Existence of L^q Dimensions and Entropy Dimension for Self-conformal Measures

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ABSTRACT. We prove the existence of limits in the definitions of L^q dimensions, $q > 0$, $q \neq 1$, and the entropy dimension for any self-conformal measure, without any separation assumptions. We also show the existence of order-two densities for a class of self-similar measures with overlap.

1. INTRODUCTION

The L^q spectrum of a measure is one of the basic ingredients in the study of multi-fractal phenomena. It is closely related to the family of L^q dimensions, introduced by Rényi in [22], [23, p. 588] and studied in [14, 24, 25, 21, 11] and the references therein. Let ν be a compactly supported Borel probability measure on $\mathbb{R}^d$. Let $\mathcal{D}_n$ be the partition of $\mathbb{R}^d$ into grid boxes $\prod_{i=1}^d [k_i 2^{-n}, (k_i + 1)2^{-n})$ with $k_i \in \mathbb{Z}$. For $q > 0$, $q \neq 1$, denote

$$\tau_n^{(q)}(\nu) = \sum_{Q \in \mathcal{D}_n} (\nu Q)^q \quad \text{and} \quad h_n(\nu) = \sum_{Q \in \mathcal{D}_n} \nu Q \log \frac{1}{\nu Q}.$$

The L^q spectrum of ν is defined by

$$(1.1) \quad \tau(q) = \tau(\nu, q) = \lim_{n \rightarrow \infty} \frac{\log \tau_n^{(q)}(\nu)}{-n \log 2} \quad \text{for } q > 0, q \neq 1,$$

if the limit exists; otherwise one considers the upper and lower limits. If the limit defining $\tau(\nu, q)$ exists, then the ratio $D_q(\nu) = \tau(\nu, q)/(q - 1)$ is called the L^q dimension of ν . (The quantity $D_2(\nu)$ is also called the *correlation dimension* of ν .) See Section 4 for other definitions of L^q dimension. The *entropy dimension* of ν is

$$(1.2) \quad h(\nu) = \lim_{n \rightarrow \infty} \frac{h_n(\nu)}{n \log 2},$$

if the limit exists; otherwise one considers the upper and lower entropy dimensions.

In general, the limits in (1.1) and (1.2) need not exist. Our main result is the existence of these limits for an arbitrary self-conformal measure, in particular, for any self-similar measure. These limits are known to exist under some separation conditions, see, e.g., [3, 16]. The point of our result is that it holds without any separation conditions. Recently, the dimension properties of self-similar measures with overlap have been studied intensively, see, e.g. [9, 10, 20].

Let $U \subset \mathbb{R}^d$ be an open set. A differentiable map $S : U \rightarrow \mathbb{R}^d$ is *conformal* if $S'(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies $|S'(x)\gamma| = |S'(x)| \cdot |\gamma| \neq 0$ for all $x \in U$ and $\gamma \in \mathbb{R}^d$, $\gamma \neq 0$. We say that $\{S_i : X \rightarrow X\}_{i \leq \ell}$ is a *conformal iterated system* on a compact set $X \subset \mathbb{R}^d$ if each S_i extends to an injective conformal map $S_i : U \rightarrow U$ on an open set $U \supset X$ and the following two conditions are satisfied.

Contractivity: $\sup\{|S'_i(x)| : x \in U\} < 1$;

Bounded distortion: There exists $L_1 \geq 1$ such that for every $n \in \mathbb{N}$ and for every word $u = (u_1, \dots, u_n) \in \{1, \dots, \ell\}^n$,

$$(1.3) \quad L_1^{-1} \leq \frac{\|S'_u(x)\|}{\|S'_u(y)\|} \leq L_1 \quad \text{for all } x, y \in U,$$

where $S_u = S_{u_1} \circ \dots \circ S_{u_n}$.

It is well-known that the bounded distortion property (1.3) follows from the other assumptions if $d \geq 2$; for $d = 1$, a sufficient condition for (1.3) is Hölder continuity of the derivatives S'_i . More generally, if the moduli of continuity $w_i(\varepsilon) = \sup_{|x-y|<\varepsilon} |S'_i(x) - S'_i(y)|$ satisfy the Dini condition $\int_0^1 w_i(\varepsilon)\varepsilon^{-1} d\varepsilon < \infty$ for $i = 1, \dots, \ell$, then (1.3) holds, see [2].

Let $\{S_i\}_{i \leq \ell}$ be a conformal iterated function system on a compact set X . It is well-known, see [7], that there is a unique non-empty compact set $\mathcal{K} \subset X$ such that $\mathcal{K} = \bigcup_{j=1}^{\ell} S_j \mathcal{K}$. Given a probability vector $(p_1, \dots, p_{\ell})$, there is a unique Borel probability measure ν satisfying

$$(1.4) \quad \nu = \sum_{i=1}^{\ell} p_i (\nu \circ S_i^{-1}).$$

This measure is supported on $\mathcal{K}$; it is called *self-conformal*. We always assume that S_i do not all have the same fixed point; then $\mathcal{K}$ is perfect and it is easy to see that ν is non-atomic.

Theorem 1.1. *If ν is a self-conformal measure, then the limits in (1.1) and (1.2) exist.*

Remarks. (1) In the special case where $q = 2$ and $\nu = \nu_\lambda$ is the classical Bernoulli convolution, Theorem 1.1 was proved in the survey Peres, Schlag and Solomyak [18, Prop.4.1]. Recall that ν_λ is the distribution of the random series $\sum \pm \lambda^n$ where the signs form an i.i.d. sequence of $(\frac{1}{2}, \frac{1}{2})$ Bernoulli random variables. The existence of L^2 dimension for ν_λ was used in [18, Prop.4.1] to show that ν_λ has L^2 dimension and Hausdorff dimension equal to one for a residual set of $\lambda \in (\frac{1}{2}, 1)$. The same method, combined with Theorem 1.1 and [20, Cor.1.4], yields the following corollary:

Consider the biased Bernoulli convolutions ν_λ^p , where the signs are chosen with probabilities $(p, 1-p)$ for some $p \in [\frac{1}{3}, \frac{2}{3}]$. Then for every $q \in (1, 2)$ there is a residual set $\mathcal{F}_q \subset ((p^q + (1-p)^q)^{1/(q-1)}, 1)$ such that the L^q dimension and the Hausdorff dimension of ν_λ^p equal one for all $\lambda \in \mathcal{F}_q$.

(2) The existence of L^2 dimension $D_2(\nu)$ for a measure ν in $\mathbb{R}^d$ implies that

$$\lim_{R \rightarrow \infty} \frac{\log \int_{|x| \leq R} |\hat{\nu}(x)|^2 dx}{\log R} = d - D_2(\nu),$$

see Strichartz [26, Th.6.6]. Moreover, the existence of L^q dimension, for $q > 1$, has implications for the spectral asymptotics of the measure, see [26, Th.6.5].

