Canonical Factorization in Subalgebras of H^∞ Associated with Douglas Algebras

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ABSTRACT. Given a Douglas algebra D , we study the canonical (Riesz-Nevanlinna) factorization in the subalgebra $QD_A := H^\infty \cap \bar{D}$. The inner and outer factors of QD_A -functions are completely characterized by explicit quantitative conditions. The classical example $D = H^\infty + C$ is discussed in more detail.

1. INTRODUCTION

Let H^∞ stand for the algebra of all bounded holomorphic functions on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and L^∞ for the algebra of all essentially bounded functions on the circle $\mathbb{T} := \partial\mathbb{D}$. As usual, H^∞ is regarded as a subspace of L^∞ , and both of them are endowed with the natural norm $\|\cdot\|_\infty$. A *Douglas algebra* is, by definition, a closed subalgebra of L^∞ that contains H^∞ . According to a well-known theorem of Chang and Marshall (see [2] and [19]), every Douglas algebra D is completely determined by the set

$$\mathcal{B}_D := \{b \in H^\infty : b \text{ is inner and } \bar{b} \in D\}.$$

(An *inner function* is, of course, an H^∞ -function whose modulus equals 1 a.e. on $\mathbb{T}$.) Precisely speaking, the Chang-Marshall theorem says that $D = [H^\infty, \overline{\mathcal{B}_D}]$, where $[\cdot, \cdot]$ stands for the smallest closed subalgebra of L^∞ containing the two sets, while the bar denotes complex conjugation. We refer to [10], Chap. IX, for a systematic treatment of Douglas algebras and, in particular, for a unified presentation of the Chang-Marshall results.

Along with a given Douglas algebra D one considers the corresponding *Sarason algebra* $QD := D \cap \bar{D}$, i.e., the largest self-adjoint subalgebra of D , and its analytic part $QD_A := QD \cap H^\infty$. The purpose of this paper is to study the canonical (inner-outer) factorization in QD_A . To make further discussion possible, let us recall that every nontrivial function $f \in H^\infty$ can be uniquely factored as $f = I\mathcal{O}_{|f|}$, where I is an inner function and $\mathcal{O}_{|f|}$ is the *outer function* with modulus $|f|$. In general, given a nonnegative function φ on $\mathbb{T}$ with $\log \varphi \in L^1$, the corresponding outer function $\mathcal{O}_\varphi$ is defined by

$$\mathcal{O}_\varphi(z) := \exp \left(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) |d\zeta| \right), \quad z \in \mathbb{D},$$

and has the crucial property that $|\mathcal{O}_\varphi| = \varphi$ a.e. on $\mathbb{T}$ (see [10], Chapter II).

We shall be chiefly concerned with outer functions of class QD_A . However, we begin by characterizing the inner functions I by which a given $f \in H^\infty$ can be multiplied, so as to ensure $fI \in QD_A$. This preliminary result, contained in Section 2 below, is not completely original (see Remarks following Proposition 2.1) but, at the same time, it does not seem to have been stated explicitly elsewhere. Besides, it has a tidy appearance and a short proof. We have therefore included it for the sake of completeness. Actually, the purpose of Section 2 is to get rid of inner factors and to reduce the general factorization problem for QD_A to that dealing with outer functions only.

Our main result appears in Section 3. Here, an explicit description is given of those positive functions φ on the circle for which $\mathcal{O}_\varphi \in QD_A$. The crucial condition, to be imposed on φ along with the *a-priori* assumptions that $\varphi \in QD$ and $\log \varphi \in L^1$, involves a certain integral of $|\log \varphi|$ against the harmonic measure; it can be viewed as a refinement of the mere fact that $\log \varphi$ must be integrable.

In Section 4, we deal with outer functions whose moduli are in $QC := (H^\infty + C) \cap (\overline{H^\infty + C})$. Let us recall that $H^\infty + C$ is the smallest nontrivial Douglas algebra, and so QC is the smallest among the nontrivial QD 's. Our starting point is an example, due to Guillory and Sarason [14], of a function $\varphi \geq 0$ such that $\varphi \in QC$ and $\log \varphi \in L^1$, but $\mathcal{O}_\varphi \notin QC$. We push the Guillory-Sarason construction further, showing that an outer function with QC modulus need not even belong to QD_ζ , where the Douglas algebra D_ζ is generated by (the complex conjugates of) the inner functions that are analytic at a fixed point $\zeta \in \mathbb{T}$. The algebra QD_ζ is of course much larger than QC , whose invertible inner functions (the finite Blaschke products) are analytic *everywhere* on $\mathbb{T}$. Thus, the *a-priori* hypotheses $\varphi \in QD$ and $\log \varphi \in L^1$, alone, are quite a “far cry” from the complete characterization, as given in Section 3, of the φ 's with $\mathcal{O}_\varphi \in QD_A$.

Finally, in Section 5 we exhibit two simple sufficient conditions on a function $\varphi \geq 0$ that imply $\mathcal{O}_\varphi \in QD_A$. One of these is stated in terms of continuity, and the other in terms of mean oscillations.

2. MULTIPLYING BY INNER FUNCTIONS IN QD_A

We begin by introducing some notation. For $z \in \mathbb{D}$, we let μ_z denote the corresponding harmonic measure:

$$d\mu_z(\zeta) = \frac{1 - |z|^2}{|\zeta - z|^2} \frac{|d\zeta|}{2\pi}, \quad \zeta \in \mathbb{T}.$$

Further, with each function $h \in L^\infty$ we associate its Poisson integral

$$h(z) := \int h d\mu_z, \quad z \in \mathbb{D},$$

the arising level sets

$$G_\delta(h) := \{z \in \mathbb{D} : |h(z)| \geq \delta\}, \quad \delta > 0,$$

and the (nonnegative) quantity

$$\Phi(h; z) := \int |h|^2 d\mu_z - |h(z)|^2 \quad \left(= \int |h - h(z)|^2 d\mu_z \right), \quad z \in \mathbb{D}.$$

We also need the following ‘‘Garsia-type’’ characterization of the algebra QD .

