Restoration of Unimodular Functions from their Holomorphic Projections

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ABSTRACT. Let P be the Bergman projection on the unit disk. Suppose that h is a real valued harmonic function and that $P(e^{ih})$ is in a diagonal Besov space. We show that, under appropriate *a-priori* constraints, we can conclude that h is in the same Besov space. Similar results are also obtained for the Hardy and Dirichlet projections.

1. Introduction and summary. Let P be the projection of a Hilbert space of functions X onto the subspace of holomorphic functions. Suppose u is a unimodular function in X . We want to know if we can recover information about u from information about $P(u)$. This restoration problem and variations have been studied extensively on the Lebesgue and Hardy spaces of the unit circle. See Sections 3.5 and 3.6 of [PK], (or Chapter III of Appendix 5 of [N]), [D], and, more recently, [Sh]. Here we look at questions of this sort on the Dirichlet, Hardy, and Bergman spaces.

We start with an informal discussion. Let P_+ be the orthogonal projection of the Lebesgue space $L^2(\mathbb{T})$ to the Hardy space H^2 and let $P_- = I - P_+$. Suppose u is in $\text{VMO}(\mathbb{T})$, $|u| = 1$ a.e.. It is a result of Sarason that we may write $u = z^k \exp(ih)$ with k an integer and h a real valued function in VMO and, conversely, such a u must be in VMO . We restrict attention to the case $k = 0$. For $0 < p < \infty$, let B_{Re}^p denote the real diagonal Besov spaces on $\mathbb{T}$. (Definitions of functions spaces are given in the next section.)

Theorem 1.1 (*Peller-Krushchev*). *Suppose $u = \exp(ih)$ for a real valued function h in VMO . Given p , $0 < p < \infty$, the following are equivalent:*

- (i) $h \in B_{\text{Re}}^p$,
- (ii) $u \in B_{\text{Re}}^p$,
- (iii) $P_+(u) \in B_{\text{Re}}^p$,
- (iv) $P_-(u) \in B_{\text{Re}}^p$.

Proof. See [N], p. 439. The discussion there is for $p \geq 1$, but the ideas extend to $p < 1$.

The facts that $(1) \Rightarrow (2) \Rightarrow (3), (4)$ are standard. The fact that $(4) \Rightarrow (2)$ is referred to as restoration of unimodular functions.

The methods of Peller and Krushchev make substantial use of the function theory of the Hardy spaces, in particular the properties of the operator of best approximation. Here we use a different approach, which seems less flexible for the Hardy space but which extends to the full range of Bergman spaces and to some Dirichlet type spaces. Although we don't use the delicate function theory of the Hardy space, we still do make fundamental use of function theory. The proof uses the relation between a harmonic function and its harmonic conjugate to move from information about a unimodular function to information about a positive weight function.

Here is a particular case of our general result. Let P be the orthogonal projection of the Lebesgue space of the disk, $L^2(\mathbb{D}, dx dy)$, to the Bergman space, A^2 , and let $\bar{P}$ be the projection onto the conjugate holomorphic functions. Let b_{Re}^∞ be the (real variable) little Bloch space on the disk.

Theorem 1.2 *Suppose $u = \exp(ih)$ for h a real valued harmonic function in b_{Re}^∞ . Given p , $0 < p < \infty$, the following are equivalent:*

- (i) $h \in B_{\text{Re}}^p$,
- (ii) $u \in B_{\text{Re}}^p$,
- (iii) $P(u) \in B_{\text{Re}}^p$,
- (iv) $\bar{P}(u) \in B_{\text{Re}}^p$.

Although there are technical details to be tended to, the basic flow of the proof is straightforward and we outline it here for the particular case $p = 2$. As mentioned, it is rather direct that $(1) \Rightarrow (2) \Rightarrow (3), (4)$. Also, once $(1), (2)$ and (3) are known to be equivalent, the fact that $\bar{P}(u) = \overline{P(u)}$ insures that (4) can be included. Hence the central issue is showing that $(3) \Rightarrow (1)$. The form of u allows us to reduce the issue to a question of boundedness of the formal inverse of a Toeplitz operator or, what is roughly the same thing, the availability of a weighted norm inequality for a particular projection. The *a-priori* assumption that h is in b_{Re}^∞ insures that the needed weighted norm inequalities are available.

We begin with a real harmonic function h in b^∞ . Let $\tilde{h}$ denote the conjugate harmonic function. Set $\psi = i(h + i\tilde{h})/2$. Hence $\psi \in b_{\text{Re}}^\infty$ and $ih = \psi - \bar{\psi}$. Condition (3) then becomes the statement that

$$e^{\psi - \bar{\psi}} = a + r$$

for some $a \in B^2$ and $r \perp A^2$. Thus

$$e^\psi = e^{\bar{\psi}}a + e^{\bar{\psi}}r = e^{\bar{\psi}}a + \tilde{r}$$

for some $\tilde{r}$ which is also orthogonal to A^2 . Projection onto A^2 gives

$$(1) \quad e^\psi = P(e^{\bar{\psi}}a).$$

Differentiation gives

$$\psi' e^\psi = P(e^{\bar{\psi}}a').$$

Let P_1 be the orthogonal projection of the weighted Lebesgue space $L^2(\mathbb{D}, (1 - |z|^2)dx dy)$ onto the weighted Bergman space A_1^2 . A simple calculation shows $P(e^{\bar{\psi}}a)' = P_1(e^{\bar{\psi}}a')$. Hence we can continue with

$$\psi' = e^{-\psi} P_1(e^{\bar{\psi}}a') = e^{-ih/2} e^{\bar{h}/2} P_1(e^{-\bar{h}/2} e^{-ih/2} a').$$

(Although we won't use the fact, it is interesting to note that the operator taking a' to ψ' is the formal inverse of the Toeplitz operator $T_{\exp(ih)}$ acting on A_1^2 .) Denoting the unimodular function $e^{-ih/2}$ by α we have

$$\psi' = \alpha e^{\bar{h}/2} P_1(e^{-\bar{h}/2} \alpha a').$$

Our hypothesis insures that a' is in A^2 ; the desired conclusion is that ψ' is in A^2 . Hence, we will be done if we know that the weighted projection operator which takes g to $e^{\bar{h}/2} P_1(e^{-\bar{h}/2} g)$ is bounded on L^2 . That boundedness follows because the hypothesis that $h \in b_{\text{Re}}^\infty$ allows us to apply results of Békollé and Bonami about weighted norm inequalities for Bergman projections.

We can see where the need for an *a-priori* constraint arises. If we start with h and proceed to (1) then smoothness of h will lead to smoothness of e^ψ . However that leaves open the possibility that e^ψ might tend to 0 at the boundary leading to a singularity for ψ . For example, if $h(z) = 2 \arg(1 + z)$ then one quickly finds that $a = 1/2 + z$ and (1) takes the form

$$\frac{1}{2} + z = P\left(\frac{1+z}{1+\bar{z}}\right).$$

Although a and also $e^\psi = 1 + z$ are as smooth as can be, when we take logarithms to recapture h we obtain a logarithmic singularity on the boundary.

Here are the contents. Background information about function spaces and integral operators is in Section 2. Also in Section 2 is a discussion of the question about Friedrichs operators on the Bergman space which led to this work. The

results about unimodular functions and their proofs are in Section 3. The starting point for the analysis in Section 3 is writing $u = f/\bar{f}$ with f holomorphic. More generally, if we begin with $u = f/\bar{g}$ with f, g holomorphic then we can still follow the same pattern. We find, for instance, using the earlier notation, that if $f = \exp(h + i\tilde{h})$ then $P(|f|) \in B^p$ implies $f \in B^p$. We will also see that it is possible to perturb away from the particular structure $u = f/\bar{g}$. All that is in Section 4. The brief final section contains questions.

2. Background

2.1. Function spaces

2.1.1. Besov, Bloch, BMO. For $p > 0$ we denote by B^p the diagonal Besov space, the space of f in $\text{Hol}(\mathbb{D})$ for which

$$\int_{\mathbb{D}} \left| (1 - |z|^2)^n f^{(n)}(z) \right|^p \frac{dx dy}{(1 - |z|^2)^2} < \infty$$

for some (and hence any) $n > 1/p$. (In the context of more general Besov spaces, the spaces just described are denoted $B_p^{1/p}$ or $B_{p,p}^{1/p}$.) Given p and n , B^p is a (quasi)-normed space in the natural way and the (quasi)-norms are independent of n . (We include $\sum_1^n |f^{(k)}(0)|^p$ in the (quasi)-norm to avoid having to work modulo polynomials.)

