The Toeplitz Algebra of a Hilbert Bimodule

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ABSTRACT. Suppose a C^* -algebra A acts by adjointable operators on a Hilbert A -module X . Pimsner constructed a C^* -algebra $\mathcal{O}_X$ which includes, for particular choices of X , crossed products of A by $\mathbb{Z}$, the Cuntz algebras $\mathcal{O}_n$, and the Cuntz-Krieger algebras $\mathcal{O}_B$. Here we analyse the representations of the corresponding Toeplitz algebra. One consequence is a uniqueness theorem for the Toeplitz-Cuntz-Krieger algebras of directed graphs, which includes Cuntz's uniqueness theorem for $\mathcal{O}_\infty$.

A Hilbert bimodule X over a C^* -algebra A is a right Hilbert A -module with a left action of A by adjointable operators. The motivating example comes from an automorphism α of A : take $X_A = A_A$, and define the left action of A by $a \cdot b := \alpha(a)b$. In [23], Pimsner constructed a C^* -algebra $\mathcal{O}_X$ from a Hilbert bimodule X in such a way that the $\mathcal{O}_X$ corresponding to an automorphism α is the crossed product $A \rtimes_\alpha \mathbb{Z}$. He also produced interesting examples of bimodules which do not arise from automorphisms or endomorphisms, including bimodules over finite-dimensional commutative C^* -algebras for which the corresponding $\mathcal{O}_X$ are the Cuntz-Krieger algebras. The Cuntz algebra $\mathcal{O}_n$ is $\mathcal{O}_X$ when ${}_C X_C$ is a Hilbert space of dimension n and the left action of $\mathbb{C}$ is by multiples of the identity.

Here we use methods developed in [18, 9] for analysing semigroup crossed products to study Pimsner's algebras. These methods seem to apply more directly to Pimsner's analogue of the Toeplitz-Cuntz algebras rather than his analogue $\mathcal{O}_X$ of the Cuntz algebras. Nevertheless, our results yield new information about the Cuntz-Krieger algebras of some infinite graphs, giving a whole class of these algebras which behave like $\mathcal{O}_\infty$.

The uniqueness theorems for C^* -algebras generated by algebraic systems of isometries say, roughly speaking, that all examples of a given system in which the isometries are non-unitary generate isomorphic C^* -algebras. We can approach such a theorem by introducing a C^* -algebra which is universal for systems of the given type, and then characterising its faithful representations. Here the systems consist of representations ψ of X and π of A on the same Hilbert space

which convert the module actions and the inner product to operator multiplication; we call these Toeplitz representations of X . (The partial isometries and isometries appearing in more conventional systems are obtained by applying ψ to the elements of a basis for X .) In Section 1, we discuss these Toeplitz representations, show that there is a universal C^* -algebra $\mathcal{T}_X$ generated by a Toeplitz representation, and prove some general results relating these representations to the induced representations of Rieffel.

Our first theorem is very much in the spirit of other theorems about C^* -algebras generated by systems of isometries: it gives a condition on a Toeplitz representation (ψ, π) which implies that the corresponding representation $\psi \times \pi$ of $\mathcal{T}_X$ is faithful (Theorem 2.1). In broad terms, this condition says that the ranges of all the operators $\psi(x)$ should leave enough room for A to act faithfully. The proof follows standard lines: we use a canonical gauge action γ to construct an expectation onto a core $\mathcal{T}_X^\gamma$, and then show that both the core and the expectation are implemented faithfully in the given Toeplitz representation.

When the bimodule ${}_C X_C$ is an infinite-dimensional Hilbert space, Theorem 2.1 says that a family $\{S_i : i \in \mathbb{N}\}$ of isometries on $\mathcal{H}$ with orthogonal ranges generates a faithful representation of $\mathcal{O}_\infty$ if the ranges $S_i \mathcal{H}$ do not span $\mathcal{H}$. However, more is true: Cuntz proved that *every* family of isometries with orthogonal ranges generates a faithful representation of $\mathcal{O}_\infty$. Our main theorem is an improvement of Theorem 2.1 which gives the full strength of Cuntz's result (Theorem 3.1): we assume that X has a direct-sum decomposition $X = \bigoplus_\lambda X^\lambda$, but only ask that A acts faithfully on $(\bigoplus_{\lambda \in F} \psi(X^\lambda) \mathcal{H})^\perp$ for every finite subset F of indices. For ${}_C X_C$, the decomposition is parametrised by a basis of X , and the hypothesis asks that $\sum_{i=1}^n S_i S_i^* < 1$ for all finite n , which is trivially true if there are infinitely many S_i . To prove Theorem 3.1, we use the direct-sum decomposition to go further into the core; we need the special case in Theorem 2.1 to construct the expectation which does this.

The new applications of our theorem involve the C^* -algebras of directed graphs. For a locally finite graph E , the C^* -algebra $C^*(E)$ is by definition universal for Cuntz-Krieger E -families: families $\{S_f\}$ of partial isometries, parametrised by the edge set E^1 of the graph, and satisfying in particular

$$S_e^* S_e = \sum_{\{f: s(f)=r(e)\}} S_f S_f^*,$$

where $r, s : E^1 \rightarrow E^0$ send edges to their range and source vertices [17, 16]. The graph algebra $C^*(E)$ can be realised in a very natural way as the Cuntz-Pimsner algebra $\mathcal{O}_X$ of a bimodule X over the algebra $A = c_0(E^0)$ (see [24, 14] and Example 1.2 below). For graphs in which vertices can emit infinitely many edges, the Cuntz-Krieger relations involve infinite sums which do not make sense in a C^* -algebra, and it is not clear how to best define a useful notion of graph C^* -algebra. We show that this problem disappears if all vertices emit infinitely many

edges: all families satisfying $S_e^* S_e \geq \sum_{\{f: s(f)=r(e)\}} S_f S_f^*$ generate isomorphic C^* -algebras (Theorem 4.1). If the graph is also transitive, this C^* -algebra is simple (Corollary 4.5).

Since Hilbert bimodules are a relatively new field of study, and since they arise in so many different ways, the precise axioms are not yet standard. Thus different authors have assumed that $\varphi : A \rightarrow \mathcal{L}(X)$ is injective, that A acts by compact operators on X , that A acts nondegenerately on X , or that X is full. We have been careful to avoid such assumptions, and in our final section we illustrate using the bimodules of graphs why we believe this to be helpful. We also give a couple of new applications involving other classes of Hilbert bimodules.

1. TOEPLITZ REPRESENTATIONS AND THE TOEPLITZ ALGEBRA

By a *Hilbert bimodule* over a C^* -algebra A we shall mean a right Hilbert A -module X together with an action of A by adjointable operators on X . The left action gives a homomorphism of A into the C^* -algebra $\mathcal{L}(X)$ of adjointable operators, which we denote by φ .

A *Toeplitz representation* (ψ, π) of a Hilbert bimodule X in a C^* -algebra B consists of a linear map $\psi : X \rightarrow B$ and a homomorphism $\pi : A \rightarrow B$ such that

$$(1.1) \quad \psi(x \cdot a) = \psi(x)\pi(a),$$

$$(1.2) \quad \psi(x)^* \psi(y) = \pi(\langle x, y \rangle_A), \quad \text{and}$$

$$(1.3) \quad \psi(a \cdot x) = \pi(a)\psi(x)$$

for $x, y \in X$ and $a \in A$. When $B = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we call (ψ, π) a Toeplitz representation of X on $\mathcal{H}$.

Remark 1.1. *In fact Condition (1.2) implies that ψ is linear, as in [1, p.8]. It also implies that ψ is bounded: for $x \in X$ we have*

$$\|\psi(x)\|^2 = \|\psi(x)^* \psi(x)\| = \|\pi(\langle x, x \rangle_A)\| \leq \|\langle x, x \rangle_A\| = \|x\|^2.$$

If π is injective, then we have equality throughout, and ψ is isometric.

While many important examples of Hilbert bimodules are given in [23, Section 1], [21, Example 22] and [20, Section 3], the examples of most interest to us are associated to an infinite directed graph. These are not entirely new: it is shown in [23, p.193] how to build a bimodule from a finite $\{0, 1\}$ -matrix A , and that bimodule can be obtained by applying the following construction to the finite graph with incidence matrix A . However, the simplicity of the formulas in the next Example suggests that it may be more natural to think in terms of graphs rather than $\{0, 1\}$ -matrices.

Example 1.2 (The Cuntz-Krieger bimodule). Suppose $E = (E^0, E^1, r, s)$ is a directed graph with vertex set E^0 , edge set E^1 , and $r, s : E^1 \rightarrow E^0$ describing

the range and source of edges. Let $X = X(E)$ be the vector space of functions $x : E^1 \rightarrow \mathbb{C}$ for which the function

$$v \in E^0 \mapsto \sum_{\{f \in E^1 : r(f)=v\}} |x(f)|^2$$

belongs to $A := c_0(E^0)$. Then with the operations

$$\begin{aligned} (x \cdot a)(f) &:= x(f)a(r(f)) \quad \text{for } f \in E^1, \\ \langle x, y \rangle_A(v) &:= \sum_{\{f \in E^1 : r(f)=v\}} \overline{x(f)}y(f) \quad \text{for } v \in E^0, \text{ and} \\ (a \cdot x)(f) &:= a(s(f))x(f) \quad \text{for } f \in E^1, \end{aligned}$$

X is a Hilbert bimodule over A .

Both the module X and the algebra A are spanned in an appropriate sense by point masses δ_f , δ_v , and we have

$$\langle \delta_e, \delta_f \rangle_A = \begin{cases} \delta_{r(e)} & \text{if } e = f \\ 0 & \text{otherwise;} \end{cases}$$

the elements δ_v are a family of mutually orthogonal projections in the C^* -algebra A . If (ψ, π) is a Toeplitz representation of this Hilbert bimodule X on $\mathcal{H}$, then the operators $P_v := \pi(\delta_v)$ are mutually orthogonal projections on $\mathcal{H}$, and (1.2) implies that the operators $S_f := \psi(\delta_f)$ are partial isometries with initial projection $P_{r(f)}$ and mutually orthogonal range projections; (1.3) implies that these range projections satisfy

$$(1.4) \quad \sum_{\{f \in E^1 : s(f)=v\}} S_f S_f^* \leq P_v \quad \text{for } v \in E^0.$$

We say that $\{S_f, P_v\}$ is a *Toeplitz-Cuntz-Krieger family* for the graph E . Conversely, given any such family on $\mathcal{H}$, we can define a representation $\pi : A \rightarrow B(\mathcal{H})$ by $\pi(a) := \sum_v a(v)P_v$, and a linear map $\psi : C_c(E^1) \rightarrow B(\mathcal{H})$ by $\psi(x) := \sum_f x(f)S_f$; routine calculations show that ψ is isometric for the A -norm on $C_c(E^1) \subset X$ and hence extends to a linear map on all of X , and that (ψ, π) is a Toeplitz representation of X .