Lemma 2.1. [see [10], Chap. IX, Sect. 4] *Let $f \in L^\infty$, and let D be a Douglas algebra. Then $f \in QD$ if and only if for any $\varepsilon > 0$ there exist $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ so that*

$$\Phi(f; z) < \varepsilon \quad \text{for } z \in G_\eta(b).$$

Now we are in a position to state and prove the following result:

Proposition 2.1. *Let $f \in H^\infty$ and I be an inner function. Given a Douglas algebra D , we have $fI \in QD_A$ if and only if the two conditions hold:*

- (a) $f \in QD_A$,
- (b) *for any $\varepsilon > 0$ and $\delta \in (0, 1)$ there are $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ such that*

$$|f(z)| < \varepsilon \quad \text{for } z \in G_\eta(b) \setminus G_\delta(I).$$

Remarks.

- (1) The “if” part of Proposition 2.1 can be actually deduced from Luecking’s Theorem 1 in [18]. Some related multiplication/division criteria for Douglas algebras can be found in [1], [13], and [26]. However, in the QD_A context, the formulation seems to be particularly clear-cut, and we have found it worthwhile to supply a short independent proof.
- (2) If $I \in \mathcal{B}_D$ (equivalently, if $I \in QD_A$), then of course $fI \in QD_A$ for any $f \in QD_A$, since QD_A is an algebra. In this case, (b) holds automatically: indeed, letting $b = I$ and $\eta = \delta$, we have $G_\eta(b) \setminus G_\delta(I) = \emptyset$. On the other hand, for $I \notin \mathcal{B}_D$, condition (b) becomes crucial and describes the interplay between the two factors.
- (3) Proposition 2.1 implies, in particular, that QD_A enjoys the f -property of Havin (cf. [15]), i.e., division by inner factors preserves membership in QD_A . This fact was previously pointed out in [12], p. 511.

In the proof below, we write $[\varepsilon \rightarrow b, \eta]$ to mean “for any $\varepsilon > 0$ there are $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ such that”. The b ’s and η ’s corresponding to a given ε are, generally speaking, different in different places, but we always have in mind (and tacitly use) the following fact: If $b_1, b_2 \in \mathcal{B}_D$ and $\eta_1, \eta_2 \in (0, 1)$, then $b = b_1 b_2 \in \mathcal{B}_D$ and $G_\eta(b) \subset G_{\eta_1}(b_1) \cap G_{\eta_2}(b_2)$, where $\eta = \max(\eta_1, \eta_2)$. Thus, if one condition is valid for $z \in G_{\eta_1}(b_1)$ and another for $z \in G_{\eta_2}(b_2)$, then both of them hold true for $z \in G_\eta(b)$.

Proof of Proposition 2.1. By Lemma 2.1, $fI \in QD_A$ if and only if

$$[\varepsilon \rightarrow b, \eta] : \Phi(fI; z) < \varepsilon \quad \text{for } z \in G_\eta(b).$$

In view of the elementary identity

$$\Phi(fI; z) = \Phi(f; z) + |f(z)|^2(1 - |I(z)|^2), \quad z \in \mathbb{D},$$

the preceding condition splits into the following two:

$$[\varepsilon \rightarrow b, \eta] : \Phi(f; z) < \varepsilon \quad \text{for } z \in G_\eta(b)$$

and

$$[\varepsilon \rightarrow b, \eta] : |f(z)|^2(1 - |I(z)|^2) < \varepsilon \quad \text{for } z \in G_\eta(b).$$

The first of these is equivalent to (a) (use Lemma 2.1 again), while the second can be rewritten as (b) (indeed, the product $|f|^2(1 - |I|^2)$ is small iff $|f|$ is small wherever $|I|$ is bounded away from 1). $\square$

In the special case where f is an outer function, Proposition 2.1 amounts to the following.

Proposition 2.1'. *Let $\varphi \in L^\infty$ be a nonnegative function with $\log \varphi \in L^1$, and let I be inner. Given a Douglas algebra D , we have $I\mathcal{O}_\varphi \in QD_A$ if and only if the two conditions hold:*

(a') $\mathcal{O}_\varphi \in QD_A$,

(b') for any $A > 0$ and $\delta \in (0, 1)$ there are $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ such that

$$\int \log \frac{1}{\varphi} d\mu_z > A \quad \text{for } z \in G_\eta(b) \setminus G_\delta(I).$$

In the next section (and, in fact, in the rest of the paper) we shall study the question of when (a') holds. Remarkably, the characterization of the φ 's with $\mathcal{O}_\varphi \in QD_A$ will again involve certain integrals of $|\log \varphi|$ against the harmonic measure; the condition obtained will thus be similar in appearance to (b'). Meanwhile, let us consider, as an example, the simplest (and the smallest) nontrivial Douglas algebra $H^\infty + C$, where $C = C(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$. When $D = H^\infty + C$, the algebras QD and QD_A are traditionally denoted by QC and QA , respectively. It is well known that $QC = L^\infty \cap VMO$, and hence $QA = H^\infty \cap VMOA$. Let us also recall the equality $H^\infty + C = [H^\infty, \bar{z}]$ and the fact that $\mathcal{B}_{H^\infty + C}$ coincides with the set of finite Blaschke products (see [21], [22], and/or [10], Chapter IX). Thus, for the current choice of D , one can simplify the statements of Propositions 2.1 and 2.1' by replacing the level sets $G_\eta(b)$ with the annuli $\{\eta < |z| < 1\}$. This yields the following (known) result.

Corollary 2.1. *Let $f \in H^\infty$ and I be inner. In order that $fI \in QA$, it is necessary and sufficient that $f \in QA$ and*

$$f(z) \rightarrow 0 \quad \text{as } |z| \rightarrow 1, \quad z \in \mathbb{D} \setminus G_\delta(I),$$

for every $\delta \in (0, 1)$.

