quad \cdot \left\{ 1 + \frac{\pi m^{3/2} 2^p}{\lambda^{1/2}a} \left(1 + \frac{\pi m^{1/2}}{2^{1+1/m}} \right)^m \right\} + n'_{p(\lambda)} \tilde{N}_{a2^{-p(\lambda)}}(\lambda).
 \end{aligned}$$

Note that

$$(2.33) \quad \sum_{p=1}^{p(\lambda)-1} n_p a^m 2^{-mp} \leq |D|_m.$$

Hence by (2.32) and (2.33),

$$(2.34) \quad N_D(\lambda) \leq W_D(\lambda) + B_1 + B_2,$$

where

$$\begin{aligned}
 (2.35) \quad B_1 &= \sum_{p=1}^{p(\lambda)-1} n_p \pi m^{3/2} \left\{ \frac{1}{2\pi^{1/2}} + \frac{(\pi m)^{1/2}}{2^{2+1/m}} \right\}^m \\
 &\quad \cdot \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} a^{m-1} 2^{-(m-1)p} \lambda^{(m-1)/2},
 \end{aligned}$$

and

$$(2.36) \quad B_2 = n'_{p(\lambda)} \tilde{N}_{a2^{-p(\lambda)}}(\lambda).$$

By (2.29), and by the definition of $p(\lambda)$,

$$\begin{aligned}
 (2.37) \quad \tilde{N}_{a2^{-p(\lambda)}}(\lambda) &= \#\{(k_1, \dots, k_m) \in (\mathbb{N} \cup \{0\})^m : \pi^2(k_1^2 + \dots + k_m^2) < \lambda a^2 2^{-2p(\lambda)}\} \\
 &\leq \#\{(k_1, \dots, k_m) \in (\mathbb{N} \cup \{0\})^m : \pi^2(k_1^2 + \dots + k_m^2) < 4\} = 1.
 \end{aligned}$$

Hence for B_2 we obtain, by (2.36), (2.37), (2.28), and (2.21) and by the definition of $p(\lambda)$,

$$\begin{aligned}
 (2.38) \quad B_2 &= n'_{p(\lambda)} \leq a^{-m} 2^{mp(\lambda)} C(a 2^{1-p(\lambda)} \sqrt{m})^{m-d} \\
 &= (4m)^{(m-d)/2} C(a^{-1} 2^{p(\lambda)})^d \leq C(4m)^{(m-d)/2} \lambda^{d/2}.
 \end{aligned}$$

To estimate B_1 we first suppose $d = m - 1$. By (2.28), (2.21), and (2.35),

$$\begin{aligned}
 (2.39) \quad B_1 &= \sum_{p=1}^{p(\lambda)-1} C 2\pi m^2 \left\{ \frac{1}{2\pi^{1/2}} + \frac{(\pi m)^{1/2}}{2^{2+1/m}} \right\}^m \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{-1} \lambda^{(m-1)/2} \\
 &\leq C 2\pi m^2 \left\{ \frac{1}{2\pi^{1/2}} + \frac{(\pi m)^{1/2}}{2^{2+1/m}} \right\}^m \\
 &\quad \cdot (\log 4)^{-1} \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{-1} \lambda^{(m-1)/2} \log\{(2|D|_m)^{2/m} \lambda\}.
 \end{aligned}$$

By (2.38) and (2.39) we have, for $d = m - 1$ and for $\lambda \geq 4|D|_m^{-2/m}$,

$$(2.40) \quad B_1 + B_2 \leq C 10^m \lambda^{(m-1)/2} \log\{(2|D|_m)^{2/m} \lambda\}.$$

Suppose $m - 1 < d < m$. By (2.28), (2.21), and (2.35),

$$\begin{aligned}
 (2.41) \quad B_1 &= C a^{m-d-1} \lambda^{(m-1)/2} 2^{m-d} \pi m^{3/2} \left\{ \frac{1}{2\pi^{1/2}} + \frac{(\pi m)^{1/2}}{2^{2+1/m}} \right\}^m \\
 &\quad \cdot \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{-1} m^{(m-d)/2} \sum_{p=1}^{p(\lambda)-1} 2^{p(d+1-m)}.
 \end{aligned}$$

By the definition of $p(\lambda)$,

$$(2.42) \quad \sum_{p=1}^{p(\lambda)-1} 2^{p(d+1-m)} \leq \frac{2^{(p(\lambda)-1)(d+1-m)}}{1 - 2^{m-d-1}} \leq \frac{(a\lambda^{1/2})^{d+1-m}}{2^{d+1-m} - 1}.$$

Combining (2.38), (2.41), and (2.42), we obtain

$$(2.43) \quad B_1 + B_2 \leq C\lambda^{d/2}(4m)^{1/2} \left\{ 1 + \pi m \left(\frac{1}{2\pi^{1/2}} + \frac{(\pi m)^{1/2}}{2^{2+1/m}} \right)^m \cdot \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} \frac{1}{(1+d-m)\log 2} \right\} \\ \leq 10^m C\lambda^{d/2}(1+d-m)^{-1}.$$

The result now follows by combining (2.40), (2.43), and (2.34). $\square$

3. PROOF OF THEOREM 1.1 FOR $\alpha < m-1$

Since Ω is bounded in $\mathbb{R}^{m-1}$ ($m = 3, 4, \dots$) and $\partial\Omega$ is piecewise smooth, or Ω is an open, bounded interval in $\mathbb{R}^1$, there exist $\Lambda < \infty$, $\nu < \infty$, and $\kappa < \infty$ such that

$$(3.1) \quad \Lambda = \max\{|\gamma - \gamma'| : \gamma \in \partial\Omega, \gamma' \in \partial\Omega\},$$

$$(3.2) \quad |N_\Omega(\mu) - W_\Omega(\mu)| \leq \nu\mu^{(m-2)/2}, \quad \mu > 0,$$

$$(3.3) \quad \mu_\Omega(\varepsilon) = |\{\gamma \in \Omega : \text{dist}(\gamma, \partial\Omega) < \varepsilon\}|_{m-1} \leq \kappa\varepsilon, \quad \varepsilon > 0.$$

Assertion (3.2) (or (1.4)) follows from the results in [14], [19].

Theorem 3.1. *Suppose $f : [0, \infty) \rightarrow (0, 1]$ is monotonically decreasing to zero, right continuous, and $f(0) = 1$. Suppose that Ω is open, bounded and star-shaped with respect to $(0, \dots, 0)$, and suppose that (3.1)-(3.3) hold with $\Lambda < \infty$, $\nu < \infty$, and $\kappa < \infty$. Then for all $L > \Lambda$ and all λ sufficiently large (but independent of L),*

$$(3.4) \quad \left| N_{D(\Omega, f)}(\lambda) - \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \right| \\ \leq (10\Lambda^{m-2} + 6\kappa + 2\nu)L\lambda^{(m-1)/2} \log(2\Lambda L^2) \\ + \Lambda^{m-1}(f(L))^{(m-1)/2} \lambda^{(2m-1)/4} \\ \cdot \left\{ \int_{\{x \geq L : \Lambda f(x) \geq \pi\lambda^{-1/2}\}} dx (f(x))^{m-1} \right\}^{1/2}.$$

Proof. Since Ω is star-shaped with respect to $(0, \dots, 0)$, $D(\Omega, f)$ is horn-shaped in $\mathbb{R}^m$. By [4, Estimate (29)],

$$(3.5) \quad \left| N_{D(\Omega, f)}(\lambda) - \pi^{-1} \int_{[0, \infty)} dx \sum_{k=1}^{\infty} \{(\lambda - \mu_k(x))_+\}^{1/2} \right| \\ \leq (1 + \delta\lambda^{1/2}\pi^{-1})N_\Omega(\lambda) + \sum_{i=1}^{\infty} \#\{k \in \mathbb{Z}^+ : \mu_k(i\delta) < \lambda\},$$

where $\mu_k(x)$ is the k^{th} eigenvalue of the $(m-1)$ -dimensional Dirichlet laplacian on $D(\Omega, f) \cap \{(\xi, \eta) \in \mathbb{R}^m : \xi = x\}$. By scaling we have

$$(3.6) \quad \mu_k(x) = (f(x))^{-2} \mu_k.$$

By (3.6), and by monotonicity of f and N_Ω , we obtain

$$(3.7) \quad \sum_{i=1}^{\infty} \#\{k \in \mathbb{Z}^+ : \mu_k(i\delta) < \lambda\} = \sum_{i=1}^{\infty} N_\Omega(\lambda(f(i\delta))^2) \\ \leq \delta^{-1} \int_{[0, \infty)} N_\Omega(\lambda(f(x))^2) dx.$$