The second topic of this paper is the order-two (average) densities of self-similar measures. The *order-two density* of a measure ν is defined by

$$(1.5) \quad A^s(\nu, x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\nu B(x, e^{-t})}{(2e^{-t})^s} dt$$

if the limit exists. Bedford and Fisher [1] introduced order-two densities, and established their existence for Hausdorff measure on hyperbolic Cantor sets (see also [4, Ch. 6]). Patzschke and Zähle [17] computed the order-two density $\mathcal{H}^s$ -a.e. for the Hausdorff measure $\mathcal{H}^s$ on a self-similar set of dimension s satisfying the open set condition. Recently, Zähle [27] extended this to certain self-conformal measures.

Here we prove the existence of order-two density for natural measures on a class of self-similar sets with overlap. In conjunction with the results of [19], we obtain the existence of positive and finite order-two density for the packing measure on a typical self-similar set in some parameterized families. Here we state this for the family of projections; a more general result is presented in Section 5.

Let $\mathcal{K} = \bigcup_{j=1}^\ell (r_j \mathcal{K} + a_j)$ be a self-similar set in $\mathbb{R}^2$. Here $r_j \in (0, 1)$ are contraction ratios and $a_j \in \mathbb{R}^2$; note that the self-similar set does not involve rotations. We assume that the (closed) cylinders $\mathcal{K}_j = r_j \mathcal{K} + a_j$ of the self-similar set are disjoint. The orthogonal projection proj_ϑ onto the line $y \cos \vartheta =$

$x \sin \vartheta$ maps $\mathcal{K}$ into a self-similar set $\mathcal{K}_\vartheta \subset \mathbb{R}$. Let s be the similarity dimension: $\sum_{j=1}^\ell r_j^s = 1$. It is shown in [19] that the s -dimensional Hausdorff measure of $\mathcal{K}_\vartheta$ can vanish for a.e. ϑ in a certain interval, but the s -dimensional packing measure of $\mathcal{K}_\vartheta$ is positive for a.e. $\vartheta \in [0, 2\pi]$.

Proposition 1.2. *Suppose that $s < 1$, and denote by $\mathcal{P}_\vartheta^s$ the s -dimensional packing measure restricted to $\mathcal{K}_\vartheta$. Then for a.e. $\vartheta \in [0, 2\pi)$, the order-two density $A^s(\mathcal{P}_\vartheta^s, x) = A^s(\mathcal{P}_\vartheta^s)$ exists $\mathcal{P}_\vartheta^s$ -a.e., and $0 < A^s(\mathcal{P}_\vartheta^s) < \infty$.*

Section 2 contains preliminaries and Theorem 1.1 is proved in Section 3. Several alternative definitions of L^q dimension are compared in Section 4, and order-two densities are treated in Section 5.

2. PRELIMINARIES

Here we collect some easy lemmas used in the proof of Theorem 1.1.

Definition 2.1. Let K be a compact set in $\mathbb{R}^d$. Fix $M, \varepsilon > 0, N \in \mathbb{N}$. Say that a covering $\{G_i\}_{i=1}^k$ of K by Borel sets is (M, ε, N) -good if $\text{diam}(G_i) \leq M\varepsilon$, for $i \leq k$, and any cube of side ε intersects at most N elements of the covering.

The following lemma will be used in the proof of the existence of L^q dimensions.

Lemma 2.2. *Let $M > 0, N, d \in \mathbb{N}, q > 0, q \neq 1$. There exists $C_1 = C_1(M, N, d, q) > 1$ such that for any $n \in \mathbb{N}$ and any compactly supported measure ν on $\mathbb{R}^d$, for any $(M, 2^{-n}, N)$ -good covering $\{G_i\}_{i=1}^k$ of $\text{supp}(\nu)$, we have*

$$(2.1) \quad C_1^{-1} \tau_n^{(q)}(\nu) \leq \sum_{i=1}^k (\nu G_i)^q \leq C_1 \tau_n^{(q)}(\nu).$$

Proof. Any set of diameter $\leq M2^{-n}$ intersects at most $(M+1)^d$ grid boxes in $\mathcal{D}_n$. Using the inequality $(\sum a_i)^q \leq \sum a_i^q$ for $a_i > 0$ and $0 < q < 1$, and the Hölder inequality for $q > 1$, we obtain the right estimate in (2.1). Similarly, any grid box in $\mathcal{D}_n$ intersects at most N elements of the covering, and this yields the left estimate in (2.1). $\square$

The next lemma will be used in the proof of the existence of the entropy dimension.

Lemma 2.3. *Let ν be any compactly supported measure in $\mathbb{R}^d$. Then*

(a)

$$(2.2) \quad h_n(\nu) \leq h_{n+1}(\nu) \leq h_n(\nu) + 2^d \quad \text{for all } n \in \mathbb{N};$$

(b) there exists $C_2 > 0$ such that

$$(2.3) \quad \left| \int \log \frac{1}{\nu B(x, 2^{-n})} d\nu(x) - h_n(\nu) \right| \leq C_2 \quad \text{for all } n \in \mathbb{N}.$$

Proof. (a) Let $Q_n(x)$ be the grid box in $\mathcal{D}_n$ containing x . Then

$$(2.4) \quad h_n(\nu) = \int \log \frac{1}{\nu Q_n(x)} d\nu(x).$$

Thus, $h_{n+1}(\nu) \geq h_n(\nu)$. Next,

$$\begin{aligned} h_{n+1}(\nu) - h_n(\nu) &= \int \log \frac{\nu Q_n(x)}{\nu Q_{n+1}(x)} d\nu(x) \\ &\leq \int \frac{\nu Q_n(x)}{\nu Q_{n+1}(x)} d\nu(x) \\ &= \sum_{Q \in \mathcal{D}_{n+1}} \nu \tilde{Q} \leq 2^d \end{aligned}$$

where $\tilde{Q} \in \mathcal{D}_n$ is such that $Q \subset \tilde{Q}$. We have verified (2.2).

