Tangential Markov Inequalities on Real Algebraic Varieties

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ABSTRACT. We say that a smooth compact submanifold M of $\mathbb{R}^n$ admits a tangential Markov inequality of exponent ℓ if there is a constant $C > 0$ such that for all polynomials $P \in \mathbb{R}[x_1, \dots, x_n]$ and points $a \in M$

$$|D_T P(a)| \leq C(\deg(P))^\ell \sup_{x \in M} |P(x)|.$$

In a previous paper, the authors have shown that M admits a tangential Markov inequality of exponent $\ell = 1$ if and only if M is algebraic. Here we show that if M is a smooth, locally closed manifold, then M is algebraic iff M admits a local weighted version of this inequality with $\ell = 1$. We also give an example to show that in the non-smooth case, the exponent ℓ must depend on the singularities.

In recent years there has been a great deal of interest in Markov-Bernstein type inequalities on the derivatives of polynomials restricted to algebraic varieties (cf. [BLMT], [FN], [RY], [B]). The simplest case arises for smooth compact varieties.

Definition 1. Suppose that M is a smooth compact submanifold of $\mathbb{R}^n$. We say that M admits a tangential Markov (or Bernstein) inequality of exponent ℓ if there is a constant $C > 0$ such that for all polynomials $P \in \mathbb{R}[x_1, \dots, x_n]$ and points $a \in M$

$$|D_T P(a)| \leq C(\deg(P))^\ell \|P\|_M.$$

Here $D_T P$ denotes any (unit) tangential derivative of P and $\|P\|_M$ is the supremum norm of P on M .

A remarkable fact is that such tangential Markov inequalities of exponent one characterize M being algebraic.

Theorem 1 ([BLMT]). *Suppose that M is a smooth (C^∞) compact m -dimensional submanifold of $\mathbb{R}^n$. Then M admits a tangential Markov inequality of exponent $\ell = 1$ iff M is algebraic (i.e., a subset of an algebraic variety of the same dimension).*

A priori, there is some possible confusion about the hypothesis that M be C^∞ smooth. The proof of Theorem 1 makes use of the fact that in this case M must also be locally analytic. For completeness we include a proof.

Theorem 2. *Suppose that M is a smooth (C^∞) m -dimensional submanifold of $\mathbb{R}^n$ which is also algebraic. Then M must be locally analytic.*

Proof. Suppose that M is given locally by C^∞ equations $y = g(x)$ for x in some ball contained in $\mathbb{R}^m$. We will show that $g(x)$ is actually locally analytic, thus establishing the result. We proceed to show this as follows.

Without loss we may assume that $g(0) = 0 \in M$. Let $I = I(M)$ be the ideal of polynomials in (x, y) which vanish on M . We begin by establishing that there are n polynomials $P_j \in I$, $1 \leq j \leq n$, depending only on x and y_j respectively, such that the Jacobian

$$\det \left[\frac{\partial P}{\partial y}(x, G(x)) \right] \neq 0,$$

where $G(x)$ is the formal Taylor series of $g(x)$ at $x = 0$. Indeed, we may take P_j to be a polynomial in I , depending only on x and y_j , of lowest degree when considered as a polynomial in y_j . Such a P_j must exist for otherwise all polynomials depending only on x and y_j are linearly independent when restricted to M . It would follow that the dimension of the space of polynomials of degree at most k , restricted to M , would be at least of order k^{m+1} contradicting the fact that, from the definition of the dimension of an algebraic set, it must be of at most order k^m (see e.g. [CLO, Chapt. 9]). Moreover, P_j must be of positive degree in y_j for otherwise $\dim(M) \leq m - 1$.