Proposition 1.3. *Let X be a Hilbert bimodule over A . Then there is a C^* -algebra $\mathcal{T}_X$ and a Toeplitz representation $(i_X, i_A) : X \rightarrow \mathcal{T}_X$ such that*

- (a) *for every Toeplitz representation (ψ, π) of X , there is a homomorphism $\psi \times \pi$ of $\mathcal{T}_X$ such that $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$; and*
- (b) *$\mathcal{T}_X$ is generated as a C^* -algebra by $i_X(X) \cup i_A(A)$.*

The triple $(\mathcal{T}_X, i_X, i_A)$ is unique: if (B, i'_X, i'_A) has similar properties, there is an isomorphism $\theta : \mathcal{T}_X \rightarrow B$ such that $\theta \circ i_X = i'_X$ and $\theta \circ i_A = i'_A$. Both maps i_X and i_A are injective. There is a strongly continuous action $\gamma : \mathbb{T} \rightarrow \text{Aut } \mathcal{T}_X$ such that $\gamma_z(i_A(a)) = i_A(a)$ and $\gamma_z(i_X(x)) = zi_X(x)$ for $a \in A$, $x \in X$.

We call $\mathcal{T}_X$ the *Toeplitz algebra* of X and γ the *gauge action*. To prove the existence of $\mathcal{T}_X$, we need to know that the bimodule has lots of nontrivial representations. Here the fundamental example is a modification of Fock space, due essentially to Pimsner [23].

Example 1.4 (The Fock representation). For $n \geq 1$, the n -fold internal tensor product $X^{\otimes n} := X \otimes_A \cdots \otimes_A X$ is naturally a right Hilbert A -module, and A acts on the left by

$$a \cdot (x_1 \otimes_A \cdots \otimes_A x_n) := (a \cdot x_1) \otimes_A \cdots \otimes_A x_n;$$

if we need a name for the operator we call it $\varphi(a) \otimes_A 1^{n-1}$, and we continue to write x for a typical element of $X^{\otimes n}$. For $n = 0$, we take $X^{\otimes 0}$ to be the Hilbert module A with left action $a \cdot b := ab$. Then the Hilbert-module direct sum $F(X) := \bigoplus_{n=0}^{\infty} X^{\otimes n}$ carries a diagonal left action of A in which $a \cdot (x_n) := (a \cdot x_n)$. We can induce a representation $\pi_0 : A \rightarrow B(\mathcal{H})$ to a representation $F(X)\text{-Ind}_A^{\mathcal{L}(F(X))} \pi_0$ of $\mathcal{L}(F(X))$ on $F(X) \otimes_A \mathcal{H}$, which restricts to a representation $\pi := F(X)\text{-Ind}_A^A \pi_0$ of A .

For each $x \in X$, we can define a *creation operator* $T(x)$ on $F(X)$ by

$$T(x)y = \begin{cases} x \cdot y & \text{if } y \in X^{\otimes 0} = A \\ x \otimes_A y & \text{if } y \in X^{\otimes n} \text{ for some } n \geq 1; \end{cases}$$

routine calculations show that $T(x)$ is adjointable with

$$T(x)^*z = \begin{cases} 0 & \text{if } z \in X^{\otimes 0} = A \\ \langle x, x_1 \rangle_A \cdot y & \text{if } z = x_1 \otimes_A y \in X \otimes_A X^{\otimes n-1} = X^{\otimes n}. \end{cases}$$

If we now define $\psi : X \rightarrow B(F(X) \otimes_A \mathcal{H})$ by

$$\psi(x) := F(X)\text{-Ind}_A^{\mathcal{L}(F(X))} \pi_0(T(x)),$$

then (ψ, π) is a Toeplitz representation of X , called the *Fock representation induced from π_0* . Note that, since A acts faithfully on $X^{\otimes 0} = A$ and the representation $F(X)\text{-Ind}_A^{\mathcal{L}(F(X))} \pi_0$ is faithful whenever π_0 is, the representation π is faithful whenever π_0 is; by Remark 1.1, so is ψ .

Remark 1.5. If we denote by φ_∞ the diagonal embedding of A in $\mathcal{L}(F(X))$, then (T, φ_∞) is a Toeplitz representation of X in the C^* -algebra $\mathcal{L}(F(X))$. Pimsner's Toeplitz algebra of X is by definition the C^* -subalgebra of $\mathcal{L}(F(X))$ generated by $T(X) \cup \varphi_\infty(A)$ [23, Definition 1.1], which is precisely the image of $\mathcal{T}_X$ under $T \times \varphi_\infty$. In Corollary 2.2, we will show that our Toeplitz algebra is isomorphic to his by proving that $T \times \varphi_\infty$ is faithful.

Proof of Proposition 1.3. Say that a (Toeplitz) representation (ψ, π) of X on a Hilbert space $\mathcal{H}$ is *nondegenerate* (resp. *cyclic*) if the C^* -algebra $C^*(\psi, \pi)$ generated by $\psi(X) \cup \pi(A)$ acts nondegenerately (resp. cyclically). For an arbitrary representation (ψ, π) of X , let P be the orthogonal projection onto the essential subspace $\mathcal{K} := \overline{C^*(\psi, \pi)\mathcal{H}}$; then $(P\psi, P\pi)$ is a nondegenerate representation of X on $P\mathcal{H}$, and $((I - P)\psi, (I - P)\pi)$ is the zero representation. By the usual Zorn's lemma argument, $\mathcal{K}$ decomposes as a direct sum of subspaces on which $C^*(\psi, \pi)$ acts cyclically. Hence every representation is the direct sum of a zero representation and a collection of cyclic representations.

Let S be a set of cyclic representations of X such that every cyclic representation of X is unitarily equivalent to an element of S . (It can be shown that such a set S exists by fixing a Hilbert space $\mathcal{H}$ of sufficiently large dimension, and considering only cyclic representations on subspaces of $\mathcal{H}$. The set S is nonempty because the Fock representations must have nonzero cyclic summands.) Let

$$\mathcal{H} := \bigoplus_{(\psi, \pi) \in S} \mathcal{H}_{\psi, \pi}, \quad i_X := \bigoplus_{(\psi, \pi) \in S} \psi, \quad \text{and} \quad i_A := \bigoplus_{(\psi, \pi) \in S} \pi$$

(the direct sum defining i_X makes sense because every ψ is contractive). Then (i_X, i_A) is a representation of X in $\mathcal{T}_X := C^*(i_X, i_A)$; (b) is satisfied by definition, and (a) can be routinely verified.

The uniqueness follows by a standard argument, and the maps i_X and i_A are injective because the Fock representations factor through (i_X, i_A) by (a). To establish the existence of the gauge automorphism γ_z , just note that $(\mathcal{T}_X, z i_X, i_A)$ is also universal, and invoke the uniqueness. The continuity of the gauge action follows from a straightforward $\varepsilon/3$ -argument. $\square$

Whenever a C^* -algebra C acts by adjointable operators on a Hilbert A -module, one can use the module to induce representations of A to representations of C . If the representation π of A is half of a Toeplitz representation, we can realise the induced representation on the Hilbert space of π :

Proposition 1.6. *Let X be a right Hilbert A -module, and suppose (ψ, π) is a representation of X on $\mathcal{H}$; that is, $\psi : X \rightarrow B(\mathcal{H})$ is linear, $\pi : A \rightarrow B(\mathcal{H})$ is a representation, and (1.1) and (1.2) hold.*

- (1) There is a *unique* representation $\rho = \rho^{\psi, \pi}$ of $\mathcal{L}(X)$ on $\mathcal{H}$ with essential subspace $\overline{\psi(X)\mathcal{H}} := \overline{\text{span}}\{\psi(x)h : x \in X, h \in \mathcal{H}\}$ such that

$$\rho^{\psi, \pi}(S)(\psi(x)h) = \psi(Sx)h \quad \text{for } S \in \mathcal{L}(X), x \in X, \text{ and } h \in \mathcal{H},$$

and we then have $\rho(\Theta_{x,y}) = \psi(x)\psi(y)^*$.

- (2) If $\mathcal{K}$ is a subspace of $\mathcal{H}$ which is invariant for π , then the subspace $\mathcal{M} = \overline{\psi(X)\mathcal{K}}$ is invariant for ρ . If $\pi|_{\mathcal{K}}$ is faithful, so is $\rho|_{\mathcal{M}}$.

Proof.

(1) The map $(x, h) \mapsto \psi(x)h$ is bilinear, and hence there is a linear map $U : X \odot \mathcal{H} \rightarrow \mathcal{H}$ such that $U(x \otimes h) = \psi(x)h$. Since

$$\begin{aligned} (U(x \otimes h) \mid U(y \otimes k)) &= (\psi(x)h \mid \psi(y)k) = (h \mid \psi(x)^* \psi(y)k) \\ &= (h \mid \pi(\langle x, y \rangle_A)k) = (x \otimes h \mid y \otimes k), \end{aligned}$$

U extends to an isometry from $X \otimes_A \mathcal{H}$ to $\mathcal{H}$ such that $U(x \otimes_A h) = \psi(x)h$. For $S \in \mathcal{L}(X)$ we have

$$U \text{Ind } \pi(S) U^*(\psi(x)h) = U \text{Ind } \pi(S)(x \otimes_A h) = U(Sx \otimes_A h) = \psi(Sx)h,$$

so we can define $\rho := \text{Ad } U \circ \text{Ind } \pi$.

If $x, y, z \in X$, then

$$\rho(\Theta_{x,y})\psi(z) = \psi(x \cdot \langle y, z \rangle_A) = \psi(x)\pi(\langle y, z \rangle_A) = \psi(x)\psi(y)^*\psi(z),$$

so $\rho(\Theta_{x,y})$ and $\psi(x)\psi(y)^*$ agree on $\overline{\psi(X)\mathcal{H}}$. If k is orthogonal to $\overline{\psi(X)\mathcal{H}}$, then $\rho(\Theta_{x,y})k = 0$, so we must show that $\psi(x)\psi(y)^*k = 0$. But this follows from $(\psi(x)\psi(y)^*k \mid h) = (k \mid \psi(y)\psi(x)^*h) = 0$.

(2) The subspace $\mathcal{M}$ is invariant for ρ because $\rho(S)(\psi(x)k) = \psi(Sx)k$. The restriction of U to $X \otimes_A \mathcal{K}$ implements a unitary equivalence between $\text{Ind } \pi|_{X \otimes_A \mathcal{K}}$ and $\rho|_{\mathcal{M}}$; since the first of these is equivalent to $\text{Ind } (\pi|_{\mathcal{K}})$, it is faithful if $\pi|_{\mathcal{K}}$ is, and hence so is $\rho|_{\mathcal{M}}$. $\square$

Remark 1.7. The formula $\rho(\Theta_{x,y}) = \psi(x)\psi(y)^*$ implies that the representation ρ is the canonical extension to $M(\mathcal{K}(X)) = \mathcal{L}(X)$ of the map Pimsner would call $\pi^{(1)}$; see [23, page 202]. (We have avoided the notation $\pi^{(1)}$ because the map depends on both ψ and π .) For a representation (ψ, π) of X in a C^* -algebra B , we can represent B on a Hilbert space and apply the Proposition to obtain a homomorphism $\rho^{\psi, \pi} : \mathcal{K}(X) \rightarrow B$, but it need not extend canonically to $\mathcal{L}(X)$.

Proposition 1.8. Let A and B be C^* -algebras, let X and Y be Hilbert bimodules over A , and suppose that $\pi : A \rightarrow B$ is a homomorphism which forms part of Toeplitz representations (ψ, π) and (μ, π) of X and Y in B .