Denote by B^∞ the Bloch space, consisting of f in $\text{Hol}(\mathbb{D})$ with

$$(2) \quad \sup_{\mathbb{D}} |(1 - |z|^2)f'(z)| < \infty;$$

and by b^∞ the little Bloch space, the subspace of B^∞ consisting of functions, f , for which

$$\lim_{|z| \rightarrow 1} |(1 - |z|^2)f'(z)| = 0.$$

Both of these spaces are normed by the expression in (2). We will identify the functions in B^p , VMO, and BMO with their boundary values on $\mathbb{T}$, the unit circle.

We will also use the real variable versions of these function spaces. Set

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

and

$$\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

For $0 < p < \infty$ and an integer n with $np > 1$, define B_{Re}^p to be the space of functions f on $\mathbb{D}$ for which

$$\sum_{i+j \leq n} \int_{\mathbb{D}} |(1 - |z|^2)^n \partial^i \bar{\partial}^j f(z)|^p \frac{dx dy}{(1 - |z|^2)^2} < \infty.$$

As before the space defined is independent of n . We also use the limiting cases b_{Re}^∞ , and B_{Re}^∞ . (We should emphasize that the spaces B_{Re}^p does not consist of real-valued functions nor is it the complexification of the space of real parts of functions in B^p .)

For $0 < p < \infty$, the subspace of B_{Re}^p consisting of harmonic functions is the complex linear span of $\{\text{Re } f : f \in B_{\text{Re}}^p\}$. For those functions, membership in B_{Re}^p can be determined by testing the boundary values in various straightforward ways; however, that is not true for general functions in B_{Re}^p . Our primary interest will be in functions, f , of the form $f(z) = \exp(ih(z))$ for real-valued harmonic h . We will see in Theorem 3.1 that in certain cases, $f \in B_{\text{Re}}^p$ if and only if $h \in B_{\text{Re}}^p$.

For more information about these spaces we refer to [N] and [Z]. Here we collect several facts which we will use later.

For $0 < p < q < \infty$ we have proper inclusions:

$$\begin{array}{ccccccc} B^p & \subset & B^q & \subset & b^\infty & \subset & B^\infty \\ & & & & \cup & & \cup \\ B^p & \subset & B^q & \subset & \text{VMO} & \subset & \text{BMO}. \end{array}$$

We also have analogous inclusions for the real variable spaces.

The functions in B_{Re}^∞ are Lipschitz functions with respect to the hyperbolic geometry on $\mathbb{D}$. Recall that the pseudo-hyperbolic between points $z, w \in \mathbb{D}$ is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

and the hyperbolic distance by

$$d(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

We have the following estimate:

Proposition 2.1 *There is a constant c so that for all $f \in B^\infty$, $z, w \in \mathbb{D}$ we have*

$$|f(z) - f(w)| \leq c \|f\|_{B_{\text{Re}}^\infty} d(z, w).$$

Proof. This is Theorem 5.1.6 of [Z]. □

The following representation will be useful.

Proposition 2.2 *Given p , $0 < p \leq \infty$, and $b > \max(0, 1-1/p)$, there are points $\{z_j\}_{j=1}^\infty \subset \mathbb{D}$ and $c > 0$ so that any $f \in B^p$ may be written*

$$f(z) = \sum_{j=1}^{\infty} \lambda_j \frac{(1 - |z_j|^2)^b}{(1 - \bar{z}_j z)^b}$$

with

$$\|\{\lambda_j\}\|_{l^p} \leq c \|f\|_{B^p}.$$

Proof. This is a direct corollary of Theorem A of [RW]. □

2.1.2. Bergman and Dirichlet spaces. For $\alpha > -1$, $p > 0$ set

$$L_\alpha^p = L^p(\mathbb{D}, (1 - |z|^2)^\alpha dx dy)$$

and let A_α^p be the weighted Bergman space

$$A_\alpha^p = L_\alpha^p \cap \text{Hol}(\mathbb{D}).$$

We denote the norm on both of these spaces by $\|\cdot\|_{p,\alpha}$. Let P_α denote the self adjoint projection of L_α^2 onto A_α^2 and let $\bar{P}_\alpha$ denote the projection onto conjugate holomorphic functions. P_α can be realized by integration against a reproducing kernel. That is, with $k_{\alpha,z}(w) = c_\alpha (1 - \bar{z}w)^{-2-\alpha}$, and denoting the inner product on both of these spaces by $\langle \cdot, \cdot \rangle_\alpha$, we have, for any f in L_α^2 ,

$$(3) \quad P_\alpha(f)(z) = \langle f, k_{\alpha,z} \rangle_\alpha,$$

and for g in A_α^2

$$(4) \quad g(z) = \langle g, k_{\alpha,z} \rangle_\alpha.$$

(We will generally confound and ignore constants such as c_α .)

We now define the Dirichlet spaces. Although the Dirichlet spaces are Hilbert spaces with reproducing kernel, the containing potential spaces are not Lebesgue spaces with respect to a measure on $\mathbb{D}$. Nevertheless we will use notation similar to that of the Bergman and Lebesgue spaces to focus attention on the fact that, in this context, the Dirichlet and Bergman spaces are part of a single natural scale.

Suppose n is a positive integer. For α in the range

$$(5) \quad -1 - 2n < \alpha \leq 1 - 2n,$$

we define the potential space L_α^2 to be the closure of the smooth functions on $\mathbb{D}$ with respect to the norm given by

$$\|f\|_\alpha^2 = \sum_{i+j < n} |\partial^i \bar{\partial}^j f(0)|^2 + \sum_{i+j=n} \|\partial^i \bar{\partial}^j f\|_{2, \alpha+2n}^2.$$

The Dirichlet spaces are the subspaces of these spaces consisting of holomorphic functions; $A_\alpha^2 = L_\alpha^2 \cap \text{Hol}(\mathbb{D})$. Thus A_{-1}^2 is the same space of functions on the disk as the classical Hardy space H^2 ; similarly, A_{-2}^2 is the classical Dirichlet space (see, e.g. [Z]). The inner products are given by

$$(6) \quad \langle f, g \rangle_\alpha = \sum_{i+j < n} \partial^i \bar{\partial}^j f(0) \overline{\partial^i \bar{\partial}^j g(0)} + \sum_{i+j=n} \langle \partial^i \bar{\partial}^j f, \partial^i \bar{\partial}^j g \rangle_{\alpha+2n}.$$

The Dirichlet spaces are Hilbert spaces with reproducing kernels. That is, analogs of (4) and (3) hold. Although we do not need explicit formulas for the kernel functions we do need to know how they are related to the kernel functions for the Bergman spaces. Fix $\alpha \leq -1$, and select the integer n so that (5) holds.

Lemma 2.3

$$(7) \quad \frac{\partial^{2n}}{\partial^n \bar{z} \partial^n \zeta} k_{\alpha, z}(\zeta) = k_{\alpha+2n, z}(\zeta).$$

Proof. Fix α and the associated n . All the weights considered are radial and hence the monomials are, for any β , an orthogonal basis for A_β^2 . Thus, with $c_{\beta, j} = \|z^j\|_{A_\beta^2}^{-2}$,

$$k_{\beta, z}(\zeta) = \sum_{j=0}^{\infty} c_{\beta, j} (\bar{z}\zeta)^j.$$

We now compute these Taylor coefficients with $\beta = \alpha$. For $j \geq n$ we have

$$\begin{aligned} \|z^j\|_{A_\alpha^2}^2 &= \|\partial^n z^j\|_{A_{\alpha+2n}^2}^2 \\ &= \left(\frac{j!}{(j-n)!} \right)^2 \|z^{j-n}\|_{A_{\alpha+2n}^2}^2 \\ &= \left(\frac{j!}{(j-n)!} \right)^2 (2\pi) \int_0^1 r^{2j-2n+1} (1-r^2)^{\alpha+2n} dr. \end{aligned}$$

Thus, for $j \geq n$,

$$c_{\alpha, j} = \frac{\frac{1}{2\pi} \left(\frac{(j-n)!}{j!} \right)^2}{\int_0^1 r^{2j-2n+1} (1-r^2)^{\alpha+2n} dr}.$$

Now we compute the Taylor coefficients for the left side of (7). By direct computation, for m positive,

$$\frac{\partial^2}{\partial \bar{z} \partial \zeta} (\bar{z}\zeta)^m = m^2 (\bar{z}\zeta)^{m-1}.$$