Now let $R_j = R_j(x)$ be the resultant of P_j and $\partial P_j / \partial y_j$ considered as polynomials in y_j (cf. [CLO]). Note that $R_j \neq 0$ for otherwise P_j would have a square factor of positive degree in y_j , say $P_j = F_j^2 T_j$. But then $F_j T_j \in I$ is a polynomial, depending only on x and y_j , of lower degree than P_j . Consequently, there are non-zero polynomials A_j and B_j such that

$$A_j P_j + B_j \frac{\partial P_j}{\partial y_j} = R_j,$$

which we may rewrite as

$$B_j \frac{\partial P_j}{\partial y_j} = R_j - A_j P_j.$$

Then letting $B := \prod_{j=1}^n B_j$ and $R := \prod_{j=1}^n R_j$, we have

$$B \prod_{j=1}^n \frac{\partial P_j}{\partial y_j} = R \pmod{I}$$

for some non-zero $B(x, y)$ and $R(x)$. In other words,

$$B(x, y) \det \left[\frac{\partial P}{\partial y}(x, y) \right] = R(x)$$

for all (x, y) such that $p(x, y) = 0$ for all $p \in I$.

In particular,

$$B(x, g(x)) \det \left[\frac{\partial P}{\partial y}(x, g(x)) \right] = R(x) \neq 0$$

and our claim follows by taking formal Taylor expansions of both sides.

Next, let us introduce, for $a, b \in \mathbb{R}^n$,

$$DP(x, a, b) := \int_0^1 \frac{\partial P}{\partial y}(x, ta + (1-t)b) dt.$$

Then

$$DP(x, a, b) \cdot (a - b) = P(x, a) - P(x, b),$$

an identity that justifies referring to $DP(x, a, b)$ as “the divided difference of $P(x, y)$ at $y = a, b$ ”. In particular,

$$DP(x, y, y) = \frac{\partial P}{\partial y}(x, y)$$

so that

$$\det[DP(x, G(x), G(x))] = \det \left[\frac{\partial P}{\partial y}(x, G(x)) \right] \neq 0.$$

Then, there exists an integer N such that for any formal power series $H(x)$ that agrees with $G(x)$ up to order N , it is also true that

$$\det[DP(x, G(x), H(x))] \neq 0.$$

But, since $P(x, g(x)) = 0$, it follows that $P(x, G(x)) = 0$ and from the Artin Approximation Theorem (see e.g. [T]) there exists a function $h(x)$, analytic in

a neighbourhood of the origin, whose Taylor expansion at $x = 0$, say $H(x)$, coincides with $G(x)$ up to order N and such that $P(x, h(x)) = 0$. We claim, that in fact, $h(x) = g(x)$ near the origin, after which we will be done.

To see this, consider

$$DP(x, g(x), h(x)) \cdot (g(x) - h(x)) = P(x, g(x)) - P(x, h(x)) = 0$$

so that, multiplying by the classical adjoint of the matrix DP ,

$$f(x) \cdot (g(x) - h(x)) = 0$$

where we have set $f(x) := \det[DP(x, g(x), h(x))]$. But the formal Taylor series of $f(x)$ at $x = 0$ is just $\det[DP(x, G(x), H(x))]$ which we have established to be non-zero. It follows from continuity of the derivatives, that also the formal Taylor series of $f(x)$ at all points a in a neighbourhood of the origin, are non-zero. But then since the product of $f(x)$ and $g(x) - h(x)$ is identically zero, we must have $g(x) = h(x)$ in this same neighbourhood. $\square$

Algebraic varieties need neither be smooth nor compact and hence this characterization does not hold in general. We give an example indicating that the exponent ℓ must depend on the nature of any singularities.

Consider the curves

$$\gamma_r : y = x^r, \quad 0 \leq x \leq 1,$$

where $r \geq 1$. There is a remarkable divergence between the r rational and the r irrational cases.

Theorem 3.

- (a) For $r = q/p$ with $q \geq p$ positive integers in lowest terms, γ_r admits a tangential Markov inequality of exponent $\ell = 2p$. Moreover, this is best possible.
- (b) For r irrational, γ_r does not admit a tangential Markov inequality of any exponent.

The proof will be given later in the paper.

In general, as is well known, every algebraic variety may be decomposed into a set of smooth points which is *locally closed* (every point has a neighbourhood, closed in the induced topology) and a complementary set which is a lower dimensional variety. It is the purpose of this note to use a local form of Definition 1 that extends our characterization to this non-smooth, non-compact case.