- (1) *There is a linear map $\psi \otimes_A \mu$ of the internal tensor product $X \otimes_A Y$ into B which satisfies*

$$(1.5) \quad \psi \otimes_A \mu(x \otimes_A y) = \psi(x)\mu(y), \quad x \in X, y \in Y,$$

and $(\psi \otimes_A \mu, \pi)$ is a Toeplitz representation of $X \otimes_A Y$.

- (2) *Suppose $B = B(\mathcal{H})$. Denote by $S \mapsto S \otimes_A 1$ the canonical homomorphism of $\mathcal{L}(X)$ into $\mathcal{L}(X \otimes_A Y)$ given by the left action of $\mathcal{L}(X)$ on X , and let $P_{\psi \otimes_A \mu}$ be the projection of $\mathcal{H}$ onto $\overline{\psi \otimes_A \mu(X \otimes_A Y)(\mathcal{H})}$. Then the representations $\rho^{\psi, \pi}$ and $\rho^{\psi \otimes_A \mu, \pi}$ of Proposition 1.6 are related by*

$$\rho^{\psi \otimes_A \mu, \pi}(S \otimes_A 1) = \rho^{\psi, \pi}(S)P_{\psi \otimes_A \mu} \quad \text{for } S \in \mathcal{L}(X).$$

Proof. Since $(x, y) \mapsto \psi(x)\mu(y)$ is bilinear, it induces a linear map $\psi \odot \mu$ on the algebraic tensor product $X \odot Y$. For any $x, z \in X$ and $y, w \in Y$ we have

$$(1.6) \quad \begin{aligned} \mu(y)^* \psi(x)^* \psi(z) \mu(w) &= \mu(y)^* \pi(\langle x, z \rangle_A) \mu(w) \\ &= \mu(y)^* \mu(\langle x, z \rangle_A \cdot w) \\ &= \pi(\langle y, \langle x, z \rangle_A \cdot w \rangle_A) \\ &= \pi(\langle x \otimes_A y, z \otimes_A w \rangle_A). \end{aligned}$$

Thus for $v = \sum_i x_i \otimes y_i \in X \odot Y$ we have

$$\begin{aligned} \|\psi \odot \mu(v)\|^2 &= \|\psi \odot \mu(v)^* \psi \odot \mu(v)\| = \left\| \sum_{i,j} \mu(y_i)^* \psi(x_i)^* \psi(x_j) \mu(y_j) \right\| \\ &= \left\| \pi \left(\sum_{i,j} \langle x_i \otimes_A y_i, x_j \otimes_A y_j \rangle_A \right) \right\| \\ &\leq \left\| \sum_{i,j} \langle x_i \otimes_A y_i, x_j \otimes_A y_j \rangle_A \right\| = \|v\|^2, \end{aligned}$$

so $\psi \odot \mu$ induces a contractive linear map $\psi \otimes_A \mu$ on $X \otimes_A Y$. Routine calculations on elementary tensors show that $(\psi \otimes_A \mu, \pi)$ is a Toeplitz representation of $X \otimes_A Y$.

For part (2), note that the vectors $\overline{\psi \otimes_A \mu(x \otimes_A y)h} = \psi(x)\mu(y)h$ span a dense subspace of the essential subspace $\overline{\psi \otimes_A \mu(X \otimes_A Y)\mathcal{H}}$ of $\rho^{\psi \otimes_A \mu, \pi}$. Thus the calculation

$$\begin{aligned} \rho^{\psi \otimes_A \mu, \pi}(S \otimes_A 1)(\psi(x)\mu(y)h) &= \psi \otimes_A \mu((S \otimes_A 1)(x \otimes_A y))h \\ &= \psi(Sx)\mu(y)h \\ &= \rho^{\psi, \pi}(S)(\psi(x)\mu(y)h) \end{aligned}$$

implies the result. □

2. FAITHFUL REPRESENTATIONS

If (ψ, π) is a Toeplitz representation of a Hilbert bimodule X over A on a Hilbert space $\mathcal{H}$, then the subspace

$$\overline{\psi(X)\mathcal{H}} := \overline{\text{span}\{\psi(x)h : x \in X, h \in \mathcal{H}\}}$$

is invariant for π : $\pi(a)(\psi(x)h) = \psi(a \cdot x)h$. Thus the complement $(\psi(X)\mathcal{H})^\perp$ is also invariant for π . Our first main theorem says that if π is faithful on this complement, then $\psi \times \pi$ is faithful.

Theorem 2.1. *Let X be a Hilbert bimodule over a C^* -algebra A , and let (ψ, π) be a Toeplitz representation of X on a Hilbert space $\mathcal{H}$. If A acts faithfully on $(\psi(X)\mathcal{H})^\perp$, then $\psi \times \pi$ is a faithful representation of $\mathcal{T}_X$. If the homomorphism $\varphi : A \rightarrow \mathcal{L}(X)$ describing the left action of A on X has range in $\mathcal{K}(X)$ and if $\psi \times \pi$ is faithful, then A acts faithfully on $(\psi(X)\mathcal{H})^\perp$.*

Before we prove this theorem we deduce from it that our Toeplitz algebra is isomorphic to Pimsner's. This implies in particular that his algebra is universal for Toeplitz representations [23, Theorem 3.4].

Corollary 2.2. *The Fock representation $T \times \varphi_\infty$ of $\mathcal{T}_X$ is faithful.*

Proof. Let π_0 be a faithful representation of A on $\mathcal{H}$, and consider

$$(\psi, \pi) := ((F(X) - \text{Ind}_A^{\mathcal{L}(F(X))} \pi_0) \circ T, (F(X) - \text{Ind}_A^{\mathcal{L}(F(X))} \pi_0) \circ \varphi_\infty),$$

which is a Toeplitz representation because (T, φ_∞) is. For each $n \geq 0$ and $y \in X^{\otimes n}$, we have $\psi(x)(y \otimes_A h) = (x \otimes_A y) \otimes_A h$; thus

$$\overline{\psi(X)(F(X) \otimes_A \mathcal{H})} = \left(\bigoplus_{n=1}^{\infty} X^{\otimes n} \right) \otimes_A \mathcal{H} \cong \bigoplus_{n=1}^{\infty} (X^{\otimes n} \otimes_A \mathcal{H})$$

has complement $X^{\otimes 0} \otimes_A \mathcal{H} = A \otimes_A \mathcal{H} = \mathcal{H}$. The restriction of π to this subspace is just $A - \text{Ind}_A^A \pi_0 = \pi_0$, which is faithful. Thus Theorem 2.1 says that $\psi \times \pi = (F(X) - \text{Ind}_A^{\mathcal{L}(F(X))} \pi_0) \circ (T \times \varphi_\infty)$ is faithful, and hence $T \times \varphi_\infty$ is too. $\square$

Averaging over the gauge action gives an expectation E of $\mathcal{T}_X$ onto the fixed-point algebra $\mathcal{T}_X^\gamma$:

$$E(b) := \int_{\mathbb{T}} \gamma_w(b) dw \quad \text{for } b \in \mathcal{T}_X.$$

The map E is a positive linear idempotent of norm one, and is faithful on positive elements in the sense that $E(b^*b) = 0 \implies b = 0$. The main step in the proof of Theorem 2.1 is to show that the expectation E is spatially implemented: there is a compatible expectation $E_{\psi, \pi}$ of $\psi \times \pi(\mathcal{T}_X)$ onto $\psi \times \pi(\mathcal{T}_X^\gamma)$.

Proposition 2.3. *Let (ψ, π) be a Toeplitz representation of X such that π is faithful on $(\psi(X)\mathcal{H})^\perp$.*

(1) *There is a norm-decreasing map $E_{\psi, \pi}$ on $\psi \times \pi(\mathcal{T}_X)$ such that*

$$E_{\psi, \pi} \circ (\psi \times \pi) = (\psi \times \pi) \circ E;$$

(2) *$\psi \times \pi$ is faithful on the fixed-point algebra $\mathcal{T}_X^\gamma$.*

Before we try to construct $E_{\psi, \pi}$ we need to understand what E does, and for this we need a description of a dense subalgebra of $\mathcal{T}_X$.

Suppose (ψ, π) is a Toeplitz representation of X in a C^* -algebra B . For $n \geq 1$, Proposition 1.8 gives us a representation $(\psi^{\otimes n}, \pi)$ of the tensor power $X^{\otimes n} := X \otimes_A \cdots \otimes_A X$ such that $\psi^{\otimes n}(x_1 \otimes_A \cdots \otimes_A x_n) = \psi(x_1) \cdots \psi(x_n)$. We define $\psi^{\otimes 0} := \pi$. When $m \geq 1$, $X^{\otimes m} \otimes_A X^{\otimes n} = X^{\otimes(m+n)}$ for every $n \geq 0$, and $\psi^{\otimes m} \otimes_A \psi^{\otimes n} = \psi^{\otimes(m+n)}$. There is a slight subtlety for $m = 0$: the natural map $a \otimes_A x \mapsto a \cdot x$ identifies $X^{\otimes 0} \otimes_A X^{\otimes n} = A \otimes_A X^{\otimes n}$ with the essential submodule $A \cdot X^{\otimes n}$ of $X^{\otimes n}$, and then $\psi^{\otimes 0} \otimes_A \psi^{\otimes n}$ is the restriction of $\psi^{\otimes n}$ to this submodule.

Lemma 2.4. *With the above notation, we have*

$$\mathcal{T}_X = \overline{\text{span}}\{i_X^{\otimes m}(x)i_X^{\otimes n}(y)^* : m, n \geq 0, x \in X^{\otimes m}, y \in X^{\otimes n}\}.$$

The expectation E is given by

$$E(i_X^{\otimes m}(x)i_X^{\otimes n}(y)^*) = \begin{cases} i_X^{\otimes m}(x)i_X^{\otimes n}(y)^* & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. The algebra $\mathcal{T}_X$ is spanned by products of elements $i_X(x)$, $i_A(a)$ and $i_X(y)^*$; given a word in these generators, we can usually absorb $i_A(a)$'s into $i_X(x)$'s, and use $i_X(y)^*i_X(x) = i_A(\langle y, x \rangle_A)$ to cancel any $i_X(y)^*$ appearing to the left of an $i_X(x)$. (This is [23, Lemma 3.1].) Since $\gamma_z(i_X^{\otimes m}(x)i_X^{\otimes n}(y)^*) = z^{m-n}i_X^{\otimes m}(x)i_X^{\otimes n}(y)^*$, the second assertion is easy. $\square$

Lemma 2.4 implies that the image $\psi \times \pi(\mathcal{T}_X)$ is spanned by elements $\psi^{\otimes m}(x)\psi^{\otimes n}(y)^*$ and that $E_{\psi, \pi}$ must satisfy

$$(2.1) \quad E_{\psi, \pi}(\psi^{\otimes m}(x)\psi^{\otimes n}(y)^*) = \begin{cases} \psi^{\otimes m}(x)\psi^{\otimes n}(y)^* & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

We shall show that the formal linear extension $E_{\psi, \pi}$ of the map defined by (2.1) is norm-decreasing, and hence extends to a well-defined norm-decreasing map on $\psi \times \pi(\mathcal{T}_X)$. We analyse the norm of an element $E_{\psi, \pi}(S)$ by showing that the subspaces $\overline{\psi^{\otimes n}(X^{\otimes n})\mathcal{H}}$ form a decreasing chain of reducing subspaces, in which the differences are large enough to see operators in each $\mathcal{L}(X^{\otimes n})$ faithfully.

