$\mathbb{D}^*$ -Extension Property without Hyperbolicity

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ABSTRACT. We present an example of a complex manifold X —in fact, a pseudoconvex open set in $\mathbb{C}^2$ —such that X is not Kobayashi-hyperbolic, but any holomorphic map from the punctured unit disk to X extends to a map from the whole unit disk to X .

0. Introduction. Let X be a complex manifold. We say that X has the $\mathbb{D}^*$ -extension property ($\mathbb{D}^*$ -EP) iff for any holomorphic map f from $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ to X , there exists a map $\tilde{f} \in \text{Hol}(\mathbb{D}, X)$ (where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$) such that $\tilde{f}|_{\mathbb{D}^*} = f$. Kwack [4] and Thai [8] proved that in the case where X is compact, then the $\mathbb{D}^*$ -EP is equivalent to hyperbolicity in the sense of Kobayashi, as well as in the sense of Brody (see precise definitions below).

If X is not assumed to be compact, $X = \mathbb{D}^*$, $f(z) = z$ provides an immediate example of a hyperbolic (indeed, complete hyperbolic) manifold which does not have the $\mathbb{D}^*$ -EP. But if X has the $\mathbb{D}^*$ -EP, then it must be Brody-hyperbolic [8]. The purpose of this note is to give an example of a complex manifold X (in fact, an open set in $\mathbb{C}^2$) which has the $\mathbb{D}^*$ -EP but is not Kobayashi-hyperbolic. Note that our example is pseudo-convex, and in fact any Riemann domain spread over $\mathbb{C}^n$ having the $\mathbb{D}^*$ -EP must be pseudo-convex. Whether an arbitrary complex manifold X having the $\mathbb{D}^*$ -EP is also pseudoconvex, or verifies the *Kontinuitätssatz*, is an interesting open problem.

1. Definitions and statement of the result. (see [1, 3, 5, 6, ...] for definitions and properties of Kobayashi hyperbolicity).

Definition. For any $p \in X$, $v \in T_p X$ (the tangent space at p), the infinitesimal Kobayashi-Royden pseudo-metric is defined by

$$K_D(v) := \inf \left\{ \lambda > 0 : \exists f \in \text{Hol}(\mathbb{D}, X) : f(0) = p, f'(0) = \frac{1}{\lambda} v \right\}.$$

Definition [6]. X is Kobayashi-hyperbolic iff for any $p \in X$, there exists a neighborhood U of p and a constant $c > 0$ such that for all $q \in U$, $v \in T_q X \setminus \{0\}$, then $K_X(v) \geq c\|v\|$.

The local uniformity involved in this definition cannot be dispensed with (see e.g. [1, Remark 7.2.3]).

Definition. For $p, q \in X$ a connected complex manifold, the Kobayashi pseudo-distance is given by

$$d_X(p, q) = \inf \left\{ \int_0^1 K_X(\gamma'(t)) dt : \gamma \in \mathcal{C}^1([0, 1], X), \gamma(0) = p, \gamma(1) = q \right\}.$$

The Kobayashi pseudo-distance can also be defined by chains of holomorphic maps from $\mathbb{D}$ to X which connect p to q . In the case when $X = \mathbb{D}$, it coincides with the Poincaré metric. X is Kobayashi-hyperbolic if and only if this pseudo-distance is a distance, and it then induces the usual topology on X .

Definition. X is Brody-hyperbolic iff it contains no complex lines, i.e. if any map $g \in \text{Hol}(\mathbb{C}, X)$ must be constant.

Any Kobayashi hyperbolic manifold must be Brody hyperbolic (consider $f_n(z) = g(nz)$, $n \geq 1$, in the definition of K_X), but the converse is false, as proved by the example of Eisenmann-Taylor [3].

Let $X_u := \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < e^{-u(z)}\}$. When u is upper semicontinuous from the unit disc to $[-\infty, \infty)$, X_u is a connected open set. Such domains are called *Hartogs domains* and are pseudo convex if and only if $u \in \text{SH}(\mathbb{D})$ (see e.g. Shabat [7]).

In the Eisenmann-Taylor example, $u = u_0$ where $u_0(0) = 0$, $u_0(z) = \log|z|$ for $z \neq 0$. The set X_{u_0} does not have the $\mathbb{D}^*$ -EP (consider $f(\zeta) = (\zeta/2, 2)$), but our example will be of the same general form.