Thus, for $m \geq n$, the coefficient of $(\bar{z}\zeta)^{m-n}$ in the left side of (7) is

$$\begin{aligned} \left(\frac{m!}{(m-n)!} \right)^2 c_{\alpha, m} &= \frac{\left(\frac{m!}{(m-n)!} \right)^2 \frac{1}{2\pi} \left(\frac{(m-n)!}{m!} \right)^2}{\int_0^1 r^{2m-2n+1} (1-r^2)^{\alpha+2n} dr} \\ &= \frac{\frac{1}{2\pi}}{\int_0^1 r^{2m-2n+1} (1-r^2)^{\alpha+2n} dr}. \end{aligned}$$

Hence, the coefficient of $(\bar{z}\zeta)^r$ in the left side of (7) is

$$\left(2\pi \int_0^1 r^{2r+1} (1-r^2)^{\alpha+2n} dr \right)^{-1} = c_{\alpha+2n, r}^{-1}.$$

This is also the coefficient of $(\bar{z}\zeta)^r$ for the right hand side; hence the two are equal. $\square$

2.2. Operators

2.2.1. Differentiation and orthogonality

Proposition 2.4 *Given $a \in B^\infty$, $h \in \bigcup_{\beta > 0} A_\beta^2$, $\alpha > -1$ we have*

$$P_\alpha(a\bar{h})' = P_{\alpha+1}(a' \bar{h}).$$

For $\alpha \leq -1$ and n the integer which satisfies (5) we have

$$P_\alpha(a\bar{h})^{(n)} = P_{\alpha+2n}(a^{(n)} \bar{h}).$$

Proof. It is enough to consider the case in which a is a single term from the sum in Proposition 2.2.

First suppose $\alpha > -1$. We now use Proposition 2.2 with the choice $b = \alpha + 1$. Thus, we are reduced to considering the case $a = k_{\alpha, w}$, $w \in \mathbb{D}$. Now note that

$$\begin{aligned} P_\alpha(k_{\alpha, w} \bar{h})(z) &= \langle k_{\alpha, w} \bar{h}, k_{\alpha, z} \rangle \\ &= \langle k_{\alpha, w}, h k_{\alpha, z} \rangle \\ &= \overline{h(w) k_{\alpha, z}(w)} \\ &= \overline{h(w)} k_{\alpha, w}(z). \end{aligned}$$

That is,

$$(8) \quad P_\alpha(k_{\alpha,w} \bar{h}) = \overline{h(w)} k_{\alpha,w}.$$

Furthermore, $k'_{\alpha,w} = c \bar{w} k_{\alpha+1,w}$ for some constant $c = c(\alpha)$. Thus

$$P_\alpha(k_{\alpha,w} \bar{h})' = \overline{cwh(w)} k_{\alpha+1,w}.$$

Also

$$(9) \quad P_{\alpha+1}((k_{\alpha,w})' \bar{h}) = P_{\alpha+1}(c \bar{w} k_{\alpha+1,w} \bar{h})$$

$$(10) \quad = c \bar{w} P_{\alpha+1}(k_{\alpha+1,w} \bar{h})$$

$$(11) \quad = \overline{cwh(w)} k_{\alpha+1,w};$$

with the last equality a consequence of the analog of (8) with index $\alpha + 1$. Comparing this and the previous equation gives what we want.

The computation we just did fails if $\alpha \leq -1$; in that range it is no longer true that for any three functions, a, h, r we have $\langle ah, r \rangle_\alpha = \langle a, \bar{h}r \rangle_\alpha$. However if we differentiate we can return to a range where such a computation can be done. Select a . The terms in the Taylor series for a up to z^{n-1} don't influence either side of the desired equality and hence we may assume $a(0) = a'(0) = \dots = a^{(n-1)}(0) = 0$. For such a we have

$$\begin{aligned} P_\alpha(a\bar{h})(w) &= \langle a\bar{h}, k_{\alpha,w} \rangle_\alpha \\ &= \left\langle a^{(n)} \bar{h}, \frac{\partial^n}{\partial z^n} k_{\alpha,w} \right\rangle_{\alpha+2n} \\ &= \int a^{(n)}(z) \overline{h(z)} \frac{\partial^n}{\partial z^n} k_{\alpha,w}(z) (1 - |z|^2)^{\alpha+2n} dx dy. \end{aligned}$$

Differentiating both sides gives

$$P_\alpha(a\bar{h})^{(n)}(w) = \int a^{(n)}(z) \overline{h(z)} \frac{\partial^n}{\partial \bar{w}^n} \frac{\partial^n}{\partial z^n} k_{\alpha,w}(z) (1 - |z|^2)^{\alpha+2n} dx dy$$

which, in light of (7), is the required result. $\square$

We will also use the following observation.

Proposition 2.5 Fix $\alpha \in \mathbb{R}$. Given $r \in L_\alpha^2$, $r \perp A_\alpha^2$, $h \in \bigcup_{\beta > 0} A_\beta^2$, we have $\bar{h}r \perp A_\alpha^2$.

Of course we are extending the definitions. We don't expect $\bar{h}r$ to be in L_α^2 , however with respect to the integral formula for the inner product we have $\bar{h}r$ orthogonal to a dense subclass of A_α^2 .

Proof. If $\alpha > -1$ this follows immediately from the fact that for any, a, h, r we have $\langle ah, r \rangle_\alpha = \langle a, \bar{h}r \rangle_\alpha$. Now consider smaller α , with n given by (5). For holomorphic a , and h, r as in the statement we have

$$\langle a, r \rangle_\alpha = \sum_{i < n} \partial^i a(0) \overline{\partial^i r(0)} + \langle \partial^n a, \partial^n r \rangle_{\alpha+2n}$$

and

$$\langle a, \bar{h}r \rangle_\alpha = \sum_{i < n} \partial^i a(0) \overline{\bar{h}(0) \partial^i r(0)} + \langle \partial^n a, \bar{h} \partial^n r \rangle_{\alpha+2n}.$$

Because the first inner product must vanish for all polynomials a we must have $\partial^i r(0) = 0$ for $0 \leq i < n$. Thus the sum in the second inner product vanishes. We now use the fact that $\langle \partial^n f, \bar{h} \partial^n r \rangle_{\alpha+2n} = \langle h \partial^n f, \partial^n r \rangle_{\alpha+2n} = 0$. $\square$

2.2.2. Weighted norm inequalities. For $z \in \mathbb{D}$ set $\varepsilon(z) = 1 - |z|^2$. Set $dV = dx dy$ and, for $\alpha > -1$, set $d\mu_\alpha = \varepsilon(z)^\alpha dV$. We will work with the Bergman projection realized as an integral operator. The integral kernel is $k_\alpha(z, \zeta) = (1 - z\bar{\zeta})^{-2-\alpha}$. Thus, ignoring a constant factor,

$$P_\alpha(f)(z) = \int_{\mathbb{D}} f(\zeta) k_\alpha(z, \zeta) d\mu_\alpha(\zeta).$$

For $1 < p < \infty$ and $w(z) \geq 0$ defined on $\mathbb{D}$, set

$$L_{\alpha, w}^p = L^p(\mathbb{D}, w(z) d\mu_\alpha) = L^p(\mathbb{D}, w(z) \varepsilon(z)^\alpha dV).$$

P_α is originally defined on L_α^2 . We now recall the results of Békollé [Be] extending earlier work by Békollé and Bonami [BeBo] which tell us when P_α extends to a continuous map of $L_{\alpha, w}^p$ to itself.

For any $z \in \mathbb{D}$ we define S_z , the Carleson box indexed by z :

$$S_z = \{w \in \mathbb{D} : |w| > |z| \text{ and } |\arg z - \arg w| < 2\pi(1 - |z|)\}.$$

(The discussion in [BeBo] is in terms of disks interior to $\mathbb{D}$ which meet the boundary rather than Carleson boxes but the difference is inessential.) Let q be the conjugate index to p ; $q^{-1} + p^{-1} = 1$. We say a positive function ω on $\mathbb{D}$ is in the class $B_p(d\mu_\alpha)$ if

$$\sup_{z \in \mathbb{D}} \left(\frac{1}{\mu_\alpha(S_z)} \int_{S_z} \omega d\mu_\alpha \right)^{1/p} \left(\frac{1}{\mu_\alpha(S_z)} \int_{S_z} \omega^{-q/p} d\mu_\alpha \right)^{1/q} < \infty.$$

Direct computation shows $\mu_\alpha(S_z) \approx \varepsilon(z)^{2+\alpha}$. Hence we can rewrite this as

$$(12) \quad \sup_{z \in \mathbb{D}} \varepsilon(z)^{-(2+\alpha)} \left(\int_{S_z} \omega d\mu_\alpha \right)^{1/p} \left(\int_{S_z} \omega^{-q/p} d\mu_\alpha \right)^{1/q} < \infty.$$

The fundamental difference between these classes and the more familiar A_p classes of Muckenhoupt is that, for B_p , the only small boxes considered are those which meet the boundary.