The choice

$$(3.8) \quad \delta = \pi^{1/2} \lambda^{-1/4} \{N_\Omega(\lambda)\}^{-1/2} \left\{ \int_{[0, \infty)} N_\Omega(\lambda(f(x))^2) dx \right\}^{1/2}$$

yields, by (3.5), (3.7), and (3.8),

$$(3.9) \quad \left| N_{D(\Omega, f)}(\lambda) - \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \right| \\ \leq N_\Omega(\lambda) + 2\pi^{-1/2} \lambda^{1/4} \{N_\Omega(\lambda)\}^{1/2} \left\{ \int_{[0, \infty)} N_\Omega(\lambda(f(x))^2) dx \right\}^{1/2}.$$

For $L > 0$ we define

$$(3.10) \quad D_L^1 = D(\Omega, f) \cap \{(x, y) \in \mathbb{R}^m : 0 < x < L\}, \\ D_L^2 = D(\Omega, f) \cap \{(x, y) \in \mathbb{R}^m : x > L\}.$$

By putting Dirichlet boundary conditions on the set $D(\Omega, f) \cap \{x = L\}$, we obtain

$$(3.11) \quad N_{D(\Omega, f)}(\lambda) \geq N_{D_L^1}(\lambda) + N_{D_L^2}(\lambda).$$

We use Theorem 2.1 to obtain a lower bound for $N_{D_L^1}$: define

$$(3.12) \quad \mu_{D_L^1}(\varepsilon) = |\{z \in D_L^1 : \text{dist}(z, \partial D_L^1) < \varepsilon\}|_m.$$

Lemma 3.2. *Suppose Ω and f satisfy the conditions of Theorem 3.1, (3.1), and (3.3). Then for all $\varepsilon > 0$*

$$(3.13) \quad \mu_{D_L^1}(\varepsilon) \leq (3\Lambda^{m-1} + L\kappa)\varepsilon.$$

Proof. Note that

$$(3.14) \quad |\{z \in D_L^1 : z = (x, y), x < \varepsilon \text{ or } x > L - \varepsilon\}|_m \leq 2\varepsilon|\Omega|_{m-1} \leq 2\varepsilon\Lambda^{m-1}.$$

Let

$$(3.15) \quad A = \{(x, y) \in D_L^1 : \\ \varepsilon < x < L - \varepsilon, y \in f(x)\Omega, \text{dist}((x, y), \partial D(\Omega, f)) < \varepsilon\}.$$

Suppose $z = (x, y) \in A$. Then there exists $(x', y') \in \partial D(\Omega, f)$ such that $|x - x'| < \varepsilon$. In fact, since f is monotonically decreasing, $x \leq x' \leq x + \varepsilon$. Since $y' \in f(x')\partial\Omega$, it follows that $y' \notin f(x')\Omega$. Hence $y' \notin f(x + \varepsilon)\Omega$. Then $y' \in f(x)\bar{\Omega} \setminus f(x + \varepsilon)\Omega$, so that $y \in f(x)\bar{\Omega} \setminus f(x + \varepsilon)\Omega$ or $y \in f(x + \varepsilon)\Omega$ and $\text{dist}(y, f(x + \varepsilon)\partial\Omega) < \varepsilon$. Hence

$$(3.16) \quad \begin{aligned} \int_A dz &\leq \\ &\leq \int_{[\varepsilon, L-\varepsilon]} dx \{ |f(x)\bar{\Omega} \setminus f(x + \varepsilon)\Omega|_{m-1} \\ &\quad + |\{y \in f(x + \varepsilon)\Omega : \text{dist}(y, f(x + \varepsilon)\partial\Omega) < \varepsilon\}|_{m-1} \} \\ &\leq \int_{[\varepsilon, L-\varepsilon]} dx \{ \{(f(x))^{m-1} - (f(x + \varepsilon))^{m-1}\} |\Omega|_{m-1} + \mu_\Omega(\varepsilon) \} \\ &\leq |\Omega|_{m-1} \left\{ \int_{[\varepsilon, 2\varepsilon]} (f(x))^{m-1} dx - \int_{[L-\varepsilon, L]} (f(x))^{m-1} dx \right\} + \int_{[\varepsilon, L-\varepsilon]} \kappa \varepsilon dx \\ &\leq (\Lambda^{m-1} + L\kappa)\varepsilon, \end{aligned}$$

where we have used various scaling properties and (3.3). The lemma follows by (3.14) and (3.16). $\square$

Lemma 3.3. For $L > \Lambda$ and

$$(3.17) \quad \lambda \geq 4(\Lambda(f(\Lambda))^{m-1}|\Omega|_{m-1})^{-2/m},$$

$$(3.18) \quad \begin{aligned} N_{D_L^1}(\lambda) &\geq \pi^{-1} \int_{[0, L]} dx \int_{[0, \infty)} \left\{ \{\lambda - \mu(f(x))^{-2}\}_+ \right\}^{1/2} dN_\Omega(\mu) \\ &\quad - (9\Lambda^{m-2} + 3\kappa + \nu)L\Lambda^{(m-1)/2} \log(2\lambda L^2). \end{aligned}$$

Proof. For $L > \Lambda$ we have

$$(3.19) \quad |D_L^1|_m \geq \Lambda(f(\Lambda))^{m-1}|\Omega|_{m-1}.$$

Lemma (3.2) combined with (3.17) and (3.19) guarantee that Theorem 2.1 holds with constants $d = m - 1$ and $C = 3\Lambda^{m-1} + L\kappa$. By Theorem 2.1, we thus have that for $L > \Lambda$ and λ satisfying (3.17),

$$(3.20) \quad N_{D_L^1}(\lambda) \geq W_{D_L^1}(\lambda) - (9\Lambda^{m-1} + 3L\kappa)\lambda^{(m-1)/2} \log(2\lambda L^2).$$

Moreover, using (3.2) we obtain the following:

$$(3.21) \quad \begin{aligned} & \frac{1}{\pi} \int_{[0,L]} dx \int_{[0,\infty)} \{(\lambda - (f(x))^{-2}\mu)_+\}^{1/2} dN_{\Omega}(\mu) \\ &= \frac{1}{2\pi} \int_{[0,L]} dx \int_{[0,\lambda(f(x))^2)} (\lambda - (f(x))^{-2}\mu)^{-1/2} (f(x))^{-2} N_{\Omega}(\mu) d\mu \\ &\leq \frac{1}{2\pi} \int_{[0,L]} dx \int_{[0,\lambda(f(x))^2)} (\lambda - (f(x))^{-2}\mu)^{-1/2} (f(x))^{-2} \\ &\quad \cdot \{W_{\Omega}(\mu) + \nu\mu^{(m-2)/2}\} d\mu \\ &\leq W_{D_L^1}(\lambda) + \nu L\lambda^{(m-2)/2}. \end{aligned}$$

By (3.17) and (3.1)

$$(3.22) \quad \lambda \geq 4(\Lambda|\Omega|_{m-1})^{-2/m} \geq 4\Lambda^{-2} \geq 4L^{-2}.$$

Hence

$$(3.23) \quad \nu L\lambda^{(m-2)/2} \leq 2^{-1}\nu L\Lambda^{(m-1)/2} \leq \nu L\lambda^{(m-1)/2} \log(2\lambda L^2),$$

and the Lemma follows by (3.20), (3.21), and (3.23). $\square$

In order to obtain a lower bound for $N_{D_L^2}$ we note that D_L^2 is a region of the form (1.6), with Ω and f being replaced by $f(L)\Omega$ and $f(x+L)/f(L)$ respectively. Before we use (3.9) we prove the following lemma.