(b) Since $Q_n(x) \subset B(x, 2^{-n}\sqrt{d})$, it follows from (2.4) that

$$h_n(\nu) \geq \int \log \frac{1}{\nu B(x, 2^{-n}\sqrt{d})} d\nu(x) \quad \text{for all } n \in \mathbb{N}.$$

Thus, if $2^{-k}\sqrt{d} \leq 1$, then by (2.2),

$$h_n(\nu) \geq h_{n+k} - k \cdot 2^d \geq \int \log \frac{1}{\nu B(x, 2^{-n})} d\nu(x) - k \cdot 2^d.$$

On the other hand, denoting by $\hat{Q}_n(x)$ the union of $Q_n(x)$ with the boxes in $\mathcal{D}_n$ whose closure intersects the closure of $Q_n(x)$, we have

$$\begin{aligned} h_n(\nu) - \int \log \frac{1}{\nu B(x, 2^{-n})} d\nu(x) &= \int \log \frac{\nu B(x, 2^{-n})}{\nu Q_n(x)} d\nu(x) \\ &\leq \int \frac{\nu \hat{Q}_n(x)}{\nu Q_n(x)} d\nu(x) = \sum_{Q_n \in \mathcal{D}_n} \nu \hat{Q}_n(x) \leq 3^d, \end{aligned}$$

and the lemma is proved. $\square$

Notation. Let $\mathcal{A} = \{1, \dots, \ell\}$. Recall that S_i , for $i \in \mathcal{A}$, are injective conformal contractions on an open set $U \subset \mathbb{R}^d$ containing the compact X . For $u = u_1 \dots u_k$ we write $S_u = S_{u_1} \circ \dots \circ S_{u_k}$ and $p_u = p_{u_1} \dots p_{u_k}$. Recall that $\mathcal{K}$ is the support of the measure ν and let $\mathcal{K}_u = S_u(\mathcal{K})$. Denote $\mathcal{A}^* = \bigcup_{n \geq 1} \mathcal{A}^n$; for $u \in \mathcal{A}^*$ let $\bar{u}$ be the word obtained by dropping the last letter. Consider the “cut-set”

$$W_n := \{u \in \mathcal{A}^* : \text{diam}(\mathcal{K}_u) \leq 2^{-n}, \text{diam}(\mathcal{K}_{\bar{u}}) > 2^{-n}\}.$$

It follows from (1.4) that

$$(2.5) \quad \nu = \sum_{u \in W_n} p_u(\nu \circ S_u^{-1}) \quad \text{for all } n \in \mathbb{N}.$$

Lemma 2.4. *There exist $M > 0$ and $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$ and any $u \in W_n$, the collection*

$$\{S_u^{-1}Q \cap \mathcal{K} : Q \in \mathcal{D}_{m+n}, Q \cap \mathcal{K}_u \neq \emptyset\}$$

is an $(M, 2^{-m}, N)$ -good covering of $\mathcal{K}$.

Proof. This is obviously a covering. Denote $\|S'_u\| = \sup_{\zeta \in U} |S'_u(\zeta)|$. We will use the well-known consequences of bounded distortion (1.3), see e.g. [16]:

There is a constant $C \geq 1$ such that for all $u \in \mathcal{A}^*$,

$$(2.6) \quad C^{-1}\|S'_u\| \leq \text{diam}(\mathcal{K}_u) \leq C\|S'_u\|;$$

$$(2.7) \quad C^{-1}\|S'_u\| \cdot |x - y| \leq |S_u(x) - S_u(y)| \leq C\|S'_u\| \cdot |x - y|$$

for all $x, y \in X$.

We have by (2.6), (1.3), and conformality:

$$(2.8) \quad 2^{-n} < \text{diam}(\mathcal{K}_{\bar{u}}) \leq C\|S'_{\bar{u}}\| \leq CL_1\lambda_0^{-1}\|S'_u\|,$$

where $\lambda_0 := \min\{|S'_i(\zeta)| : i \leq \ell, \zeta \in X\} > 0$. Therefore, we obtain from (2.7) and (2.8) for $Q \in \mathcal{D}_{m+n}$:

$$\begin{aligned} \text{diam}(S_u^{-1}Q \cap \mathcal{K}) &\leq \|S'_u\|^{-1} \cdot \text{diam}(Q) \\ &\leq C^2 L_1 \lambda_0^{-1} 2^n \cdot 2^{-m-n} \sqrt{d} =: M 2^{-m}. \end{aligned}$$

We have verified the first property of the good covering. Now we need to show that there exists $N \in \mathbb{N}$ such that any cube of side 2^{-m} intersects at most N

elements of the covering. Let G be such a cube intersecting $\mathcal{K}$. Without loss of generality, we can assume that m is so large that $G \subset U$. If $G \cap S_u^{-1}Q \neq \emptyset$, then $S_u G \cap Q \neq \emptyset$. We have by the mean value inequality and (2.6):

$$\text{diam}(S_u G) \leq \|S'_u\| \cdot \text{diam}(G) \leq C \text{diam}(\mathcal{K}_u) 2^{-m} \sqrt{d} \leq C 2^{-m-n} \sqrt{d}.$$

Hence $S_u G$ intersects at most $(C\sqrt{d} + 1)^d$ grid cubes in $\mathcal{D}_{m+n}$. The lemma is proved. $\square$

3. PROOF OF THEOREM 1.1

Existence of L^q dimensions. It is convenient to consider the two cases of $q > 1$ and $0 < q < 1$ separately, although the arguments are parallel in many places. Combining Lemmas 2.2 and 2.4 we obtain that there exists $C_1 > 1$ independent of $m, n \in \mathbb{N}$ such that for any $u \in W_n$,

$$(3.1) \quad C_1^{-1} \tau_m^{(q)}(\nu) \leq \sum_{Q \in \mathcal{D}_{m+n}} \nu(S_u^{-1}Q)^q \leq C_1 \tau_m^{(q)}(\nu).$$

For $Q \in \mathcal{D}_{m+n}$ let $\tilde{Q} \in \mathcal{D}_n$ be such that $Q \subset \tilde{Q}$.

Case 1: $q > 1$. We are going to show that there exists $C \geq 1$ such that

$$(3.2) \quad \tau_{m+n}^{(q)}(\nu) \leq C \tau_m^{(q)}(\nu) \tau_n^{(q)}(\nu) \quad \text{for all } m, n \in \mathbb{N}.$$

Then the sequence $\{C \tau_n^{(q)}(\nu)\}$ is sub-multiplicative, hence the limit $\tau(\nu, q)$ in (1.1) exists, and moreover, $\tau(\nu, q) \geq -\log(C \tau_n^{(q)}(\nu))/n \log 2$ for any n .

