Stability of Solutions of Varying Degenerate Elliptic Equations

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ABSTRACT. Let $1 < p_0 < \infty$, $s > p_0$ and $p_i \rightarrow p_0$. For fixed $\psi, \theta \in W^{1,s}(\Omega)$ consider the solution u_i to the obstacle problem associated with a second order quasilinear degenerate elliptic equation $\nabla \cdot A_{p_i}(x, \nabla u) = 0$ with obstacle ψ and boundary values θ . Here $|A_{p_i}(x, \xi)| \approx |\xi|^{p_i}$ and the functions A_{p_i} have uniformly bounded structure constants. If for a.e. $x \in \Omega$, $A_{p_i}(x, \xi) \rightarrow A_{p_0}(x, \xi)$ uniformly on compact subsets of $\mathbb{R}^n$, then it is shown that $u_i \rightarrow u_0$ in $W^{1,t}(\Omega)$ where u_0 is the corresponding solution to $\nabla \cdot A_{p_0}(x, \nabla u) = 0$ and $t > p_0$ depends on n, s, p_0 , the structure constants, and the regularity of Ω .

1. Introduction. Many applications lead to differential equations which are only known approximately. In most cases this concerns the coefficients but in the nonlinear case other parameters could be uncertain as well. In this paper we are interested in the nonlinear degenerate elliptic equation

$$(1.1) \quad \nabla \cdot A(x, \nabla u) = 0$$

in divergence form and with measurable coefficients where $|A(x, h)| \approx |h|^{p-1}$. We study the dependence of some classes of solutions to (1.1) on A and this includes the dependence on $p > 1$ as well.

A typical situation where the equation (1.1) has a unique solution is the Dirichlet boundary value problem and the obstacle problem. We keep the boundary values and the obstacle fixed; the dependence on this data is studied in [LM2] and the combination of the results in [LM2] and the results here give a rather complete description on the stability of the solutions with respect to all data. Let $p_i \rightarrow p_0 \in (1, \infty)$ and $A_{p_i} \rightarrow A_{p_0}$ in the sense specified in Section 2. Consider the solution u_i to the equation $\nabla \cdot A_{p_i}(x, \nabla u) = 0$, $i = 0, 1, \dots$, with respect to the given data in a bounded domain Ω of $\mathbb{R}^n$. Our result says that $u_i \rightarrow u_0$ in the Sobolev space $W^{1,p_0+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ whose dependence on A_i , $i = 0, 1,$

..., and the data will also become clear. The proof consists of two parts. First we show that $u_i \rightarrow \bar{u}$ in $W^{1,p_0+\varepsilon}(\Omega)$ and then that $\bar{u} = u_0$. It is rather easy to see that $\bar{u}$ is a local solution or a supersolution to $\nabla \cdot A_{p_0}(x, \nabla u) = 0$. For $\bar{u} = u_0$ the cases $p_i \nearrow p_0$ and $p_i \searrow p_0$ impose different difficulties.

The higher integrability of the gradients of solutions and supersolutions to (1.1) plays an essential role. Since we are interested in the global situation, the regularity of Ω as well as the regularity of the boundary data and the obstacle are important. These questions have been studied for a fixed equation in [G, HKM, I, and LM1], see also references therein. We employ uniform estimates from the previous works.

The behavior of extremals of the variational integral

$$(1.2) \quad \int_{\Omega} |\nabla u|^p dx$$

with respect to p has been studied in [Li]. This corresponds to the p -harmonic case $A_p(x, h) = |h|^{p-2}h$. The obstacle problem, essential to the theory of supersolutions, has not been considered previously.

Section 2 contains the assumptions and the main results. Section 3 is devoted to the preliminary lemmas and the main result is proved in Section 4.

2. Main results. To formulate our results we need some notation. Let $1 < p < \infty$, $0 < \alpha \leq \beta < \infty$, and Ω a bounded open set in $\mathbb{R}^n$. We consider the second order quasilinear degenerate elliptic equation

$$(2.1) \quad \nabla \cdot A_p(x, \nabla u) = 0$$

where the mapping $A_p : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the assumptions:

(a) the mapping $x \mapsto A_p(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$.