Lemma 3.4. *Suppose Ω satisfies (3.1). Then for all $\lambda > 0$,*

$$(3.24) \quad N_{\Omega}(\lambda) \leq [\pi^{-1}\Lambda^{m-1}\lambda^{(m-1)/2}] \leq \pi^{-1}\Lambda^{m-1}\lambda^{(m-1)/2},$$

$$(3.25) \quad \mu_1 \geq \left(\frac{\pi}{\Lambda}\right)^2.$$

Proof. There exists an $(m-1)$ -dimensional hypercube H with volume Λ^{m-1} such that $\Omega \subset H$. By monotonicity of the Dirichlet eigenvalues,

$$\begin{aligned}
 (3.26) \quad N_{\Omega}(\lambda) &\leq N_H(\lambda) \\
 &= \# \left\{ (k_1, \dots, k_{m-1}) \in \mathbb{Z}_+^{m-1} : k_1^2 + \dots + k_{m-1}^2 < \frac{\lambda \Lambda^2}{\pi^2} \right\} \\
 &\leq 2^{1-m} \frac{\pi^{(m-1)/2}}{\Gamma((m+1)/2)} \left(\frac{\lambda \Lambda^2}{\pi^2} \right)^{(m-1)/2} \\
 &\leq \pi^{-1} \Lambda^{m-1} \lambda^{(m-1)/2}.
 \end{aligned}$$

Since $N_{\Omega}(\lambda)$ is an integer, we also have the first inequality in (3.25). Finally

$$(3.27) \quad \mu_1 \geq (m-1) \left(\frac{\pi}{\Lambda} \right)^2 \geq \left(\frac{\pi}{\Lambda} \right)^2.$$

By (3.9),

$$\begin{aligned}
 (3.28) \quad N_{D_L^2}(\lambda) &\geq \pi^{-1} \int_{[L, \infty)} dx \int_{[0, \infty)} \left\{ \left\{ \lambda - \mu(f(x))^{-2} \right\}_+ \right\}^{1/2} dN_{\Omega}(\mu) - N_{f(L)\Omega}(\lambda) \\
 &\quad - 2\pi^{-1/2} \lambda^{1/4} \left\{ N_{f(L)\Omega}(\lambda) \int_{[0, \infty)} N_{f(L)\Omega} \left(\lambda \left(\frac{f(x+L)}{f(L)} \right)^2 \right) dx \right\}^{1/2}.
 \end{aligned}$$

By Lemma 3.4,

$$(3.29) \quad N_{f(L)\Omega}(\lambda) = N_{\Omega}(\lambda(f(L))^2) \leq \pi^{-1} \Lambda^{m-1} (f(L))^{m-1} \lambda^{(m-1)/2},$$

and

$$\begin{aligned}
 (3.30) \quad &\left\{ N_{f(L)\Omega}(\lambda) \int_{[0, \infty)} N_{f(L)\Omega} \left(\lambda \left(\frac{f(x+L)}{f(L)} \right)^2 \right) dx \right\}^{1/2} \\
 &= \left\{ N_{\Omega}(\lambda(f(L))^2) \int_{[L, \infty)} N_{\Omega}(\lambda(f(x))^2) dx \right\}^{1/2} \\
 &\leq \left\{ N_{\Omega}(\lambda(f(L))^2) \int_{\{x \geq L, \Lambda f(x) \geq \pi \lambda^{-1/2}\}} N_{\Omega}(\lambda(f(x))^2) dx \right\}^{1/2} \\
 &\leq (f(L))^{(m-1)/2} \lambda^{(m-1)/2} \left\{ \int_{\{x \geq L, \Lambda f(x) \geq \pi \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2}.
 \end{aligned}$$

Combining (3.11), Lemma (3.3), and (3.28)-(3.30), we obtain that for $L > \Lambda$ and λ large,

$$(3.31) \quad N_{D(\Omega, f)}(\lambda) \geq \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \infty)} \{ \{ \lambda - \mu(f(x))^{-2} \}_+ \}^{1/2} dN_{\Omega}(\mu) \\ - (10\Lambda^{m-2} + 3\kappa + \nu)L\lambda^{(m-1)/2} \log(2\lambda L^2) \\ - \Lambda^{m-1}(f(L))^{(m-1)/2} \lambda^{(2m-1)/4} \\ \cdot \left\{ \int_{\{x \geq L, \Lambda f(x) \geq \pi \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2}.$$

To prove the upper bound in Theorem 3.1, we put Neumann boundary conditions on $D(\Omega, f) \cap \{(x, y) : x = L\}$. Then

$$(3.32) \quad N_{D(\Omega, f)}(\lambda) \leq \tilde{N}_{D_L^1}(\lambda) + \tilde{N}_{D_L^2}(\lambda),$$

where $\tilde{N}_{D_L^1}$, $\tilde{N}_{D_L^2}$ are the counting functions for D_L^1 , D_L^2 with Neumann boundary conditions at $x = L$ and Dirichlet boundary conditions elsewhere. Following [4, Theorem 8 and its proof], (3.5)-(3.9), and (3.28)-(3.30),

$$(3.33) \quad \tilde{N}_{D_L^2}(\lambda) \leq \pi^{-1} \int_{[L, \infty)} dx \int_{[0, \infty)} \{ \{ \lambda - \mu(f(x))^{-2} \}_+ \}^{1/2} dN_{\Omega}(\mu) \\ + \pi^{-1} \Lambda^{m-1}(f(L))^{m-1} \lambda^{(m-1)/2} \\ + \Lambda^{m-1}(f(L))^{(m-1)/2} \lambda^{(2m-1)/4} \\ \cdot \left\{ \int_{\{x \geq L, \Lambda f(x) \geq \pi \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2}. \quad \square$$

Lemma 3.5. For $L > \Lambda$ and λ sufficiently large (independently of L),

$$(3.34) \quad \tilde{N}_{D_L^1}(\lambda) \leq \pi^{-1} \int_{[0, L)} dx \int_{[0, \infty)} \{ \{ \lambda - \mu(f(x))^{-2} \}_+ \}^{1/2} dN_{\Omega}(\mu) \\ + ((9 + \pi^{-1})\Lambda^{m-1} + 6L\kappa + 2\nu L)\lambda^{(m-1)/2} \log(2\lambda L^2).$$

Proof. Let $R = \Omega \times (0, L)$, $H_c = R \setminus \overline{D_L^1}$, and denote by $\tilde{N}_{H_c}$ the counting function for the Laplace operator for H_c with Neumann boundary conditions at $x = L$ and Dirichlet conditions elsewhere. Then

$$(3.35) \quad \tilde{N}_{D_L^1}(\lambda) \leq \tilde{N}_{D_L^1}(\lambda) + \tilde{N}_{H_c}(\lambda) - N_{H_c}(\lambda) \leq \tilde{N}_R(\lambda) - N_{H_c}(\lambda),$$

where $\tilde{N}_R$ is the counting function for the Laplace operator for R with Neumann boundary conditions at $x = L$ and Dirichlet conditions elsewhere. The second inequality in (3.35) follows from the fact that removing the Dirichlet boundary conditions on $\partial D_L^1 \cap \partial H_c$ decreases the spectrum, and hence increases the counting function to $\tilde{N}_R$. By separation of variables,

$$(3.36) \quad \begin{aligned} \tilde{N}_R(\lambda) &= \# \left\{ (k_1, k_2) \in \mathbb{Z}_+^2 : \mu_{k_1} + \frac{\pi^2(k_2 - \frac{1}{2})^2}{L^2} < \lambda \right\} \\ &\leq N_\Omega(\lambda) + \sum_{k_2 \geq 2} N_\Omega \left(\lambda - \frac{\pi^2(k_2 - \frac{1}{2})^2}{L^2} \right). \end{aligned}$$

Since N_Ω is monotone,

$$(3.37) \quad \begin{aligned} \tilde{N}_R(\lambda) &\leq N_\Omega(\lambda) + \int_{[0, \infty)} dx N_\Omega \left(\lambda - \frac{\pi^2 x^2}{L^2} \right) \\ &= N_\Omega(\lambda) + \frac{L}{2\pi} \int_{[0, \lambda)} d\xi N_\Omega(\xi) (\lambda - \xi)^{-1/2}. \end{aligned}$$

By Lemma (3.4) and (3.2),

$$(3.38) \quad \begin{aligned} \tilde{N}_R(\lambda) &\leq \\ &\leq \pi^{-1} \Lambda^{m-1} \lambda^{(m-1)/2} + \frac{L}{2\pi} \int_{[0, \lambda)} d\xi \{W_\Omega(\xi) + \nu \xi^{(m-2)/2}\} (\lambda - \xi)^{-1/2} \\ &= \pi^{-1} \Lambda^{m-1} \lambda^{(m-1)/2} + W_R(\lambda) + \frac{\nu L}{(4\pi)^{1/2}} \cdot \frac{\Gamma(m/2)}{\Gamma((m+1)/2)} \lambda^{(m-1)/2} \\ &\leq W_R(\lambda) + (\pi^{-1} \Lambda^{m-1} + \nu L) \lambda^{(m-1)/2}. \end{aligned}$$

It remains to find a lower bound for N_{H_c} . In order to apply Theorem 2.1 we need a bound on $\mu_{H_c}(\varepsilon)$.