Using (2.5), we have for any $Q \in \mathcal{D}_{m+n}$:

$$(3.3) \quad \nu Q = \sum_{u \in W_n} p_u \nu(S_u^{-1}Q) = \sum_{u \in W_n: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} p_u \nu(S_u^{-1}Q).$$

We will omit (but assume) $u \in W_n$ in the subscripts for summation signs to simplify notation. Denote

$$(3.4) \quad p_+(\tilde{Q}) = \sum_{u: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} p_u.$$

Using convexity of x^q for $q > 1$, we obtain from (3.3):

$$\begin{aligned} (\nu Q)^q &= p_+(\tilde{Q})^q \left(\sum_{u: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} \frac{p_u}{p_+(\tilde{Q})} \nu(S_u^{-1}Q) \right)^q \\ &\leq p_+(\tilde{Q})^{q-1} \sum_{u: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} p_u \nu(S_u^{-1}Q)^q. \end{aligned}$$

Summing over $Q \subset \tilde{Q}$, $Q \in \mathcal{D}_{m+n}$, we obtain

$$\begin{aligned} \sum_{Q \subset \tilde{Q}} (\nu Q)^q &\leq p_+(\tilde{Q})^{q-1} \sum_{u: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} p_u \sum_{Q \subset \tilde{Q}} \nu(S_u^{-1}Q)^q \\ &\leq p_+(\tilde{Q})^{q-1} \sum_{u: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} p_u \cdot C_1 \tau_m^{(q)}(\nu) \\ &= C_1 p_+(\tilde{Q})^q \cdot \tau_m^{(q)}(\nu), \end{aligned}$$

where in the second displayed line we used the right inequality in (3.1). Summing over $\tilde{Q} \in \mathcal{D}_n$ yields

$$(3.5) \quad \tau_{m+n}^{(q)}(\nu) \leq C_1 \tau_m^{(q)}(\nu) \sum_{\tilde{Q} \in \mathcal{D}_n} p_+(\tilde{Q})^q.$$

Write $\tilde{B} \sim \tilde{Q}$ if $\tilde{B} \in \mathcal{D}_n$ and the closures of $\tilde{B}$ and $\tilde{Q}$ intersect. Clearly, there are 3^d such boxes. Recall that $\text{diam}(\mathcal{K}_u) \leq 2^{-n}$ for $u \in W_n$, hence $\mathcal{K}_u \cap \tilde{Q} \neq \emptyset$ implies $\mathcal{K}_u \subset \bigcup_{\tilde{B} \sim \tilde{Q}} \tilde{B}$. By (3.4),

$$p_+(\tilde{Q})^q \leq \left(\sum_{\tilde{B} \sim \tilde{Q}} \nu \tilde{B} \right)^q \leq 3^{d(q-1)} \sum_{\tilde{B} \sim \tilde{Q}} (\nu \tilde{B})^q,$$

by Hölder's inequality, using again that $q > 1$. Therefore,

$$\sum_{\tilde{Q} \in \mathcal{D}_n} p_+(\tilde{Q})^q \leq 3^d 3^{d(q-1)} \sum_{\tilde{B} \in \mathcal{D}_n} (\nu \tilde{B})^q = 3^{dq} \tau_n^{(q)}(\nu).$$

Combining this with (3.5) yields (3.2), and the proof is finished. $\square$

Case 2: $0 < q < 1$. We are going to show that there exists $C \geq 1$ such that

$$(3.6) \quad \tau_{m+n}^{(q)}(\nu) \geq C^{-1} \tau_m^{(q)}(\nu) \tau_n^{(q)}(\nu) \quad \text{for all } m, n \in \mathbb{N}.$$

Then the sequence $\{C^{-1} \tau_n^{(q)}(\nu)\}$ is super-multiplicative, hence the limit $\tau(\nu, q)$ in (1.1) exists, and moreover, $\tau(\nu, q) \leq -\log(C^{-1} \tau_n^{(q)}(\nu))/n \log 2$ for any n .

For $u \in W_n$ and $\tilde{Q} \in \mathcal{D}_n$ let

$$w(u, \tilde{Q}) := \sum_{Q \in \mathcal{D}_{m+n}: Q \subset \tilde{Q}} \nu(S_u^{-1}Q)^q.$$

The box $\tilde{Q}$ on which $w(u, \tilde{Q})$ attains its maximum will be called *q-heavy* for $u \in W_n$. We will denote the *q-heavy* box by $\tilde{Q} = H(u)$; if there are several boxes with equal sums, we choose one of them arbitrarily.

Since any $\mathcal{K}_u$, with $u \in W_n$, intersects at most 3^d boxes in $\mathcal{D}_n$, it follows from (3.1) that

$$(3.7) \quad \sum_{\substack{Q \in \mathcal{D}_{m+n} \\ Q \subset H(u)}} \nu(S_u^{-1}Q)^q \geq 3^{-d} C_1^{-1} \tau_m^{(q)}(\nu) \quad \text{for all } u \in W_n.$$

By (2.5), for any $Q \in \mathcal{D}_{m+n}$,

$$\nu Q = \sum_{u \in W_n} p_u \nu(S_u^{-1}Q) \geq \sum_{u \in W_n: \tilde{Q} = H(u)} p_u \nu(S_u^{-1}Q).$$

Below we omit (but assume) $u \in W_n$ in summation signs. Denote

$$(3.8) \quad p_-(\tilde{Q}) = \sum_{u: \tilde{Q} = H(u)} p_u.$$

By concavity of x^q for $0 < q < 1$,

$$\begin{aligned} (\nu Q)^q &= p_-(\tilde{Q})^q \left(\sum_{u: \tilde{Q} = H(u)} \frac{p_u}{p_-(\tilde{Q})} \nu(S_u^{-1}Q) \right)^q \\ &\geq p_-(\tilde{Q})^{q-1} \sum_{u: \tilde{Q} = H(u)} p_u \nu(S_u^{-1}Q)^q. \end{aligned}$$

Summing over $Q \subset \tilde{Q}$, $Q \in \mathcal{D}_{m+n}$, we obtain

$$\begin{aligned} \sum_{Q \subset \tilde{Q}} (\nu Q)^q &\geq p_-(\tilde{Q})^{q-1} \sum_{u: \tilde{Q} = H(u)} p_u \sum_{Q \subset \tilde{Q}} \nu(S_u^{-1}Q)^q \\ &\geq p_-(\tilde{Q})^{q-1} \sum_{u: \mathcal{K}_u \cap \tilde{Q} \neq \emptyset} p_u \cdot 3^{-d} C_1^{-1} \tau_m^{(q)}(\nu) \\ &= 3^{-d} C_1^{-1} p_-(\tilde{Q})^q \cdot \tau_m^{(q)}(\nu), \end{aligned}$$

where in the second displayed line we used (3.7). Summing over $\tilde{Q} \in \mathcal{D}_n$ yields

$$(3.9) \quad \tau_{m+n}^{(q)}(\nu) \geq 3^{-d} C_1^{-1} \tau_m^{(q)} \sum_{\tilde{Q} \in \mathcal{D}_n} p_-(\tilde{Q})^q.$$

As in Case 1, write $\tilde{B} \sim \tilde{Q}$ if $\tilde{B} \in \mathcal{D}_n$ and the closures of $\tilde{B}$ and $\tilde{Q}$ intersect. Observe that for any $\tilde{Q} \in \mathcal{D}_n$,

$$\nu \tilde{Q} \leq p_+(\tilde{Q}) \leq \sum_{\tilde{B} \sim \tilde{Q}} p_-(\tilde{B}).$$