In the case that $f = \mathcal{O}_\varphi$, the latter condition becomes

$$\int \log \frac{1}{\varphi} d\mu_z \rightarrow \infty \quad \text{as } |z| \rightarrow 1, \quad z \in \mathbb{D} \setminus G_\delta(I),$$

for every $\delta \in (0, 1)$.

A somewhat stronger version of Corollary 2.1 was established in [4] (the proof was largely based on an earlier result from [13]). One refinement is that the word “every” in the above statement can be actually replaced by “some”.

3. OUTER FUNCTIONS IN QD_A

Before stating the result, we have to introduce yet another type of level set associated with a function $\varphi \geq 0$ on $\mathbb{T}$. Namely, we put

$$E_\delta(\varphi) := \left\{ \zeta \in \mathbb{T} : \varphi(\zeta) < \frac{\delta}{2} \right\}, \quad \delta > 0.$$

Theorem 3.1. *Let D be a Douglas algebra, and let $\varphi \in QD$ be a nonnegative function with $\log \varphi \in L^1$. The following conditions are equivalent.*

- (i) $\mathcal{O}_\varphi \in QD_A$.
- (ii) Whenever $\varepsilon > 0$ and $\delta > 0$ are small enough, there exist $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ such that

$$\int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_z < \varepsilon \quad \text{for all } z \in G_\eta(b) \cap G_\delta(\varphi).$$

Proof. Our plan is to insert a number of equivalent conditions between (i) and (ii), thereby proving the equivalence of the endpoint statements. The intermediate conditions will begin either with $[\varepsilon \rightarrow b, \eta]$ (which has the same meaning as before) or with $[\varepsilon, \delta \rightarrow b, \eta]$, which stands for the phrase “for any $\varepsilon > 0$ and $\delta > 0$ that are sufficiently small, there are $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ such that”. Now, whenever such a condition is numbered as (k), we write (k') to refer to its “main body” obtained by dropping the part [...].

By Lemma 2.1, condition (i) holds if and only if

$$(3.1) \quad [\varepsilon \rightarrow b, \eta] : \Phi(\mathcal{O}_\varphi; z) = \int \varphi^2 d\mu_z - |\mathcal{O}_\varphi(z)|^2 < \varepsilon \quad \text{for } z \in G_\eta(b).$$

On the other hand, the hypothesis $\varphi \in QD$ tells us that

$$(3.2) \quad [\varepsilon \rightarrow b, \eta] : \Phi(\varphi; z) = \int \varphi^2 d\mu_z - \varphi(z)^2 < \varepsilon \quad \text{for } z \in G_\eta(b),$$

and so (3.1) is equivalent to

$$(3.3) \quad [\varepsilon \rightarrow b, \eta] : \varphi(z)^2 - |\mathcal{O}_\varphi(z)|^2 < \varepsilon \quad \text{for } z \in G_\eta(b).$$

Since $|\mathcal{O}_\varphi(z)| = \exp(\int \log \varphi d\mu_z)$, (3.3) can in turn be rewritten as

$$(3.4) \quad [\varepsilon \rightarrow b, \eta] : \varphi(z)^2(1 - e^{-2J(z)}) < \varepsilon \quad \text{for } z \in G_\eta(b),$$

where

$$J(z) := \log \varphi(z) - \int \log \varphi d\mu_z, \quad z \in \mathbb{D}.$$

(Note that $J(z) \geq 0$ on $\mathbb{D}$ by Jensen's inequality.) Next, we claim that (3.4) is equivalent to the following condition:

$$(3.5) \quad [\varepsilon, \delta \rightarrow b, \eta] : 1 - e^{-2J(z)} < \varepsilon \quad \text{for } z \in G_\eta(b) \cap G_\delta(\varphi).$$

Indeed, once (3.4) is given, we pick $\varepsilon > 0$ and $\delta > 0$ and then find $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ so as to make (3.4') hold with $\varepsilon\delta^2$ in place of ε . This done, (3.5') holds with the same b and η . Conversely, to derive (3.4) from (3.5), we pick an $\varepsilon > 0$ and use (3.5) to obtain $b \in \mathcal{B}_D$ and $\eta \in (0, 1)$ such that

$$1 - e^{-2J(z)} < \frac{\varepsilon}{\|\varphi\|_\infty} \quad \text{for } z \in G_\eta(b) \cap G_{\sqrt{\varepsilon}}(\varphi).$$

Combining this last inequality with the obvious fact that $\varphi(z)^2 < \varepsilon$ for $z \in \mathbb{D} \setminus G_{\sqrt{\varepsilon}}(\varphi)$, we arrive at (3.4'). The equivalence of (3.4) and (3.5) is thus established, and we can further restate (3.5) as

$$(3.6) \quad [\varepsilon, \delta \rightarrow b, \eta] : J(z) < \varepsilon \quad \text{for } z \in G_\eta(b) \cap G_\delta(\varphi).$$

(Just note that $1 - e^{-2x}$ is small iff x is small.)

Now, in order to find out when (3.6) holds, let us temporarily fix $\delta > 0$ with $\delta/2 < \min(1, \|\varphi\|_\infty)$ and a point $z \in G_\delta(\varphi)$. We have then

$$(3.7) \quad J(z) = \int \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) = \int_E + \int_{E'},$$

where $E := E_\delta(\varphi)$ and $E' := \mathbb{T} \setminus E$. The first integral on the right can be written in the form

$$(3.8) \quad \int_E \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) = \mu_z(E) \cdot \log \varphi(z) + \int_E \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta).$$

Since

$$\varphi(z) - \varphi(\zeta) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} \quad \text{for } z \in G_\delta(\varphi), \zeta \in E,$$

it follows that

$$(3.9) \quad \begin{aligned} \mu_z(E) &\leq 2\delta^{-1} \int_E (\varphi(z) - \varphi(\zeta)) d\mu_z(\zeta) \\ &\leq 2\delta^{-1} \int |\varphi(z) - \varphi(\zeta)| d\mu_z(\zeta) \leq 2\delta^{-1} \Phi(\varphi; z)^{1/2}. \end{aligned}$$