In retrospect, our condition is already evident in the relation between the classical Bernstein inequality on the circle (compact case) and the classical

Markov inequality on an interval (viewed as a ball in the non-compact line). To be more precise, we briefly review this point of view.

Consider the interval $K := [-1, 1] \subset \mathbb{R} =: M$. We wish to bound the derivatives of polynomials $P'(x)$ at points $a \in K$ in terms of $\|P\|_K$, i.e., establish the classical Markov inequality. Our approach is to reduce this problem to the smooth case (i.e., the classical Bernstein inequality) by the following technique. Extend $P(x)$ to a bivariate polynomial in the trivial way, i.e. to $\tilde{P}(x, y) := P(x)$. Then $\tilde{P}$, restricted to the unit circle, $T := \{(x, y) \mid x^2 + y^2 = 1\}$, is a trigonometric polynomial of the same degree. By the classical Bernstein inequality

$$\left| \frac{d}{d\theta} \tilde{P}(\cos(\theta), \sin(\theta)) \right| \leq \deg(\tilde{P}) \|\tilde{P}\|_T.$$

In other words,

$$\left| \frac{d}{d\theta} P(\cos(\theta)) \right| \leq \deg(P) \|P\|_K,$$

which expressed in terms of the original variable x becomes

$$(1) \quad |P'(x)| \leq \frac{\deg(P)}{\sqrt{1-x^2}} \|P\|_K$$

for $|x| < 1$, i.e. $x \in \text{int } K$ (relative to $M = \mathbb{R}$). It is important to note that the factor $\sqrt{1-x^2}$ is the size of the projection of the unit tangent vector of the circle at the point $(x, \sqrt{1-x^2})$ onto the real line, M .

The classical Markov inequality states that, in fact,

$$|P'(x)| \leq (\deg(P))^2 \|P\|_K$$

which follows easily from (1) by means of an analytical “trick” which we will not pursue here. The interested reader will find the details in, for example, [C].

This generalizes in a natural way. Suppose first that $M \subset \mathbb{R}^n$ is an algebraic variety. We emphasize that M need not be compact and may have singular points. Let $a \in M$ be a smooth point of M and $\rho := \text{dist}(a, \text{Sing}(M))$ be the distance from a to the singular set (used here in the sense of M an analytic set). We embed M in its natural cylinder

$$\tilde{M} := \{x \in \mathbb{R}^{n+1} \mid \pi(x) \in M\},$$

where $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the natural projection. Now choose $r < \rho$ so that

$$\mathcal{S}(a, r) := \tilde{M} \cap S(a, r),$$

where $S(a, r) \subset \mathbb{R}^{n+1}$ is the Euclidean sphere of radius r centered at a , is *smooth*. Clearly then, $S(a, r)$ is a smooth compact manifold of the same dimension as M which is also algebraic. We will denote by $K(a, r)$ the corresponding “ball” on M , i.e.,

$$K(a, r) := \pi(S(a, r)).$$

Then each point $x \in \text{int}(K(a, r))$ is covered by exactly two points x_+ , $x_- \in S(a, r)$ such that $\pi(x_{\pm}) = x$. In fact, we may write

$$x_{\pm} = (x, x_{n+1})$$

where $x_{n+1} = \pm \sqrt{r^2 - |x - a|^2}$.

Furthermore, it is easy to see that T is a vector tangent to M at $x \in \text{int}(K(a, r))$ iff $\tilde{T} := (T, t_{n+1})$, where

$$t_{n+1} := -\frac{T \cdot (x - a)}{x_{n+1}},$$

is tangent to $S(a, r)$ at $x_{\pm}$. If T is assumed to be a unit vector then

$$|\tilde{T}|^2 = |T|^2 + t_{n+1}^2 = 1 + \frac{(T \cdot (x - a))^2}{r^2 - |x - a|^2}.$$

Hence we define

$$(2) \quad w_T(x) := \sqrt{1 + \frac{(T \cdot (x - a))^2}{r^2 - |x - a|^2}}$$

which will play the role of (and indeed reduces to) the factor $1/\sqrt{1 - x^2}$ in (1).