Indeed, every $\mathcal{K}_u$ that hits $\tilde{Q}$ must have a box $\tilde{B}$ which is q -heavy for u , and clearly, $\tilde{B} \sim \tilde{Q}$. Thus,

$$(\nu\tilde{Q})^q \leq \left(\sum_{\tilde{B} \sim \tilde{Q}} p_-(\tilde{B}) \right)^q \leq \sum_{\tilde{B} \sim \tilde{Q}} p_-(\tilde{B})^q,$$

using that $0 < q < 1$. Summing over $\tilde{Q} \in \mathcal{D}_n$ implies

$$\tau_n^{(q)}(\nu) \leq 3^d \sum_{\tilde{Q} \in \mathcal{D}_n} p_-(\tilde{Q})^q.$$

Combining this with (3.9) yields (3.6), and the proof is finished. $\square$

Existence of the entropy dimension. By (2.2), the sequence $\{h_n(\nu)\}$ is monotone increasing, and we can assume, without loss of generality, that $h_n(\nu) \rightarrow \infty$. We are going to show that

$$(3.10) \quad h_{n+m} \geq h_n + h_m - C \quad \text{for all } m, n \in \mathbb{N},$$

for some $C > 0$. Then the sequence $\{h_n - C\}$ is super-additive, hence the limit $h(\nu)$ in (1.2) exists and satisfies $h(\nu) \geq (h_n - C)/n \log 2$ for all n .

For $Q \in \mathcal{D}_{m+n}$ denote

$$(3.11) \quad p(Q) = \sum_{u: \mathcal{K}_u \cap Q \neq \emptyset} p_u.$$

Let $f(x) = x \log(1/x)$ and observe that $f(xy) = f(x)y + xf(y)$ for all $x, y \neq 0$. Applying this to the identity $\nu Q = p(Q) \cdot \nu Q / p(Q)$ and using (2.5), we obtain

$$\begin{aligned} f(\nu Q) &= f(p(Q)) \frac{\nu Q}{p(Q)} + p(Q) f\left(\sum_{u: \mathcal{K}_u \cap Q \neq \emptyset} \frac{p_u}{p(Q)} \nu(S_u^{-1}Q) \right) \\ &=: A_1(Q) + A_2(Q). \end{aligned}$$

As before, the summation is over $u \in W_n$. Clearly,

$$(3.12) \quad A_1(Q) = f(p(Q)) \frac{\nu Q}{p(Q)} = \nu Q \log \frac{1}{p(Q)}.$$

By concavity of f ,

$$\begin{aligned} (3.13) \quad A_2(Q) &\geq p(Q) \sum_{u: \mathcal{K}_u \cap Q \neq \emptyset} \frac{p_u}{p(Q)} f(\nu(S_u^{-1}Q)) \\ &= \sum_{u: \mathcal{K}_u \cap Q \neq \emptyset} p_u f(\nu(S_u^{-1}Q)). \end{aligned}$$

We have

$$(3.14) \quad h_{m+n} = \sum_{Q \in \mathcal{D}_{m+n}} f(\nu Q) = \sum_{Q \in \mathcal{D}_{m+n}} [A_1(Q) + A_2(Q)].$$

Since $\text{diam}(\mathcal{K}_u) \leq 2^{-n}$, for any $x \in Q \in \mathcal{D}_{m+n}$ we have $p(Q) \leq \nu B(x, 2^{-n}(\sqrt{d} + 1))$. Therefore, for $k \in \mathbb{N}$ such that $\sqrt{d} + 1 \leq 2^k$,

$$(3.15) \quad \begin{aligned} \sum_{Q \in \mathcal{D}_{m+n}} A_1(Q) &\geq \int \log \frac{1}{\nu B(x, 2^{-n}(\sqrt{d} + 1))} d\nu(x) \\ &\geq h_{n-k} - C_2 \geq h_n - C_2 - k \cdot 2^d, \end{aligned}$$

by (3.12) and Lemma 2.3. On the other hand, by (3.13),

$$\sum_{Q \in \mathcal{D}_{m+n}} A_2(Q) \geq \sum_{u \in W_n} p_u \sum_{Q: \mathcal{K}_u \cap Q \neq \emptyset} f(\nu(S_u^{-1}Q)).$$

By Lemma 2.4, for any $u \in W_n$ there exist $M > 0$ and $N \in \mathbb{N}$ such that $\{S_u^{-1}Q : Q \in \mathcal{D}_{m+n}, Q \cap \mathcal{K}_u \neq \emptyset\}$ is an $(M, 2^{-m}, N)$ -good covering of $\mathcal{K}$. It follows that $S_u^{-1}Q \subset B(x, M \cdot 2^{-m})$ for any $u \in W_n$ and $x \in S_u^{-1}Q$. Thus, we obtain from Lemma 2.3 that

$$(3.16) \quad \begin{aligned} \sum_{Q: \mathcal{K}_u \cap Q \neq \emptyset} f(\nu(S_u^{-1}Q)) &= \sum_{Q: \mathcal{K}_u \cap Q \neq \emptyset} \nu(S_u^{-1}Q) \cdot \log \frac{1}{\nu(S_u^{-1}Q)} \\ &\geq \int \log \frac{1}{\nu B(x, M \cdot 2^{-m})} d\nu(x) \\ &\geq h_m - C_2 - k \cdot 2^d, \end{aligned}$$

where k is such that $M \leq 2^k$. Finally, we note that $\sum_{u \in W_n} p_u = 1$, so combining (3.14), (3.15), and (3.16) yields (3.10), and the desired statement follows. $\square$

4. ALTERNATIVE DEFINITIONS OF L^q DIMENSION

Here we compare some of the definitions of L^q dimension that can be found, e.g., in [14, 24, 25, 21, 11]. To the best of our knowledge, for $0 < q < 1$ the relation (4.1) below has not been justified in the literature.

For a compactly supported Borel probability measure ν on $\mathbb{R}^d$ and $q > 0$, $q \neq 1$, denote

$$I_r^{(q)}(\nu) = \int (\nu B(x, r))^{q-1} d\nu.$$

A collection of balls $\{B(x_i, r)\}_i$ is said to be a centered packing of a set X if $x_i \in X$ for all i and the balls are disjoint. A cover $\{B(x_i, r)\}_i$ of X is a centered cover if

$x_i \in X$ for all i . Let

$$T_{r,\text{pack}}^{(q)}(\nu) = \sup \left\{ \sum_i (\nu B(x_i, r))^q : \{B(x_i, r)\}_i \text{ is a centered packing of } \text{supp}(\nu) \right\}$$

and

$$T_{r,\text{cov}}^{(q)}(\nu) = \inf \left\{ \sum_i (\nu B(x_i, r))^q : \{B(x_i, r)\}_i \text{ is a centered cover of } \text{supp}(\nu) \right\}.$$

These quantities should be compared to the expression $\tau_{n(r)}^{(q)}(\nu)$, used in this paper, where $n = n(r)$ is defined by $2^{-n-1} \leq r < 2^{-n}$. Each of the quantities above has been used to define upper and lower dimensions (taking log, dividing by $\log r$ and considering $\limsup$ and $\liminf$, as $r \rightarrow 0$).