Theorem 2.6 (Békollé-Bonami [BeBo]) P_α extends to a continuous map of $L^p_{\alpha,w}$ to itself if and only if $w \in B_p(d\mu_\alpha)$, that is, if and only if (12) holds.

Corollary 2.7 For $\gamma, \delta > -1$ and a real valued function h , a necessary and sufficient condition for the mapping $f \rightarrow e^{-h} P_\gamma(e^h f)$ to extend to a continuous map of L^δ_δ to itself is that

$$\sup_{z \in \mathbb{D}} \varepsilon^{-(2+\gamma)} \left(\int_{S_z} e^{-ph} \varepsilon^\delta dV \right)^{1/p} \left(\int_{S_z} e^{qh} \varepsilon^{q\gamma-q\delta/p} dV \right)^{1/q} < \infty.$$

Proof. The required estimate is

$$\int |e^{-h} P_\gamma(e^h f)|^p \varepsilon^\delta dV \leq C \int |f|^p \varepsilon^\delta dV.$$

Writing $e^h f = g$ this is equivalent to

$$\int |P_\gamma(g)|^p e^{-ph} \varepsilon^{\delta-\gamma} \varepsilon^\gamma dV \leq C \int |g|^p e^{-ph} \varepsilon^{\delta-\gamma} \varepsilon^\gamma dV.$$

Recalling that $d\mu_\gamma = \varepsilon^\gamma dV$ we see that this is equivalent to the boundedness of P_γ on $L^p_{\gamma,w}$ with $w = e^{-ph} \varepsilon^{\delta-\gamma}$. The result now follows from the theorem and bookkeeping. $\square$

Theorem 2.8 Suppose we are given p, γ , and δ with $1 < p < \infty$, $\gamma, \delta > -1$, and $p > (\delta + 1)/(\gamma + 1)$. In that case there is a positive $c = c(\gamma, \delta, p)$ so that, if $h \in B^\infty_{\text{Re}}$, $\|h\| < c$ then the mapping $f \rightarrow e^{-h} P_\gamma(e^h f)$ extends to a continuous map of L^δ_δ to itself.

Proof. We will show that we can choose c so as to insure that h meets the condition in the previous corollary.

Pick h and set $\theta = \|h\|$. For a fixed $z \in \mathbb{D}$, we will estimate the integrals over S_z . For $w \in \mathbb{D}$ we have

$$d(z, w) \leq c(1 + \log \frac{\varepsilon(z)}{\varepsilon(w)}).$$

Hence, by Proposition 2.1 we have

$$|h(z) - h(w)| \leq c\theta \left(1 + \log \frac{\varepsilon(z)}{\varepsilon(w)} \right).$$

Thus

$$\begin{aligned} e^{-ph(w)} &= e^{-ph(z)} e^{-p(h(w)-h(z))} \\ &\leq e^{-ph(z)} \exp \left(pc\theta \left(1 + \log \left(\frac{\varepsilon(z)}{\varepsilon(w)} \right) \right) \right) \\ &\leq ce^{-ph(z)} \left(\frac{\varepsilon(z)}{\varepsilon(w)} \right)^{pc\theta}. \end{aligned}$$

Similarly

$$e^{qh(w)} \leq ce^{qh(z)} \left(\frac{\varepsilon(z)}{\varepsilon(w)} \right)^{qc\theta}.$$

Thus, by the corollary, we need to estimate

$$\varepsilon^{-(2+\gamma)} \left(e^{-ph(z)} \varepsilon^{pc\theta} \int_{S_z} \varepsilon^{\delta-pc\theta} dV \right)^{1/p} \left(e^{qh(z)} \varepsilon^{qc\theta} \int_{S_z} \varepsilon^{q\gamma-q\delta/p-qc\theta} dV \right)^{1/q}.$$

If $\tau > -1$, then $\int_{S_z} \varepsilon^\tau dV = O(\varepsilon^{\tau+2})$. Hence, for the same indices, the previous expression is dominated by

$$\varepsilon^{-(2+\gamma)} \left(e^{-ph(z)} \varepsilon^{pc\theta} \varepsilon^{\delta-pc\theta+2} \right)^{1/p} \left(e^{qh(z)} \varepsilon^{qc\theta} \varepsilon^{q\gamma-q\delta/p-qc\theta+2} \right)^{1/q} = O(1).$$

The constraints on the indices are $\delta - pc\theta > -1$ and $q\gamma - q\delta/p - qc\theta > -1$. Because $\delta > -1$ the first will hold if θ is sufficiently small. If θ is small then the second will hold if $q\gamma - q\delta/p > -1$ or, equivalently $p > (\delta + 1)/(\gamma + 1)$. $\square$

Corollary 2.9 *Suppose we are given p , δ , and γ with $1 < p < \infty$ and γ , $\delta > -1$. If $h \in b_{\text{Re}}^\infty$, then the mapping $f \rightarrow e^{-h} P_\gamma(e^h f)$ extends to a continuous map of L_δ^p to itself.*

Proof. Recall that b_{Re}^∞ is the closure in B_{Re}^∞ of the smooth functions. Hence we can write $h \in b_{\text{Re}}^\infty$ as $h = k + j$ with $\|k\| < c(\gamma, \delta, p)$ and j continuous on $\mathbb{D}$. The map of interest takes f to $e^{-j} e^{-k} P_\gamma(e^k e^j f)$. Because e^j is bounded above and below, it suffices to apply the previous theorem to the mapping of g to $e^{-k} P_\gamma(e^k g)$. $\square$

We will also need a related result for $p \leq 1$.

Theorem 2.10 *Suppose we are given α, p, n with $\alpha > -4$, $0 < p \leq 1$, n an integer, and $np > 1$. There is a positive constant $c = c(\alpha, p, n)$ so that if $\varphi \in B^\infty$, $\|\varphi\| \leq c$, then the map $f \rightarrow e^{-\varphi} \partial^n P_\alpha (f e^{\bar{\varphi}})$ is a bounded map of B^p to A_{np-2}^p .*

We will use this result as stated. However we should mention that the proof actually demonstrates the following "weighted norm inequality" for A^p spaces with $p \leq 1$.

Corollary 2.11 *Given p, γ, δ with $0 < p \leq 1$, $-1 < \delta$, $-3 < \gamma$ and $1 < p(\gamma + 4)$. There is a $c = c(\gamma, \delta, p)$ so that if $\varphi \in B^\infty$, $\|\varphi\| < c$ then the mapping $f \rightarrow e^{-\varphi} P_\gamma (f e^{\bar{\varphi}})$ extends to a continuous map of A_δ^p to itself.*

Proof. Recall that B^p is p -convex; that is, instead of the triangle inequality we have $\|\sum f_i\|_{B^p}^p \leq \sum \|f_i\|_{B^p}^p$. Thus, recalling Proposition 2.2, the theorem will follow if we can show that, given α and p , we can find $\theta > 0$ and an integer $n, np > 1$, so that

$$\sup_{a \in \mathbb{D}} \left\| e^{-\varphi} \partial^n P_\alpha \left(\frac{(1 - |a|^2)^\theta}{(1 - \bar{a}z)^\theta} e^{\bar{\varphi}} \right) \right\|_{A_{np-2}^p}^p < \infty.$$

Suppose for now that $\alpha > -1$. Pick and fix a . Set $\theta = \alpha + 2$. By Proposition 2.4 we need to estimate

$$\left\| e^{-\varphi} P_{\alpha+n} \left(\frac{(1 - |a|^2)^\theta}{(1 - \bar{a}z)^{\theta+n}} e^{\bar{\varphi}} \right) \right\|_{A_{np-2}^p}^p.$$

We can evaluate the projection using (8). We find that we need to estimate

$$\begin{aligned} \varepsilon(a)^{\theta p} & \left\| \frac{e^{\bar{\varphi}(a)} e^{-\varphi(z)}}{(1 - \bar{a}z)^{\theta+n}} \right\|_{A_{np-2}^p}^p \\ &= \varepsilon(a)^{\theta p} \int \left| e^{\bar{\varphi}(a)} e^{-\varphi(z)} (1 - \bar{a}z)^{-(\theta+n)} \right|^p d\mu_{np-2} \\ &= \varepsilon(a)^{\theta p} \int \left| e^{\bar{\varphi}(a)} e^{-\varphi(z)} (1 - \bar{a}z)^{-(\theta+n)} \right|^p d\mu_{np-2} \\ &= \varepsilon(a)^{\theta p} \int \left| e^{\bar{\varphi}(a)p/\lambda} e^{-\varphi(z)p/\lambda} (1 - \bar{a}z)^{-(\theta+n)p/\lambda} \right|^\lambda d\mu_{np-2} \end{aligned}$$