Finally, any polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$ may trivially be lifted to

$$\tilde{P}(x) := P(\pi(x)).$$

Hence, by the characterization in Theorem 1, we have, for $x \in \text{int}(K(a, r))$ and T a unit vector tangent to M at x ,

$$\begin{aligned} (3) \quad |D_T P(x)| &= |D_{\tilde{T}} \tilde{P}(x_+)| \\ &\leq |\tilde{T}| C_{a,r} \deg(\tilde{P}) \|\tilde{P}\|_{S(a,r)} \\ &= C_{a,r} w_T(x) \deg(P) \|P\|_{K(a,r)} \end{aligned}$$

for some local constant $C_{a,r}$ depending on a, r (and M) where $w_T(x)$ is given in (2).

The converse is also true, when expressed for manifolds analogous to algebraic varieties with their singular sets removed, i.e., those *locally closed* manifolds; every point has a neighbourhood closed in the induced topology. Then, with the same notation as above, we have the following result.

Theorem 4. *Suppose that $M \subset \mathbb{R}^n$ is a smooth (C^∞), locally closed manifold with finitely many connected components. Then M is algebraic iff M admits a weighted Markov-Bernstein inequality of the form (3), i.e., iff for every $a \in M$ there is an $r_0 > 0$ such that for all $0 < r \leq r_0$ there is a constant $C_{a,r}$ such that for all $x \in \text{int}(K(a,r))$ and all polynomials P ,*

$$(4) \quad |D_T P(x)| \leq C_{a,r} w_T(x) \deg(P) \|P\|_{K(a,r)}.$$

Proof. We have already shown that (4) holds for M algebraic. Hence suppose that M admits such a Markov-Bernstein type inequality. Without loss of generality we may assume that M is connected.

For $a \in M$ fix an r so that (4) holds and so that $\mathcal{S}(a,r)$ is smooth. The polynomials of degree d restricted to $K(a,r)$, $\mathcal{P}_d(K(a,r))$, form a vector space of a certain dimension, D say. We will show that $D = O(d^m)$ where m is the dimension of M . From this it follows easily that M is algebraic. In fact, the argument is a slight modification of one given in [BLMT]. The key point is that (4) is equivalent to a tangential Markov inequality of exponent $\ell = 1$ on $\mathcal{S}(a,r)$ for polynomials lifted from $\mathcal{P}_d(K(a,r))$.

For each $t > 0$ let A_t denote a maximal subset of $\mathcal{S}(a,r)$ such that the collection of *geodesic* balls

$$\mathcal{B}(x,t) := \{y \in \mathcal{S}(a,r) \mid \text{dist}(x,y) < t\}$$

with centres $x \in A_t$, are disjoint. Let N_t be the cardinality of A_t . It follows that

$$(5) \quad \mathcal{S}(a,r) \subset \bigcup_{x \in A_t} \mathcal{B}(x, 2t)$$

and

$$(6) \quad N_t t^m \approx \text{vol}_m(\mathcal{S}(a,r)),$$

the m -dimensional volume of $\mathcal{S}(a,r)$. This last property implies that, in particular,

$$N_t \uparrow \infty \quad \text{as } t \downarrow 0.$$

Hence,

$$t^* := \sup\{t \mid N_t > D - 1\} > 0,$$

and consequently

$$N_{2t^*} \leq D - 1 \leq N_{t^*}.$$

From this and (6) it follows that there is a constant c such that

$$(7) \quad t^* \leq cN_{t^*}^{-1/m} \leq c(D - 1)^{-1/m}$$

and that

$$(8) \quad \mathcal{S}(a, r) \subset \bigcup_{x \in A_{2t^*}} \mathcal{B}(x, 4t^*).$$

But, since $N_{2t^*} \leq D - 1 < D$ we can find a polynomial $P \in \mathcal{P}_d(K(a, r))$ so that the polynomial lifted to $\mathcal{S}(a, r)$, $\tilde{P}$, has the properties that $\tilde{P}(x) = 0$, $\forall x \in A_{2t^*}$ but $\tilde{P}$ is *not* identically zero on $\mathcal{S}(a, r)$.