Besides, the inequalities $\delta \leq \varphi(z) \leq \|\varphi\|_\infty$ imply

$$(3.10) \quad |\log \varphi(z)| \leq \log \left(\frac{1}{\delta} + \|\varphi\|_\infty \right).$$

Taken together, (3.9) and (3.10) give

$$\mu_z(E) \cdot |\log \varphi(z)| \leq \frac{2}{\delta} \Phi(\varphi; z)^{1/2} \cdot \log \left(\frac{1}{\delta} + \|\varphi\|_\infty \right),$$

and so (3.8) yields

$$(3.11) \quad \int_E \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) = O(\Phi(\varphi; z)^{1/2}) + \int_E \log \frac{1}{\varphi(\zeta)} d\mu_z(\zeta),$$

$$z \in G_\delta(\varphi).$$

We now turn to $\int_{E'}$, the second term appearing in (3.7). For $\zeta \in E'$, we obviously have

$$\frac{\varphi(\zeta)}{\varphi(z)} \geq \frac{\delta}{2\|\varphi\|_\infty},$$

and so the elementary inequality $|\log u| \leq (1/\beta)|u - 1|$, valid for $\beta \in (0, 1)$ and $u \geq \beta$, yields

$$(3.12) \quad \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| \leq \frac{2\|\varphi\|_\infty}{\delta} \left| \frac{\varphi(\zeta)}{\varphi(z)} - 1 \right| \leq \frac{2\|\varphi\|_\infty}{\delta^2} |\varphi(\zeta) - \varphi(z)|.$$

(To verify the last step, recall that $z \in G_\delta(\varphi)$.) Integrating (3.12), we get

$$\begin{aligned} \left| \int_{E'} \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) \right| &\leq \int_{E'} \left| \log \frac{\varphi(\zeta)}{\varphi(z)} \right| d\mu_z(\zeta) \\ &\leq 2\delta^{-2}\|\varphi\|_\infty \int |\varphi(\zeta) - \varphi(z)| d\mu_z(\zeta) \\ &\leq 2\delta^{-2}\|\varphi\|_\infty \Phi(\varphi; z)^{1/2}. \end{aligned}$$

Thus,

$$(3.13) \quad \int_{E'} \log \frac{\varphi(z)}{\varphi(\zeta)} d\mu_z(\zeta) = O(\Phi(\varphi; z)^{1/2}), \quad z \in G_\delta(\varphi).$$

Finally, substituting (3.11) and (3.13) into (3.7), we obtain

$$(3.14) \quad J(z) = O(\Phi(\varphi; z)^{1/2}) + \int_E \log \frac{1}{\varphi} d\mu_z, \quad z \in G_\delta(\varphi),$$

where the constant in the “big oh” condition actually depends only on δ and $\|\varphi\|_\infty$. Combining (3.14) with (3.2), we now readily conclude that (3.6) holds if and only if

$$[\varepsilon, \delta \rightarrow b, \eta] : \int_E \log \frac{1}{\varphi} d\mu_z < \varepsilon \quad \text{for } z \in G_\eta(b) \cap G_\delta(\varphi),$$

and the latter condition coincides with (ii). Since (3.6) was also shown to be a restatement of (i), the proof is complete. $\square$

Remarks.

- (1) While Theorem 3.1 describes the moduli of QD_A -functions, one might also ask which nonnegative functions $\varphi \in L^\infty$ are eligible as majorants for these moduli (i.e., satisfy $\varphi \geq |g|$ for some nontrivial $g \in QD_A$). This time the necessary condition $\log \varphi \in L^1$ turns out to be also sufficient. Indeed, given $\varphi \in L^\infty$ with $\log \varphi \in L^1$, set $F := \mathcal{O}_\varphi$ and use Wolff's theorem (see [25], Theorem 1) to find an outer function $q \in QA$ with $\|q\|_\infty = 1$ so that $Fq \in QA$. Clearly, $g := Fq$ is then a nonzero function with $|g| \leq \varphi$ a.e. on $\mathbb{T}$. Now since $D \supset H^\infty + C$, provided that $D \neq H^\infty$, we have $QD_A \supset QA$ and so $g \in QD_A$. (If $D = H^\infty$, then $QD_A = \mathbb{C}$, and this trivial case was excluded from consideration.)
- (2) The preceding remark has at least one amusing consequence: *A function $f \in QD_A$, with $\|f\|_\infty = 1$, is an extreme point of $\text{ball}(QD_A)$ if and only if $\log(1 - |f|) \notin L^1$.* Indeed, the “if” part follows at once from the corresponding fact for $\text{ball}(H^\infty)$ (see [3]); conversely, if $\log(1 - |f|) \in L^1$ then $f = \frac{1}{2}(f + g) + \frac{1}{2}(f - g)$, where $g \in QD_A$ is a nonzero function with $|g| \leq 1 - |f|$. We do not know whether this fact about extreme points in QD_A has been pointed out elsewhere. Let us mention, however, the description given in [20] of the so-called strong extreme points in QD_A (and in generic uniform algebras). For other connections between extreme points and Douglas algebras, see e.g. [16].

We conclude this section by applying Theorem 3.1 to the case $D = H^\infty + C$. Once again, the level sets $G_\eta(b)$ can be replaced by annuli adjacent to $\mathbb{T}$, and we arrive at the following result.

Corollary 3.1. *Let $\varphi \in QC$ be a nonnegative function with $\log \varphi \in L^1$. The following are equivalent.*

- (i) $\mathcal{O}_\varphi \in QA$.
- (ii) Whenever $\delta > 0$ is small enough, one has

$$\int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_z \rightarrow 0 \quad \text{as } |z| \rightarrow 1, \ z \in G_\delta(\varphi).$$

In this connection, we mention our previous results in [7], [8] characterizing the outer functions in $BMOA$ and in some of its weighted analogues. In

fact, a similar characterization exists¹ for $VMOA$, and when restricted to bounded functions, it yields Corollary 3.1. Results on other smoothness spaces and outer functions therein can be found in [5], [6], and [23].