Set $\beta = -2 + (\theta + n)p/\lambda$. If

$$(13) \quad \beta > -1$$

then we can use (8) again to rewrite the integrand and continue the equality with

$$= \varepsilon(a)^{\theta p} \left\| e^{-p\varphi(z)/\lambda} P_\beta \left((1 - \bar{a}z)^{-(\beta+2)} e^{p\bar{\varphi}/\lambda} \right) \right\|_{A_{np-2}^\lambda}^\lambda.$$

We now wish to use Theorem 2.8 with λ chosen to be p of the theorem, $\delta = np-2$, and $\gamma = \beta$. To do that we need the conditions

$$\lambda > 1, \quad \beta > -1, \quad np-2 > 1, \quad \text{and} \quad \lambda > (np-1)/(\beta+1).$$

The selection of n insures that $np-1 > 0$. Hence the second of the inequalities holds. After using the definition of β and doing algebra, the other three inequalities, which include (13), can be rewritten as

$$(14) \quad \begin{aligned} \lambda - 1 &> 0, \\ (\alpha + 2)p + (np - 1) &> \lambda - 1, \\ (\alpha + 2)p &> \lambda - 1. \end{aligned}$$

Recalling that $\alpha > -1$ we see that we can select λ so that the inequalities hold. Hence, by Theorem 2.8, the quantity we need to estimate is dominated by

$$c\varepsilon(a)^{\theta p} \left\| (1 - \bar{a}z)^{-(\beta+2)} \right\|_{A_{np-2}^\lambda}^\lambda = c\varepsilon(a)^{\theta p} \int |1 - \bar{a}z|^{-\lambda(\beta+2)} d\mu_{np-2}.$$

We now estimate the integral (Lemma 4.2.2 of [Z]) and continue with

$$\leq c\varepsilon(a)^{\theta p} \varepsilon(a)^{np-\lambda(\beta+2)} = O(1).$$

Little needs to be changed for $-4 < \alpha \leq -1$. We need to use the second part of Proposition 2.4. When we arrive at (14), rather than $\alpha + 2$ we obtain $\alpha + 3$ when $-3 < \alpha \leq 2$ and $\alpha + 4$ when $-4 < \alpha \leq -3$. In these cases $\alpha + 3$, respectively $\alpha + 4$, is strictly positive and hence we can find the required λ . $\square$

This proof does not extend to $\alpha \leq -4$. To follow the same pattern for $\alpha \leq -4$ we would select $\theta = \alpha + 4$. However we also need $\theta > 0$.

As with Corollary 2.9 we have the following:

Corollary 2.12 *The conclusions of the previous theorem and corollary continue to hold if, instead of assuming that φ is in B^∞ with small norm, we assume φ is in b^∞ .*

2.3. The Friedrichs operator. Let Ω be a domain in the plane and let $L^2(\Omega)$ and $A^2(\Omega)$ be the associated Lebesgue and Bergman spaces (with respect to area measure). Let $\bar{P}_\Omega$ be the orthogonal projection onto the conjugate holomorphic functions. The Friedrichs operator, T_Ω , is the map from $A^2(\Omega)$ to $L^2(\Omega)$ given by $T_\Omega(f) = \bar{P}_\Omega(f)$. If $\Omega = \mathbb{D}$, then T_Ω is a rank one operator. However this example is misleading; it was shown by Friedrichs [F] that T_Ω need not even be compact if Ω does not have smooth boundary. The relation between the size of T_Ω and the smoothness of $\partial\Omega$ was studied further in [LR]. For more about T_Ω see [S].

If Ω is simply connected then the Riemann map φ of $\mathbb{D}$ to Ω can be used to reduce the study of T_Ω to the study of the (small) Hankel operator on A^2 with symbol $\varphi'/\bar{\varphi}'$ (Lemma 2.1 of [LR]). The size of such a Hankel operator (compactness, Schatten ideal membership, etc.) is determined by the smoothness of the holomorphic function $P(\varphi'/\bar{\varphi}')$. The global smoothness of φ' encodes information about the smoothness of $\partial\Omega$ and thus we have a relationship between smoothness of $\partial\Omega$ and the size of T_Ω . Although the relationship between the smoothness of $\partial\Omega$ and the smoothness of φ' is well understood, the relation between the smoothness of φ' and that of $P(\varphi'/\bar{\varphi}')$ is not as clear. The size of φ'' is the same as the size of $\nabla(\varphi'/\bar{\varphi}')$ and this allows smoothness of φ' to be used to get information about smoothness of $P(\varphi'/\bar{\varphi}')$. The attempt to understand the passage from information about smoothness of $P(\varphi'/\bar{\varphi}')$ to information about the smoothness of φ' was the starting point for this work. The consequences for the Friedrichs operator of our main theorem are in Corollary 3.3.

3. Unimodular restoration

Theorem 3.1 Suppose $0 < p < \infty$ and:

- (i) $-2 < \alpha$, or
- (ii) $-3 < \alpha \leq -2$ and $p < -1/(\alpha + 2)$, or
- (iii) $-4 < \alpha \leq -3$ and $p < -1/(\alpha + 3)$.

There is a positive constant $c = c(\alpha, p)$ so that for a real valued harmonic function h with $\|h\|_{B_{\text{Re}}^\infty} < c(\alpha, p)$ and for $u = \exp(ih)$ the following are equivalent:

- (i) $h \in B_{\text{Re}}^p$,
- (ii) $u \in B_{\text{Re}}^p$,
- (iii) $P_\alpha(u) \in B_{\text{Re}}^p$,
- (iv) $\bar{P}_\alpha(u) \in B_{\text{Re}}^p$.

In this case the four functions have comparable B_{Re}^p norms.

Note. I don't know what restrictions on p and α are actually necessary.

Proof. We will begin by proving $(1) \iff (2) \implies (3)$. For these implications we don't need the assumption that h have small Bloch norm; we only need the inequalities on p, α and enough integrability to insure that the formulas in the proof make sense. We know that $u \in B_{\text{Re}}^p$ if and only if $\bar{u} \in B_{\text{Re}}^p$ and that $\overline{P_\alpha(\bar{u})} = \bar{P}_\alpha(u)$. Hence, once we have the equivalence of the first three statements, we can include the fourth. Thus the remaining effort will be to establish that $(3) \implies (1)$. We will do that by considering the cases: $p > 1$ and $-1 < \alpha$; $p > 1$ and $-2 < \alpha \leq -1$; $p > 1$ and $-3 < \alpha \leq -2$; $p > 1$ and $-4 < \alpha \leq -3$; $0 < p < 1$ and $-4 < \alpha$.

Throughout this proof when we refer to "integrability" we will mean with respect to the measure $(1 - |z|^2)^{-2} dx dy$.

(1) implies (2): If $p > 1$ we need only note $|\nabla h| = |\nabla u|$. For $p \leq 1$ we let n be the smallest integer such that $np > 1$, we just dealt with $n = 1$. Suppose now that $\frac{1}{2} < p \leq 1$ and thus $n = 2$. Going to the definition of B_{Re}^p , doing the differentiation, and recalling that u is unimodular, we find we need to establish the p^{th} power integrability of $(1 - |z|^2)^2(\partial^2 h + (\partial h)^2)$, $(1 - |z|^2)^2(\partial h \bar{\partial} h)$, and $(1 - |z|^2)^2(\bar{\partial}^2 h + (\bar{\partial} h)^2)$. By applying the Cauchy-Schwarz inequality to the second term and noting that $\bar{\partial}^j h = \overline{\partial^j \bar{h}}$ we see that we need to establish the p^{th} power integrability of $(1 - |z|^2)^2 \partial^2 h$ and of $(1 - |z|^2)^2 (\partial h)^2$. For the first term that follows from the hypotheses on h . To understand the second term we rewrite it as

$$|(1 - |z|^2)^2 (\partial h)^2|^p = |(1 - |z|^2) (\partial h)|^{2p}.$$

Now recall that the B_{Re}^p scale increases with p , hence h is in B_{Re}^{2p} which establishes the integrability of the right hand side.