Let $y \in \mathcal{S}(a, r)$ be a point where $|\tilde{P}(y)| = \|\tilde{P}\|_{\mathcal{S}(a, r)}$ and let $x \in A_{2t^*}$ be a point closest to y . Then, letting $\gamma(x, y)$ be the geodesic curve connecting x and y and letting $\tilde{T} = \tilde{T}_z$ denote a unit tangent to γ at $z \in \gamma(x, y)$, from (2) and (4) we obtain

$$\begin{aligned} \|\tilde{P}\|_{\mathcal{S}(a, r)} &= |\tilde{P}(y) - \tilde{P}(x)| \\ &\leq \text{dist}(x, y) \max_{z \in \gamma(x, y)} |D_{\tilde{T}} \tilde{P}(z)| \\ &\leq \text{dist}(x, y) C_{a, r} \deg(\tilde{P}) \|\tilde{P}\|_{\mathcal{S}(a, r)} && \text{(by (3))} \\ &\leq 4t^* C_{a, r} \deg(\tilde{P}) \|\tilde{P}\|_{\mathcal{S}(a, r)} && \text{(by (8))} \\ &\leq 4C_{a, r} c(D - 1)^{-1/m} \deg(\tilde{P}) \|\tilde{P}\|_{\mathcal{S}(a, r)}, && \text{(by (7))} \end{aligned}$$

so that

$$(D - 1)^{1/m} \leq 4C_{a, r} c \deg(P)$$

which clearly implies that $D \leq C(\deg(P))^m$ for some constant $C > 0$.

We may therefore conclude that every point $a \in M$ has an open neighbourhood which is algebraic. Consequently, M itself is algebraic. $\square$

We now proceed to the proof of the example given as Theorem 3. Recall that we consider the curves

$$\gamma_r : y = x^r, \quad 0 \leq x \leq 1,$$

where $r \geq 1$.

Theorem 3.

- (a) For $r = q/p$ with $q \geq p$ positive integers in lowest terms, γ_r admits a tangential Markov inequality of exponent $\ell = 2p$. Moreover, this is best possible.
- (b) For r irrational, γ_r does not admit a tangential Markov inequality of any exponent.

Proof. Suppose first that $r = p/q$ is in lowest terms and $q \geq p$. Then γ_r may be parameterized by

$$x = t^p, \quad y = t^q, \quad 0 \leq t \leq 1.$$

Any bivariate polynomial, $P(x, y)$, restricts on γ_r to

$$\tilde{P}(t) = P(t^p, t^q),$$

a univariate polynomial of $\deg(\tilde{P}) \leq q \deg(P)$. The tangential derivative of P may then be expressed as

$$\begin{aligned} D_T P(x, y) &= \frac{d}{dt} \tilde{P}(t) / \left\| \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \right\|_2 \\ &= \frac{\frac{d}{dt} \tilde{P}(t)}{\{(pt^{p-1})^2 + (qt^{q-1})^2\}^{1/2}} \\ &= \frac{\frac{1}{t^{p-1}} \frac{d}{dt} \tilde{P}(t)}{\{p^2 + q^2 t^{2(q-p)}\}^{1/2}} \end{aligned}$$

so that

$$\frac{1}{\sqrt{p^2 + q^2}} \left\| \frac{1}{t^{p-1}} \frac{d}{dt} \tilde{P}(t) \right\|_{[0,1]} \leq \|D_T P\|_{\gamma_r} \leq \frac{1}{p} \left\| \frac{1}{t^{p-1}} \frac{d}{dt} \tilde{P}(t) \right\|_{[0,1]}.$$

Hence, it suffices to show that there is a constant $C > 0$ such that

$$(9) \quad \left\| \frac{1}{t^{p-1}} \frac{d}{dt} \tilde{P}(t) \right\|_{[0,1]} \leq C (\deg \tilde{P})^\ell \|\tilde{P}\|_{[0,1]}$$

for all polynomials $\tilde{P}(t)$ of the form $\tilde{P}(t) = a_0 + a_p t^p + \cdots$.