Lemma 2.5. *Suppose that (ψ, π) is a Toeplitz representation of X on $\mathcal{H}$. For $n \geq 1$, let P_n denote the projection of $\mathcal{H}$ onto $\overline{\psi^{\otimes n}(X^{\otimes n})\mathcal{H}}$, and let $P_0 = 1$. Write ρ_n for the representation $\rho^{\psi^{\otimes n}, \pi}$ of $\mathcal{L}(X^{\otimes n})$ (so that ρ_0 is the extension of π on its essential subspace).*

- (1) *We have $P_n \geq P_{n+1}$ for all $n \geq 0$, so $Q_n := P_n - P_{n+1}$ is also a projection for $n \geq 0$.*
- (2) *For every $n \geq 0$, $k \geq 0$, and $x, y \in X^{\otimes n}$ we have*

$$(2.2) \quad \psi^{\otimes n}(x)P_k = P_{n+k}\psi^{\otimes n}(x), \quad \text{and}$$

$$(2.3) \quad P_k\psi^{\otimes n}(x)\psi^{\otimes n}(y)^* = \psi^{\otimes n}(x)\psi^{\otimes n}(y)^*P_k.$$

- (3) *If π is faithful on $(\psi(X)\mathcal{H})^\perp$, then each ρ_n restricts to a faithful representation of $\mathcal{L}(X^{\otimes n})$ on $Q_n(\mathcal{H})$.*

Proof. For part (1), observe that the vectors $\psi^{\otimes n}(z)\psi(w)h = \psi^{\otimes(n+1)}(z \otimes_A w)h$ span the range of P_{n+1} , and are clearly in the range of P_n .

Equation (2.2) is trivially true for $k = 0$ and/or $n = 0$. If $k \geq 1$, $n \geq 1$, and $w \in X^{\otimes k}$, then

$$\psi^{\otimes n}(x)P_k\psi^{\otimes k}(w) = \psi^{\otimes n}(x)\psi^{\otimes k}(w) = P_{n+k}\psi^{\otimes n}(x)\psi^{\otimes k}(w),$$

so $\psi^{\otimes n}(x)P_k$ and $P_{n+k}\psi^{\otimes n}(x)$ agree on $P_k(\mathcal{H})$. If $f \in P_k(\mathcal{H})^\perp$, then for any $z \in X^{\otimes n}$, $w \in X^{\otimes k}$, and $h \in \mathcal{H}$ we have

$$\begin{aligned} (\psi^{\otimes n}(x)f \mid \psi^{\otimes n}(z)\psi^{\otimes k}(w)h) &= (f \mid \pi(\langle x, z \rangle_A)\psi^{\otimes k}(w)h) \\ &= (f \mid \psi^{\otimes k}(\langle x, z \rangle_A \cdot w)h) = 0, \end{aligned}$$

which implies $P_{n+k}\psi^{\otimes n}(x)f = 0$ because the vectors

$$\psi^{\otimes n}(z)\psi^{\otimes k}(w)h = \psi^{\otimes(n+k)}(z \otimes_A w)h$$

span the range of P_{n+k} . This gives (2.2). When $k < n$, both sides of (2.3) reduce to $\psi^{\otimes n}(x)\psi^{\otimes n}(y)^*$; for $k \geq n$, (2.3) follows from two applications of (2.2).

Part (3) is trivial for $n = 0$. For $n \geq 1$, we apply Proposition 1.6(2): since $\pi|_{(I-P_1)\mathcal{H}}$ is faithful, ρ_n is a faithful representation of $\mathcal{L}(X^{\otimes n})$ on

$$\overline{\psi^{\otimes n}(X^{\otimes n})(1 - P_1)\mathcal{H}}.$$

But this space is precisely $Q_n(\mathcal{H})$, because (2.2) implies that $\psi^{\otimes n}(x)(1 - P_1) = (P_n - P_{n+1})\psi^{\otimes n}(x)$. □

Proof of Proposition 2.3.

(1) We have to prove that for every finite sum

$$S := \sum_j i_X^{\otimes m_j}(x_j) i_X^{\otimes n_j}(y_j)^*$$

we have $\|\psi \times \pi(E(S))\| \leq \|\psi \times \pi(S)\|$; equivalently, we have to prove

$$\left\| \sum_{\{j: m_j = n_j\}} \psi^{\otimes n_j}(x_j) \psi^{\otimes n_j}(y_j)^* \right\| \leq \left\| \sum_j \psi^{\otimes m_j}(x_j) \psi^{\otimes n_j}(y_j)^* \right\|.$$

We know from (2.3) that the projections Q_k commute with every summand in $\psi \times \pi(E(S))$. If $m > k$, we have $Q_k \psi^{\otimes m}(x) = Q_k P_m \psi^{\otimes m}(x) = 0$, and if $m \leq k$ and $n \leq k$, (2.2) gives

$$Q_k \psi^{\otimes m}(x) \psi^{\otimes n}(y)^* Q_k = \psi^{\otimes m}(x) Q_{k-m} Q_{k-n} \psi^{\otimes n}(y)^*,$$

which is 0 unless $m = n$. Let $K := \max n_j$. Then $\rho_K(T \otimes_A 1^{K-n}) = \rho_n(T) P_K$ by Proposition 1.8(2), so we have

$$(2.4) \quad P_K(\psi \times \pi(E(S))) = P_K \rho_K \left(\sum_{\{j: m_j = n_j\}} \Theta_{x_j, y_j} \otimes_A 1^{K-n_j} \right);$$

because $Q_K \rho_K$ is faithful on $\mathcal{L}(X^{\otimes K})$ by the previous lemma, it follows that

$$\|P_K(\psi \times \pi(E(S)))\| = \|Q_K(\psi \times \pi(E(S)))\|.$$

Since $Q_0 + \cdots + Q_{K-1} + P_K = 1$, this gives

$$\begin{aligned} \|\psi \times \pi(E(S))\| &= \sup\{\|Q_k(\psi \times \pi(E(S)))\| : 0 \leq k \leq K\} \\ &= \sup\{\|Q_k(\psi \times \pi(E(S)))Q_k\| : 0 \leq k \leq K\} \\ &= \sup\{\|Q_k(\psi \times \pi(S))Q_k\| : 0 \leq k \leq K\} \\ &\leq \|\psi \times \pi(S)\|. \end{aligned}$$

Thus $E_{\psi, \pi}$ extends to a norm-decreasing map on $\psi \times \pi(\mathcal{T}_X)$, giving (1).

Next let $R := \sum_j i_X^{\otimes n_j}(x_j) i_X^{\otimes n_j}(y_j)^*$ be a typical finite sum in the core $\mathcal{T}_X^\gamma$; such sums are dense because E is continuous and maps finite sums to finite sums. For $k < K := \max n_j$, Proposition 1.8(2) implies that

$$Q_k(\psi \times \pi(R)) = Q_k \rho_k \left(\sum_{\{j: n_j \leq k\}} \Theta_{x_j, y_j} \otimes_A 1^{k-n_j} \right),$$

and hence

$$\|Q_k(\psi \times \pi(R))\| \leq \left\| \sum_{\{j:n_j \leq k\}} \Theta_{x_j, y_j} \otimes_A 1^{k-n_j} \right\|.$$

There is a similar formula for $\|P_K(\psi \times \pi(R))\|$ (see (2.4)), so

$$(2.5) \quad \begin{aligned} \|\psi \times \pi(R)\| &= \max \{ \|P_K(\psi \times \pi(R))\|; \|Q_k(\psi \times \pi(R))\| : 0 \leq k < K \} \\ &\leq \max \left\{ \left\| \sum_{\{j:n_j \leq k\}} \Theta_{x_j, y_j} \otimes_A 1^{k-n_j} \right\| : 0 \leq k \leq K \right\} \end{aligned}$$

for every Toeplitz representation (ψ, π) . Applying this to a faithful representation shows that (2.5) is an upper bound for $\|R\|$.

When π is faithful on $(\psi(X)\mathcal{H})^\perp$, the representations $Q_k\rho_k$ and ρ_K are faithful too, so we actually have

$$(2.6) \quad \|\psi \times \pi(R)\| = \max \left\{ \left\| \sum_{\{j:n_j \leq k\}} \Theta_{x_j, y_j} \otimes_A 1^{k-n_j} \right\| : 0 \leq k \leq K \right\}.$$

In particular, this implies that $\|R\|$ is at least (2.5); since we have already seen that $\|R\|$ is at most (2.5), we deduce that $\|R\| = (2.5)$, and (2.6) implies that $\psi \times \pi$ is isometric on the core. $\square$

Proof of Theorem 2.1. Suppose π is faithful on $(\psi(X)\mathcal{H})^\perp$ and $S \in \ker \psi \times \pi$. Then by Proposition 2.3(1) we have $\psi \times \pi(E(S^*S)) = E_{\psi, \pi}(\psi \times \pi(S^*S)) = 0$, which by Proposition 2.3(2) implies that $E(S^*S) = 0$. Because E is faithful, this forces $S^*S = 0$ and $S = 0$.

Now suppose that $\varphi(A) \subset \mathcal{K}(X)$. Proposition 1.6 gives a homomorphism $\rho^{i_X, i_A} : \mathcal{K}(X) \rightarrow \mathcal{T}_X$ (see Remark 1.7), and we claim that, for any Toeplitz representation (ψ, π) ,

$$(2.7) \quad \psi \times \pi(i_A(a) - \rho^{i_X, i_A}(\varphi(a))) = \pi(a)(1 - P_1) = \pi(a)|_{(\psi(X)\mathcal{H})^\perp}.$$

For any rank-one operator $\Theta_{x, y}$ we have

$$\psi \times \pi(\rho^{i_X, i_A}(\Theta_{x, y})) = \psi \times \pi(i_X(x)i_X(y)^*) = \psi(x)\psi(y)^* = \rho^{\psi, \pi}(\Theta_{x, y}),$$

and hence $(\psi \times \pi) \circ \rho^{i_X, i_A} = \rho^{\psi, \pi}$ on $\mathcal{K}(X)$. On the other hand, since $\rho^{\psi, \pi}(\varphi(a))$ agrees with $\pi(a)$ on $\psi(X)\mathcal{H}$, we have $\rho^{\psi, \pi}(\varphi(a)) = \pi(a)P_1$. These two observations give the claim (2.7).