Lemma 3.6. *Suppose Ω and f satisfy the conditions of Theorem 3.1, (3.1), and (3.3). Then*

$$(3.39) \quad \mu_{H_c}(\varepsilon) \leq (3\Lambda^{m-1} + 2L\kappa)\varepsilon.$$

Proof. We note the following:

$$(3.40) \quad |\{(x, y) \in H_c : x \geq L - \varepsilon\}|_m \leq |\Omega|_{m-1} \varepsilon \leq \Lambda^{m-1} \varepsilon,$$

$$(3.41) \quad |\{(x, y) \in H_c : 0 < x \leq \varepsilon\}|_m \leq |\Omega|_{m-1} \varepsilon \leq \Lambda^{m-1} \varepsilon,$$

$$(3.42) \quad |\{(x, y) \in H_c : \text{dist}(y, \partial\Omega) < \varepsilon\}|_m \leq L\kappa\varepsilon.$$

Let

$$(3.43) \quad B = \{(x, y) \in H_c : \varepsilon < x < L - \varepsilon, y \in \Omega \setminus f(x)\Omega, \\ \text{dist}((x, y), \partial D(\Omega, f)) < \varepsilon\}.$$

Suppose $(x, y) \in B$. Then there exists $(x', y') \in \partial D(\Omega, f)$ such that $|x - x'| < \varepsilon$. Since f is monotonically decreasing, $x - \varepsilon \leq x' \leq x$. Since $y' \in f(x')\partial\Omega$, it follows that $y' \notin \Omega \setminus f(x - \varepsilon)\overline{\Omega}$. Then $y' \in (\overline{\Omega} \setminus f(x)\Omega) \setminus (\Omega \setminus f(x - \varepsilon)\overline{\Omega})$, so that $y \in (\overline{\Omega} \setminus f(x)\Omega) \setminus (\Omega \setminus f(x - \varepsilon)\overline{\Omega})$, or $y \in \Omega \setminus f(x - \varepsilon)\overline{\Omega}$ and $\text{dist}(y, f(x - \varepsilon)\partial\Omega) < \varepsilon$. Hence

$$(3.44) \quad \int_B dz \leq \int_{[\varepsilon, L - \varepsilon]} dx \{ |(\overline{\Omega} \setminus f(x)\Omega) \setminus (\Omega \setminus f(x - \varepsilon)\overline{\Omega})|_{m-1} \\ + |\{y \in f(x - \varepsilon)\Omega : \text{dist}(y, f(x - \varepsilon)\partial\Omega) < \varepsilon\}|_{m-1} \} \\ \leq \int_{[\varepsilon, L - \varepsilon]} dx \{ \{(f(x - \varepsilon))^{m-1} - (f(x))^m\} |\Omega|_{m-1} + \mu_\Omega(\varepsilon) \} \\ \leq \varepsilon |\Omega|_{m-1} + (L - 2\varepsilon)\kappa\varepsilon \\ \leq (|\Lambda|^{m-1} + L\kappa)\varepsilon.$$

We obtain (3.39) by adding up the contributions from (3.40), (3.41), (3.42), and (3.44). $\square$

Since $|H_c|_m \rightarrow \infty$ as $L \rightarrow \infty$, we have that for λ sufficiently large, $\lambda \geq 4(|H_c|_m)^{-2/m}$. Therefore by Theorem 2.1, for all λ sufficiently large and $L > \Lambda$,

$$(3.45) \quad N_{H_c}(\lambda) \geq W_{H_c}(\lambda) - 3(3\Lambda^{m-1} + 2L\kappa)\lambda^{(m-1)/2} \log\{\lambda(2|H_c|_m)^{2/m}\} \\ \geq W_{H_c}(\lambda) - 3(3\Lambda^{m-1} + 2L\kappa)\lambda^{(m-1)/2} \log(2\lambda L^2).$$

For all λ sufficiently large, we have by (3.35), (3.38), and (3.45)

$$(3.46) \quad \tilde{N}_{D_L^1}(\lambda) \leq W_R(\lambda) - W_{H_c}(\lambda) + (9\Lambda^{m-1} + 6L\kappa + \nu L)\lambda^{(m-1)/2} \log(2\lambda L^2) \\ = W_{D_L^1}(\lambda) + ((9 + \pi^{-1})\Lambda^{m-1} + 6L\kappa + \nu L)\lambda^{(m-1)/2} \log(2\lambda L^2).$$

To complete the proof of Lemma 3.5 we note that by (3.2),

$$(3.47) \quad \pi^{-1} \int_{[0, L]} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x)))^{-2}\}_+^{1/2} dN_\Omega(\mu) \\ \geq \frac{1}{2\pi} \int_{[0, L]} dx \int_{[0, \lambda(f(x))^2)} (\lambda - (f(x))^{-2}\mu)^{-1/2} (f(x))^{-2} \\ \cdot \{W_\Omega(\mu) - \nu\mu^{(m-2)/2}\} d\mu \geq$$

$$\geq W_{D_L^1}(\lambda) - \nu L \lambda^{(m-1)/2}.$$

Lemma 3.5 follows by (3.46) and (3.47). $\square$

From (3.33) and (3.34), we obtain that for $L > \Lambda$ and λ sufficiently large,

$$\begin{aligned} (3.48) \quad N_{D(\Omega, f)}(\lambda) &\leq \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_{\Omega}(\mu) \\ &\quad + (10\Lambda^{m-2} + 6\kappa + 2\nu)L\lambda^{(m-1)/2} \log(2\lambda L^2) \\ &\quad + \Lambda^{m-1} (f(L))^{(m-1)/2} \lambda^{(2m-1)/4} \\ &\quad \cdot \left\{ \int_{\{x \geq L: \Lambda f(x) \geq \pi \lambda^{-1/2}\}} dx (f(x))^{m-1} \right\}^{1/2}. \end{aligned}$$

Theorem 3.1 follows from (3.31) and (3.48). $\square$

Remark 3.7. By using (3.2) it is easily seen that, if $D(\Omega, f)$ has finite volume, then the double integral in the left hand side of (3.4) contains the leading (Weyl) term, whereas the right hand side of (3.4) is $O(\lambda^{(m-1)/2} \log \lambda)$ with the choice $L = 2\Lambda$. However, in order to extract two-term asymptotics from (3.4) for finite volume horns, we require that f decays in a regular fashion, and not faster than the critical power for α in Theorem 1.1 (v).

Proof of Theorem 1.1 (v) and (vi). Since

$$(3.49) \quad f_{\alpha}(L) = (1 + L)^{-1/\alpha},$$

we obtain that the right hand side of (3.4) is bounded by

$$(3.50) \quad c \{L\lambda^{(m-1)/2} + L^{(\alpha+2-2m)/(2\alpha)} \lambda^{(2m-1)/4}\} \log(2\lambda L^2),$$

for some constant $c < \infty$. Choose λ sufficiently large, and

$$(3.51) \quad L = \lambda^{\alpha/(2\alpha+4m-4)};$$

then (3.50) and (3.51) show that the right hand side of (3.4) is

$$O(\lambda^{(m\alpha+2m^2-4m+2)/(2\alpha+4m-4)}).$$

Finally, we compute the integral in the left hand side of (3.4):

$$\begin{aligned}
 (3.52) \quad & \pi^{-1} \int_{[0,\infty)} dx \int_{[0,\infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \\
 &= (2\pi)^{-1} \int_{[0,\infty)} dx (f(x))^{-2} \int_{[0,\lambda(f(x))^2)} d\mu (\lambda - \mu(f(x))^{-2})^{-1/2} W_\Omega(\mu) \\
 &\quad + (2\pi)^{-1} \int_{[1,\infty)} dx x^{2/\alpha} \int_{[0,\lambda x^{-2/\alpha})} d\mu (\lambda - \mu x^{2/\alpha})^{-1/2} (N_\Omega(\mu) - W_\Omega(\mu)).
 \end{aligned}$$

The first term in the right hand side of (3.52) can be computed explicitly, by using (1.3) for the open bounded set Ω in $\mathbb{R}^{m-1}$, if we make the change of variables $\mu = \xi \lambda (f(x))^2$. This yields

$$(3.53) \quad (4\pi)^{-m/2} \left(\Gamma \left(1 + \frac{m}{2} \right) \right)^{-1} |\Omega|_{m-1} \int_{[0,\infty)} (f(x))^{m-1} dx = W_{D(\Omega,f)}(\lambda).$$