For the moment let $Q(t) = \tilde{P}'(t)$. Then $Q(0) = Q'(0) = \dots = Q^{(p-2)}(0) = 0$ and we may write

$$\frac{Q(t)}{t^{p-1}} = \int_0^1 \int_0^{t_{p-2}} \dots \int_0^{t_1} Q^{(p-1)}(t_0 t) dt_0 dt_1 \dots dt_{p-2}$$

or

$$\frac{\tilde{P}'(t)}{t^{p-1}} = \int_0^1 \int_0^{t_{p-2}} \dots \int_0^{t_1} \tilde{P}^{(p)}(t_0 t) dt_0 dt_1 \dots dt_{p-2}$$

so that

$$\left\| \frac{\tilde{P}'(t)}{t^{p-1}} \right\|_{[0,1]} \leq \|\tilde{P}^{(p)}\|_{[0,1]} \leq 2^p (\deg(\tilde{P}))^{2p} \|\tilde{P}\|_{[0,1]}.$$

We have therefore established the first part of (a).

To show that the exponent $2p$ is best possible we require some information about which univariate polynomials $\tilde{P}$ may be realized as the restriction of a bivariate polynomial $P(x, y)$.

Lemma 1. *There is a constant $c = c(p, q)$ such that for all integers $d \geq pq$,*

$$\{\tilde{P}(t) = P(t^p, t^q) \mid \deg(P) \leq cd\} \supset \{t^k \mid pq \leq k \leq d\}.$$

Proof. Since by assumption $\gcd(p, q) = 1$, there are integers i_0, j_0 such that

$$i_0 p + j_0 q = 1$$

so that

$$(ki_0 + nq)p + (kj_0 - np)q = k$$

for any integer n .

First note that

$$\begin{aligned} k \geq pq &\implies k(i_0 p + j_0 q) \geq pq \implies k \frac{i_0}{p} + k \frac{j_0}{q} \geq 1 \\ &\implies k \frac{j_0}{p} - \left(-k \frac{i_0}{q} \right) \geq 1 \end{aligned}$$

so that the interval $[-ki_0/q, kj_0/p]$ contains an integer n . It is easy to verify that for such an n ,

$$i := ki_0 + nq \geq 0 \quad \text{and} \quad j := kj_0 - np \geq 0.$$

Moreover,

$$i \leq k|i_0| + q|n| \leq k \left\{ |i_0| + q \max \left(\frac{|i_0|}{q}, \frac{|j_0|}{p} \right) \right\}$$

and, similarly,

$$j \leq k \left\{ |j_0| + p \max \left(\frac{|i_0|}{q}, \frac{|j_0|}{p} \right) \right\}.$$

Hence, for $k \geq pq$, with $x = t^p$ and $y = t^q$,

$$t^k = x^i y^j$$

with $0 \leq i, j$ and $\deg(x^i y^j) = i + j \leq c(p, q)k$, where

$$c(p, q) := |i_0| + |j_0| + (q + p) \max \left(\frac{|i_0|}{q}, \frac{|j_0|}{p} \right). \quad \square$$

Now, by Lemma 2 below, there is a constant C such that

$$\inf_{\deg(Q) \leq d} \|t^p - t^{pq}Q(t)\|_{[0,1]} \leq Cd^{-2p}.$$

Let Q_d be the polynomial of degree d for which the infimum is attained and let $P_d(x, y)$ be a bivariate polynomial (of degree at most $c(p, q)(pq + d)$) such that $P_d(t^p, t^q) = t^p - t^{pq}Q_d(t)$. Then

$$D_T P_d(0, 0) = \frac{\partial}{\partial x} P_d(0, 0) = \frac{1}{p} \left(\frac{1}{t^{p-1}} \frac{d}{dt} (t^p - t^{pq}Q_d(t)) \right)_{t=0} = 1.$$