Since there are Toeplitz representations (ψ, π) in which π is faithful on $(\psi(X)\mathcal{H})^\perp$ (for example, the Fock representation induced from a faithful representation of A) and $\psi \times \pi$ is then faithful, (2.7) implies that $\alpha : a \mapsto i_A(a) - \rho^{i_X, i_A}(\varphi(a))$ is an injective homomorphism of A into $\mathcal{T}_X$. (Warning: it is crucial

here that $\varphi(A) \subset \mathcal{K}(X)$.) Thus if $\psi \times \pi$ is faithful, so is the composition with α , and (2.7) gives the result. $\square$

3. DIRECT SUMS OF HILBERT BIMODULES

If $\{X^\lambda : \lambda \in \Lambda\}$ is a family of Hilbert bimodules over the same C^* -algebra A , then the algebraic direct sum X_0 is a pre-Hilbert A -module with $(x_\lambda) \cdot a := (x_\lambda \cdot a)$ and $\langle (x_\lambda), (y_\lambda) \rangle_A := \sum_\lambda \langle x_\lambda, y_\lambda \rangle_A$. We can therefore complete X_0 to obtain a Hilbert A -module X , which we denote by $\bigoplus_{\lambda \in \Lambda} X^\lambda$ (see [25, Lemma 2.16]). There is a left action of A on X_0 defined by $a \cdot (x_\lambda) := (a \cdot x_\lambda)$, which we claim extends to an action of A by adjointable operators on $\bigoplus X^\lambda$. To see this, note that the left action of A on each X^λ satisfies $\langle a \cdot x_\lambda, a \cdot x_\lambda \rangle_A \leq \|a\|^2 \langle x_\lambda, x_\lambda \rangle_A$, and since the sum of positive elements is positive, we deduce that

$$\langle a \cdot (x_\lambda), a \cdot (x_\lambda) \rangle_A \leq \|a\|^2 \left(\sum_\lambda \langle x_\lambda, x_\lambda \rangle_A \right) = \|a\|^2 \langle (x_\lambda), (x_\lambda) \rangle_A.$$

Thus the map $(x_\lambda) \mapsto a \cdot (x_\lambda)$ is bounded for the norm on X_0 induced by $\langle \cdot, \cdot \rangle_A$, and extends to a map on all of X , which is adjointable with adjoint $(x_\lambda) \mapsto a^* \cdot (x_\lambda)$, as claimed. We have now shown that $X = \bigoplus_{\lambda \in \Lambda} X^\lambda$ is itself a Hilbert bimodule over A , which we call the *direct sum* of the Hilbert bimodules X^λ .

Theorem 3.1. *Let $\{X^\lambda : \lambda \in \Lambda\}$ be a family of Hilbert bimodules over a C^* -algebra A , let $X := \bigoplus_{\lambda \in \Lambda} X^\lambda$, and let (ψ, π) be a Toeplitz representation of X on a Hilbert space $\mathcal{H}$. If A acts faithfully on $(\psi(\bigoplus_{\lambda \in F} X^\lambda)\mathcal{H})^\perp$ for every finite subset F of Λ , then $\psi \times \pi$ is faithful on $\mathcal{T}_X$. If A acts by compact operators on the left of each X^λ and if $\psi \times \pi$ is faithful, then π acts faithfully on every $(\psi(\bigoplus_{\lambda \in F} X^\lambda)\mathcal{H})^\perp$.*

The proof of this Theorem exploits a grading of $\mathcal{T}_X$ by the free group $\mathbb{F}_\Lambda$ on Λ : picking off the e -graded piece gives an expectation E^Λ which goes further into the core $\mathcal{T}_X^\gamma$ than the expectation E used in Section 2. Such gradings are usually formalised in terms of a coaction of $\mathbb{F}_\Lambda$ on $\mathcal{T}_X$, but because $\mathbb{F}_\Lambda$ is not amenable, it would not be obvious from such a formalisation that the associated expectation E^Λ is faithful (see, for example, [18, Section 4]). Here we shall construct the expectation directly using the Fock representation of $\mathcal{T}_X$, which we know is faithful by Corollary 2.2.

First we need some notation. Let $\mathbb{F}_\Lambda^+$ be the subsemigroup of $\mathbb{F}_\Lambda$ generated by Λ and the identity e . For $s, t \in \mathbb{F}_\Lambda^+$, we write $s \leq t$ if t has the form sr for some $r \in \mathbb{F}_\Lambda^+$, and we define

$$s \vee t := \begin{cases} t & \text{if } s \leq t, \\ s & \text{if } t \leq s, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

(The pair $(\mathbb{F}_\Lambda, \mathbb{F}_\Lambda^+)$ is an example of a *quasi-lattice ordered group* ([22, 18]): the subsemigroup defines a left-invariant partial order on $\mathbb{F}_\Lambda$ in which $s \leq t$ if and only if $s^{-1}t \in \mathbb{F}_\Lambda^+$, and, loosely speaking, every finite bounded subset has a least upper bound.)

For a reduced word $s = \lambda_1 \cdots \lambda_n$ in $\mathbb{F}_\Lambda^+ \setminus \{e\}$, we write $|s| := n$. We can identify the tensor power $X^s := X^{\lambda_1} \otimes_A \cdots \otimes_A X^{\lambda_n}$ with a submodule of $X^{\otimes n}$. If (ψ, π) is a Toeplitz representation of X , we can define $\psi^\lambda := \psi|_{X^\lambda}$ and $\psi^s := \psi^{\lambda_1} \otimes_A \cdots \otimes_A \psi^{\lambda_n}$, and then (ψ^s, π) is a Toeplitz representation of X^s by Proposition 1.8. The associativity of $\otimes_A$ gives an isomorphism of $X^s \otimes_A X^t$ onto X^{st} which carries $\psi^s \otimes_A \psi^t$ into ψ^{st} , and that ψ^s agrees with the restriction of $\psi^{\otimes |s|}$ to $X^s \subseteq X^{\otimes |s|}$.

Proposition 3.2. *Let (ψ, π) be a Toeplitz representation of X in a C^* -algebra.*

- (1) *Suppose $s, t \in \mathbb{F}_\Lambda^+$ and $s \leq t$. Then for every $x, y_1 \in X^s$ and $y_2 \in X^{s^{-1}t}$ we have $\psi^s(x)^* \psi^t(y_1 \otimes_A y_2) = \psi^{s^{-1}t}(\langle x, y_1 \rangle_A \cdot y_2)$.*
- (2) *Suppose $s, t \in \mathbb{F}_\Lambda^+$ and $s \vee t = \infty$. Then for every $x \in X^s$ and $y \in X^t$ we have $\psi^s(x)^* \psi^t(y) = 0$.*
- (3) *$\psi \times \pi(\mathcal{T}_X) = \overline{\text{span}}\{\psi^s(x) \psi^t(y)^* : x \in X^s, y \in X^t, s, t \in \mathbb{F}_\Lambda^+\}$.*
- (4) *There is a norm-decreasing linear map E^Λ on $\mathcal{T}_X$ which satisfies*

$$E^\Lambda(i_X^s(x) i_X^t(y)^*) = \begin{cases} i_X^s(x) i_X^t(y)^* & \text{if } s = t \text{ in } \mathbb{F}_\Lambda^+, \\ 0 & \text{otherwise,} \end{cases}$$

and which is faithful on positive elements.

Proof.

Part (1) is a straightforward computation.

For (2), let r be the longest common initial segment in s and t , so that $s = r\lambda s_1$ and $t = r\mu t_1$ for $r, s_1, t_1 \in \mathbb{F}_\Lambda^+$ and $\lambda \neq \mu \in \Lambda$. Then $X^{r\lambda}$ and $X^{r\mu}$ are orthogonal submodules of $X^{\otimes(|r|+1)}$. Since vectors of the form $x \otimes_A y \in X^{r\lambda} \otimes_A X^{s_1}$ span X^s and similarly for X^t , the calculation

$$\begin{aligned} \psi^s(x \otimes y)^* \psi^t(w \otimes z) &= \psi^{s_1}(y)^* \psi^{\otimes(|r|+1)}(x)^* \psi^{\otimes(|r|+1)}(w) \psi^{t_1}(z) \\ &= \psi^{s_1}(y)^* \pi(\langle x, w \rangle_A) \psi^{t_1}(z) = 0 \end{aligned}$$

implies (2).

For (3), we show that $C := \overline{\text{span}}\{\psi^s(x)\psi^t(y)^*\}$ is a C^* -subalgebra of $\psi \times \pi(\mathcal{T}_X)$ which contains $\psi(X)$ and $\pi(A)$. It is clearly closed under taking adjoints. To see that it is a subalgebra, consider $\psi^s(x)\psi^t(y)^*$ and $\psi^u(z)\psi^v(w)^*$. Part (2) implies that $\psi^t(y)^*\psi^u(z) = 0$ if $t \vee u = \infty$. Otherwise, (1) implies that $\psi^t(y)^*\psi^u(z)$ has the form $\psi^{t^{-1}u}(z')$ (if $t \leq u$) or $\psi^{u^{-1}t}(y')^*$ (if $u \leq t$). Absorbing this element into either $\psi^s(x)$ or $\psi^v(w)^*$ shows that the product $\psi^s(x)\psi^t(y)^*\psi^u(z)\psi^v(w)^*$ belongs to C .

Since X is essential as a right A -module, every element has the form $y \cdot a$ for $y \in X$ and $a \in A$. Approximating y by a finite sum of the form $\sum y_\lambda$ shows that $\psi(y \cdot a) = \psi(y)\pi(a) \sim \sum_\lambda \psi^\lambda(y_\lambda)\psi^e(a^*)^*$ belongs to C . Similarly, writing an arbitrary element of A as bc^* shows that $\pi(bc^*) = \psi^e(b)\psi^e(c)^* \in C$.

(4) Part (2) implies that the subspaces X^s of $X^{\otimes n}$ corresponding to different words of length n are orthogonal; thus the natural map is an isomorphism of the Fock bimodule $\bigoplus_{s \in \mathbb{F}_\Lambda^+} X^s$ onto $F(X)$. For $r \in \mathbb{F}_\Lambda^+$, let R_r be the orthogonal projection of $F(X)$ onto X^r . Then for each $S \in \mathcal{L}(F(X))$, the sum $\sum_{r \in \mathbb{F}_\Lambda^+} R_r S R_r$ converges $*$ -strongly to an adjointable operator $\Phi(S)$; the resulting linear mapping Φ on $\mathcal{L}(F(X))$ is idempotent, norm-decreasing, and faithful on positive operators. Let $T \times \varphi_\infty$ be the Fock representation of $\mathcal{T}_X$, which is faithful by Corollary 2.2. We want to define $E^\Lambda := (T \times \varphi_\infty)^{-1} \circ \Phi \circ (T \times \varphi_\infty)$; before we can do this, we need to know that Φ leaves the range of $T \times \varphi_\infty$ invariant. Both this and the formula in (4) will follow if we can show that

$$(3.1) \quad \Phi(T^s(x)T^t(y)^*) = \begin{cases} T^s(x)T^t(y)^* & \text{if } s = t \text{ in } \mathbb{F}_\Lambda^+, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in X^s$ and $y \in X^t$, and note that X^r is spanned by vectors of the form $T^r(z)a$, where $a \in A = X^{\otimes 0}$. If $t \vee r = \infty$, then (2) gives $T^s(x)T^t(y)^*T^r(z)a = 0$. If $r < t$, then (1) implies that $T^t(y)^*T^r(z)a = T^{r^{-1}t}(y')^*a$ for some $y' \in X^{r^{-1}t}$, and this vanishes because $T^{r^{-1}t}(y')^*$ kills $A = X^{\otimes 0} \subset F(X)$. If $t \leq r$, then $T^t(y)^*T^r(z) = T^{t^{-1}r}(z')$ for some $z' \in X^{t^{-1}r}$, and $T^s(x)T^t(y)^*T^r(z)a = T^{st^{-1}r}(x \otimes z')a \in X^{st^{-1}r}$. Thus

$$R_r T^s(x)T^t(y)^* R_r = \begin{cases} T^s(x)T^t(y)^* R_r & \text{if } s = t \text{ in } \mathbb{F}_\Lambda^+, \\ 0 & \text{otherwise,} \end{cases}$$

and summing over $r \in \mathbb{F}_\Lambda^+$ gives (3.1). $\square$

Now suppose that (ψ, π) is a Toeplitz representation of X on $\mathcal{H}$. As in the previous section, we aim to show that if π satisfies the hypothesis of Theorem 3.1, then the expectation E^Λ is spatially implemented. The analogues of

the projections P_n are the projections P_s onto the subspaces $\overline{\psi^s(X^s)\mathcal{H}}$, and in the next Lemma we write down some of their properties. The analogues of the projections $Q_n = P_n - P_{n+1}$ are the projections Q_s^F described in Lemma 3.4, which is based on [18, Lemma 1.4]; in Lemma 3.5 we show that Q_s^F is large enough to see $\mathcal{L}(X^s)$ faithfully.