For all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$:

(b) $A_p(x, \xi) \cdot \xi \geq \alpha |\xi|^p$,

(c) $|A_p(x, \xi)| \leq \beta |\xi|^{p-1}$,

(d) $(A_p(x, \xi_1) - A_p(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$ whenever $\xi_1 \neq \xi_2$.

In the sequel we will consider operators A_p with varying p . Note that we shall keep α and β fixed.

For $1 < p < \infty$ we let $W^{1,p}(\Omega)$ denote the Sobolev space of functions $u \in L^p(\Omega)$ whose distributional gradient ∇u belongs to $L^p(\Omega)$. In $W^{1,p}(\Omega)$ we use the standard norm

$$\|u\|_{1,p} = \|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p$$

where $\|u\|_p = \|u\|_{L^p(\Omega)}$. The space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Next we recall the obstacle problem associated with (2.1).

Let $\theta \in W^{1,p}(\Omega)$ and let $\psi : \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be measurable. Write

$$\mathcal{K}_{\psi,\theta}^p = \{u \in W^{1,p}(\Omega) : u - \theta \in W_0^{1,p}(\Omega), u \geq \psi \text{ a.e.}\}.$$

A function $u \in W^{1,p}(\Omega)$ is a solution to the $\mathcal{K}_{\psi,\theta}^p$ -obstacle problem of the equation (2.1) if $u \in \mathcal{K}_{\psi,\theta}^p$ and if

$$(2.2) \quad \int_{\Omega} A_p(x, \nabla u(x)) \cdot \nabla(v - u)(x) dx \geq 0$$

for all $v \in \mathcal{K}_{\psi,\theta}^p$. If $\mathcal{K}_{\psi,\theta}^p \neq \emptyset$, then there is a unique solution to the $\mathcal{K}_{\psi,\theta}^p$ -obstacle problem, see [HKM, pp. 60–61].

Let Ω be a bounded domain in $\mathbb{R}^n$. We say that Ω^c is thick if for each cube $Q(r)$ with $Q(3r/2) \cap \Omega^c \neq \emptyset$ we have

$$(2.3) \quad |\Omega^c \cap Q(2r)| \geq \mu |Q(2r)|$$

where $\mu > 0$ is independent of the cube $Q(r)$. Here and in the sequel $Q(\lambda r)$ denotes the cube parallel to $Q(r)$ with the same center and side length λr . We let $|A|$ stand for the n -measure of $A \subset \mathbb{R}^n$.

Let $1 < p_0 < \infty$ and $s > p_0$. Suppose that $\psi, \theta \in W^{1,s}(\Omega)$ where Ω is a bounded domain satisfying (2.3) and $\mathcal{K}_{\psi,\theta}^s \neq \emptyset$. Next suppose that (p_i) is a sequence such that $p_i \rightarrow p_0$ and that for each $i = 0, 1, \dots$ there is an operator $A_i = A_{p_i}$ satisfying (a)–(d) for $p = p_i$ and for a.e. $x \in \Omega$

$$(2.4) \quad A_i(x, \xi) \rightarrow A_0(x, \xi)$$

uniformly on compact subsets of $\mathbb{R}^n$. Now for $i = 0$ and for large i , in fact for $p_i \leq s$, there is a unique solution u_i to the $\mathcal{K}_{\psi,\theta}^{p_i}$ -obstacle problem for the equation $\nabla \cdot A_i(x, \nabla u) = 0$. Our main theorem now takes the following form.

2.5. Theorem. *There is $t > p_0$ such that $u_i \rightarrow u_0$ in $W^{1,t}(\Omega)$. Moreover, t depends only on n, p_0, s, α, β , and μ , where α and β are as in (b), (c), and μ as in (2.3).*

2.6. Remarks.

- (a) It is possible to consider the cases $p_i \nearrow p_0$ and $p_i \searrow p_0$ separately. In the case $p_i \nearrow p_0$ there is no need to assume that $\psi, \theta \in W^{1,s}(\Omega)$ for $s > p_0$. However, then it is not possible to achieve a global convergence in $W^{1,t}(\Omega)$, $t > p_0$. For simplicity, we consider both cases simultaneously.
- (b) The condition (2.3) for Ω^c is too strong. In fact, this condition is only used for the optimal Sobolev inequality and for a conclusion that a certain function belongs to $W_0^{1,p_0}(\Omega)$, see Lemma 3.25. For these purposes some capacity thickness conditions for Ω^c suffice. These conditions depend on the

exponent p , and the condition (2.3) has been chosen because it provides a simple sufficient condition independent of $p > 1$. Note that (2.3) is satisfied if $\partial\Omega$ is smooth.