Any bounded domain in $\mathbb{C}^n$ is hyperbolic, so we shall need an unbounded u . In fact:

Lemma 1. X_u is hyperbolic iff u is locally bounded below, i.e. for any $K \Subset \mathbb{D}$,

$$\inf_K u > -\infty.$$

On the other hand, X must not contain any complex line, so in particular we must require $u(z) \neq -\infty$, $z \in \mathbb{D}$.

Lemma 2. Let $\{z_k\} \subset \mathbb{D}$, $\{z_k\} \rightarrow 0$, and $\{\alpha_k\}, \{\delta_k\} \subset (0, 1)$ such that

$$(i) \sum_{k \geq 1} \alpha_k \log \frac{1}{|z_k|} < \infty, \text{ and}$$

$$(ii) \lim_{k \rightarrow \infty} \alpha_k \log \frac{1}{\delta_k} = +\infty.$$

Then $u(z) := \sum_k \alpha_k \log(\delta_k^2 + |z - z_k|^2)$ defines a function in $\text{SH}(\mathbb{D})$,
 $u \in \mathcal{C}^\infty(\mathbb{D} \setminus \{0\})$, $u(0) > -\infty$, and $\lim_{z \rightarrow 0} u(z) = -\infty$.

Note that our example uses a subharmonic function which is discontinuous in one point, as does the (distinct) example of Kerzman and Rosay [2] of a domain that is bounded and pseudo-convex without being taut.

Theorem. There exist $u \in \text{SH}(\mathbb{D})$, as above, such that X_u is not Kobayashi hyperbolic and X_u has the $\mathbb{D}^*$ -EP.

2. Proofs.

Proof of Lemma 1. If K is compact in $\mathbb{D}$ and $\inf_K u = -\infty$, then there exists $z_0 \in K$ such that $\lim_{z \rightarrow z_0} u(z) = -\infty$. Choose $\{z_k\}_{k \geq 1} \rightarrow z_0$ such that $u(z_k) \rightarrow -\infty$. Then for any $w, w' \in \mathbb{C}$, $d_K((z_k, w), (z_k, w')) \leq \rho(e^{u(z_k)} w, e^{u(z_k)} w')$, where ρ denotes the Poincaré distance in the unit disc. Pick $w_0 \in \mathbb{C}$ such that $|w_0| < e^{-u(z_0)}$ then

$$\begin{aligned} d_{X_u}((z_0, 0), (z_0, w_0)) &\leq d_{X_u}((z_0, 0), (z_k, 0)) + d_{X_u}((z_k, 0), (z_k, w_0)) \\ &\quad + d_{X_u}((z_k, w_0), (z_0, w_0)) \end{aligned}$$

Take $r > 0$ small enough so that the disc $\{(z, w_0) : |z - z_0| < r\}$ is included in X_u . Then, for k large enough, $|z_k - z_0| < r$ and

$$d_{X_u}((z_0, 0), (z_0, w_0)) \leq \rho(z_0, z_k) + \rho(0, e^{u(z_k)} w) + \frac{1}{r} \rho(z_0, z_k),$$

which tends to 0 as k goes to ∞ , so $d_K((z_0, 0), (z_0, w)) = 0$ and X_u is not Kobayashi hyperbolic.

Conversely, let $\varphi(r) = -\inf_{\overline{D}(0, r)} u < +\infty$ for $r < 1$. Let $\nu \in \mathbb{C}^2$, and $f = (f_1, f_2) \in \text{Hol}(\mathbb{D}, X_u)$ with $f_1(0) = z_0$, $f_2(0) = w_0$, $f'(0) = \lambda \nu$, $\lambda \in \mathbb{C}$. Since $f_1 \in \text{Hol}(\mathbb{D}, \mathbb{D})$, the (invariant) Schwarz Lemma implies $|f'_1(0)| \leq 1 - |z_0|^2$. Let $r_0 \in (|z_0|, 1)$, $\Omega_{r_0} = \{\zeta \in \mathbb{D} : f_1(\zeta) < r_0\}$. Take $r_1 > 0$ such that $D(0, r_1) \subset \Omega_{r_0}$. Then $\sup_{\Omega_{r_0}} |f_2| \leq e^{\varphi(r_0)}$, and $|f'_2(0)| \leq e^{\varphi(r_0)}/r_1$.