Suppose $m - 1 > \alpha > m - 2$. We write the second term in the right hand side of (3.52) as

$$\begin{aligned}
 (3.54) \quad & (2\pi)^{-1} \int_{[0,\infty)} dx x^{2/\alpha} \int_{[0,\lambda x^{-2/\alpha})} d\mu (\lambda - \mu x^{2/\alpha})^{-1/2} (N_\Omega(\mu) - W_\Omega(\mu)) \\
 & - (2\pi)^{-1} \int_{[0,1)} dx x^{2/\alpha} \int_{[0,\lambda x^{-2/\alpha})} d\mu (\lambda - \mu x^{2/\alpha})^{-1/2} (N_\Omega(\mu) - W_\Omega(\mu)).
 \end{aligned}$$

It is straightforward to check that the first double integral in (3.54) converges absolutely, since $m - 2 < \alpha < m - 1$. By Fubini's theorem this term is equal to $\eta(\alpha, \Omega) \lambda^{(1+\alpha)/2}$. The second term is bounded in absolute value by

$$\begin{aligned}
 (3.55) \quad & (2\pi)^{-1} \int_{[0,\lambda)} d\mu (\lambda - \mu)^{-1/2} |N_\Omega(\mu) - W_\Omega(\mu)| \\
 & \leq (2\pi)^{-1} \nu \int_{[0,\lambda)} d\mu (\lambda - \mu)^{-1/2} \mu^{(m-2)/2} \leq \nu \lambda^{(m-1)/2}.
 \end{aligned}$$

Suppose $m > 2$ and $\alpha \leq m - 2$. The absolute value of the second term in the right-hand side of (3.52) is bounded from above by

$$\begin{aligned}
 (3.56) \quad & (2\pi)^{-1} \int_{[0,\lambda)} d\mu \int_{[1,(\lambda/\mu)^{\alpha/2})} dx x^{2/\alpha} (\lambda - \mu x^{2/\alpha})^{-1/2} |N_\Omega(\mu) - W_\Omega(\mu)| \\
 & \leq \frac{\alpha}{2\pi} \lambda^{\alpha/2} \int_{[0,\lambda)} d\mu \mu^{-(\alpha+2)/2} (\lambda - \mu)^{1/2} |N_\Omega(\mu) - W_\Omega(\mu)|.
 \end{aligned}$$

The contribution from the interval $[0, \mu_1)$ in (3.56) is $O(\lambda^{(1+\alpha)/2})$, since $N_\Omega(\mu) = 0$ for $0 \leq \mu \leq \mu_1$, and W_Ω is given by (1.3). The contribution from the interval

$[\mu_1, \lambda)$ in (3.56) is, by (3.2), bounded from above by

$$(3.57) \quad \frac{\alpha v}{2\pi} \lambda^{(1+\alpha)/2} \int_{[\mu_1, \lambda)} d\mu \mu^{(m-\alpha-4)/2} = O(\lambda^{(m-1)/2}).$$

To complete the proof, we note that $m-1 > (1-m+[(m-1)(9m-17)]^{1/2})/2 \geq m-2$, and so we separately consider two cases:

Case I: $m-1 > \alpha > (1-m+[(m-1)(9m-17)]^{1/2})/2$,

Case II: $(1-m+[(m-1)(9m-17)]^{1/2})/2 > \alpha \geq m-2$.

In Case I, we have that

$$\frac{1+\alpha}{2}$$

is strictly greater than both $(m\alpha+2m^2-4m+2)/(2\alpha+4m-4)$ and $(m-1)/2$; and so $\eta(\alpha, \Omega)\lambda^{(1+\alpha)/2}$ is the second term in the asymptotic expansion. In Case II (which can only occur for $m > 2$), we have that

$$\frac{m\alpha+2m^2-4m+2}{2\alpha+4m-4}$$

is greater or equal to both $(1+\alpha)/2$ and $(m-1)/2$. This completes the proof of (v) and (vi). $\square$

4. PROOF OF THEOREM 1.1 FOR $\alpha = m-1$

In this section we use Theorem 3.1 to prove Theorem 1.1 for $\alpha = m-1$. Substitution of

$$(4.1) \quad f(x) = (1+x)^{-1/(m-1)}$$

into the right hand side of (3.4) yields a bound

$$(4.2) \quad c\{L\lambda^{(m-1)/2} \log(2\lambda L^2) + L^{-1/2}\lambda^{(2m-1)/4} \log \lambda\}$$

for some $c < \infty$. Choose λ sufficiently large and $L = \lambda^{1/6}$. Then (4.2) gives the remainder estimate in Theorem 1.1 (iv). To compute the double integral in the left hand side of (3.4), we write

$$\begin{aligned} (4.3) \quad & \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \infty)} dN_{\Omega}(\mu) \{(\lambda - \mu(1+x)^{2/(m-1)})_+\}^{1/2} \\ &= \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \lambda)} dN_{\Omega}(\mu) \{(\lambda - \mu x^{2/(m-1)})_+\}^{1/2} \\ &\quad - \pi^{-1} \int_{[0, \lambda)} d(N_{\Omega}(\mu) - W_{\Omega}(\mu)) \int_{[0, 1)} dx \{(\lambda - \mu x^{2/(m-1)})_+\}^{1/2} \\ &\quad - \pi^{-1} \int_{[0, \lambda)} dW_{\Omega}(\mu) \int_{[0, 1)} dx \{(\lambda - \mu x^{2/(m-1)})_+\}^{1/2}. \end{aligned}$$

To compute the third term in the right hand side of (4.3), we make the substitution $x = y^{(m-1)/2}$, and subsequently

$$\mu = (\lambda/y)\xi.$$

This gives

$$\begin{aligned} (4.4) \quad & -\frac{m-1}{2\pi} \int_{[0,\lambda)} dW_{\Omega}(\mu) \int_{[0,1)} dy y^{(m-3)/2} (\lambda - \mu y)^{1/2} \\ & = -\frac{m-1}{2} (4\pi)^{-m/2} \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{-1} |\Omega|_{m-1} \\ & \quad \cdot \left[\psi\left(\frac{m+2}{2}\right) - \psi\left(\frac{m-1}{2}\right) \right] \lambda^{m/2}, \end{aligned}$$

where we have used [7, 4.253.1], and where ψ is defined in [7, 8.360]. By [7, 8.365.1, 8.365.2, and 8.375.2], we obtain

$$\begin{aligned} (4.5) \quad & \psi\left(\frac{m+2}{2}\right) - \psi\left(\frac{m-1}{2}\right) = \frac{2}{m} + \psi\left(\frac{m}{2}\right) - \psi\left(\frac{m-1}{2}\right) \\ & = 2 \left[(-1)^m \log 2 + \sum_{k=1}^m \frac{(-1)^{k+m}}{k} + \frac{1}{m-1} \right]. \end{aligned}$$

To estimate the second term in the right hand side of (4.3) we integrate by parts to obtain

$$\begin{aligned} (4.6) \quad & - (2\pi)^{-1} \int_{[0,\lambda)} d\mu (N_{\Omega}(\mu) - W_{\Omega}(\mu)) \\ & \quad \cdot \int_{[0,1)} dx x^{2/(m-1)} (\lambda - \mu x^{2/(m-1)})^{-1/2} \\ & - \pi^{-1} (N_{\Omega}(\lambda) - W_{\Omega}(\lambda)) \int_{[0,1)} dx (\lambda - \lambda x^{2/(m-1)})^{1/2}. \end{aligned}$$

The second term in (4.6) is estimated by (3.2) and is $O(\lambda^{(m-1)/2})$. The first term in (4.6) is bounded using (3.2) by

$$\begin{aligned} (4.7) \quad & (2\pi)^{-1} \int_{[0,\lambda)} d\mu |N_{\Omega}(\mu) - W_{\Omega}(\mu)| \int_{[0,1)} dx x^{2/(m-1)} (\lambda - \mu x^{2/(m-1)})^{-1/2} \\ & \leq (2\pi)^{-1} \int_{[0,\lambda)} d\mu |N_{\Omega}(\mu) - W_{\Omega}(\mu)| (\lambda - \mu)^{-1/2} \\ & = O(\lambda^{(m-1)/2}). \end{aligned}$$