A much simplified version of the proof for Theorem 2.5 produces a stability result for solutions of the equations $\nabla \cdot A_i(x, \nabla u) = 0$. We recall the necessary concepts. Let $\theta \in W^{1,p}(\Omega)$. A function $u \in W^{1,p}(\Omega)$ is called the solution of $\nabla \cdot A_p(x, \nabla u) = 0$ with boundary value θ if $u - \theta \in W_0^{1,p}(\Omega)$ and if

$$\int_{\Omega} A_p(x, \nabla u(x)) \cdot \nabla \varphi(x) dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. If A_p satisfies (a)–(d), then there is a unique solution u with boundary values θ , see [HKM].

To formulate the result let $p_0 > 1$ and $p_i \rightarrow p_0$. Suppose that $\theta \in W^{1,s}(\Omega)$, $s > p_0$, where Ω satisfies (2.3). Let u_i be the solution of $\nabla \cdot A_{p_i}(x, \nabla u) = 0$ with boundary values θ ; we assume that $A_i = A_{p_i}$ satisfies (a)–(d), $i = 0, 1, \dots$ and that (2.4) holds.

2.8. Theorem. *There is $t > p_0$ such that $u_i \rightarrow u_0$ in $W^{1,t}(\Omega)$. The constant t depends on the same parameters as in Theorem 2.5.*

We shall not prove Theorem 2.8. It is enough to note that in this case the obstacle ψ can be completely omitted from considerations and as a test function φ in (2.7) it is possible to use any φ in $W_0^{1,p}(\Omega)$.

2.9. Remark. The method of Theorem 2.5 can be employed to produce other stability results as well. In particular, local stability results are much easier since a local solution or supersolution for $\nabla \cdot A_p(x, \nabla u) = 0$ is not uniquely defined and hence a stability result of this kind would only state that a sequence of solutions or supersolutions (u_i) for $\nabla \cdot A_{p_i}(x, \nabla u) = 0$ converges to some solution or supersolution for $\nabla \cdot A_{p_0}(x, \nabla u) = 0$.

3. Preliminary results: Higher integrability, zero boundary values. In this section we present a global higher inequality result for solutions to the obstacle problem associated with the equation $\nabla \cdot A(x, \nabla u) = 0$. There is an abundance of results of this kind. However, global results are more difficult to obtain and we need results which are uniform with respect to the exponent $p > 1$. Our result is an extension of [LM1, Theorem B] in this sense. The most convenient real analytic lemma for our purposes is [I, Proposition 6.1], which gives explicit bounds for higher integrability exponents. We have also taken some care to collect all the other estimates in such a form that the dependence on p is uniform.

Our second preliminary result gives a sufficient condition such that each function $u \in W^{1,p}(\Omega)$ with $u \in W_0^{1,t}(\Omega)$, $t < p$, also belongs to $W_0^{1,p}(\Omega)$. Here we use a recent result from [KinM].

We shall use the following convention. Suppose that the parameter t_i belongs to an interval Δ_i , $i = 1, 2, \dots$. We say that a constant $C = C(t_1, t_2, \dots)$ satisfies $C < \infty$ uniformly for $t_1 \in \Delta_1$, say, if for each $t_2, t_3, \dots$ and for each compact subset $F \subset \Delta_1$ there is $c_0 < \infty$ such that $C \leq c_0$ for all $t_1 \in F$. Obviously this definition extends to several parameters and to the case $C > 0$ as well. Various constants, denoted usually by C , may differ even on the same line.

For the higher integrability result in the obstacle problems let Ω be a bounded domain with the property (2.3) and let $A = A_p$ satisfy (a)–(d) in Ω . Suppose that $\psi, \theta \in W^{1,p}(\Omega)$ and that u is a solution to the $\mathcal{K}_{\psi,\theta}^p$ -obstacle problem; the existence requires that $\mathcal{K}_{\psi,\theta}^p \neq \emptyset$. We let $d(\Omega)$ denote the diameter of Ω .