By the invariant Schwarz Lemma, if $|f_1(\zeta)| \geq r_0$,

$$|\zeta| \geq \left| \frac{z_0 - f_1(\zeta)}{1 - \bar{z}_0 f_1(\zeta)} \right| \geq \frac{|f_1(\zeta)| - |z_0|}{1 - |f_1(\zeta)||z_0|} \geq \frac{r_0 - |z_0|}{1 - r_0|z_0|} \geq \frac{1}{2}$$

for r_0 large enough. So this choice of r_0 implies that we can pick $r_1 \geq \frac{1}{2}$, and thus $|f'_2(0)| \leq 2e^{\varphi(r_0)}$. Then

$$|\lambda| \leq \min \left(\frac{1 - |z_0|^2}{|\nu_1|}, \frac{2e^{\varphi(r_0)}}{|\nu_2|} \right) < +\infty \quad \text{for } \nu \neq (0,0),$$

and the bound depends only on (z_0, w_0) and $\|\nu\|$ (not on f). Furthermore, it is easy to see that those estimates are locally uniform in (z_0, w_0) , which implies the hyperbolicity of X_u . $\square$

Proof of Lemma 2. For any $z \neq 0$, there exists k_0 such that for any $k \geq k_0$, $|z_k| \leq \frac{1}{2}|z|$, so in a neighborhood U of z , the p -th derivative of $\log(\delta_k^2 + |z - z_k|^2)$ is bounded above by a constant depending only on p and U . The smoothness of u on U follows from the convergence of $\sum \alpha_k$. Each term of the series is subharmonic, and $\log(\delta_k^2 + |z - z_k|^2) - \log 5 < 0$, $\sum \alpha_k \log 5 < +\infty$, so that $u \in \text{SH}(\mathbb{D}) \cup \{-\infty\}$ as the limit of the decreasing sequence of subharmonic functions

$$\sum_{k=1}^n \alpha_k \log \left(\frac{\delta_k^2 + |z - z_k|^2}{5} \right) + \left(\sum_1^\infty \alpha_k \right) \log 5,$$

and

$$u(0) = \sum_k \alpha_k \log(\delta_k^2 + |z_k|^2) \geq 2 \sum \alpha_k \log |z_k| > -\infty$$

by (i). This also proves that $u \not\equiv -\infty$. Finally,

$$u(z_k) \leq \alpha_k \log(\delta_k^2) + \left(\sum_{j \neq k} \alpha_j \right) \log 5,$$

which goes to $-\infty$ as k goes to ∞ . $\square$

Proof of the Theorem. Let $f = (f_1, f_2) \in \text{Hol}(\mathbb{D}^*, X_u)$. If each f_j extends to $\tilde{f}_j \in \text{Hol}(\mathbb{D}, \mathbb{C})$ then $v(\zeta) := \log|\tilde{f}_1(\zeta)| + u(\tilde{f}_2(\zeta)) < 0$ for any $\zeta \in \mathbb{D}^*$, and since $v \in \text{SH}(\mathbb{D})$, $v(0) < 0$. This means that $(\tilde{f}_1(0), \tilde{f}_2(0)) \in X_u$ and $\tilde{f} \in \text{Hol}(\mathbb{D}, X_u)$.

By Riemann's removable singularity theorem, f_1 extends to $\tilde{f}_1 \in \text{Hol}(\mathbb{D}, \mathbb{D})$. If $\tilde{f}_1(0) \neq 0$, then there exists $\varepsilon > 0$ such that for $|\zeta| < \varepsilon$, $|f_1(\zeta)| > \frac{1}{2}|\tilde{f}_1(0)|$ and so f_2 is bounded on $D(0, \varepsilon)$ by $\exp\{-\inf_{|z| \geq |\tilde{f}_1(0)|/2} u(z)\}$, so it must extend as well and we are done.

Now suppose that $\tilde{f}_1(0) = 0$. Then there exists $m \in \mathbb{Z}_+^*$ such that $\tilde{f}_1(\zeta) = \sum_{k \geq m} a_k \zeta^k$, with $a_m \neq 0$, and thus there is $r_0 > 0$ such that $\tilde{f}_1(\zeta) = h(\zeta)^m$ where $h \in \text{Hol}(D(0, r_0))$, $h(0) = 0$, and h is one-to-one on a neighborhood U of 0. It will be enough to show that $f_2|_{U \setminus \{0\}}$ extends to U , or equivalently that $f_2 \circ h^{-1}(\xi)$ extends from $h(U) \setminus \{0\}$ to $h(U)$.