Comparing these quantities as functions of r , we write $\preceq$ and $\succeq$ in the sense that the corresponding inequality holds up to a multiplicative constant independent of r (of course, the constants will depend of q and d). The symbol $\asymp$ means that both $\preceq$ and $\succeq$ are true. In [11, Prop.3.1] it is proved that $\tau_{n(r)}^{(q)}(\nu) \asymp T_{r,\text{pack}}^{(q)}(\nu)$ for all q .

Lemma 4.1. *Let ν be an arbitrary compactly supported Borel probability measure on $\mathbb{R}^d$.*

(i) *For all $q > 0$, $q \neq 1$,*

$$(4.1) \quad \tau_{n(r)}^{(q)}(\nu) \asymp I_r^{(q)}(\nu).$$

(ii) *We have*

$$\tau_{n(r)}^{(q)}(\nu) \asymp T_{r,\text{cov}}^{(q)}(\nu) \text{ for } 0 < q < 1 \quad \text{and} \quad T_{r,\text{cov}}^{(q)}(\nu) \leq \tau_{n(r)}^{(q)}(\nu) \text{ for } q > 1.$$

However, $\tau_{n(r)}^{(q)}(\nu) \leq T_{r,\text{cov}}^{(q)}(\nu)$ for $q > 1$ need not hold; in fact, the dimensions arising from these quantities may differ.

Proof. (i) The proof of $\tau_{n(r)}^{(q)}(\nu) \asymp I_r^{(q)}(\nu)$ for $q > 1$ can be found in [21, Th.18.2]. For completeness, we provide the proof for the case $0 < q < 1$.

We want to show that for $2^{-n-1} \leq r < 2^{-n}$, the quantities $\tau_n^{(q)}(\nu)$ and $I_r^{(q)}(\nu)$ are comparable. We drop ν from the notation as it is fixed.

Let $Q_n(x)$ be the grid box in $\mathcal{D}_n$ containing x . Writing

$$\tau_n^{(q)} = \int (\nu Q_n(x))^{q-1} d\nu,$$

we see that $I_r^{(q)} \leq \tau_{n+k}^{(q)} \leq C(k, q, d) \tau_n^{(q)}$ where $k = 1 + \log_2 d$, say, is chosen to ensure that the diameter of a cube in $\mathcal{D}_{n+k}$ is at most r .

For the proof of the other direction, say that a cube Q in $\mathcal{D}_n$ is fat (and write $Q \in \text{fat}_n$) if for every cube $Q' \in \mathcal{D}_n$ that intersects Q , we have $\nu Q' \leq 2\nu Q$; otherwise, say Q is thin. If $x \in Q$ and $Q \in \text{fat}_n$, then $\nu B(x, r) \leq 2 \cdot 3^d \nu Q$, whence $(\nu B(x, r))^{q-1} \geq (2 \cdot 3^d \nu Q)^{q-1}$. Thus we can bound the sum over fat cubes:

$$(4.2) \quad \sum_{Q \in \text{fat}_n} (\nu Q)^q \leq (2 \cdot 3^d)^{1-q} I_r^{(q)}.$$

It remains to bound the contribution of thin cubes. For every thin cube $Q_0 \in \mathcal{D}_n$, we can find a sequence of cubes $Q_1, Q_2, \dots, Q_L$ in $\mathcal{D}_n$ such that each Q_i intersects Q_{i-1} and $\nu Q_i > 2\nu Q_{i-1}$, where Q_L is fat. For fixed L and Q_L , there are at most $(2L+1)^d$ choices of Q_0 , and each has $\nu Q_0 < 2^{-L} \nu Q_L$. Summing the q 'th powers, we conclude that

$$\tau_n^{(q)} \leq \left(1 + \sum_{L \geq 1} (2L+1)^d 2^{-Lq}\right) \sum_{Q \in \text{fat}_n} (\nu Q)^q.$$

Combining this with (4.2) completes the proof of part (i).

(ii) The inequality $T_{r,\text{cov}}^{(q)}(\nu) \leq \tau_{n(r)}^{(q)}(\nu)$ is straightforward for all q . Indeed, fix k such that $2^k \geq \sqrt{d}$. Then any box in $\mathcal{D}_{n(r)+k}$ intersecting $\text{supp}(\nu)$ can be covered by an r -ball centered in $\text{supp}(\nu)$. Since such a ball will intersect finitely many grid boxes, it follows that $T_{r,\text{cov}}^{(q)} \leq \tau_{n(r)+k}^{(q)}(\nu)$, and $\tau_{n(r)+k}^{(q)}(\nu) \asymp \tau_{n(r)}^{(q)}(\nu)$ by Lemma 2.2.

An example, showing that $\tau_{n(r)}^{(q)}(\nu) \leq T_{r,\text{cov}}^{(q)}(\nu)$ for $q > 1$ need not hold, is a minor modification of [21, Ex.8.3]: just take ν to be the (normalized) sum of the two-dimensional Lebesgue measure on $[0, 1]^2$ and the one-dimensional Lebesgue measure on $[0, 1] \times \{0\}$. Then $D_2(\nu) = 1$ but the dimension associated with $T_{r,\text{cov}}^{(q)}(\nu)$ is equal to 2.

For $0 < q < 1$ one can proceed as follows: for any covering $\{B(x_i, r)\}_i$ of $\text{supp}(\nu)$ and $q \in \mathcal{D}_{n(r)}$, write $\nu Q \leq \sum_i \nu B(x_i, r)$, where the sum is over balls which intersect Q . Then $(\nu Q)^q \leq \sum_i (\nu B(x_i, r))^q$ since $0 < q < 1$. We have $\tau_{n(r)}^{(q)}(\nu) \leq T_{r,\text{cov}}^{(q)}(\nu)$ using that an r -ball intersects a uniformly bounded number of boxes. Note that for $q > 1$ the argument breaks down: we would have to use the Hölder inequality which involves the number of balls intersecting a given box, and this number can be unbounded. $\square$

5. EXISTENCE OF ORDER-TWO DENSITIES FOR SELF-SIMILAR MEASURES WITH OVERLAP

Let $\{S_1, \dots, S_\ell\}$ be a family of similitudes in $\mathbb{R}^d$, with $r_i = \|S_i\| \in (0, 1)$. The similarity dimension is $s > 0$ such that $\sum_{i=1}^\ell r_i^s = 1$. We let $p_i = r_i^s$ and consider the natural self-similar measure associated with the iterated function system: $\nu = \sum_{i=1}^\ell p_i(\nu \circ S_i^{-1})$. The *order-two* (or average) density of ν is defined by (1.5), if the limit exists.