3.1. Lemma. *There exists a constant $\varepsilon_0 = \varepsilon_0(n, p, \alpha, \beta, \mu) > 0$ uniformly for $p > 1$ such that for each $\varepsilon \in [0, \varepsilon_0]$*

$$(3.2) \quad \int_{\Omega} |\nabla u|^{p+\varepsilon} dx \leq C \left[\left(\int_{\Omega} |\nabla u|^p dx \right)^{1+\varepsilon/p} + \int_{\Omega} |\nabla \psi|^{p+\varepsilon} dx + \int_{\Omega} |\nabla \theta|^{p+\varepsilon} dx \right]$$

where $C = C(n, p, \alpha, \beta, \mu, d(\Omega)) < \infty$ uniformly for $p > 1$.

3.3. Remark. The proof for Lemma 3.1 shows that the constants ε_0 and C satisfy $\varepsilon_0 > 0$ and $C < \infty$ uniformly for $p > 1$, $0 < \alpha \leq \beta < \infty$, $\mu \in (0, 1]$ and $d(\Omega) > 0$ and not for $p > 1$ only. Note that the right hand side of (3.2) is ∞ , if $\nabla \psi$ or $\nabla \theta$ does not belong to $L^{p+\varepsilon}$. Lemma 3.1 will be used when ψ and θ belong to the higher Sobolev space.

For the proof of Lemma 3.1 various Sobolev and Poincaré type inequalities are needed. We will employ the following Poincaré inequality for a function $u \in W^{1,p}(Q)$, $p > 1$, in a cube $Q \subset \mathbb{R}^n$;

$$(3.4) \quad \left(\int_Q |u - u_Q|^q dx \right)^{1/q} \leq C \left(\int_Q |\nabla u|^p dx \right)^{1/p}$$

where u_Q is the mean value of u in Q and

$$(3.5) \quad C = c(n) \left(\frac{1 - \frac{1}{p} + \frac{1}{q}}{\frac{1}{n} - \frac{1}{p} + \frac{1}{q}} \right)^{1-1/p+1/q} |Q|^{1/n-1/p+1/q}$$

for $q \geq p$ and $q < np/(n-p)$ for $p < n$. For $p < n$ and $q = np/(n-p)$ the inequality (3.4) holds with

$$(3.6) \quad C = c(n) \left(\frac{p}{n-p} \right) \left(\frac{p}{p-1} \right)^{(n-p)/np}.$$

The inequality (3.4) follows from the potential estimate [GT, Lemma 7.16, p. 162] together with [GT, Lemma 7.12, p. 159]; for the case $q = np/(n-p)$ the aforementioned potential estimate with [Z, Theorem 2.8.4] yields the result.

If $u \in W_0^{1,p}(\Omega)$, $p \geq 1$, then

$$(3.7) \quad \|u\|_p \leq c(n) \|\nabla u\|_p,$$

see [GT, 7.44]. In fact, the method used in the proof of Lemma 3.8 below provides a simple and explicit approach to (3.4) and (3.7).

3.8. Lemma. *Suppose that Ω is a bounded domain with the property (2.3) and that $q \geq 1$. Let $Q(r)$ be a cube with side length r such that $Q(3/2r) \cap \Omega^c \neq \emptyset$. Then for each $u \in W^{1,q}(Q(2r))$*

$$(3.9) \quad \int_{Q(2r)} |u|^q dx \leq Cr^q \int_{Q(2r)} |\nabla u|^q dx$$

provided that $u = 0$ on $Q(2r) \cap \Omega^c$. The constant $C = C(n, q, \mu)$ satisfies $C < \infty$ uniformly for $q \geq 1$.

3.10. Remark. Lemma 3.8 is far from the best possible. The sharp versions of (3.9) assume that $u = 0$ on $K \cap Q(2r)$ where K has positive $(1, q)$ -capacity only. In this case the function u is defined in the capacity sense, see [AH, p. 231] for the history of the problem. The inequality (3.9) also appears for $q = 1$ in [LU, Lemma 3.4, p. 76]. However, we need an explicit bound for the constant C in order to take the dependence on q into account. Hence we present a simple proof for Lemma 3.8.