Lemma 3.3. *Let (ψ, π) be a Toeplitz representation of X on $\mathcal{H}$. For $s \in \mathbb{F}_\Lambda^+$, denote by ρ_s the representation $\rho^{\psi^s, \pi} : \mathcal{L}(X^s) \rightarrow B(\mathcal{H})$, and let P_s be the projection of $\mathcal{H}$ onto $\overline{\psi^s(X^s)\mathcal{H}}$; take $P_e = 1$ and $P_\infty = 0$.*

(1) *We have $P_s P_t = P_{s \vee t}$ for $s, t \in \mathbb{F}_\Lambda^+$.*

(2) *For $s, t \in \mathbb{F}_\Lambda^+$ and $x, y \in X^s$, we have*

$$(3.2) \quad \psi^s(x) P_t = P_{st} \psi^s(x), \quad \text{and}$$

$$(3.3) \quad P_t \psi^s(x) \psi^s(y)^* = \psi^s(x) \psi^s(y)^* P_t.$$

The proofs are like those of Part (2) of Lemma 2.5; the orthogonality of P_s and P_t when $s \vee t = \infty$ follows from Proposition 3.2(2).

Lemma 3.4. *Let F be a finite subset of $\mathbb{F}_\Lambda^+$ such that $e \in F$. For $s \in F$, let*

$$Q_s^F := P_s \left(\prod_{\{t \in F : s < t\}} (1 - P_t) \right).$$

Then $1 = \sum_{s \in F} Q_s^F$.

Proof. We proceed by induction on $|F|$. If $|F| = 1$, then $F = \{e\}$, and $Q_e^F = P_e = 1$. If $|F| \geq 2$, we remove a maximal element c from F , and apply the inductive hypothesis to $G := F \setminus \{c\}$. There is a unique longest word $b \in G$ such that $b < c$. We claim that only the summand Q_b^G in the decomposition $1 = \sum_{s \in G} Q_s^G$ is changed by adding c to G ; in other words, we claim that $Q_s^F = Q_s^G$ for $s \neq b$. Suppose $s \in G \setminus \{b\}$. Then Q_s^F and Q_s^G have the same factors except for an extra $1 - P_c$ in Q_s^F when $s < c$. But $s < c$ implies $s < b$, because b is the longest word in G with $b < c$, and $P_b P_c = P_c$ by Lemma 3.3(1); thus $1 - P_b = (1 - P_b)(1 - P_c)$ and $Q_s^F = Q_s^G$, as claimed.

We now have $\sum_{s \in F} Q_s^F = \sum_{s \in G \setminus \{b\}} Q_s^G + Q_b^F + Q_c^F$, and it suffices to show that $Q_b^G = Q_b^F + Q_c^F$. If $t \in G$ and $b < t$, the maximality of b implies that $c \vee t = \infty$, and hence $P_c P_t = 0$ by Lemma 3.3(1). Thus

$$\begin{aligned} Q_b^G &= P_b(1 - P_c) \left(\prod_{\{t \in G : b < t\}} (1 - P_t) \right) + P_b P_c \left(\prod_{\{t \in G : b < t\}} (1 - P_t) \right) \\ &= P_b \left(\prod_{\{t \in F : b < t\}} (1 - P_t) \right) + P_b P_c \\ &= Q_b^F + P_c = Q_b^F + Q_c^F, \end{aligned}$$

as required. $\square$

Lemma 3.5. *Suppose (ψ, π) is a Toeplitz representation such that A acts faithfully on $(\psi(\bigoplus_{\lambda \in F} X^\lambda)\mathcal{H})^\perp$ for every finite subset F of Λ .*

- (1) *Let G be a finite subset of $\mathbb{F}_\Lambda^+ \setminus \{e\}$, and let $s \in \mathbb{F}_\Lambda^+$. Then $\rho_s := \rho^{\psi^s, \pi}$ is a faithful representation of $\mathcal{L}(X^s)$ on $P_s \prod_{t \in G} (1 - P_{st})\mathcal{H}$.*
- (2) *If F is a finite subset of $\mathbb{F}_\Lambda^+$ with $e \in F$, then for each $s \in F$, ρ_s is a faithful representation of $\mathcal{L}(X^s)$ on $Q_s^F \mathcal{H}$.*

Proof.

(1) Each $t \in G$ has a unique decomposition $t = \lambda_t r$ with $\lambda_t \in \Lambda$ and $r \in \mathbb{F}_\Lambda^+$; write $G' := \{\lambda_t : t \in G\}$. Lemma 3.3(1) implies that the projections P_λ for $\lambda \in \Lambda$ are mutually orthogonal, so $\overline{\psi(\bigoplus_{\lambda \in G'} X^\lambda)\mathcal{H}} = \bigoplus_{\lambda \in G'} P_\lambda \mathcal{H}$, and our hypothesis says that π is faithful on the range of $1 - \sum_{\lambda \in G'} P_\lambda = \prod_{\lambda \in G'} (1 - P_\lambda)$. But $P_t \leq P_{\lambda_t}$ for each t , so $\prod_{t \in G} (1 - P_t) \geq \prod_{\lambda \in G'} (1 - P_\lambda)$, and π is also faithful on $\prod_{t \in G} (1 - P_t)\mathcal{H}$. Now Proposition 1.6(2) implies that ρ_s is faithful on

$$\mathcal{M}_s := \overline{\text{span}} \left\{ \psi^s(x) \left(\prod_{t \in G} (1 - P_t) h \right) : x \in X^s, h \in \mathcal{H} \right\},$$

which by (3.2) is precisely $\prod_{t \in G} (1 - P_{st}) P_s \mathcal{H}$, at least for $s \neq e$. When $s = e$, $\mathcal{M}_e$ is a subspace of $\prod_{t \in G} (1 - P_t)\mathcal{H}$, and the result follows.

- (2) Apply (1) with $G := \{s^{-1}t : t \in F, s < t\}$. $\square$

We can now construct our spatial implementation of the expectation E^Λ .

Proposition 3.6. *Suppose (ψ, π) is a Toeplitz representation of $\bigoplus X^\lambda$ such that A acts faithfully on $(\psi(\bigoplus_{\lambda \in F} X^\lambda)\mathcal{H})^\perp$ for every finite subset F of Λ .*

- (1) *There is a norm-decreasing linear map $E_{\psi, \pi}^\Lambda$ on $\psi \times \pi(\mathcal{T}_X)$ such that*

$$E_{\psi, \pi}^\Lambda \circ (\psi \times \pi) = (\psi \times \pi) \circ E^\Lambda;$$

- (2) *$\psi \times \pi$ is faithful on $E^\Lambda(\mathcal{T}_X)$.*

Proof. (1) We show that for each finite sum $S := \sum_j i_X^{s_j}(x_j) i_X^{t_j}(y_j)^*$, we have

$$\left\| \sum_{\{j: s_j = t_j\}} \psi^{s_j}(x_j) \psi^{s_j}(y_j)^* \right\| \leq \left\| \sum_j \psi^{s_j}(x_j) \psi^{t_j}(y_j)^* \right\|;$$

then the map $E_{\psi, \pi}^\Lambda : \psi \times \pi(S) \mapsto \psi \times \pi(E^\Lambda(S))$ extends to a well-defined norm-decreasing map on $\psi \times \pi(\mathcal{T}_X)$ with the required properties.

Let $F := \{e\} \cup \{s_j\} \cup \{t_j\}$. Equation (3.3) implies that the projections P_s and Q_s^F commute with every summand in $\psi \times \pi(E^\Lambda(S))$; it follows from Lemma 3.4 that there exists $c \in F$ such that

$$\|\psi \times \pi(E^\Lambda(S))\| = \|Q_c^F(\psi \times \pi(E^\Lambda(S)))\|.$$

If $t \in F$ with $c < t$, then $Q_c^F \psi^t(x) = Q_c^F(1 - P_t)P_t \psi^t(x) = 0$, and if $c \vee t = \infty$, then $Q_c^F \psi^t(x) = Q_c^F P_c P_t \psi^t(x) = 0$; thus compressing by Q_c^F kills all summands in $\psi \times \pi(S)$ except possibly those for which $s_j \leq c$ and $t_j \leq c$. As in the proof of Proposition 2.3, it follows from Proposition 1.8(2) that

$$Q_c^F(\psi \times \pi(E^\Lambda(S))) = Q_c^F \rho_c \left(\sum_{\{j: s_j = t_j \leq c\}} \Theta_{x_j, y_j} \otimes_A 1^{s_j^{-1}c} \right),$$

and from Lemma 3.5(2) that

$$\|\psi \times \pi(E^\Lambda(S))\| = \left\| \sum_{\{j: s_j = t_j \leq c\}} \Theta_{x_j, y_j} \otimes_A 1^{s_j^{-1}c} \right\|.$$

The idea now is to replace Q_c^F by a smaller projection Q , in such a way that compressing by Q kills the remaining off-diagonal terms of $Q_c^F(\psi \times \pi(S))Q_c^F$ but still preserves the norm of $\psi \times \pi(E^\Lambda(S))$.

For each $s, t \in F$ such that $s \neq t$, $s, t \leq c$ and $s^{-1}c \vee t^{-1}c < \infty$, we define $d_{s,t} \in \mathbb{F}_\Lambda^+$ as in [18, Lemma 3.2]:

$$d_{s,t} = \begin{cases} (s^{-1}c)^{-1}(t^{-1}c) & \text{if } s^{-1}c < t^{-1}c \\ (t^{-1}c)^{-1}(s^{-1}c) & \text{if } t^{-1}c < s^{-1}c, \end{cases}$$

noting in particular that $d_{s,t}$ is never the identity in $\mathbb{F}_\Lambda^+$. Let

$$G := \{c^{-1}t : t \in F, c < t\} \cup \{d_{s,t}\},$$

and define $Q := P_c \prod_{t \in G} (I - P_{ct})$. Notice that we have added factors to the formula for Q_c^F , so $Q \leq Q_c^F$.