Applying Fubini's Theorem to the first term in the right hand side of (4.3) gives

$$\begin{aligned}
 (4.8) \quad & \pi^{-1} \int_{[0, \lambda)} dN_{\Omega}(\mu) \int_{[0, (\lambda/\mu)^{(m-1)/2})} dx (\lambda - \mu x^{2/(m-1)})^{1/2} \\
 &= (4\pi)^{-m/2} \left(\Gamma\left(\frac{m+2}{2}\right) \right)^{-1} |\Omega|_{m-1} \lambda^{m/2} \\
 &\quad + (4\pi)^{-1/2} \Gamma\left(\frac{m+1}{2}\right) \left(\Gamma\left(\frac{m+2}{2}\right) \right)^{-1} \frac{m-1}{2} \lambda^{m/2} \\
 &\quad \cdot \int_{[\mu_1, \lambda)} \mu^{-(m+1)/2} W_{\Omega}(\mu) d\mu \\
 &\quad + (4\pi)^{-1/2} \Gamma\left(\frac{m+1}{2}\right) \left(\Gamma\left(\frac{m+2}{2}\right) \right)^{-1} (N_{\Omega}(\lambda) - W_{\Omega}(\lambda)) \lambda^{1/2} \\
 &\quad + (4\pi)^{-1/2} \Gamma\left(\frac{m+1}{2}\right) \left(\Gamma\left(\frac{m+2}{2}\right) \right)^{-1} \frac{m-1}{2} \lambda^{m/2} \\
 &\quad \cdot \int_{[\mu_1, \infty)} \mu^{-(m+1)/2} (N_{\Omega}(\mu) - W_{\Omega}(\mu)) d\mu \\
 &\quad - (4\pi)^{-1/2} \Gamma\left(\frac{m+1}{2}\right) \left(\Gamma\left(\frac{m+2}{2}\right) \right)^{-1} \frac{m-1}{2} \lambda^{m/2} \\
 &\quad \cdot \int_{[\lambda, \infty)} \mu^{-(m+1)/2} (N_{\Omega}(\mu) - W_{\Omega}(\mu)) d\mu.
 \end{aligned}$$

The second term in the right hand side of (4.8) is equal to

$$(4.9) \quad (4\pi)^{-m/2} \left(\Gamma\left(\frac{m+2}{2}\right) \right)^{-1} \frac{m-1}{2} |\Omega|_{m-1} \lambda^{m/2} \log \frac{\lambda}{\mu_1}.$$

The third and fifth terms in the right hand side of (4.8) can be estimated by (3.2) and are both $O(\lambda^{(m-1)/2})$. To complete the proof of (1.14) we combine (4.4), (4.5), (4.8), and (4.9). $\square$

5. PROOF OF THEOREM 1.1 FOR $\alpha > m - 1$

Theorem 5.1. *Suppose $f : [0, \infty) \rightarrow (0, 1]$ is monotonically decreasing to 0, right continuous, and $f(0) = 1$. Suppose that Ω is an open, bounded set in $\mathbb{R}^{m-1}$, and star-shaped with respect to $(0, \dots, 0)$. Suppose that (3.1)-(3.3) hold for $\Lambda < \infty$, $\nu < \infty$, and $\kappa < \infty$. Let $0 = L_0 < L_1 < \dots < L_{K+1} = \infty$ for some non-negative integer K . Then*

$$\begin{aligned}
 (5.1) \quad & \left| N_{D(\Omega, f)}(\lambda) - \pi^{-1} \int_{[0, \infty)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x)))^{-2}\}_+^{1/2} dN_{\Omega}(\mu) \right| \\
 & \leq \sum_{i=1}^{K+1} \Lambda^{m-1} \left\{ \lambda^{(m-1)/2} (f(L_{i-1}))^{m-1} + 3\lambda^{(2m-1)/4} (f(L_{i-1}))^{(m-1)/2} \right. \\
 & \quad \cdot \left. \left\{ \int_{\{L_{i-1} \leq x < L_i, f(x) \geq \Lambda^{-1}\lambda^{-1/2}\}} dx (f(x))^{m-1} \right\}^{1/2} \right\}.
 \end{aligned}$$

Proof. Define

$$(5.2) \quad D_i(\Omega, f) = D(\Omega, f) \cap \{(x, y) \in \mathbb{R}^m : L_{i-1} < x < L_i\}.$$

The open set D_i is a one-sided horn-shaped open set in $\mathbb{R}^m$ with $\mu_1(x) = \infty$ for $x > L_i$. By [4, Theorem 8], we have for each of the D_i

$$\begin{aligned}
 (5.3) \quad & \left| N_{D_i(\Omega, f)}(\lambda) - \pi^{-1} \int_{[L_{i-1}, L_i)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x)))^{-2}\}_+^{1/2} dN_{\Omega}(\mu) \right| \\
 & \leq (1 + \delta\lambda^{1/2}\pi^{-1}) \# \{k \in \mathbb{Z}^+ : \mu_k(L_{i-1}) < \lambda\} \\
 & \quad + \sum_{\{j \in \mathbb{Z}^+ : L_{i-1} + j\delta < L_i\}} \# \{k \in \mathbb{Z}^+ : \mu_k(L_{i-1} + j\delta) < \lambda\}.
 \end{aligned}$$

By Lemma 3.4,

$$(5.4) \quad \delta\lambda^{1/2}\pi^{-1} \# \{k \in \mathbb{Z}^+ : \mu_k(L_{i-1}) < \lambda\} \leq \delta\lambda^{m/2}\Lambda^{m-1} (f(L_{i-1}))^{m-1},$$

and

$$\begin{aligned}
 (5.5) \quad & \sum_{\{j \in \mathbb{Z}^+ : L_{i-1} + j\delta < L_i\}} \# \{k \in \mathbb{Z}^+ : \mu_k(L_{i-1} + j\delta) < \lambda\} \\
 & \leq \sum_{\{j \in \mathbb{Z}^+ : L_{i-1} + j\delta < L_i, \lambda(f(L_{i-1} + j\delta))^2 > \mu_1\}} N_{\Omega}(\lambda(f(L_{i-1} + j\delta))^2) \\
 & \leq \Lambda^{m-1} \lambda^{(m-1)/2} \sum_{\{j \in \mathbb{Z}^+ : L_{i-1} + j\delta < L_i, \lambda(f(L_{i-1} + j\delta))^2 > \mu_1\}} (f(L_{i-1} + j\delta))^{m-1} \\
 & \leq \Lambda^{m-1} \lambda^{(m-1)/2} \delta^{-1} \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1}\lambda^{-1/2}\}} (f(x))^{m-1} dx.
 \end{aligned}$$

The choice

$$\begin{aligned}
 (5.6) \quad & \delta = (f(L_{i-1}))^{-(m-1)/2} \lambda^{-1/4} \\
 & \cdot \left\{ \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1}\lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2}
 \end{aligned}$$

yields

$$\begin{aligned}
 (5.7) \quad & \left| N_{D_i(\Omega, f)}(\lambda) - \pi^{-1} \int_{[L_{i-1}, L_i]} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \right| \\
 & \leq \lambda^{(m-1)/2} \Lambda^{m-1} (f(L_{i-1}))^{m-1} + 2\lambda^{(2m-1)/4} \Lambda^{m-1} (f(L_{i-1}))^{(m-1)/2} \\
 & \quad \cdot \left\{ \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1} \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2}.
 \end{aligned}$$

By Dirichlet bracketing,

$$(5.8) \quad N_{D(\Omega, f)}(\lambda) \geq \sum_{i=1}^{K+1} N_{D_i(\Omega, f)}(\lambda),$$

which yields the lower bound in (5.1) after summing (5.7) over $i = 1, \dots, K+1$.