Proof for Lemma 3.8. Write $Q = Q(2r) = \{x : 0 < x_i < 2r\}$ and choose a sequence of functions $u_i \in C^1(Q)$ such that $u_i \rightarrow u$ a.e. and $u_i \rightarrow u$ in $W^{1,p}(Q)$. Fix $\varepsilon > 0$ and let

$$A_i = \{x \in Q : |u_i(x)| < \varepsilon\}.$$

Since $\bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i$ contains almost all the points of the set $A = \{u(x) = 0\}$ and since $\bigcap_{i=j}^{\infty} A_i \subset \bigcap_{i=j+1}^{\infty} A_i$,

$$|A| \leq \left| \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i \right| = \lim_{j \rightarrow \infty} \left| \bigcap_{i=j}^{\infty} A_i \right| \leq \liminf_{j \rightarrow \infty} |A_j|.$$

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be points in Q . Integrating the partial derivatives of u_j along a path consisting of line segments parallel to coordinate

axis we obtain

$$\begin{aligned} |u_j(x) - u_j(y)| &\leq |u_j(x) - u_j(y)| \\ &\leq \sum_{i=1}^n \int_0^{2r} |\nabla u_j(y_1, \dots, y_{i-1}, s, x_{i+1}, \dots, x_n)| ds. \end{aligned}$$

The Hölder inequality yields

$$|u_j(x)|^q \leq (n+1)^{q-1} \left[\sum_{i=1}^n (2r)^{q-1} \int_0^{2r} |\nabla u_j(y_1, \dots, x_n)|^q ds + |u_j(y)|^q \right].$$

Integrating with respect to $y \in A_j$ and then with respect to $x \in Q$ we have by the Fubini theorem

$$|A_j| \int_Q |u_j|^q dx \leq (n+1)^{q-1} \left[\sum_{i=1}^n (2r)^{q-1+n+1} \int_Q |\nabla u_j|^q dx + \varepsilon^q |A_j| |Q| \right].$$

Hence

$$|A_j| \int_Q |u_j|^q dx \leq (n+1)^{q-1} \left[n(2r)^{q+n} \int_Q |\nabla u_j|^q dx + \varepsilon^q |A_j| |Q| \right]$$

and thus by (2.3)

$$\begin{aligned} \int_Q |u_j|^q dx &\leq (n+1)^{q-1} \left[n \frac{|Q|}{|A_j|} (2r)^q \int_Q |\nabla u_j|^q dx + \varepsilon^q |Q| \right] \\ &\leq (n+1)^{q-1} \left[\frac{n}{\mu} (2r)^q \int_Q |\nabla u_j|^q dx + \varepsilon^q |Q| \right]. \end{aligned}$$

Letting $j \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ we obtain

$$\int_Q |u|^q dx \leq \frac{(n+1)^q}{\mu} (2r)^q \int_Q |\nabla u|^q dx$$

as required. $\square$

Now the previous lemma and (3.4) easily provide the following result:

3.11. Lemma. *Let $Q(r)$ and Ω be as in Lemma 3.8 with $r \leq 2d(\Omega) \leq 2$. If $u \in W^{1,p}(Q(2r))$, $p > 1$, and if $u = 0$ in $Q(2r) \cap \Omega^c$, then*

$$(3.12) \quad \left(\int_{Q(2r)} |u|^p dx \right)^{1/p} \leq C \left(\int_{Q(2r)} |\nabla u|^q dx \right)^{1/q}$$

where $q = pn/(n+p)$ and the constant $C = C(n, p, \mu)$ satisfies $C < \infty$ uniformly for $p > 1$.

Proof. Since $q \leq p$, Lemma 3.8 yields

$$(3.13) \quad \int_{Q(2r)} |u|^q dx \leq Cr^q \int_{Q(2r)} |\nabla u|^q dx.$$

Now let c_u be the mean value of u in $Q(2r)$. By (3.4)

$$(3.14) \quad \left(\int_{Q(2r)} |u - c_u|^p dx \right)^{1/p} \leq C \left(\int_{Q(2r)} |\nabla u|^q dx \right)^{1/q}$$

where the constant C in (3.5) can be estimated from above in terms of $d(\Omega)$ because $r \leq 4d(\Omega)$. From (3.13) and (3.14) we obtain

$$\begin{aligned} \left(\int_{Q(2r)} |u|^p dx \right)^{1/p} &\leq \left(\int_{Q(2r)} |u - c_u|^p dx \right)^{1/p} + |Q(2r)|^{1/p} |c_u| \\ &\leq C \left(\int_{Q(2r)} |\nabla u|^q dx \right)^{1/q} + |Q(2r)|^{-1/n} \left(\int_{Q(2r)} |u|^q dx \right)^{1/q} \\ &\leq C \left(\int_{Q(2r)} |\nabla u|^q dx \right)^{1/q}. \end{aligned}$$

The inspection of the constant C on the last line easily gives the desired property. The lemma follows. $\square$

Our last lemma is the important higher integrability condition. The formulation due to Iwaniec [I, Proposition 6.1] is convenient for our purposes because the constants have been computed explicitly.