To see that Q has the required properties, fix $s, t \in F$ satisfying $s \neq t$, $s \leq c$, and $t \leq c$. Then from (3.2) we have

$$Q\psi^s(x)\psi^t(y)^*Q = QP_c\psi^s(x)\psi^t(y)^*P_cQ = Q\psi^s(x)P_{s^{-1}c}P_{t^{-1}c}\psi^t(y)^*Q,$$

which certainly vanishes if $s^{-1}c \vee t^{-1}c = \infty$. But if $s^{-1}c \vee t^{-1}c < \infty$, then $Q \leq P_c - P_{cd_{s,t}}$, so

$$\begin{aligned} Q\psi^s(x)\psi^t(y)^*Q &= Q(P_c - P_{cd_{s,t}})\psi^s(x)\psi^t(y)^*(P_c - P_{cd_{s,t}})Q \\ &= Q\psi^s(x)(P_{s^{-1}c} - P_{s^{-1}cd_{s,t}})(P_{t^{-1}c} - P_{t^{-1}cd_{s,t}})\psi^t(y)^*Q, \end{aligned}$$

which vanishes because either $s^{-1}cd_{s,t} = t^{-1}c$ or $t^{-1}cd_{s,t} = s^{-1}c$. We deduce that

$$Q(\psi \times \pi(S))Q = Q\rho_c\left(\sum_{\{j:s_j=t_j \leq c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\right).$$

Since $Q\rho_c$ is faithful by Lemma 3.5(1), we have

$$\begin{aligned} \|\psi \times \pi(E^\Lambda(S))\| &= \left\| \sum_{\{j:s_j=t_j \leq c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c} \right\| \\ &= \left\| Q\rho_c\left(\sum_{\{j:s_j=t_j \leq c\}} \Theta_{x_j,y_j} \otimes_A 1^{s_j^{-1}c}\right) \right\| \\ &= \|Q(\psi \times \pi(S))Q\| \\ &\leq \|\psi \times \pi(S)\|, \end{aligned}$$

giving (1).

Applying the argument of Proposition 2.3(2) to the partition $\{Q_s^F\}$ of 1 gives (2). $\square$

Proof of Theorem 3.1. The first part follows from Proposition 3.6 just as Theorem 2.1 follows from Proposition 2.3. Suppose A acts by compact operators on each summand X^λ . Then A acts by compact operators on $\bigoplus_{\lambda \in F} X^\lambda$ for any finite set F of indices, giving maps $\varphi_F : A \rightarrow \mathcal{K}(X)$. An argument like that in the proof of Theorem 2.1 shows that

$$\psi \times \pi(i_A(a) - \rho^{i_X, i_A}(\varphi_F(a))) = \pi(a)|_{(1 - \sum_{\lambda \in F} P_\lambda)\mathcal{H}}.$$

Applying this with (ψ, π) satisfying the hypothesis of the first part implies that $\alpha_F : a \mapsto i_A(a) - \rho^{i_X, i_A}(\varphi_F(a))$ is an injection of A in $\mathcal{T}_X$. If now (ψ, π) is a Toeplitz representation for which $\psi \times \pi$ is faithful, then composing with α_F shows that the hypothesis is necessary. $\square$

4. THE TOEPLITZ ALGEBRA OF A DIRECTED GRAPH

Let $E = (E^0, E^1, r, s)$ be a directed graph and $X(E)$ the Hilbert bimodule over $A = c_0(E^0)$ discussed in Example 1.2. Recall that $X(E)$ consists of functions on the edge set E^1 , and that $X(E)$ and A are spanned by point masses $\{\delta_f : f \in E^1\}$ and $\{\delta_v : v \in E^0\}$, respectively.

Theorem 4.1. *The Toeplitz algebra $\mathcal{T}_{X(E)}$ is generated by a Toeplitz-Cuntz-Krieger E -family $\{i_X(\delta_f), i_A(\delta_v) : f \in E^1, v \in E^0\}$. It is universal for such families: if $\{S_f, P_v\}$ is a Toeplitz-Cuntz-Krieger E -family on a Hilbert space $\mathcal{H}$, there is a representation $\pi^{S,P} : \mathcal{T}_{X(E)} \rightarrow B(\mathcal{H})$ such that $\pi^{S,P}(i_X(\delta_f)) = S_f$ and*

$\pi^{S,P}(i_A(\delta_v)) = P_v$. The representation $\pi^{S,P}$ is faithful if and only if every P_v is nonzero (and hence every S_f is nonzero), and

$$P_v > \sum_{\{f \in E^1: s(f)=v\}} S_f S_f^*$$

for every vertex v which emits at most finitely many edges.

Proof. Write $X := X(E)$. We proved in Example 1.2 that $\{\psi(\delta_f), \pi(\delta_v)\}$ is a Toeplitz-Cuntz-Krieger E -family for any Toeplitz representation (ψ, π) , and this applies in particular to the canonical representation (i_X, i_A) in $\mathcal{T}_X$. The family generates $\mathcal{T}_X$ because $i_X(X)$ and $i_A(A)$ do, because δ_f and δ_v span dense subspaces of X and A , and because i_X and i_A are isometric. We saw in Example 1.2 how the family $\{S_f, P_v\}$ generates a Toeplitz representation (ψ, π) with $\psi(\delta_f) = S_f$ and $\pi(\delta_v) = P_v$, so $\pi^{S,P} := \psi \times \pi$ has the required property.

For the final statement, we apply Theorem 3.1. For each $f \in E^1$, we let X_f be the bimodule $\mathbb{C}$ in which $a \cdot z = a(s(f))z$, $z \cdot a = za(r(f))$ and $\langle z, w \rangle_A = \bar{z}w\delta_{r(f)}$, and note that $(z_f) \mapsto \sum_f z_f \delta_f$ induces an isomorphism of $\bigoplus_{f \in E^1} X_f$ onto X . (It is easy to check on the algebraic direct sum that the map is a bimodule homomorphism which preserves the inner products.) Since $\mathcal{K}(X_f) = \mathcal{L}(X_f)$ for each f , A acts by compact operators on each X_f , and Theorem 3.1 says that $\pi^{S,P}$ is faithful if and only if A acts faithfully on each $(\bigoplus_{f \in F} \mathcal{H}_f)^\perp$, where $\mathcal{H}_f = \pi^{S,P}(i_X(\delta_f)\mathcal{H}) = S_f\mathcal{H}$. The action of $A = c_0(E^0)$ on any space is faithful iff every δ_v acts nontrivially, so A acts faithfully on $(\bigoplus_{f \in F} \mathcal{H}_f)^\perp$ if and only if

$$0 \neq P_v \left(1 - \sum_{f \in F} S_f S_f^*\right) = P_v - \sum_{\{f \in F: s(f)=v\}} S_f S_f^*.$$

If each P_w is nonzero and v emits infinitely many edges, this holds since $P_v \geq \sum_{\{f \in E^1: s(f)=v\}} S_f S_f^*$, so the result follows. $\square$

Corollary 4.2. *Let E be a directed graph, and suppose that $\{S_f, P_v\}$ and $\{T_f, Q_v\}$ are Toeplitz-Cuntz-Krieger E -families such that each P_v and Q_v is nonzero, and such that*

$$P_v > \sum_{\{f \in E^1: s(f)=v\}} S_f S_f^* \quad \text{and} \quad Q_v > \sum_{\{f \in E^1: s(f)=v\}} T_f T_f^*$$

for every vertex v which emits at most finitely many edges. Then there is an isomorphism θ of $C^*(S_f, P_v)$ onto $C^*(T_f, Q_v)$ such that $\theta(S_f) = T_f$ for all $f \in E^1$ and $\theta(P_v) = Q_v$ for all $v \in E^0$.

Proof. Take $\theta := \pi^{T,Q} \circ (\pi^{S,P})^{-1}$. $\square$

Corollary 4.3. *Let E be a directed graph with at least one edge. Then $\mathcal{T}_{X(E)}$ is simple if and only if every vertex emits infinitely many edges and every pair of vertices are joined by a finite path.*

Proof. First we show that the hypotheses imply simplicity. Suppose θ is a representation of $\mathcal{T}_{X(E)}$ with a nontrivial kernel, and let $S_f := \theta(s_f)$ and $P_v := \theta(p_v)$. Since each vertex emits infinitely many edges, Theorem 4.1 implies that $P_v = 0$ for some v . If $s(f) = v$, then $S_f = P_v S_f = 0$, and hence $P_{r(f)} = S_f^* S_f = 0$ as well. Since every pair of vertices are joined by a finite path, it follows that $P_w = 0$ for every $w \in E^0$. But then $S_f = S_f S_f^* S_f = S_f P_{r(f)} = 0$ for every $f \in E^1$, and $\theta = 0$.

Conversely, suppose $\mathcal{T}_{X(E)}$ is simple. We prove that we can reach every vertex from a given vertex v by considering the ideal $\langle p_v \rangle$ generated by p_v , which is all of $\mathcal{T}_{X(E)}$ by simplicity. As usual, we write $s_\mu := s_{f_1} \cdots s_{f_n}$ for a finite path $\mu = f_1 \cdots f_n$, define $s_w := p_w$ for each vertex w , and verify that $\mathcal{T}_{X(E)} = \overline{\text{span}}\{s_\mu s_\nu^*\}$. The ideal $\langle p_v \rangle$ is spanned by products of the form $s_\mu s_\nu^* p_v s_\sigma s_\tau^*$, which satisfy

$$s_\mu s_\nu^* p_v s_\sigma s_\tau^* = \begin{cases} s_\mu s_{\sigma'} s_\tau^* & \text{if } s(\nu) = s(\sigma) = v \text{ and } \sigma = \nu \sigma', \\ s_\mu s_{\nu'} s_\tau^* & \text{if } s(\nu) = s(\sigma) = v \text{ and } \nu = \sigma \nu', \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $r(\mu) = r(\tau)$ can be reached from v , say by α , then $s_\mu s_\tau^* = s_\mu s_\alpha^* s_\alpha s_\tau^* = s_\mu s_\alpha^* p_v s_\alpha s_\tau^*$ belongs to $\langle p_v \rangle$. Thus

$$\langle p_v \rangle = \overline{\text{span}}\{s_\mu s_\tau^* : r(\mu) = r(\tau) \in H(v)\},$$

where $H(v)$ is the set of vertices w for which there is a path from v to w .

We want to prove that $H(v)$ is all of E^0 . Suppose there exists $w \in E^0 \setminus H(v)$. We shall show that $\|p_w - b\| \geq 1$ for all $b \in \langle p_v \rangle$, which contradicts $\langle p_v \rangle = \mathcal{T}_{X(E)}$. Suppose $b = \sum \lambda_i s_{\mu_i} s_{\tau_i}^*$ is a typical finite sum in $\langle p_v \rangle$. Let F be the (finite) set of edges which start at w and are the initial edge of some μ_i . Theorem 4.1 implies that the projection $q := p_w - \sum_{f \in F} s_f s_f^*$ is nonzero. But $p_w s_{\mu_i} = 0$ unless $s(\mu_i) = w$, and then $s_f s_f^* s_{\mu_i} = s_{\mu_i}$ for the one f which starts μ_i . Thus

$$qb = \sum_i \lambda_i p_w s_{\mu_i} s_{\tau_i}^* - \sum_i \lambda_i \left(\sum_{f \in F} s_f s_f^* \right) s_{\mu_i} s_{\tau_i}^* = 0,$$

and $\|p_w - b\| \geq \|q(p_w - b)\| = \|q\| = 1$, as required.

The transitivity we have just proved implies that each vertex v emits at least one edge. If v emits only finitely many edges, then $q := p_v - \sum_{\{f: s(f)=v\}} s_f s_f^*$ is nonzero by Theorem 4.1. However, one can easily construct Toeplitz-Cuntz-Krieger E -families on Hilbert space such that $P_v = \sum_{\{f: s(f)=v\}} S_f S_f^*$, and then

q would be in the kernel of the corresponding representation of $\mathcal{T}_{X(E)}$. Thus each vertex must emit infinitely many edges. $\square$

In passing from the Toeplitz algebra $\mathcal{T}_X$ to the Cuntz-Pimsner algebra $\mathcal{O}_X$, an important role is played by the ideal $J := \varphi^{-1}(\mathcal{K}(X))$; the theory simplifies when this ideal is either $\{0\}$ or A , and authors have often imposed hypotheses which force $J = A$. (This is done, for example, in [20] and [13].) For the bimodules of graphs, one can identify the ideal J explicitly.