Lemma 3. *There exist sequences $\{z_k\}$, $\{\alpha_k\}$, and $\{\delta_k\}$ such that for any $\varepsilon > 0$, $\varlimsup_{|z| \rightarrow 0} (-u(z) + \varepsilon \log |z|) < 0$. For instance, $z_k = 2^{-k}$, $\delta_k = k^{-1/\alpha_k}$ will do.*

Taking u as in the Lemma, and $\xi \in h(U) \setminus \{0\}$,

$$|f_2 \circ h^{-1}(\xi)| \leq |\exp(-u(f_1 \circ h^{-1}(\xi)))| = |\exp(-u(\xi^m))| \leq |\xi^m|^{-\varepsilon} = |\xi|^{-m\varepsilon}$$

for $|\xi|$ small enough. Taking $\varepsilon m < 1$, we can apply the classical removable singularity theorem to obtain the desired extension of $f_2 \circ h^{-1}$.

Proof. Given $z \in \mathbb{D}$,

$$\begin{aligned} -u(z) &= \sum_{k: |z-z_k| < |z|/2} \alpha_k |\log(\delta_k^2 + |z-z_k|^2)| \\ &\quad + \sum_{k: |z-z_k| \geq |z|/2} \alpha_k |\log(\delta_k^2 + |z-z_k|^2)| \end{aligned}$$

If $|z| \leq \frac{1}{2}|z_k|$, then $|z-z_k| \geq |z_k| - |z| \geq \frac{1}{2}|z_k|$, while if $|z| \geq \frac{1}{2}|z_k|$ and $|z-z_k| \geq \frac{1}{2}|z|$, then $|z-z_k| \geq \frac{1}{4}|z_k|$, so

$$\sum_{k: |z-z_k| \geq |z|/2} \alpha_k |\log(\delta_k^2 + |z-z_k|^2)| \leq \sum_k \alpha_k \left| \log \left(\frac{1}{16} |z_k|^2 \right) \right| < +\infty.$$

Choosing $z_k = 2^{-k}$, we see that $\sum_{k: |z-z_k| < |z|/2}$ has at most 2 terms, each bounded by $2\alpha_k |\log \delta_k|$, so $-u(z) \leq C_1 \alpha_k |\log \delta_k| + C_2$.

Finally, $|z-z_k| < \frac{1}{2}|z| \implies |z| > \frac{2}{3}|z_k|$ and $|z| < 2|z_k|$, so $k \leq C_3 |\log |z|| + C_4$. Our choice of δ_k means that $\alpha_k |\log \delta_k| = |\log k|$, so

$$-u(z) \leq C_1 |\log k| + C_2 \leq C'_1 \log(|\log |z||) + C'_2,$$

which implies the desired inequality. $\square$

REFERENCES

- [1] M. JARNICKI & P. PFLUG, *Invariant Distances and Metrics in Complex Analysis*, vol. 9, de Gruyter, Expositions in Mathematics, Berlin, New York, 1993.
- [2] N. KERZMAN & J. P. ROSAY, *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut*, Math. Ann. **257** (1981), 171-184.
- [3] S. KOBAYASHI, *Hyperbolic Manifolds and Holomorphic Mappings*, N. Y. Dekker (1970).
- [4] M. KWACK, *Generalization of the big Picard theorem*, Ann. Math. **90** (1969), 9-22.
- [5] J. NOGUCHI & T. OCHIAI, *Geometric Function Theory in Several Complex Variables*, vol. 80, Amer. Math. Soc., Translation of Math. Monographs, Providence, 1990.

- [6] H.L. ROYDEN, *Remark on the Kobayashi metric*, Lecture Notes in Mathematics no 185, Springer Verlag, New York, Heidelberg, Berlin, 1971, pp. 125-137.
- [7] B.V. SHABAT, *Introduction to Complex Analysis, Part II. Functions of Several Complex Variables*, vol. 110, Amer. Math. Soc., Translations of Math. Monographs, Providence.
- [8] DO DUC THAI, *On the D^* -extension and the Hartogs extension*, Annali della Scuo. Nor. Sup. di Pisa, Scienze Fisiche e Mate., Serie IV **8** (1991), 13-38.

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