3.15. Lemma ([I, Proposition 6.1]). *Let $Q_0 = Q_0(l)$ be a cube in $\mathbb{R}^n$ and let $g, h \in L^q(Q_0)$, $1 < q < \infty$, be nonnegative functions satisfying*

$$(3.16) \quad \left(\int_{Q(r)} g^q dx \right)^{1/q} \leq K \int_{Q(2r)} g dx + \left(\int_{Q(2r)} h^q dx \right)^{1/q}$$

for all cubes $Q(r) \subset Q(2r) \subset Q_0$ parallel to Q_0 . Then for each $\sigma \in (0, 1)$ and

$$q \leq s < q + \frac{q-1}{10^{n-q} 4^n K^q}$$

we have

$$(3.17) \quad \left(\int_{Q_0(\sigma l)} g^s dx \right)^{1/2} \leq C \left[\left(\int_{Q_0} g^q dx \right)^{1/q} + \left(\int_{Q_0} h^s dx \right)^{1/s} \right]$$

where

$$C = \frac{100^n}{\sigma^{n/s}(1-\sigma)^{n/q}}.$$

Proof for Lemma 3.1. We prove the lemma for $d(\Omega) \leq 1$ with the constant C in (3.2) independent of $d(\Omega)$. For $d(\Omega) > 1$ an application of a dilation map produces the required result with the same ε_0 and with a constant C depending on $d(\Omega)$ as well.

Pick a cube $Q_0 = Q_0(2r_0)$ such that $\Omega \subset Q_0(r_0)$ and $r_0 \leq d(\Omega) \leq 1$. Next take an arbitrary cube $Q(2r)$ in Q_0 . There are two possibilities

- (i) $Q(3r/2) \subset \Omega$ or
- (ii) $Q(3r/2) \cap \Omega^c \neq \emptyset$.

In the case (i) we can use the previous estimates just as in [LM1, Proof for Theorem B, p. 31] to obtain

$$(3.18) \quad \int_{Q(r)} |\nabla u|^p dx \leq C \left[\left(\int_{Q(3r/2)} |\nabla u|^{np/(n+p)} dx \right)^{(p+n)/n} + \int_{Q(3r/2)} |\nabla \psi|^p dx \right]$$

where $C = C(n, p, \alpha, \beta) < \infty$ uniformly for $p > 1$.

In the case (ii) we can proceed similarly as in [LM1, Proof for Theorem B, pp. 31–32] and obtain

$$(3.19) \quad \int_{\Omega} \varphi^p |\nabla u|^p dx \leq C \left[r^{-p} \left(\int_{Q(2r) \cap \Omega} |\nabla u|^{np/(n+p)} dx \right)^{(p+n)/n} + \int_{Q(2r) \cap \Omega} |\nabla \theta|^p dx \right]$$

where $C = C(n, p, \alpha, \beta, \mu) < \infty$ uniformly for $p > 1$ and where φ is a cut off function such that $\varphi \in C_0^\infty(Q(2r))$, $0 \leq \varphi \leq 1$, $|\nabla \varphi| \leq C(n)/r$ and $\varphi = 1$ on $Q(r)$.

For the inequalities (3.18) and (3.19) we have employed the uniform estimates (3.4) and (3.12).