The general case is the same but with more notation. We write $\partial^j e^{ih} = Q_j(h) e^{ih}$. We need to establish the p^{th} power integrability of terms of the form

$$(1 - |z|^2)^n Q_i(h) \overline{Q_j(h)} e^{ih}$$

with $i + j = n$. By applying Hölder's inequality with exponents n/i and n/j , we find that we need to establish the 1^{st} power integrability of $\left| (1 - |z|^2)^i Q_i(h) \right|^{np/i}$ for $1 \leq i \leq n$. By the formula of Faà de Bruno

$$(15) \quad \partial^j e^{ih} = Q_j(h) e^{ih} = \left\{ \sum c(k_1, \dots, k_j) (\partial h)^{k_1} (\partial^2 h)^{k_2} \dots (\partial^j h)^{k_j} \right\} e^{ih}.$$

Here the c 's are combinatoric numbers and the sum is over sets of integers with

$$(16) \quad k_1 + 2k_2 + \dots + jk_j = j.$$

Hence, to establish integrability for Q_j it suffices to establish it for each monomial in the sum. We now select a monomial and apply the generalized (i.e. multi-factor) Hölder inequality with exponents $i/k_1, i/(2k_2), \dots, i/(ik_i)$. We then see that it suffices to establish 1^{st} power integrability of

$$(17) \quad \left| (1 - |z|^2)^{sk_s} (\partial^s h)^{k_s} \right|^{(np/i)(i/(sk_s))} = \left| (1 - |z|^2)^s (\partial^s h) \right|^{(np/s)}$$

where now $s \leq i \leq n$. First suppose $s < n$. In that case the integrability follows from the fact that $h \in B_{\text{Re}}^p \subset B_{\text{Re}}^{np/s}$ which, because $s(np/s) > 1$, insures that the right hand expression is integrable. If $s = i = n$ then we need the 1^{st} power integrability of $\left| (1 - |z|^2)^n (\partial^n h) \right|^p$. That is, by definition, a consequence of (1) in the hypotheses.

(2) implies (1): Again we select n as small as possible subject to $np > 1$. We work by induction on n , the case $n = 1$ being trivial as before. The assumption that (2) holds insures p^{th} power integrability of terms of the form $(1 - |z|^2)^n Q_n(h)$. Now note that $Q_n(h) = \partial^n h + R_n(h)$ where R_n is again a polynomial in derivatives of h , but now only involving derivatives of order less than n . If we can establish p^{th} power integrability of $(1 - |z|^2)^n R_n(h)$, then, by subtraction, we will have what we want. As in the proof of the previous implication it suffices to consider the monomials forming R_n and then by Hölder's inequality it suffices to consider terms such as the one in (17). In obtaining R_n from Q_n we set aside the one term which, in the notation of that equation, corresponded to $s = i = n$. Hence, as we saw in the previous proof, the terms in R_n can be controlled if we know $h \in B_{\text{Re}}^{np/s}$. In fact, this follows from the induction hypothesis as soon as we show that n^* , the integer used with $p^* = np/s$, is strictly less than n . To do that, we must show $(n - 1)p^* > 1$. But $(n - 1)p^* = ((n - 1)/s)np$, we know $np > 1$, and, because we are avoiding the possibility that $s = n$, we have $s \leq n - 1$.

(2) implies (3): Let $\tilde{h}$ denote the harmonic conjugate of h . Set $\psi = (i/2)(h + i\tilde{h})$. Hence $\psi \in B^\infty$, $ih = \psi - \bar{\psi}$, and $u = e^{\psi - \bar{\psi}}$.

First suppose $p > 1$. Suppose $\alpha > -1$. We need to show $P_\alpha(u)' \in L_{p-2}^p$. Using Proposition 2.4 $P_\alpha(u)' = P_\alpha(e^{\psi - \bar{\psi}})' = P_{\alpha+1}(\psi' e^{\psi - \bar{\psi}})$. The assumption on h insures $\psi' \in L_{p-2}^p$ and Theorem 2.8 insures that P_α is bounded on L_{p-2}^p . Hence $P_\alpha(u)' \in L_{p-2}^p$ and this case is done. Next, suppose $\alpha \in (-3, -1]$. Proposition 2.4 leads to $P_\alpha(u)' = P_{\alpha+2}(\psi' e^{\psi - \bar{\psi}})$ and we will be done if $P_{\alpha+2}$ is bounded on L_{p-2}^p . By Theorem 2.8 this will happen if $p > (p - 2 + 1)/(\alpha + 2 + 1)$ which will hold if $(\alpha + 2)p > -1$. The constraints on α and p insure that this condition is met. Finally we consider $\alpha < -3$. We use the argument just used but we

differentiate twice. We then need to show $P_{\alpha+4}((\psi'' + \psi'^2)e^{\psi-\bar{\psi}}) \in L_{2p-2}^p$. The hypotheses insure that $(\psi'' + \psi'^2) \in L_{2p-2}^p$ and again we appeal to Theorem 2.8 and the constraints on α and p .

Now suppose $p \leq 1$. We first look at $\alpha > -1$. We need to show that for an integer n with $np > 1$ we have $P_\alpha(e^{\psi-\bar{\psi}})^{(n)} \in L_{np-2}^p$. By Proposition 2.4 $P_\alpha(e^{\psi-\bar{\psi}})^{(n)} = P_{\alpha+n}(Q_n(\psi)e^{\psi-\bar{\psi}})$. In the proof that (1) implies (2) we saw that $Q_n(\psi) \in A_{np-2}^p$. Thus $Q_n(\psi)$ is the n^{th} derivative of a function in B^p . Hence, by Proposition 2.2 we can, for given $b > 0$, write

$$Q_n(\psi) = \sum_{j=1}^{\infty} \lambda_j \frac{(1 - |z_j|^2)^b}{(1 - \bar{z}_j z)^{b+n}}$$

with $\|\{\lambda_j\}\|_{l^p} \leq c \|Q_n(\psi)\|_{B^p}$. By the p -convexity of A_{np-2}^p , it now suffices to show that

$$(18) \quad (*) = P_{\alpha+n} \left(\frac{(1 - |z_j|^2)^b}{(1 - \bar{z}_j z)^{b+n}} e^{\psi-\bar{\psi}} \bar{z}^n \right) \in A_{np-2}^p$$

and that this holds uniformly if $|z_j| < 1$. We begin by using the p -convexity of A_{np-2}^p .

$$\begin{aligned} \|(*)\|_{A_{np-2}^p}^p &= \left\| \iint_{\mathbb{D}} \frac{(1 - |z_j|^2)^b}{(1 - \bar{z}_j z)^{b+n}} e^{\psi-\bar{\psi}} \bar{z}^n \frac{(1 - |w|^2)^{\alpha+n}}{(1 - \bar{z}w)^{\alpha+n+2}} dx dy \right\|_{A_{np-2}^p}^p \\ &\leq \iint_{\mathbb{D}} \left\| \frac{(1 - |z_j|^2)^b}{(1 - \bar{z}_j z)^{(b+n)}} e^{\psi-\bar{\psi}} \bar{z}^n \frac{(1 - |w|^2)^{\alpha+n}}{(1 - \bar{z}w)^{\alpha+n+2}} \right\|_{A_{np-2}^p}^p dx dy \\ &\leq \iint_{\mathbb{D}} \frac{(1 - |z_j|^2)^{pb}}{|1 - \bar{z}_j z|^{p(b+n)}} \left\| \frac{(1 - |w|^2)^{\alpha+n}}{(1 - \bar{z}w)^{\alpha+n+2}} \right\|_{A_{np-2}^p}^p dx dy. \end{aligned}$$

By definition

$$\left\| \frac{(1 - |w|^2)^{\alpha+n}}{(1 - \bar{z}w)^{\alpha+n+2}} \right\|_{A_{np-2}^p}^p = \iint_{\mathbb{D}} \left| \frac{(1 - |w|^2)^{\alpha+n}}{(1 - \bar{z}w)^{\alpha+n+2}} \right|^p (1 - |w|^2)^{np-2} du dv.$$

We can estimate that integral using Lemma 4.2.2 of [Z]:

$$\begin{aligned} \left\| \frac{(1 - |w|^2)^{\alpha+n}}{(1 - \bar{z}w)^{\alpha+n+2}} \right\|_{A_{np-2}^p}^p &\leq c(1 - |z|^2)^{np-2+p(\alpha+n)-p(\alpha+n+2)+2} \\ &\leq c(1 - |z|^2)^{(n-2)p}. \end{aligned}$$

Thus

$$\|(\ast)\|_{A_{np-2}^p}^p \leq c \int \int_{\mathbb{D}} \frac{(1 - |z_j|^2)^{pb}}{|1 - \bar{z}_j z|^{p(b+n)}} (1 - |z|^2)^{(n-2)p} dx dy.$$

We now use Lemma 4.2.2 of [Z] again and continue with

$$\begin{aligned} \|(\ast)\|_{A_{np-2}^p}^p &\leq c(1 - |z_j|^2)^{(n-2)p+pb-p(b+n)+2} \\ &\leq c(1 - |z_j|^2)^{2(1-p)}. \end{aligned}$$

Because $p < 1$ and $|z_j| < 1$, this is bounded uniformly, as we wanted.