4. OUTER FUNCTIONS WITH QC MODULI

Let us recall the assumptions

$$(4.15) \quad \varphi \in QD \quad \text{and} \quad \log \varphi \in L^1$$

imposed on φ in Theorem 3.1. Of course, both of them are necessary in order that $\mathcal{O}_\varphi \in QD_A$. Indeed, the modulus of a QD -function must itself be in QD , as follows from Lemma 2.1 and the inequality $\Phi(|f|; z) \leq \Phi(f; z)$, $z \in \mathbb{D}$. Besides, the classical Hardy spaces theory (see [10], Chapter II) tells us that $\log |f| \in L^1$ for any nontrivial $f \in H^\infty$. On the other hand, one might ask if the necessary conditions (4.15) are also sufficient for $\mathcal{O}_\varphi$ to be in QD_A . In light of Theorem 3.1, this is equivalent to asking whether or not condition (ii) of that theorem follows automatically from the hypotheses (4.15).

For the trivial Douglas algebras $D = H^\infty$ (resp., $D = L^\infty$), we have $QD = \mathbb{C}$ (resp., $QD = L^\infty$) and $QD_A = \mathbb{C}$ (resp., $QD_A = H^\infty$). In both of these endpoint cases, the necessary conditions (4.15) alone do characterize the nonnegative φ 's with $\mathcal{O}_\varphi \in QD_A$. However, the first nontrivial Douglas algebra $H^\infty + C$ exhibits a different kind of behavior: as was shown by Guillyory and Sarason [14], there is a nonnegative function $\varphi \in QC$ with $\log \varphi \in L^1$ such that $\mathcal{O}_\varphi \notin QA$. Thus, the answer to the above question (concerning the sufficiency of (4.15)) is “no” in general.

On the other hand, it is known (see [22]) that the conditions $\varphi \in C$ and $\log \varphi \in L^1$ do imply $\mathcal{O}_\varphi \in QA$. Motivated by these results, we raise the following question: *Given a function $\varphi \geq 0$ lying in QC (i.e., in the smallest nontrivial QD -algebra) and satisfying $\log \varphi \in L^1$, must $\mathcal{O}_\varphi$ belong to QD_A for any fixed Douglas algebra $D \neq L^\infty$?* We strongly suspect that the answer is no. However, we are only able to prove that $\mathcal{O}_\varphi$ need not belong to any of the QD_A 's associated with some special D 's, all of them being fairly large.

Let us fix some point $\zeta \in \mathbb{T}$ and denote by $\mathcal{J}_\zeta$ the set of those inner functions which are analytic at ζ . Precisely speaking, an inner function b is in $\mathcal{J}_\zeta$ iff it extends analytically into some disk $\{|z - \zeta| < r\}$, with $r = r(b) > 0$. Consider the Douglas algebra $D_\zeta := [H^\infty, \overline{\mathcal{J}_\zeta}]$. It follows from Tolokonnikov's results (see Theorem 3.1 and Example 3.2 in [24]) that $\mathcal{J}_\zeta = \mathcal{B}_{D_\zeta}$, i.e., $\mathcal{J}_\zeta$ is precisely the set of all inner functions invertible in D_ζ . When $D = D_\zeta$, the corresponding

¹This was pointed out to me by David Walsh.

QD_A will be denoted by QA_ζ ; in other words, $QA_\zeta := \overline{D_\zeta} \cap H^\infty$. Finally, let us remark that $\bigcap_{\zeta \in \mathbb{T}} J_\zeta$ is the set of finite Blaschke products. Thus, when passing from QA to QA_ζ , we replace this intersection by an individual J_ζ ; this results in a considerable enlargement of the algebra involved. The following theorem is therefore a refinement of the Guillory-Sarason result mentioned above.

Theorem 4.1. *Given $\zeta \in \mathbb{T}$, there exists a nonnegative function $\varphi \in QC$ with $\log \varphi \in L^1$ such that $\mathcal{O}_\varphi \notin QA_\zeta$.*

The construction will be carried out on the real line, $\mathbb{R}$, instead of $\mathbb{T}$. The disk gets replaced by the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$, while the role of QC is now played by $QC(\mathbb{R}) := L^\infty(\mathbb{R}) \cap VMO(\mathbb{R})$. Further, for a fixed $x \in \mathbb{R}$, we consider the set J_x of those inner functions on $\mathbb{C}_+$ which are analytic at x , and we let $D_x(\mathbb{R})$ denote the Douglas algebra generated by $H^\infty(\mathbb{C}_+)$ and $\overline{J_x}$. In view of the above discussion, we know that J_x coincides with the set of all invertible inner functions in $D_x(\mathbb{R})$. Finally, we set $QA_x(\mathbb{R}) := \overline{D_x(\mathbb{R})} \cap H^\infty(\mathbb{C}_+)$; this is the analog of the algebra QA_ζ , $\zeta \in \mathbb{T}$, as described above.

Our plan is to produce, for a given $x \in \mathbb{R}$, a nonnegative function $\varphi \in QC(\mathbb{R})$ with $\log \varphi \in L^1(\mathbb{R})$ such that the outer function

$$\mathcal{O}_\varphi(z) = \exp \left(\frac{i}{\pi} \int_{\mathbb{R}} \frac{\log \varphi(t)}{z - t} dt \right), \quad z \in \mathbb{C}_+,$$

is not in $QA_x(\mathbb{R})$. Of course, the condition $\log \varphi \in L^1(\mathbb{R})$ will imply $\log \varphi \in L^1(\mathbb{R}, dt/(1+t^2))$, the usual logarithmic integral condition on the line. In fact, our φ will be constant (and equal to 1) outside some finite interval. A conformal mapping from $\mathbb{C}_+$ onto $\mathbb{D}$ can be used then to transplant φ to the circle, so as to furnish the required example.