To complete the proof for the inequality (3.2) we let

$$g = |\nabla u|^{np/(n+p)}, \quad f = (|\nabla \theta| + |\nabla \psi|)^{np/(n+p)}$$

in Ω and $g = 0 = f$ in Ω^c . By (3.18) and (3.19) for any $Q(2r) \subset Q_0$ we have

$$(3.20) \quad \int_{Q(r)} g^q dx \leq C_1 \left[\left(\int_{Q(2r)} g dx \right)^q + \int_{Q(2r)} f^q dx \right]$$

where $q = (p+n)/n$ and $C_1 = C_1(n, p, \alpha, \beta, \mu) < \infty$ uniformly for $p > 1$; this is also true for all the constants C which will appear in the rest of the proof. Rewriting (3.20) in the form

$$(3.21) \quad \left(\int_{Q(r)} g^q dx \right)^{1/q} \leq C_1^{1/q} \int_{Q(2r)} g dx + \left(\int_{Q(2r)} (Cf)^q dx \right)^{1/q}$$

we obtain from Lemma 3.15 with $\sigma = 1/2$ that

$$(3.22) \quad \left(\int_{Q_0(r_0)} g^s dx \right)^{1/s} \leq C \left[\left(\int_{Q_0} g^q dx \right)^{1/q} + \left(\int_{Q_0} f^s dx \right)^{1/s} \right]$$

for any $s \in [q, q + \varepsilon_1)$ where $\varepsilon_1 = \varepsilon_1(n, p, \alpha, \beta, \mu) > 0$ uniformly for $p > 1$. Writing (3.22) with the original exponents we obtain

$$(3.23) \quad \int_{\Omega} |\nabla u|^{nps/(n+p)} dx \leq C \left[\left(\int_{\Omega} |\nabla u|^{npq/(n+p)} dx \right)^{s/q} + \int_{\Omega} (|\nabla \theta| + |\nabla \psi|)^{nps/(n+p)} dx \right].$$

We fix $\varepsilon_0 = np\varepsilon_1/2(n+p) > 0$ and let $\varepsilon \in [0, \varepsilon_0]$. For

$$s = \frac{n+p}{n} \left(1 + \frac{\varepsilon}{p} \right) \leq q + \varepsilon_1/2$$

(3.23) yields

$$(3.24) \quad \int_{\Omega} |\nabla u|^{p+\varepsilon} dx \leq C \left[\left(\int_{\Omega} |\nabla u|^p dx \right)^{1+\varepsilon/p} + \int_{\Omega} (|\nabla \theta| + |\nabla \psi|)^{p+\varepsilon} dx \right],$$

and this is the desired inequality (3.2) for the case $d(\Omega) \leq 1$. $\square$

Our last lemma concerns functions with zero boundary values in the Sobolev sense. The formulation is more general than what is needed in Section 4.

3.25. Lemma. Suppose that a bounded open set Ω satisfies (2.3). Let $p_i > 1$ and $u_i \in W^{1,p_i}(\Omega)$, $i = 0, 1, \dots$, and suppose that $p_i \rightarrow p_0$ and $u_i \rightarrow u_0$ a.e. in Ω . If $\theta \in W^{1,p_0+\varepsilon}(\Omega)$, $\varepsilon > 0$, and if $u_i - \theta \in W_0^{1,p_i}(\Omega)$ with

$$(3.26) \quad \int_{\Omega} |\nabla(u_i - \theta)|^{p_i} dx \leq M,$$

$i = 1, 2, \dots$, where $M < \infty$ is independent of i , then $u_0 - \theta \in W_0^{1,p_0}(\Omega)$.

3.27. Remarks.

- (a) The lemma is trivial if $p_i \searrow p_0$. In the case $p_i \nearrow p_0$ the condition $\theta \in W^{1,p_0+\varepsilon}(\Omega)$ can be replaced by $\theta \in W^{1,p_0}(\Omega)$.
- (b) We employ Lemma 3.25 in the situation where $u_i \in W^{1,p_0+\varepsilon}(\Omega)$, $i = 1, 2, \dots$, with $\|\nabla u_i\|_{p_0+\varepsilon} \leq C$. Then (3.26) is clear but the conclusion of the lemma is not trivial for $p_i \nearrow p_0$.

Proof for Lemma 3.25. Since Ω satisfies (2.3), Ω also satisfies the uniform Hardy inequality

$$(3.28) \quad \int_{\Omega} \left(\frac{|u(x)|}{d(x, \partial\Omega)} \right)^p dx \leq C \int_{\Omega} |\nabla u(x)|^p dx$$

for all $u \in W_0^{1,p}(\Omega)$ where $C = C(n, p, \mu) < \infty$ uniformly for $p > 1$. This is due to the fact that condition (2.3) gives a uniform lower estimate for the $(1, p)$ -capacity of $B(x, r) \cap \Omega^c$ in terms of the $(1, p)$ -capacity of $B(x, r)$ at each boundary point $x \in \partial\Omega$. See [Le, Mi, and KinM].