But,

$$\|P_d\|_{\gamma_r} = \|t^p - t^{pq}Q_d(t)\|_{[0,1]} \leq Cd^{-2p}.$$

Hence,

$$\frac{\|D_T P_d\|_{\gamma_r}}{\|P_d\|_{\gamma_r}} \geq \frac{1}{C} d^{2p}$$

from which it follows that the Markov exponent must be at least $2p$. □

Lemma 2. For $i, j \geq 0$ there are constants $C_{ij} \geq 0$ such that

$$\inf_{\deg(Q) \leq d} \|t^i - t^j Q(t)\|_{[0,1]} \leq C_{ij} d^{-2i}.$$

Proof. If $j \leq i$, then we may trivially take $C_{ij} = 0$, so suppose that $j > i$. Consider first this “distance” in the L^2 -norm. By the formula given, for example, in [D, Lemma 11.3.2, p. 269],

$$\begin{aligned} \inf_{\deg(Q) \leq d} \|t^i - t^j Q(t)\|_{L^2[0,1]} &= \frac{1}{\sqrt{2i+1}} \prod_{k=j}^{j+d} \frac{k-i}{k+i+1} \\ &= \frac{1}{\sqrt{2i+1}} \frac{(j-i+d)!}{(j-i-1)!} \frac{(j+i)!}{(j+i+d+1)!} \\ &= \frac{1}{\sqrt{2i+1}} \frac{(j+i)!}{(j-i-1)!} \frac{(j-i+d)!}{(j+i+d+1)!} \\ &= \frac{1}{\sqrt{2i+1}} \frac{(j+i)!}{(j-i-1)!} \frac{1}{(j+i+d+1) \cdots (j-i+d+1)} \\ &\leq \frac{1}{\sqrt{2i+1}} \frac{(j+i)!}{(j-i-1)!} \frac{1}{d^{2i+1}}, \end{aligned}$$

there being $2i+1$ terms in the above product, each greater than d . But, as is well known, there is a constant $C > 0$ such that

$$\|P\|_{[0,1]} \leq C(\deg P) \|P\|_{L^2[0,1]}$$

so that

$$\begin{aligned} \inf_{\deg(Q) \leq d} \|t^i - t^j Q(t)\|_{[0,1]} &\leq C(d+j) \inf_{\deg(Q) \leq d} \|t^i - t^j Q(t)\|_{L^2[0,1]} \\ &\leq C_{ij} d^{-2i}. \end{aligned} \quad \square$$

We now proceed to the proof of (b). Hence we suppose that $r > 1$ is irrational. We construct a sequence of bivariate polynomials, P_d , of increasing degree, such that $D_T P_d(0,0) = 1$ but $\|P_d\|_{\gamma_r}$ is exponentially small in d . The argument is similar to that just given above.

Let

$$V_r^d := \left\{ \sum_{2 \leq i+j \leq d} a_{ij} t^{i+rj} \mid a_{ij} \in \mathbb{R} \right\}$$

be the set of restrictions of the monomials $\{x^i y^j \mid 2 \leq i+j \leq d\}$ to γ_r . Since r is irrational, the exponents $i+rj$ are distinct and

$$\dim V_r^d = \binom{d+2}{2} - 3 \approx d^2/2.$$

We will be done if we can show that

$$\inf_{P \in V_r^d} \|t - P(t)\|_{[0,1]}$$

decreases exponentially in the degree d . For then, the bivariate polynomial $P_d(x, y)$ for which $P_d(t, t^r) = t - P(t)$ has the desired properties mentioned above. Now, by [BE, Theorem 6.13], if $Q(t) = \sum_{i=0}^n a_i t^{\lambda_i}$ for some $\lambda_i \geq -1/2$,

$$(10) \quad \|\sqrt{t}Q(t)\|_{[0,1]} \leq 13 \left(\sum_{j=0}^n (\lambda_j + 2) \right)^{1/2} \|Q\|_{L^2[0,1]}.$$