The harmonic measure, denoted again by μ_z , is in the current setting given by

$$d\mu_z(t) = \frac{1}{\pi} \frac{\Im z}{|z - t|^2} dt \quad (z \in \mathbb{C}_+, t \in \mathbb{R}).$$

We identify each function $h \in L^\infty(\mathbb{R})$ with its Poisson integral $h(z) := \int h d\mu_z$, and we define the level sets $G_\delta(h)$ as before (but with $\mathbb{C}_+$ in place of $\mathbb{D}$).

Finally, we need an auxiliary result. When stating it, we write $|I|$ to denote the length of an interval $I \subset \mathbb{R}$ and use the standard notation $\|\cdot\|_*$ for the BMO -norm.

Lemma 4.1. *Suppose J and J' are (finite and nondegenerate) subintervals of $\mathbb{R}$ having the same center and satisfying $|J'| = \sigma|J|$, with $0 < \sigma < 1$. Then there exists a function $\psi \in VMO(\mathbb{R})$ such that*

$$0 \leq \psi \leq 1 \text{ on } \mathbb{R}, \quad \psi = 1 \text{ on } J', \quad \psi = 0 \text{ on } \mathbb{R} \setminus J,$$

and

$$\|\psi\|_* \leq C \left(\log \frac{1}{\sigma} \right)^{-1},$$

where $C > 0$ is an absolute constant.

This lemma is known. In fact, it also emerges and plays a crucial role in [14]. Besides, a more general version can be found in [11] (see Lemma 2.2) and in [10] (see Exercise 19 in Chap. VI). However, we include a short proof.

Proof of Lemma 4.1. By a change of scale argument, it suffices to consider the case where $J = [-1, 1]$ and $J' = [-\sigma, \sigma]$. In this case, put $\psi = 1$ on J' , $\psi = 0$ on $\mathbb{R} \setminus J$ and

$$\psi(t) = \frac{\log(1/|t|)}{\log(1/\sigma)} \quad \text{for } t \in J \setminus J'.$$

Since ψ is continuous and compactly supported, it is in $VMO(\mathbb{R})$. To verify the required estimate on $\|\psi\|_*$, one only has to know that $\log|t| \in BMO(\mathbb{R})$ and that truncations do not essentially increase the BMO -norm (cf. [10], Exercise 1 in Chap. VI). $\square$

Proof of Theorem 4.1 (on the line). For $k = 1, 2, \dots$, set $\sigma_k := \exp(-k^2)$ and $\alpha_k := \exp(-1/\sigma_k)$. We are free to assume that our point x (to which the algebra $QA_x(\mathbb{R})$ is associated) is given by $x = 2 \sum_k \sigma_k^2$. Now define the numbers

$$0 = a_1 < b_1 < a_2 < b_2 < \dots$$

inductively by

$$a_{k+1} - b_k = b_k - a_k = \sigma_k^2 \quad (k = 1, 2, \dots).$$

This done, consider the intervals $I_k := [a_k, b_k]$ and $J_k := [b_k, a_{k+1}]$, and let J'_k be the interval with center $(b_k + a_{k+1})/2$ of length $|J'_k| = \sigma_k |J_k| = \sigma_k^3$. Clearly, the union $\bigcup_k (I_k \cup J_k)$ coincides with the finite interval $[0, x)$.

By Lemma 4.1, we can find, for each k , a function $\psi_k \in VMO(\mathbb{R})$ satisfying

$$0 \leq \psi_k \leq 1 \text{ on } \mathbb{R}, \quad \psi_k = 1 \text{ on } J'_k, \quad \psi_k = 0 \text{ on } \mathbb{R} \setminus J_k,$$

and

$$\|\psi_k\|_* \leq C \left(\log \frac{1}{\sigma_k} \right)^{-1} = \frac{C}{k^2}.$$

The sought-after function φ can now be defined by

$$\varphi := 1 - \sum_k (1 - \alpha_k) \psi_k.$$

The series on the right converges everywhere on $\mathbb{R}$ because each ψ_k is supported on its own interval J_k , and the J_k 's are disjoint. One easily verifies that $0 \leq \varphi \leq 1$ on $\mathbb{R}$; in fact,

$$(4.16) \quad \varphi = 1 \text{ on } \mathbb{R} \setminus \bigcup_k J_k \quad \text{and} \quad 1 \geq \varphi \geq \alpha_k \text{ on } J_k.$$

Furthermore, the inequalities

$$\|(1 - \alpha_k) \psi_k\|_* \leq \|\psi_k\|_* \leq \frac{C}{k^2}$$

show that the series $\sum (1 - \alpha_k) \psi_k$ is norm convergent in $VMO(\mathbb{R})$, and since $VMO(\mathbb{R})$ is complete, it follows that $\varphi \in VMO(\mathbb{R})$. We have already seen that φ is bounded on $\mathbb{R}$, so now we conclude that $\varphi \in QC(\mathbb{R})$.

Next, using (4.16), we observe that

$$\int_{\mathbb{R}} |\log \varphi(t)| dt = \sum_k \int_{J_k} \log \frac{1}{\varphi(t)} dt \leq \sum_k |J_k| \log \frac{1}{\alpha_k} = \sum_k \sigma_k < \infty.$$

Thus, $\log \varphi \in L^1(\mathbb{R})$.