To prove the upper bound in Theorem 5.1 we denote by $\tilde{N}_{D_i(\Omega, f)}$ the counting function for the Laplace operator on $D_i(\Omega, f)$ with Neumann boundary conditions at $x = L_{i-1}$ and $x = L_i$, and Dirichlet boundary conditions elsewhere. First suppose that $\lambda(L_i - L_{i-1})^2 < \pi^2$. Since $D_i(\Omega, f) \subset f(L_{i-1})\Omega \times (L_{i-1}, L_i)$, we have by Lemma 3.4,

$$\begin{aligned}
 (5.9) \quad & \tilde{N}_{D_i(\Omega, f)}(\lambda) \leq N_\Omega((f(L_{i-1}))^2 \lambda) \tilde{N}_{(L_{i-1}, L_i)}(\lambda) \\
 & = N_\Omega((f(L_{i-1}))^2 \lambda) \\
 & \leq \lambda^{(m-1)/2} \Lambda^{m-1} (f(L_{i-1}))^{m-1} \\
 & \leq \pi^{-1} \int_{[L_{i-1}, L_i]} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \\
 & \quad + \lambda^{(m-1)/2} \Lambda^{m-1} (f(L_{i-1}))^{m-1},
 \end{aligned}$$

where $\tilde{N}_{(L_{i-1}, L_i)}$ is the counting function for an interval (L_{i-1}, L_i) in $\mathbb{R}$ with Neumann boundary conditions at the endpoints. Next suppose that $\lambda(L_i - L_{i-1})^2 \geq \pi^2$. Fix a positive integer M and let

$$(5.10) \quad \delta = \frac{L_i - L_{i-1}}{M},$$

$$(5.11) \quad D_i^+(\Omega, f) = \{(x, y) \in \mathbb{R}^m : L_{i-1} < x < L_i, y \in f^+(x)\Omega\},$$

where f^+ is the piecewise constant approximation of f defined by

$$(5.12) \quad f^+(x) = f\left(L_{i-1} + \left\lceil \frac{x - L_{i-1}}{\delta} \right\rceil \delta\right).$$

By putting additional Neumann boundary conditions at

$$x = L_{i-1} + j\delta, \quad j = 1, \dots, M-1$$

(if $M \geq 2$) and following [4, Estimate (30) in proof of Theorem 8], we obtain

$$\begin{aligned}
 (5.13) \quad & \tilde{N}_{D_i^+(\Omega, f)}(\lambda) \\
 & \leq \sum_{j=0}^{M-1} \#\{(k, \ell) \in (\mathbb{Z}^+ \cup \{0\}) \times \mathbb{Z}^+ : \frac{\pi^2 k^2}{\delta^2} + (f^+(L_{i-1} + j\delta))^{-2} \mu_\ell < \lambda\} \\
 & \leq \#\{k \in \mathbb{Z}^+ \cup \{0\} : \frac{\pi^2 k^2}{\delta^2} < \lambda\} \cdot \#\{\ell \in \mathbb{Z}^+ : \mu_\ell < \lambda(f^+(L_{i-1}))^2\} \\
 & \quad + \sum_{j=1}^{M-1} \#\{\ell \in \mathbb{Z}^+ : \mu_\ell < \lambda(f^+(L_{i-1} + j\delta))^2\} \\
 & \quad + \sum_{j=1}^{M-1} \#\{(k, \ell) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \frac{\pi^2 k^2}{\delta^2} + (f^+(L_{i-1} + j\delta))^{-2} \mu_\ell < \lambda\} \\
 & \leq (1 + \delta\lambda^{1/2}\pi^{-1})\pi^{-1}\Lambda^{m-1}(f(L_{i-1}))^{m-1}\lambda^{(m-1)/2} \\
 & \quad + \sum_{j=1}^{M-1} [\pi^{-1}(f(L_{i-1} + j\delta))^{m-1}\Lambda^{m-1}\lambda^{(m-1)/2}] \\
 & \quad + \int_{[L_{i-1}, L_i)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \\
 & \leq (1 + \delta\lambda^{1/2})\Lambda^{m-1}(f(L_{i-1}))^{m-1}\lambda^{(m-1)/2} \\
 & \quad + \delta^{-1} \int_{[L_{i-1}, L_i)} [\pi^{-1}(f(x))^{m-1}\Lambda^{m-1}\lambda^{(m-1)/2}] dx \\
 & \quad + \int_{[L_{i-1}, L_i)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \\
 & \leq (1 + \delta\lambda^{1/2})\Lambda^{m-1}(f(L_{i-1}))^{m-1}\lambda^{(m-1)/2} \\
 & \quad + \delta^{-1}\Lambda^{m-1}\lambda^{(m-1)/2} \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1}\lambda^{1/2}\}} (f(x))^{m-1} dx \\
 & \quad + \int_{[L_{i-1}, L_i)} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu).
 \end{aligned}$$

We wish to minimize the right hand side of (5.13) by choosing δ –or equivalently M – appropriately. Let

$$(5.14) \quad M = \left[\lambda^{1/4} (L_i - L_{i-1}) (f(L_{i-1}))^{(m-1)/2} \cdot \left\{ \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1} \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{-1/2} \right].$$

Since

$$(5.15) \quad \left\{ \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1} \lambda^{1/2}\}} (f(x))^{m-1} dx \right\}^{1/2} \leq (f(L_{i-1}))^{(m-1)/2} (L_i - L_{i-1})^{1/2},$$

we conclude that

$$(5.16) \quad M \geq [\lambda^{1/4} (L_i - L_{i-1})^{1/2}] \geq [\pi^{1/2}] = 1.$$

Since $x \geq [x] \geq x/2$ for $x \geq 1$,

$$(5.17) \quad \delta \leq 2\lambda^{-1/4} (f(L_{i-1}))^{-(m-1)/2} \cdot \left\{ \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1} \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2} \leq 2\delta.$$

Combining (5.13) and (5.17), we obtain for $\lambda(L_i - L_{i-1})^2 \geq \pi^2$

$$(5.18) \quad \begin{aligned} \tilde{N}_{D_i^+(\Omega, f)} &\leq \lambda^{(m-1)/2} \Lambda^{m-1} (f(L_{i-1}))^{m-1} \\ &\quad + 3(f(L_{i-1}))^{(m-1)/2} \Lambda^{m-1} \lambda^{(2m-1)/4} \\ &\quad \cdot \left\{ \int_{\{L_{i-1} < x < L_i, f(x) \geq \Lambda^{-1} \lambda^{-1/2}\}} (f(x))^{m-1} dx \right\}^{1/2} \\ &\quad + \pi^{-1} \int_{[L_{i-1}, L_i]} dx \int_{[0, \infty)} \{(\lambda - \mu(f(x))^{-2})_+\}^{1/2} dN_\Omega(\mu). \end{aligned}$$

By Neumann bracketing,

$$(5.19) \quad \begin{aligned} N_{D(\Omega, f)}(\lambda) &\leq \sum_{\{i: \lambda(L_i - L_{i-1})^2 < \pi^2\}} \tilde{N}_{D_i(\Omega, f)}(\lambda) + \sum_{\{i: \lambda(L_i - L_{i-1})^2 \geq \pi^2\}} \tilde{N}_{D_i^+(\Omega, f)}(\lambda), \end{aligned}$$

and the upper bound in Theorem 5.1 follows from (5.9), (5.18), and (5.19). $\square$

We now turn to the proof of (1.11)-(1.13) in Theorem 1.1. We first show how the two leading terms occur in (1.11)-(1.13).

Lemma 5.2. *Let f_α be given by (1.7), and suppose that Ω is an open, bounded set in $\mathbb{R}^{m-1}$ which satisfies (3.1) and (3.2) with $\Lambda < \infty$ and $\nu < \infty$. Then for $\alpha > m - 1$,*

$$(5.20) \quad \pi^{-1} \int_{[0,\infty)} dx \int_{[0,\infty)} \{(\lambda - \mu(f_\alpha(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \\ = \zeta(\alpha, \Omega) \lambda^{(1+\alpha)/2} - \frac{\alpha}{\alpha + 1 - m} (4\pi)^{-m/2} \left(\Gamma\left(1 + \frac{m}{2}\right) \right)^{-1} |\Omega|_{m-1} \lambda^{m/2} \\ + O(\lambda^{(m-1)/2}).$$

Proof.

$$(5.21) \quad \pi^{-1} \int_{[0,\infty)} dx \int_{[0,\infty)} \{(\lambda - \mu(f_\alpha(x))^{-2})_+\}^{1/2} dN_\Omega(\mu) \\ = \pi^{-1} \int_{[0,\infty)} dx \int_{[0,\infty)} \{(\lambda - \mu x^{2/\alpha})_+\}^{1/2} dN_\Omega(\mu) \\ - \pi^{-1} \int_{[0,1)} dx \int_{[0,\infty)} \{(\lambda - \mu x^{2/\alpha})_+\}^{1/2} dW_\Omega(\mu) \\ - (2\pi)^{-1} \int_{[0,1)} dx x^{2/\alpha} \int_{[0,\infty)} (N_\Omega(\mu) - W_\Omega(\mu)) \\ \cdot \{(\lambda - \mu x^{2/\alpha})_+\}^{-1/2} d\mu.$$

Using (3.2), we can bound the absolute value of the third term in the right hand side of (5.21) by

$$(5.22) \quad \int_{[0,1)} dx x^{2/\alpha} \int_{[0,\infty)} |N_\Omega(\mu) - W_\Omega(\mu)| \{(\lambda - \mu x^{2/\alpha})_+\}^{-1/2} d\mu \\ \leq \int_{[0,\infty)} |N_\Omega(\mu) - W_\Omega(\mu)| \{(\lambda - \mu)_+\}^{-1/2} d\mu \\ \leq \nu \int_{[0,\lambda)} \mu^{(m-2)/2} (\lambda - \mu)^{-1/2} d\mu \\ = O(\lambda^{(m-1)/2}).$$