Theorem 5.1. *Let ν be as above. Then $A^s(\nu, x) = A^s(\nu) > 0$ exists for ν -a.e. x . Moreover, $A^s(\nu) < \infty$ provided that*

$$(5.1) \quad \Gamma^s(\nu) := \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{(\nu \times \nu)\{(x, y) : |x - y| \leq e^{-t}\}}{e^{-ts}} dt < \infty.$$

With the help of a result from [19], this theorem may be applied to some families of self-similar measures. In order to state the result we introduce some notation.

Let $\Omega = \{1, \dots, \ell\}^{\mathbb{N}}$. For $\omega \in \Omega$ we write $S_{\omega, n} = S_{\omega_1} \circ \dots \circ S_{\omega_n}$ and $r_{\omega, n} = r_{\omega_1} \cdot \dots \cdot r_{\omega_n}$. The natural “projection” map $\pi : \Omega \rightarrow \mathbb{R}$ is defined by

$$\pi\omega = \lim_{n \rightarrow \infty} S_{\omega, n}(x_0)$$

(the limit is independent of x_0). Now consider a one-parameter family of i.f.s. $\{S_1^\lambda, \dots, S_\ell^\lambda\}_{\lambda \in J}$, where $S_i^\lambda(x) = r_i(\lambda)x + a_i(\lambda)$ and $J \subset \mathbb{R}$ is a closed interval. We assume that $a_i(\lambda)$ and $r_i(\lambda)$ belong to $C^1(J)$ and

$$(5.2) \quad 0 < r_i(\lambda) < 1 \quad \text{for all } i \leq \ell \text{ and } \lambda \in J.$$

Moreover, we suppose that

$$(5.3) \quad \text{either } r_i(\lambda) = r_i \text{ for all } \lambda \in J, \text{ or } r_i(\lambda) = r(\lambda) \text{ for all } i \leq \ell.$$

Let $s(\lambda)$ be the similarity dimension, i.e., $\sum_{i=1}^\ell r_i^{s(\lambda)}(\lambda) = 1$. Let π^λ be the natural projection map corresponding to λ . Denote

$$f_{\omega, \tau}(\lambda) = \pi^\lambda(\omega) - \pi^\lambda(\tau).$$

We say that the *transversality condition* holds on J if for any $\omega, \tau \in \Omega$,

$$(5.4) \quad \text{if } \exists \lambda \in J : f_{\omega, \tau}(\lambda) = f'_{\omega, \tau}(\lambda) = 0, \quad \text{then } \omega = \tau.$$

Proposition 5.2. *Suppose that the one-parameter family of i.f.s. $\{S_1^\lambda, \dots, S_\ell^\lambda\}_{\lambda \in J}$ satisfies (5.2), (5.3) and (5.4), and $s_{\max} = \sup_{\lambda \in J} s(\lambda) < 1$. Let ν_λ be the corresponding family of natural self-similar measures. Then for all $\lambda \in J$ except a set of Hausdorff dimension $s_{\max}$, the order-two density $A^{s(\lambda)}(\nu_\lambda, x) = A^{s(\lambda)}(\nu_\lambda)$ exists ν_λ -a.e. and satisfies $0 < A^{s(\lambda)}(\nu_\lambda) < \infty$.*

Remark 5.3. It is proved in [19] that under the conditions of the proposition, the natural self-similar measure coincides, up to scaling, with the packing measure of dimension $s(\lambda)$ restricted to the corresponding self-similar set $\mathcal{K}_\lambda$ (for all $\lambda \in J$ except a set of Hausdorff dimension $s_{\max}$). Thus, we obtain the order-two density of the packing measure. It is shown in [19] that in many cases the Hausdorff measure $\mathcal{H}^{s(\lambda)}(\mathcal{K}_\lambda)$ vanishes for all λ in a dense G_δ set of full Lebesgue measure in J .

Example. Consider the s -dimensional Sierpinski gasket $G^r = \{\sum_{n=0}^\infty c_n r^n : c_n \in \{0, 1, i\}\} \subset \mathbb{R}^2$, where $s = \log 3 / |\log r|$. Let proj_ϑ be the orthogonal projection of $\mathbb{R}^2$ onto the line $y \cos \vartheta = x \sin \vartheta$. It is shown in [19, Ex.2.8] that for any $r \in (\frac{1}{5}, \frac{1}{3})$, there is a nonempty interval $J_r \subset [0, 2\pi)$ such that the family $\{\text{proj}_\vartheta G^r\}_{\vartheta \in J_r}$ satisfies the conditions of Proposition 5.2. In [19, Ex.2.8] it is shown that for Lebesgue-a.e. $\vartheta \in J_r$, the self-similar set $\text{proj}_\vartheta G^r$ has zero Hausdorff measure and positive and finite packing measure in its dimension. Proposition 5.2 complements this result: it yields that the order-two density is (typically) positive and finite.

Other examples to which Proposition 5.2 applies, in particular, those with contraction ratio as parameter, as in the “ $\{0, 1, 3\}$ -problem”, can be found in [19].

Proof of Proposition 5.2 assuming Theorem 5.1. We only need to prove that the order-two density is finite. Suppose that the dimension of the set of $\lambda \in J$, where $A^{s(\lambda)}(\nu_\lambda) = \infty$, is greater than $s_{\max}$. Then we can find a positive Frostman measure η on this set such that for some $s_1 > s_{\max}$ we have $\eta B(x, \rho) \leq \text{const} \cdot \rho^{s_1}$ for any $x \in \mathbb{R}$ and $\rho > 0$. By [19, Prop.4.4], there exists $C > 0$ such that

$$(5.5) \quad J_r := \int_J \int v_\lambda B(x, r^{1/s(\lambda)}) dv_\lambda(x) d\eta(\lambda) \leq Cr \quad \text{for all } r > 0.$$

Note that

$$J_{e^{-t}} = \int (v_\lambda \times v_\lambda) \{(x, y) : |x - y| < e^{-t/s(\lambda)}\} d\eta(\lambda).$$

Applying (5.5) and Fatou's Lemma, we obtain that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{(v_\lambda \times v_\lambda) \{(x, y) : |x - y| < e^{-t/s(\lambda)}\}}{e^{-t}} dt < \infty$$

for η -a.e. λ . For every such λ we can change the variable $u = t/s(\lambda)$, and (5.1) implies that $A^{s(\lambda)}(\nu_\lambda)$ is finite. This contradicts the choice of η which concludes the proof of the proposition. $\square$

Proof of Theorem 5.1. For $\omega \in \Omega$ let $\kappa(\omega, n) = 1/r_{\omega, n}$ and define

$$(5.6) \quad \psi_n(\omega) = \int_0^{\log \kappa(\omega, n)} \frac{\nu B(\pi \omega, e^{-t})}{e^{-st}} dt.$$