It remains to show that $\mathcal{O}_\varphi \notin QA_x(\mathbb{R})$, and we do this by invoking Theorem 3.1. To this end, put

$$x_k = \frac{a_k + b_k}{2}, \quad y_k = \frac{1}{2} \sigma_k^2, \quad \text{and} \quad z_k = x_k + i y_k.$$

Because $\varphi = 1$ on I_k , one has

$$\varphi(z_k) = \int_{\mathbb{R}} \varphi d\mu_{z_k} \geq \int_{I_k} \varphi d\mu_{z_k} = \mu_{z_k}(I_k) = \frac{1}{2}.$$

Therefore,

$$(4.17) \quad \{z_k\} \subset \{z \in \mathbb{C}_+ : \varphi(z) \geq \delta\} = G_\delta(\varphi),$$

provided that $0 < \delta < \frac{1}{2}$. On the other hand, since $\varphi = \alpha_k$ on J'_k and since $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$, we have

$$J'_k \subset \left\{ t \in \mathbb{R} : \varphi(t) < \frac{\delta}{2} \right\} =: E_\delta(\varphi),$$

whenever $\delta > 0$ is fixed and k is larger than a suitable $k_0(\delta)$. Consequently, for $k > k_0(\delta)$, we obtain

$$\int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_{z_k} \geq \int_{J'_k} \log \frac{1}{\varphi} d\mu_{z_k} = \log \frac{1}{\alpha_k} \cdot \mu_{z_k}(J'_k).$$

Noting that

$$\log \frac{1}{\alpha_k} = \frac{1}{\sigma_k} \quad \text{and} \quad \mu_{z_k}(J'_k) \geq c\sigma_k,$$

where $c > 0$ is an absolute constant, we deduce from the preceding inequality that

$$(4.18) \quad \int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_{z_k} \geq c > 0, \quad k > k_0(\delta).$$

Further, because $z_k \rightarrow x$, we infer that $z_k \in G_\eta(b)$ whenever $b \in \mathcal{J}_x$, $\eta \in (0, 1)$, and k is large enough. Indeed, every $b \in \mathcal{J}_x$ is analytic at x , whence $|b(z_k)| \rightarrow 1$.

This last observation, in conjunction with (4.17) and (4.18), leads us to the following conclusion: Given $\delta \in (0, \frac{1}{2})$, $\eta \in (0, 1)$ and $b \in \mathcal{J}_x (= \mathcal{B}_{D_x})$, there is a point $z \in G_\eta(b) \cap G_\delta(\varphi)$ such that

$$\int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_z \geq c > 0,$$

the constant c being numerical. (In fact, one can take $z = z_k$ with k sufficiently large.) In other words, condition (ii) of Theorem 3.1 (or rather its analog for the half-plane, with $D = D_x$) fails. The theorem now tells us that $\mathcal{O}_\varphi \notin QA_x(\mathbb{R})$, and the proof is complete. $\square$

Remark 4.1. Using similar techniques, we constructed in [8] a positive function $\varphi \in BMO$ with $\log \varphi \in L^1$ so that $\mathcal{O}_\varphi \notin BMOA$.

5. SUFFICIENT CONDITIONS FOR $\mathcal{O}_\varphi \in QD_A$

In this section, we are looking for simple sufficient conditions on a function $\varphi \geq 0$ that imply, in conjunction with (4.15), condition (ii) of Theorem 3.1 (and hence the inclusion $\mathcal{O}_\varphi \in QD_A$). The first of these is given in terms of continuity.

Proposition 5.1. *Let D be a Douglas algebra, and let $\varphi \in QD$ be a nonnegative function with $\log \varphi \in L^1$. Assume in addition that, for some $\sigma > 0$, the set $\{\zeta \in \mathbb{T} : \varphi(\zeta) \leq \sigma\}$ is closed and consists of continuity points for φ . Then $\mathcal{O}_\varphi \in QD_A$.*

Proof. We may assume that $\sigma \leq 1$; then we fix $\varepsilon > 0$ and $\delta \in (0, 2\sigma)$. Clearly, $E_\delta(\varphi)$ is contained in $\{\zeta \in \mathbb{T} : \varphi(\zeta) \leq \sigma\}$, and our hypotheses (together with well-known continuity properties of the Poisson integral) imply that the sets $E_\delta(\varphi)$ and $G_\delta(\varphi)$ have disjoint closures. Consequently,

$$(5.19) \quad \gamma = \gamma(\delta) := \text{dist}(E_\delta(\varphi), G_\delta(\varphi)) > 0.$$

Using (5.19) to estimate the Poisson kernel $d\mu_z(\zeta)/|d\zeta|$, we get

$$\int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_z \leq \frac{1 - |z|^2}{\gamma^2} \int_{E_\delta(\varphi)} \log \frac{1}{\varphi(\zeta)} \frac{|d\zeta|}{2\pi} \leq \frac{1 - |z|^2}{\gamma^2} \|\log \varphi\|_{L^1}$$

whenever $z \in G_\delta(\varphi)$. Therefore,

$$(5.20) \quad \int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_z < \varepsilon \quad \text{for } z \in G_\delta(\varphi) \cap \{|z| \geq r_0\},$$

where $r_0 \in (0, 1)$ is a suitable number depending on ε and δ (as well as on φ , of course). Now pick any nonconstant inner function $b \in \mathcal{B}_D$ and put $\eta := \max\{|b(z)| : |z| \leq r_0\}$. This done, we obviously have

$$G_\eta(b) \subset \{z : r_0 \leq |z| < 1\},$$

and so (5.20) yields

$$\int_{E_\delta(\varphi)} \log \frac{1}{\varphi} d\mu_z < \varepsilon \quad \text{for } z \in G_\delta(\varphi) \cap G_\eta(b).$$

Thus we arrive at condition (ii) of Theorem 3.1, and the theorem gives $\mathcal{O}_\varphi \in QD_A$. $\square$

The following corollary is immediate.

Corollary 5.1. *Let D be a Douglas algebra. If $\varphi \in QD$ is a positive function with $\text{ess-inf}_{\mathbb{T}} \varphi > 0$, then $\mathcal{O}_\varphi \in QD_A$.*

Remark 5.1. When $D = H^\infty + C$, Corollary 5.1 is also a consequence of Guillory and Sarason's results (cf. Proposition 4 in [14]).

In what follows, we restrict ourselves to the case $D = H^\infty + C$, and we are now concerned with BMO -type conditions on $\varphi \in QC$ ensuring that $\mathcal{O}_\varphi \in QA$. After all, it is in terms of mean oscillations that QC -functions are characterized among all L^∞ -functions, so it seems natural to start with a positive $\varphi \in L^\infty$ and to look for BMO -type (or VMO -type) refinements of the necessary conditions $\varphi \in VMO$ and $\log \varphi \in L^1$ that would make them sufficient.