Proposition 4.4. *Let $X(E)$ be the Hilbert bimodule of a directed graph E , and let $\varphi : A \rightarrow \mathcal{L}(X(E))$ be the homomorphism describing the left action of $A = c_0(E^0)$. Then*

$$\varphi^{-1}(\mathcal{K}(X(E))) = \overline{\text{span}}\{\delta_v : v \text{ emits at most finitely many edges}\}.$$

Proof. Write $X := X(E)$. Since $\mathcal{K}(X)$ is an ideal in $\mathcal{L}(X)$, $J := \varphi^{-1}(\mathcal{K}(X))$ is an ideal in $A = c_0(E^0)$, and hence has the form

$$\{a \in A : a(w) = 0 \text{ for } w \notin F\} = \overline{\text{span}}\{\delta_v : v \in F\}$$

for some subset F of the discrete space E^0 . So it suffices to see that $\varphi(\delta_v)$ belongs to $\mathcal{K}(X)$ iff v emits finitely many edges. If v emits finitely many edges, then $\varphi(\delta_v) = \sum_{\{f:s(f)=v\}} \Theta_{\delta_f, \delta_f}$ is compact.

Suppose now that v emits infinitely many edges. Since $\text{span}\{\delta_f\}$ is dense in X and $(x, y) \mapsto \Theta_{x, y}$ is continuous, we can approximate any compact operator on X by a finite linear combination of the form $K := \sum_{e, f \in F} \lambda_{e, f} \Theta_{\delta_e, \delta_f}$. But for any such combination K , we can find an edge $g \notin F$ such that $s(g) = v$, and then $\Theta_{\delta_e, \delta_f}(\delta_g) = \delta_e \cdot \langle \delta_f, \delta_g \rangle_A = 0$ for all $e, f \in F$. Thus

$$\begin{aligned} \|\varphi(\delta_v) - K\| &= \sup\{\|(\varphi(\delta_v) - K)(x)\| : \|x\|_A \leq 1\} \\ &\geq \|\varphi(\delta_v)(\delta_g) - K(\delta_g)\| \\ &= \|\delta_g - 0\| = 1, \end{aligned}$$

and hence $\varphi(\delta_v)$ is not compact. $\square$

Corollary 4.5. *If E is a directed graph in which every vertex emits infinitely many edges, then the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ coincides with the Toeplitz algebra $\mathcal{T}_{X(E)}$, and is simple if and only if E is transitive.*

Remark 4.6. Since (at least in the absence of sources and sinks) the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ is generated by a Cuntz-Krieger family for the edge matrix B of E , one might guess that $\mathcal{O}_{X(E)}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_B$ of [8], and that this last Corollary follows from [8, Theorem 14.1]. This guess is correct, but the connection is nontrivial; since it concerns Cuntz-Pimsner algebras rather than Toeplitz algebras, we shall present the details elsewhere. We

note also that our Toeplitz algebra $\mathcal{T}_{X(E)}$ is not the Toeplitz-Cuntz-Krieger algebra $\mathcal{TC}_B$ discussed in [8]: their relations do not imply that the initial projections P_v are mutually orthogonal.

5. CONCLUDING REMARKS

To see why we have avoided placing additional hypotheses on our bimodules, consider the Cuntz-Krieger bimodules of graphs. We want to allow graphs with infinite valency, so Proposition 4.4 shows that A will not always act by compact operators. We also want to consider graphs with sinks (vertices which emit no edges) and sources (vertices which receive no edges). Since $v \in E^0$ is a sink iff $\delta_v \in c_0(E^0)$ acts trivially on the left of $X(E)$, $\varphi : A \rightarrow \mathcal{L}(X)$ may not be injective; since v is a source iff δ_v is not in the ideal $\overline{\text{span}}\{(x, y)_A\}$, X need not be full as a right Hilbert module.

Every Cuntz-Krieger bimodule $X = X(E)$ is essential, in the sense that $\overline{\text{span}} A \cdot X = X$, because $\delta_{s(f)} \cdot \delta_f = \delta_f$ for every $f \in E^1$. However, the following non-essential submodules arise in analysing the ideal structure of $\mathcal{T}_{X(E)}$. Suppose $V \subset E^0$ is hereditary in the sense that $r(f) \in V$ whenever $s(f) \in V$. Then $I := c_0(V)$ is an ideal in $c_0(E^0)$ such that $I \cdot X(E) \subseteq X(E) \cdot I$, so $X(E) \cdot I$ is a Hilbert I -bimodule. However, if there is an edge f such that $s(f) \notin V$ and $r(f) \in V$, then $\delta_f \in X(E) \cdot I$ but $a \cdot \delta_f = 0$ for all $a \in I$.

Because our modules may not be essential, we cannot require that the representations π in our Toeplitz representations (ψ, π) are nondegenerate: in the Fock representation induced from a nondegenerate representation of A , π is nondegenerate if and only if X is essential. Moreover, the essential subspace of π need not be invariant under ψ , so it is not in general possible to reduce to the nondegenerate case as one typically does when dealing with representations of a $*$ -algebra. The following Corollary illustrates an extreme case: when the left action is trivial, ψ and π have orthogonal ranges. In general, we believe the correct notion of nondegeneracy for a Toeplitz representation (ψ, π) is that the C^* -algebra generated by $\psi(X) \cup \pi(A)$ acts nondegenerately; see the proof of Proposition 1.3.

Corollary 5.1. *Suppose the left action of A on X is trivial.*

- (1) $\psi \times \pi$ is faithful if and only if π is faithful. If A is simple, so is $\mathcal{T}_X$.
- (2) $\mathcal{T}_X$ is canonically isomorphic to the algebra

$$L(X) := \mathcal{K}(X \oplus A) = \begin{pmatrix} \mathcal{K}(X) & X \\ \bar{X} & A \end{pmatrix};$$

if X_A is full, $L(X)$ is the linking algebra of the imprimitivity bimodule $\mathcal{K}(X)X_A$ (see [25, Section 3.2]).

Proof.

(1) If $\psi \times \pi$ is faithful, so is $(\psi \times \pi) \circ i_A = \pi$. On the other hand, for $a \in A$ and $x \in X$ we have $\pi(a)\psi(x) = \psi(a \cdot x) = 0$, so π acts trivially on $\overline{\psi(X)\mathcal{H}}$. Thus if π is faithful it must be faithful on $(\psi(X)\mathcal{H})^\perp$, and $\psi \times \pi$ is faithful by the Theorem.

(2) The formulas $\psi(x) := \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\pi(a) := \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ define a Toeplitz representation of X in $L(X)$ such that π is faithful and $\psi(X) \cup \pi(A)$ generates $L(X)$. Now use (1). $\square$

Our next application is a different extension of Cuntz's result on the simplicity of $\mathcal{O}_\infty$: to recover it, take each $X^\lambda = {}_\mathbb{C}\mathbb{C}_\mathbb{C}$.

Corollary 5.2. *Let X be a Hilbert bimodule over a simple C^* -algebra A . If $X = \bigoplus_{\lambda \in \Lambda} X^\lambda$ and the left action of A is nontrivial on infinitely many summands, then the Toeplitz algebra $\mathcal{T}_X$ is simple.*

Proof. If $\psi \times \pi$ is a nonzero representation of $\mathcal{T}_X$ on $\mathcal{H}$, then the simplicity of A implies that π and ψ are faithful. Since the summands in X are mutually orthogonal, this implies that the action of π in each $(\psi(\bigoplus_{\lambda \in F} X^\lambda)\mathcal{H})^\perp$ is nonzero and hence faithful. Thus the result follows from Theorem 3.1. $\square$

Our final application is motivated by Pimsner's realisation of crossed products by endomorphisms as $\mathcal{O}_X$ for suitable X . Let τ denote the forward-shift endomorphism on the C^* -algebra c of bounded sequences, and let $X := \tau(1)c$ be the Hilbert bimodule over c in which $x \cdot a := xa$, $\langle x, y \rangle_c := x^*y$ and $a \cdot x := \tau(a)x$. Since the identity operator on X is compact, Theorem 2.1 applies, and we recover a theorem of Conway, Duncan and Paterson [2] (see also [11, Theorem 1.3]). Recall that an element v in a C^* -algebra is a *power partial isometry* if v^n is a partial isometry for every $n \geq 1$.

Proposition 5.3. *$\mathcal{T}_X$ is unital, $v := i_X(\tau(1))^*$ is a power partial isometry, and $\mathcal{T}_X = C^*(1, v)$. The pair $(\mathcal{T}_X, v)$ has the following universal property: if B is a unital C^* -algebra and $V \in B$ is a power partial isometry, there is a unital homomorphism $\mathcal{T}_X \rightarrow B$ which maps v to V .*

Proof. $i_c(1)$ is an identity for $\mathcal{T}_X$, and the calculation

$$i_c(\tau(a)) = i_X(\tau(1))^* i_X(\tau(a)) = v i_X(a \cdot \tau(1)) = v i_c(a) v^*$$

shows that $v^n v^{*n} = i_c(\tau^n(1))$ is a projection. These projections and the identity generate $i_c(c)$; this and $i_X(x) = v^* i_c(x)$ show that $\mathcal{T}_X = C^*(1, v)$.

Suppose $V \in B$ is a power partial isometry. Since $V^n V^{*n} \geq V^{n+1} V^{*(n+1)}$, there is a unital homomorphism $\pi_V : c \rightarrow B$ which satisfies $\pi_V(\tau^n(1)) = V^n V^{*n}$. Define $\psi_V(x) := V^* \pi_V(x)$. We claim that (ψ_V, π_V) is a Toeplitz representation. Conditions (1.1) and (1.2) for a Toeplitz representation are easy. For (1.3) notice

that $\pi_V(\tau(a)) = V\pi_V(a)V^*$, and recall from [10] that the initial and range projections of the powers of V form a commuting family, so that $V^*V \in \pi(c)'$; thus

$$\begin{aligned}\psi_V(a \cdot x) &= \psi_V(\tau(a)x) = V^*\pi_V(\tau(a))\pi_V(x) = V^*V\pi_V(a)V^*\pi_V(x) \\ &= \pi_V(a)V^*VV^*\pi_V(x) = \pi_V(a)V^*\pi_V(x) = \pi_V(a)\psi_V(x),\end{aligned}$$

as required. Since $\psi_V \times \pi_V(i_c(1)) = \pi_V(1) = 1$ and $\psi_V \times \pi_V(v) = \psi_V(\tau(1))^* = \pi_V(\tau(1))V = VV^*V = V$, $\psi_V \times \pi_V$ is the desired map. $\square$

Corollary 5.4. *Let J_n denote the truncated shift on $\mathbb{C}^n$ (with $J_1 := 0$). Then $C^*(1, \oplus J_n)$ is the universal unital C^* -algebra generated by a power partial isometry.*

Proof. If $V := \oplus J_n$, then Theorem 2.1 implies that $\psi_V \times \pi_V$ is faithful. $\square$

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This research was supported by the Australian Research Council.

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Submitted: June 20, 1998.