By Fubini's theorem, we obtain for the first term in the right hand side of (5.21)

$$\begin{aligned}
 (5.23) \quad & \pi^{-1} \int_{[0,\infty)} dN_{\Omega}(\mu) \int_{[0,\infty)} \{(\lambda - \mu x^{2/\alpha})_+\}^{1/2} dx \\
 &= \pi^{-1} \int_{[0,\infty)} dN_{\Omega}(\mu) \int_{[0,(\lambda/\mu)^{\alpha/2})} (\lambda - \mu x^{2/\alpha})^{1/2} dx \\
 &= (4\pi)^{-1/2} \frac{\Gamma(\alpha/2 + 1)}{\Gamma(\alpha/2 + 3/2)} \int_{[0,\infty)} \mu^{-\alpha/2} dN_{\Omega}(\mu) \lambda^{(1+\alpha)/2} \\
 &= \zeta(\alpha, \Omega) \lambda^{(1+\alpha)/2}.
 \end{aligned}$$

Note that by Lemma 3.4,

$$(5.24) \quad \int_{[0,\infty)} \mu^{-\alpha/2} dN_{\Omega}(\mu) \leq \Lambda^{-\alpha} \sum_{j=1}^{\infty} j^{-\alpha/(m-1)} < \infty,$$

since $\alpha > m - 1$. To compute the second term in the right hand side of (5.21) we use (1.3) for the open, bounded set Ω in $\mathbb{R}^{m-1}$:

$$(5.25) \quad W_{\Omega}(\mu) = (4\pi)^{-(m-1)/2} \left(\Gamma\left(\frac{m+1}{2}\right) \right)^{-1} |\Omega|_{m-1} \lambda^{(m-1)/2}.$$

The second term now becomes

$$\begin{aligned}
 (5.26) \quad & -\frac{m-1}{2\pi} |\Omega|_{m-1} (4\pi)^{-(m-1)/2} \left(\Gamma\left(\frac{m+1}{2}\right) \right)^{-1} \\
 & \cdot \int_{[0,1)} dx \int_{[0,\lambda/x^{2/\alpha})} (\lambda - \mu x^{2/\alpha})^{1/2} \mu^{(m-3)/2} d\mu,
 \end{aligned}$$

which yields the second term in the right hand side of (5.20) after the change of variables $\mu = \xi \lambda x^{-2/\alpha}$.

To complete the proof of Theorem 1.1 for $\alpha > m - 1$, we let

$$(5.27) \quad 0 < \gamma_1 < \dots < \gamma_K,$$

and put (if $K \geq 1$)

$$(5.28) \quad L_i = \lambda^{\gamma_i}, \quad i = 1, \dots, K.$$

The first term in the right hand side of (5.1) is bounded from above by

$$(5.29) \quad \sum_{i=1}^{K+1} \lambda^{(m-1)/2} \Lambda^{m-1} (f(L_{i-1}))^{(m-1)/2} \leq (K+1) \Lambda^{m-1} \lambda^{(m-1)/2}.$$

The i^{th} term ($K \geq i \geq 2$) in the second part of the right hand side of (5.1) is bounded from above by

$$(5.30) \quad 3\lambda^{(2m-1)/4}\Lambda^{m-1}(1+\lambda^{y_{i-1}})^{-(m-1)/(2\alpha)} \\ \cdot \left\{ \int_{[\lambda^{y_{i-1}}, \lambda^{y_i}]} (1+x)^{-(m-1)/\alpha} dx \right\}^{1/2} \\ \leq c_1(\alpha, m)\Lambda^{m-1}\lambda^{(2m-1)/4-y_{i-1}(m-1)/(2\alpha)+y_i(\alpha+1-m)/(2\alpha)},$$

for all $\lambda \geq 1$ and some constant $c_1(\alpha, m)$ depending upon α and m only. Since $L_{K+1} = \infty$, the $(K+1)^{\text{th}}$ term in the second part of the right hand side of (5.1) is bounded from above by

$$(5.31) \quad c_2(\alpha, m)\Lambda^{m-1}\lambda^{(\alpha+m)/4-(m-1)y_K/(2\alpha)},$$

for some constant $c_2(\alpha, m)$. Finally the term with $i = 1$ in the second part of the right-hand side of (5.1) is bounded by

$$(5.32) \quad 3\Lambda^{m-1}\lambda^{(2m-1)/4} \left\{ \int_{[0, L_1]} (1+x)^{-(m-1)/\alpha} dx \right\}^{1/2} \\ \leq c_3(\alpha, m)\Lambda^{m-1}\lambda^{(2m-1)/4-y_0(m-1)/(2\alpha)+y_1(\alpha+1-m)/(2\alpha)},$$

where we have put $y_0 = 0$. Let $\beta > 0$ and choose $y_1, \dots, y_K$ such that

$$(5.33) \quad \frac{2m-1}{4} - \frac{(m-1)y_{i-1}}{2\alpha} + \frac{(\alpha+1-m)y_i}{2\alpha} = \beta, \quad i = 1, \dots, K.$$

Since $y_0 = 0$,

$$(5.34) \quad y_i = \sum_{\ell=0}^{i-1} \left(\frac{\alpha+1-m}{m-1} \right)^{-\ell} \frac{\alpha(2\beta-m+\frac{1}{2})}{\alpha+1-m}, \quad i = 1, \dots, K.$$

Hence

$$(5.35) \quad y_K = \sum_{\ell=0}^{K-1} \left(\frac{\alpha+1-m}{m-1} \right)^{-\ell} \frac{\alpha(2\beta-m+\frac{1}{2})}{\alpha+1-m}.$$

We choose β such that the $(K+1)^{\text{th}}$ term estimated in (5.31) is also $O(\lambda^\beta)$. This gives the following equation for β :

$$(5.36) \quad \frac{\alpha+m}{4} - \left(\beta - \frac{m}{2} + \frac{1}{4} \right) \sum_{\ell=0}^{K-1} \left(\frac{m-1}{\alpha+1-m} \right)^{\ell+1} = \beta.$$

Hence

$$(5.37) \quad \beta = \left\{ \frac{\alpha + m}{4} + \frac{2m - 1}{4} \sum_{\ell=0}^{K-1} \left(\frac{m - 1}{\alpha + 1 - m} \right)^{\ell+1} \right\} \cdot \left\{ 1 + \sum_{\ell=0}^{K-1} \left(\frac{m - 1}{\alpha + 1 - m} \right)^{\ell+1} \right\}^{-1}.$$

We note that, since $\alpha > m - 1$, β is a decreasing function of K , and that by (5.1) and (5.29)-(5.32), the right hand side in (5.1) is bounded from above by

$$(5.38) \quad \Lambda^{m-1}(K+1) \{ \lambda^{(m-1)/2} + (c_1(\alpha, m) + c_2(\alpha, m) + c_3(\alpha, m)) \lambda^\beta \}.$$

Since $\beta > (m - 1)/2$, the problem of minimizing (5.38) is reduced to minimizing $(K + 1) \lambda^\beta$ as a function of K . We consider the various cases for α below:

If $m - 1 < \alpha \leq 2m - 2$, then

$$(5.39) \quad \beta \leq \frac{2m - 1}{4} + 4^{-1}(m + \alpha) \left\{ \sum_{\ell=0}^{K-1} \left(\frac{m - 1}{\alpha + 1 - m} \right)^{\ell+1} \right\}^{-1} \leq \frac{2m - 1}{4} + \frac{m}{K}.$$

We choose for $\lambda > 1$

$$(5.40) \quad K = [m \log \lambda].$$

By (5.38)-(5.40), we obtain the bound $O(\lambda^{(2m-1)/4} \log \lambda)$ as advertised in (1.13).

If $\alpha > 2m - 2$, then

$$(5.41) \quad \beta \leq \frac{\alpha + 1}{4} + \frac{m - 1}{4} \left(\frac{m - 1}{\alpha + 1 - m} \right)^K.$$

For λ sufficiently large, we choose

$$(5.42) \quad K = \left\lceil \frac{\log \log \lambda}{\log((\alpha + 1 - m)/(m - 1))} \right\rceil.$$

By (5.38), (5.41), and (5.42) we obtain the bound $O(\lambda^{(\alpha+1)/4} \log \log \lambda)$ as advertised in (1.11) and (1.12). This completes the proof of Theorem 1.1. $\square$

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