Given a nondecreasing continuous function $\omega : [0, 1] \rightarrow \mathbb{R}_+$ (such ω 's will be referred to as *majorants*), let us define the class $BMO(\omega)$ as the set of those $f \in L^1$ for which

$$\sup_{z \in \mathbb{D}} \frac{1}{\omega(1 - |z|)} \int |f(\zeta) - f(z)| d\mu_z(\zeta) < \infty.$$

In connection with these weighted BMO -spaces, see e.g. [7] and [17]. Here we require the following fact.

Lemma 5.1. *Let $1 \leq p < \infty$ and suppose ω is a majorant such that*

$$\omega(st) \leq Ks^\gamma \omega(t) \quad \text{whenever } s \geq 1 \text{ and } 0 \leq st \leq 1,$$

with some fixed $K > 0$ and $\gamma \in (0, 1/p)$. Then every $f \in BMO(\omega)$ satisfies

$$\left(\int |f - f(z)|^p d\mu_z \right)^{1/p} \leq C\omega(1 - |z|), \quad z \in \mathbb{D},$$

with a constant $C > 0$ not depending on z .

This can be established by using standard BMO techniques. We also refer to Proposition 2.6 on p. 91 of [7] (which essentially coincides with the current lemma) and to the sketchy proof given there.

Proposition 5.2. *Let $\varepsilon > 0$, and let $\varphi \geq 0$ be a function of class $L^\infty \cap BMO(\omega_\varepsilon)$, where*

$$\omega_\varepsilon(t) := \left(\log \frac{2}{t} \right)^{-\varepsilon}, \quad 0 < t \leq 1.$$

Suppose also that $u := |\log \varphi|$ satisfies

$$(5.21) \quad \left(\int u^r d\mu_z \right)^{1/r} \leq C \cdot \log^N \frac{2}{1 - |z|}, \quad z \in \mathbb{D},$$

for some constants $r > 1$, $C > 0$, and $N > 0$. Then $\mathcal{O}_\varphi \in QA$.

Proof. We may assume that $\|\varphi\|_\infty \leq 1$, so that $u = \log \frac{1}{\varphi}$. Next, let us fix $\delta \in (0, 1)$ and write $E := E_\delta(\varphi)$. Taking (5.21) into account, we obtain

$$(5.22) \quad \int_E u \, d\mu_z \leq \left(\int u^r \, d\mu_z \right)^{1/r} \mu_z(E)^{1/r'} \leq C \cdot \mu_z(E)^{1/r'} \cdot \log^N \frac{2}{1-|z|}.$$

Now if $z \in G_\delta(\varphi)$, then

$$\varphi(z) - \varphi(\zeta) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}, \quad \zeta \in E.$$

Hence, by Chebyshev's inequality,

$$(5.23) \quad \mu_z(E) \leq \left(\frac{2}{\delta} \right)^p \int |\varphi(\zeta) - \varphi(z)|^p \, d\mu_z(\zeta), \quad z \in G_\delta(\varphi),$$

where $p = p(\varepsilon, r, N)$ is an exponent to be chosen later. Recalling that $\varphi \in BMO(\omega_\varepsilon)$ and noting that ω_ε satisfies the hypothesis of Lemma 5.1, whenever $p \geq 1$ and $0 < \gamma < 1/p$, we infer that

$$\int |\varphi(\zeta) - \varphi(z)|^p \, d\mu_z(\zeta) \leq C(\omega_\varepsilon(1-|z|))^p = C \left(\log \frac{2}{1-|z|} \right)^{-p\varepsilon}, \quad z \in \mathbb{D}.$$

Combining this with (5.23) gives

$$(5.24) \quad \mu_z(E) \leq \text{const} \cdot \left(\log \frac{2}{1-|z|} \right)^{-p\varepsilon}, \quad z \in G_\delta(\varphi).$$

Finally, substituting (5.24) into (5.22) yields

$$(5.25) \quad \int_E u \, d\mu_z \leq \text{const} \cdot \left(\log \frac{2}{1-|z|} \right)^{-\sigma}, \quad z \in G_\delta(\varphi),$$

where $\sigma = p\varepsilon/r' - N$. Making p sufficiently large, we get $\sigma > 0$, and (5.25) tells us that

$$\int_E u \, d\mu_z \rightarrow 0 \quad \text{as } |z| \rightarrow 1, \quad z \in G_\delta(\varphi).$$

That $\mathcal{O}_\varphi \in QA$ now follows by Corollary 3.1. □

Remarks.

(1) Another natural sufficient condition for $\mathcal{O}_\varphi \in QA$ is that

$$(5.26) \quad \varphi \in QC \quad \text{and} \quad \log \varphi \in VMO$$

(cf. Proposition 1 in [14]). It was also shown in [14], Proposition 5, that the hypothesis $\log \varphi \in BMO$, in conjunction with a certain *VMO*-type condition on φ , implies $\mathcal{O}_\varphi \in QA$. Now, when passing from (5.26) to the hypotheses of our Proposition 5.2, one slightly strengthens the assumption on φ and weakens that on $\log \varphi$, the conclusion remaining intact. Actually, (5.21) is a *growth* condition on $u = |\log \varphi|$, which is essentially weaker than saying that $u \in BMO$. Indeed,

$$\left(\int u^r d\mu_z \right)^{1/r} \leq \left(\int |u - u(z)|^r d\mu_z \right)^{1/r} + u(z) \quad (r \geq 1, z \in \mathbb{D}),$$

and if $u \in BMO$, then the quantity on the right is bounded by

$$C \left(\|u\|_* + \log \frac{2}{1 - |z|} \right).$$

(2) As a prototype of Proposition 5.2, let us mention the celebrated Havin-Shamoyan-Carleson-Jacobs theorem that provides sharp Lipschitz-type conditions on φ in order that $\mathcal{O}_\varphi$ belong to a given Lipschitz class (see e.g. [5] or [23] for the statement, [5] for a simple proof, and [9] for some multidimensional extensions).

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