The situation with $\alpha \leq -1$ differs only in the numerics. When we follow the same path we find that, instead of (18), we need to show $P_{\alpha+n+j}(\dots) \in A_{np-2}^p$, where $j = 1$ if $\alpha > -3$ and $j = 2$ otherwise. The argument then proceeds as before.

(3) implies (1); $p > 1$ and $-1 < \alpha$: The hypothesis is that $a = P_\alpha(e^{ih}) \in B^p$. Thus, for some $r \perp A_\alpha^2$, we have that $u = e^{\psi-\bar{\psi}} = a + r$. Thus $e^\psi = e^{\bar{\psi}}a + e^{\bar{\psi}}r$. By Proposition 2.5, $e^{\bar{\psi}}r$ is also orthogonal to A_α^2 . Thus, applying P_α to the expression for e^ψ gives $e^\psi = P_\alpha(e^{\bar{\psi}}a)$. We now differentiate and invoke Proposition 2.4 to obtain $\psi'e^\psi = P_{\alpha+1}(e^{\bar{\psi}}a')$. Thus

$$\psi' = e^{-\psi}P_{\alpha+1}(e^{\bar{\psi}}a') = e^{-ih/2}e^{\bar{h}/2}P_{\alpha+1}(e^{-\bar{h}/2}e^{-ih/2}a').$$

The hypothesis is that $a \in B^p$ or, equivalently, $a' \in A_{p-2}^p$ and the desired conclusion is that $\psi \in B^p$ or, equivalently, $\psi' \in A_{p-2}^p$. Thus we will be done if we know that the mapping of a' to ψ' defined by the previous equation is bounded. We can drop the unimodular factors and consider the mapping of f to $e^{\bar{h}/2}P_{\alpha+1}(e^{-\bar{h}/2}f)$ acting on A_{p-2}^p . For real harmonic functions, $h, \|\tilde{h}\|_{B_{\text{Re}}^\infty} \leq c\|h\|_{B_{\text{Re}}^\infty}$. This, together with Theorem 2.8 insures that the map is bounded.

(3) implies (1); $p > 1$ and $-2 < \alpha \leq -1$: The previous analysis is unchanged to the point at which we differentiate. Now we use the second part of Proposition 2.4. On the right hand side we get $P_{\alpha+2}(e^{\bar{\psi}}a')$. Thus we need to know if the map of f to $e^{\bar{h}/2}P_{\alpha+2}(e^{-\bar{h}/2}f)$, acting on A_{p-2}^p , is continuous. We now appeal to Theorem 2.8 with $\gamma = \alpha + 2$ and $\delta = p - 2$. To do this we need $(\delta + 1)/(\gamma + 1) < p$. This holds in this range because $(\delta + 1)/(\gamma + 1) = (p - 1)/(\alpha + 3) < p/1$.

(3) implies (1); $p > 1$ and $-3 < \alpha \leq -2$: The analysis of the previous

case goes through until the last inequality. At that point we require condition (2) from the hypotheses to insure that the inequality holds.

(3) implies (1); $p > 1$ and $-4 < \alpha \leq -3$: Now n in (5) is 2 and when we use the second part of Proposition 2.4 we obtain $(\psi')^2 + \psi'' = e^{-\psi} P_{\alpha+4}(e^{\bar{\psi}} a'')$. We know $a \in B^p$ and thus $a'' \in A_{2p-2}^p$. We now use Theorem 2.8 with $\gamma = \alpha + 4$ and $\delta = 2p - 2$. That $(\delta + 1)/(\gamma + 1) = (2p - 1)/(\alpha + 5) < p$ is insured by *iii.* from the hypotheses. Thus we know

$$(19) \quad (\psi')^2 + \psi'' \in A_{2p-2}^p$$

and $\|(\psi')^2 + \psi''\|_{A_{2p-2}^p} \leq c \|a\|_{B^p}$. We want to show that $\|\psi''\|_{A_{2p-2}^p} \leq C \|a\|_{B^p}$.

Set $\|\psi''\|_{A_{2p-2}^p} = M$ and $\|(\psi')^2 + \psi''\|_{A_{2p-2}^p} = N$. We have

$$\begin{aligned} M &\leq \|(\psi')^2 + \psi''\|_{A_{2p-2}^p} + \|(\psi')^2\|_{A_{2p-2}^p} \\ &= N + \|(\psi')^2\|_{A_{2p-2}^p} \\ &= N + \|\psi'\|_{A_{2p-2}^{2p}}^2. \end{aligned}$$

Using the relation between Besov norms and the Lebesgue norms of derivatives, and using Hölder's inequality for the B^p scale we have

$$\begin{aligned} \|\psi'\|_{A_{2p-2}^{2p}}^2 &\leq c \|\psi\|_{B^{2p}}^2 \\ &\leq c \|\psi\|_{B^p} \|\psi\|_{B^\infty} \\ &\leq c' \|\psi''\|_{A_{2p-2}^p} \|\psi\|_{B^\infty} \\ &\leq c' M \|\psi\|_{B^\infty}. \end{aligned}$$

Combining this with the previous estimate, we have

$$M \leq N + c' M \|\psi\|_{B^\infty}.$$

Thus, if we know $\|\psi\|_{B^\infty}$ is small, and the hypotheses insure that we do, this yields $M \leq cN = c \|a\|_{B^p}$, as required.

(3) implies (1); $0 < p < 1$ and $-4 < \alpha$: Pick the smallest integer n so that $np > 1$. We have $e^\psi = P_\alpha(e^{\bar{\psi}} a)$. Hence $\partial^n e^\psi = \partial^n P_\alpha(e^{\bar{\psi}} a)$ and thus, in the notation of (15), we have $Q_n(\psi) = e^{-\psi} \partial^n P_\alpha(e^{\bar{\psi}} a)$. Hence, by Proposition 2.10, we find $Q_n(\psi) \in A_{np-2}^p$.

In Q_n we now need to separate the term we want (i.e. $\psi^{(n)}$) from other terms. First suppose $\frac{1}{2} < p \leq 1$ and thus $n = 2$. $Q_2(\psi) = \psi'' + (\psi')^2$. We now follow the analysis in the previous case beginning at (19). That gives $\|\psi\|_{B^p} \leq c \|a\|_{B^p}$.

We now proceed by induction on n . We return to the notation of (15).

$$\begin{aligned}
 \|\psi\|_{B^p} &\leq c \left\| \psi^{(n)} \right\|_{A_{np-2}^p} \\
 &\leq c \|Q_n(\psi)\|_{A_{np-2}^p} + c \left\| \psi^{(n)} - Q_n(\psi) \right\|_{A_{np-2}^p} \\
 &\leq c \|a\|_{B^p} + \sum' c(k_1, \dots, k_n) \left\| (\psi^{(1)})^{k_1} (\psi^{(2)})^{k_2} \dots (\psi^{(n)})^{k_n} \right\|_{A_{np-2}^p}
 \end{aligned}$$

Here $\sum'$ denotes the sum excluding the term with $\psi^{(n)}$. We now estimate a generic term in the sum on the right

$$\begin{aligned}
 \|\cdot\|_{A_{np-2}^p}^p &= \int \prod |(\psi^{(j)})^{k_j}|^p |\varepsilon(z)^{np} \varepsilon(z)^{-2}|^p dV \\
 &= \int \prod |\varepsilon(z)^j \psi^{(j)}|^{pk_j} |\varepsilon(z)^{-2}|^p dV,
 \end{aligned}$$

the last equality using (16). If the product contains more than a single factor then, because of (16) again, $n/jk_j > 1$ and $\sum jk_j/n = 1$. We use Hölder's inequality and find

$$\|\cdot\|_{A_{np-2}^p}^p \leq \prod \left(\int |\varepsilon(z)^j \psi^{(j)}|^{pn/j} |\varepsilon(z)^{-2}|^p dV \right)^{jk_j/n}.$$