Hence, we write

$$\|t - P(t)\|_{[0,1]} = \|\sqrt{t}(\sqrt{t} - P(t)/\sqrt{t})\|_{[0,1]}$$

and first estimate

$$\inf_{Q \in (1/\sqrt{t})V_r^d} \|\sqrt{t} - Q(t)\|_{L^2[0,1]}.$$

But using once again the distance formula from [D] Lemma 11.3.2,

$$\begin{aligned} \inf_{Q \in (1/\sqrt{t})V_r^d} \|\sqrt{t} - Q(t)\|_{L^2[0,1]} &= \text{dist}_{L^2[0,1]} \left(t^{1/2}, \left\{ \sum_{2 \leq i+j \leq d} a_{ij} t^{i+rj-1/2} \right\} \right) \\ &= \frac{1}{\sqrt{2}} \prod_{2 \leq i+j \leq d} \frac{i+rj-1}{i+rj+1} \\ &= \frac{1}{\sqrt{2}} \prod_{2 \leq i+j \leq d} \left(1 - \frac{2}{i+rj+1} \right) \\ &\leq \frac{1}{\sqrt{2}} \prod_{2 \leq i+j \leq d} \left(1 - \frac{2}{dr+1} \right) \\ &= \frac{1}{\sqrt{2}} \left(1 - \frac{2}{dr+1} \right)^{\binom{d+2}{3}-3} \\ &\leq \frac{1}{\sqrt{2}} \left(1 - \frac{2}{dr+1} \right)^{(dr+1)d/2r} \quad \left(\text{for } d \geq \frac{4r}{3r-1} \right) \\ &\leq \frac{1}{\sqrt{2}} e^{-d/r}. \end{aligned}$$

Hence, by (10), for d sufficiently large,

$$\inf_{P \in V_r^d} \|t - P(t)\|_{[0,1]} \leq 13 \left(\sum_{2 \leq i+j \leq d} (i + rj + 3/2) \right)^{1/2} \frac{1}{\sqrt{2}} e^{-d/r}$$

and we are done. $\square$

We conclude with a brief discussion of an interesting related conjecture. One classical approach to proving Markov inequalities is by means of complex variables in which the so-called Bernstein-Walsh inequality

$$\max_{|z| \leq r_1} |P(z)| \leq \left(\frac{r_1}{r_0} \right)^{\deg(P)} \max_{|z| \leq r_0} |P(z)|, \quad r_0 \leq r_1$$

plays an important role. (The special case of $r_1 = 2r_0$ is often referred to as the *doubling* inequality.) This in turn follows easily from the perhaps better known, Hadamard's three circle inequality for analytic functions,

$$\|f\|_{r_1} \leq \|f\|_{r_0}^{1-\theta} \|f\|_{r_2}^{\theta}, \quad r_0 \leq r_1 \leq r_2$$

where $\|f\|_r := \max_{|z| \leq r} |f(z)|$ and

$$\theta := \frac{\log(r_1) - \log(r_0)}{\log(r_2) - \log(r_0)}.$$

In fact, just substitute the crude bound

$$\|P\|_{r_2} \leq C_P r_2^{\deg(P)}$$

and let $r_2 \rightarrow \infty$ (cf. [BM]).

By the same argument, the somewhat weaker inequality,

$$(11) \quad \|f\|_{r_1} \leq C \|f\|_{r_0}^{1-s\theta} \|f\|_{r_2}^{s\theta}$$

for some constant $C > 0$, parameter $s \geq 1$ and θ , r_0 , r_1 , r_2 as before, yields,

$$\|P\|_{r_1} \leq C \left(\frac{r_1}{r_0} \right)^{s \deg(P)} \|P\|_{r_0}$$

which is sufficient to establish Markov-type inequalities, albeit with a higher exponent ℓ .

Thus, the existence of a Hadamard inequality on an algebraic variety would be sufficient to establish Markov and related inequalities, entirely along classical lines.

We therefore ask:

Question. *Given an algebraic variety $V \subset \mathbb{C}^n$, does there exist an intrinsic metric on V so that for balls measured in this metric there holds a Hadamard inequality of the form (11)?*

In the case of V a curve in $\mathbb{C}^2$ the Arakelov metric discussed in Rumely [R, Chapt. 2] has (essentially) this property.

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