Here π is the natural projection from Ω to $\mathbb{R}$ associated with the i.f.s. Recall that $\nu = \mu \circ \pi^{-1}$ where $\mu = (p_1, \dots, p_\ell)^\mathbb{N}$ is the product measure on Ω , with $p_i = r_i^s$. Denoting by σ the left shift, we get the Bernoulli system (Ω, μ, σ) . We claim that it is enough to show

$$(5.7) \quad a_0 = \lim_{n \rightarrow \infty} n^{-1} \psi_n(\omega) \text{ exists for } \mu\text{-a.e. } \omega, \text{ and} \\ 0 < a_0 < \text{const} \cdot \Gamma^s(\nu),$$

where $\Gamma^s(\nu)$ is defined in (5.1). Indeed, $n^{-1} \log(1/r_{\omega, n}) \rightarrow \sum_{i=1}^\ell p_i \log(1/r_i)$ for μ -a.e. ω , by the Law of Large Numbers. For $x = \pi \omega$ let $T_n = \log \kappa(\omega, n)$. Then $T_n \uparrow \infty$ and $T_{n+1} - T_n \leq C_3 := \max_{i \leq \ell} \log(1/r_i)$. Once (5.7) is proved, we will have that for ν -a.e. x the limit

$$b_0 = \lim_{n \rightarrow \infty} T_n^{-1} \int_0^{T_n} \frac{\nu B(x, e^{-t})}{e^{-st}} dt$$

exists and $0 < b_0 < \text{const} \cdot \Gamma^s(\nu)$. There is no difficulty in passing from the discrete to continuous limit, and the claim follows.

We prove (5.7) by applying the Subadditive Ergodic Theorem to the sequence $\{\psi_n\}$. First we check super-additivity for ψ_n . Observe that $\kappa(\omega, n+m) = \kappa(\omega, n) \cdot \kappa(\sigma^n \omega, m)$ hence we have by (5.6):

$$(5.8) \quad \begin{aligned} \psi_{n+m}(\omega) &= \int_0^{\log \kappa(\omega, n) + \log \kappa(\sigma^n \omega, m)} \frac{\nu B(\pi \omega, e^{-t})}{e^{-st}} dt \\ &= \psi_n(\omega) + \int_{\log \kappa(\omega, n)}^{\log \kappa(\omega, n) + \log \kappa(\sigma^n \omega, m)} \frac{\nu B(\pi \omega, e^{-t})}{e^{-st}} dt. \end{aligned}$$

By self-similarity of the measure ν , for any $\omega \in \Omega$,

$$\nu B(\pi \omega, e^{-t}) \geq p_{\omega_1} \dots p_{\omega_n} \nu(S_{\omega, n}^{-1} B(\pi \omega, e^{-t})) = r_{\omega, n}^s \nu B(\pi \sigma^n \omega, r_{\omega, n}^{-1} e^{-t}).$$

Here we used that $p_i = r_i^s$, $\pi \omega = S_{\omega, n} \pi \sigma^n \omega$, and $r_{\omega, n}$ is the contraction rate of the similitude $S_{\omega, n}$. Substituting the last expression into (5.8) and changing the variable $u = t - \log \kappa(\omega, n)$ yields

$$(5.9) \quad \begin{aligned} \psi_{n+m}(\omega) &\geq \psi_n(\omega) + \int_0^{\log \kappa(\sigma^n \omega, m)} \frac{\nu B(\pi \sigma^n \omega, e^{-u})}{e^{-su}} du \\ &= \psi_n(\omega) + \psi_m(\sigma^n \omega), \end{aligned}$$

verifying super-additivity. Let $E_n = \int_{\Omega} \psi_n(\omega) d\mu(\omega)$. By the Subadditive Ergodic Theorem (see [8]) applied to $\{-\psi_n\}$, the limit $a_0 = \lim_{n \rightarrow \infty} n^{-1} \psi_n(\omega)$ exists and equals $\sup(n^{-1} E_n)$ for μ -a.e. $\omega \in \Omega$. Since $\psi_n > 0$, this implies that $a_0 > 0$. Integrating (5.9) shows that $E_{m+n} \geq E_n + E_m$ for all $m, n \in \mathbb{N}$, hence $\lim_{n \rightarrow \infty} n^{-1} E_n$ exists and equals $\sup(n^{-1} E_n)$; therefore, $a_0 = \liminf_{n \rightarrow \infty} n^{-1} E_n$. Note also that $\log \kappa(\omega, n) = \log(1/r_{\omega, n}) \leq C_3 n$ for all $\omega \in \Omega$ where $C_3 = \max_{i \leq \ell} \log(1/r_i)$. Thus,

$$\begin{aligned} a_0 &= \liminf_{n \rightarrow \infty} n^{-1} E_n \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \int_0^{C_3 n} \frac{\nu B(\pi \omega, e^{-t})}{e^{-st}} dt d\mu(\omega) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{n} \int_0^{C_3 n} e^{st} \int \nu B(x, e^{-t}) d\nu(x) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{n} \int_0^{C_3 n} e^{st} \cdot (\nu \times \nu) \{(x, y) : |x - y| < e^{-t}\} dt \\ &\leq C_3 \Gamma^S(\nu). \end{aligned}$$

This concludes the proof of (5.7) and of the theorem. $\square$

6. CONCLUDING REMARKS

- Ngai [13] (see also [6, 15]) proved that if a probability measure ν in $\mathbb{R}^d$ satisfies

$$(6.1) \quad \lim_{q \uparrow 1} D_q(\nu) = \lim_{q \downarrow 1} D_q(\nu),$$

then for ν -a.e. x the *pointwise dimension*

$$(6.2) \quad \lim_{r \downarrow 0} \frac{\log \nu B(x, r)}{\log r}$$

exists (and has the same value as (6.1)). Note that the one-sided limits in (6.1) exist by monotonicity of $q \mapsto D_q(\nu)$. We cannot determine whether (6.1) holds for all self-similar measures. Another approach to prove the a.e. existence of the limits (6.2) for self-similar ν , indicated to us by F. Ledrappier, is to adapt the methods of Ledrappier and Young [12] to this setting. To the best of our knowledge, the details of this approach have not been written down.

- A self-affine measure is the unique probability measure satisfying (1.4) where S_i are affine contractions that are not necessarily similitudes; as before, no separation conditions are assumed. Falconer [5] showed the existence (for $1 < q \leq 2$) of the L^q dimension of a self-affine measure for a.e. parameter, provided that the affine contractions determining it have Lipschitz constants less than $\frac{1}{2}$.

Do L^q dimensions exist as limits for arbitrary self-affine measures in $\mathbb{R}^d$?

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