(If there was only one factor then we were already in the desired form.) Hence

$$\|\cdot\|_{A_{np-2}^p}^p \leq c \prod \|\psi\|_{B^{pn/j}}^{pk_j}.$$

Now note that $pn/j > p$. (The only term for which that would fail was the one omitted in $\sum'$.) The induction hypothesis insures that the factors on the right are finite and hence, after summing, we find $\|\psi\|_{B^p}$ is finite. Hence we find

$$\prod \|\psi\|_{B^{pn/j}}^{pk_j} \leq c \prod \left(\|\psi\|_{B^p}^{j/n} \|\psi\|_{B^\infty}^{1-j/n} \right)^{pk_j} = c \|\psi\|_{B^p}^p \|\psi\|_{B^\infty}^\lambda$$

with some positive λ . Adding such estimates, recalling that $\|\psi\|_{B^\infty}^\lambda$ is small, and rearranging completes the proof. $\square$

Just as we had Corollary 2.9, we have the following result:

Corollary 3.2 *The theorem continues to hold if we replace the hypothesis that $\|h\|_{B^\infty} \leq c$ with the assumption that $h \in b^\infty$.*

Corollary 3.3 *Suppose Ω is a simply connected domain in $\mathbb{C}$ which is asymptotically conformal and φ is a Riemann map of Ω to $\mathbb{D}$. The Friedrichs operator T_Ω is in the Schatten-von Neuman ideal S^p , $0 < p < \infty$, if and only if $\log \varphi' \in B^p$.*

Proof. It is shown in [LR] that $T_\Omega \in S^p$ if and only if the (small) Hankel operator with symbol $\varphi'/\bar{\varphi}'$, acting on the Bergman space of $\mathbb{D}$, is in S^p . It is also noted that this holds if and only if $P_0(e^{2i \operatorname{Im} \log \varphi'}) \in B^p$. In fact, those arguments are valid for $0 < p \leq 1$ as soon as we take into account the appropriate result about Hankel operators ([Pe] [Sel]). The hypothesis that Ω is asymptotically conformal implies $\log \varphi' \in b_{\operatorname{Re}}^\infty$ (Theorem 11.1 of [Po]). Hence by the previous corollary $T_\Omega \in S^p$ exactly if $\log \varphi' \in B^p$. $\square$

4. Variations. The details of the previous proofs can be varied to produce other, similar, results. Here are two samples.

4.1. Products and moduli. The proof of the main part of Theorem 3.1 starts by writing $u = e^\psi e^{-\bar{\psi}}$. The same pattern of analysis can be used if we start with $v = e^\varphi e^{\bar{\psi}}$ with $\psi, \varphi \in B^\infty$ and having small norm. For convenience we will just look at the situation with $1 < p$, $-1 < \alpha$. Set $b = P_\alpha(v)$. Following the earlier proof we obtain

$$\varphi' e^\varphi = P_{\alpha+1}(e^{-\bar{\psi}} b')$$

and hence

$$\varphi' e^{\varphi+\psi} = e^\psi P_{\alpha+1}(e^{-\bar{\psi}} b').$$

If $b \in B^p$ we can invoke the weighted norm inequalities and conclude

$$\int \varphi' e^{\varphi+\psi} \in B^p.$$

If ψ is a real multiple of φ then we can evaluate the integral.

Theorem 4.1 *Suppose $1 < p$, $-1 < \alpha$, $r \in \mathbb{R} \setminus \{-1\}$. There is a positive $c = c(\alpha, p, r)$ so that if $\varphi \in B^\infty$, $\|\varphi\| \leq c$ then*

$$P_\alpha(e^\varphi e^{r\bar{\varphi}}) \in B^p \text{ implies } e^{(1+r)\varphi} \in B^p.$$

In particular

$$P_\alpha(|e^\varphi|) \in B^p \text{ implies } e^\varphi \in B^p.$$

This result can be viewed as a smoothness condition on the modulus of an outer function F which insure that F itself be in B^p . As such it can be compared with Theorem 4.4 of [Dyk].

4.2. Perturbations. In Theorem 3.1 we see that for a real harmonic h , and $f(x) = e^{ix}$, we can draw conclusions about the smoothness of h from that of $P(f(h))$. The previous theorem gives a similar result for the choice $f(x) = e^x$. It is not clear what (if anything) is a natural class of functions f for such results. Here we show that the earlier proof extends to mild multiplicative perturbations of $f(x) = e^{ix}$. We will show that, for real harmonic h and smooth s , if $P_\alpha(e^{ih}(1 + s(ih)))$ is smooth then so is h .

Theorem 4.2 *Let h be a real harmonic function in B_{Re}^∞ . Suppose $1 < p$, $-1 < \alpha$, and that s is a function of class C^2 on $\mathbb{D}$. There are positive $c = c(\alpha, p)$, and d so that, if $\|s\|_\infty, \|\nabla s\|_\infty \leq d$, and if $\|h\| \leq c$; then*

$$P_\alpha(e^{ih}(1 + s(ih))) \in B^p \text{ implies } h \in B_{\text{Re}}^p.$$

Proof. Set $P_\alpha(e^{ih}(1 + s(ih))) = a$. Let φ be the function in B^∞ such that $ih = \varphi - \bar{\varphi}$. When we follow the previous pattern of proof we come to

$$\varphi' e^\varphi + P_\alpha(e^\varphi s(\varphi - \bar{\varphi}))' = P_\alpha(a e^{\bar{\varphi}})'$$

We formulated Proposition 2.4 for products of the form $h\bar{k}$ with h and k holomorphic. However, the linear span of such functions is dense and hence we can use the proposition on both projections in the previous equation. In particular

$$\begin{aligned} P_\alpha(e^\varphi s(\varphi - \bar{\varphi}))' &= P_{\alpha+1}(\partial(e^\varphi s(h))) \\ &= P_{\alpha+1}(\varphi' e^\varphi s(h) + e^\varphi \partial(s(h))) \\ &= P_{\alpha+1}(\varphi' e^\varphi s(h) + \varphi' e^\varphi (\partial s)(h) + \bar{\varphi}' e^\varphi (\bar{\partial} s)(h)). \end{aligned}$$

Hence

$$\varphi' e^\varphi = P_{\alpha+1}(a' e^{\bar{\varphi}}) - P_{\alpha+1}(\varphi' e^\varphi s(h) + \varphi' e^\varphi (\partial s)(h) + \bar{\varphi}' e^\varphi (\bar{\partial} s)(h)).$$

Thus

$$\varphi' = e^{-\varphi} P_{\alpha+1}(a' e^{\bar{\varphi}}) - e^{-\varphi} P_{\alpha+1}(\varphi' e^\varphi s(h) + \varphi' e^\varphi (\partial s)(h) + \bar{\varphi}' e^\varphi (\bar{\partial} s)(h)).$$

We now take norms and use the weighted norm inequalities. This gives

$$\begin{aligned} \|\varphi'\|_{A_{p-2}^p} &\leq c \|a'\|_{A_{p-2}^p} + c \|\varphi' s(h)\|_{A_{p-2}^p} + c \|\varphi' (\partial s)(h)\|_{A_{p-2}^p} \\ &\quad + c \|\bar{\varphi}' (\bar{\partial} s)(h)\|_{A_{p-2}^p}. \end{aligned}$$

We now use the hypothesis on s to find

$$\|\varphi'\|_{A_{p-2}^p} \leq c \|a'\|_{A_{p-2}^p} + cd \|\varphi'\|_{A_{p-2}^p} + cd \|\varphi'\|_{A_{p-2}^p} + cd \|\varphi'\|_{A_{p-2}^p}.$$

Hence, choosing d small we have

$$\|\varphi'\|_{A_{p-2}^p} \leq \frac{c}{1 - 3cd} \|a'\|_{A_{p-2}^p}.$$

□

Of course there is a similar variation on Theorem 4.1.

5. Questions and comments

- The problem of deriving structural information about a real harmonic h from smoothness information about $P(e^{ih})$ without making the *a-priori* assumption about h , such as h having a small Bloch norm, seems quite interesting.
- In the example mentioned in the introduction, which shows that some *a-priori* assumption is necessary, we have that $P(u)$ is smooth but $P(\bar{u})$ is not. It would be interesting to put that intriguing asymmetry into a larger context.
- What is the right type of question for this sort of reconstruction? Suppose h is a real valued harmonic function. We noted that for some functions f we can get information about smoothness of h from information about smoothness of $P(f(h))$. It is not clear for which f we might expect such a result. Even less clear is the role of the fact that our unimodular functions were of the form $u = e^{ih}$ for harmonic h and hence satisfied $\Delta \log u = 0$. Possibly there are other classes of unimodular functions for which analogous results hold.

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