From (3.28) and (3.26) we obtain

$$\int_{\Omega} \left(\frac{|u_i - \theta|}{d(x, \partial\Omega)} \right)^{p_i} dx \leq C \int_{\Omega} |\nabla(u_i - \theta)|^{p_i} dx$$

and the Fatou Lemma gives

$$\int_{\Omega} \left(\frac{|u_0 - \theta|}{d(x, \partial\Omega)} \right)^{p_0} dx \leq \liminf_{i \rightarrow \infty} \int_{\Omega} \left(\frac{|u_i - \theta|}{d(x, \partial\Omega)} \right)^{p_i} dx \leq CM.$$

Hence

$$\frac{u_0 - \theta}{d(x, \partial\Omega)} \in L^{p_0}(\Omega)$$

and since $u_0 - \theta \in W^{1,p_0}(\Omega)$, [KinM, Theorem 3.13] implies that $u_0 - \theta \in W_0^{1,p_0}(\Omega)$ as required. $\square$

4. Proof for Theorem 2.5. Since $p_i \rightarrow p_0$ and since $s > p_0$, we may assume that $p_i < s$ for all i . Then $\max(\theta, \psi) \in \mathcal{K}_{\psi, \theta}^{p_i}$ and hence there is a unique solution u_i , $i = 0, 1, \dots$, to the $\mathcal{K}_{\psi, \theta}^{p_i}$ -obstacle problem for the equation $\nabla \cdot A_{p_i}(x, \nabla u) = 0$. We write

$$A_i = A_{p_i}, \quad i = 0, 1, \dots$$

The proof is now carried out in several steps. The constant $t > p_0$ will always depend on the parameters of the theorem although it is necessary to make it smaller in certain steps of the proof. We first show that

$$(4.1) \quad \int_{\Omega} |\nabla u_i|^{p_i} dx \leq C$$

where $C < \infty$ is independent of $i = 0, 1, \dots$. Replacing θ by $\max(\theta, \psi)$ we may assume that $\theta \in \mathcal{K}_{\psi, \theta}^{p_i}$ and hence

$$\int_{\Omega} A_i(x, \nabla u_i) \cdot \nabla(\theta - u_i) dx \geq 0.$$

This yields

$$\begin{aligned} \alpha \int_{\Omega} |\nabla u_i|^{p_i} dx &\leq \int_{\Omega} A_i(x, \nabla u_i) \cdot \nabla u_i dx \\ &\leq \int_{\Omega} A_i(x, \nabla u_i) \cdot \nabla \theta dx \leq \beta \int_{\Omega} |\nabla u_i|^{p_i-1} |\nabla \theta| dx \\ &\leq \beta \left(\int_{\Omega} |\nabla u_i|^{p_i} dx \right)^{(p_i-1)/p_i} \left(\int_{\Omega} |\nabla \theta|^{p_i} dx \right)^{1/p_i} \end{aligned}$$

and hence

$$\|\nabla u_i\|_{p_i} \leq \frac{\beta}{\alpha} \|\nabla \theta\|_{p_i}.$$

By the Hölder inequality this gives

$$\int_{\Omega} |\nabla u_i|^{p_i} dx \leq \left(\frac{\beta}{\alpha} \right)^s \max(1, |\Omega|) \int_{\Omega} |\nabla \theta|^s dx = C,$$

and (4.1) follows.

Next we employ the higher integrability estimate of Lemma 3.1 and find $t \in (p_0, s]$ and $C < \infty$ such that

$$(4.2) \quad \int_{\Omega} |\nabla u_i|^t dx \leq C \left[\left(\int_{\Omega} |\nabla u_i|^{p_i} dx \right)^{1+(t-p_i)/p_i} + \int_{\Omega} |\nabla \psi|^t dx + \int_{\Omega} |\nabla \theta|^t dx \right].$$

This estimate holds for $i = 0$ and for large i ; hence we may assume that it holds for all i . Moreover, by the uniformity of the higher integrability estimate the constants C and t are independent of i . From (4.1) and (4.2) we obtain

$$(4.3) \quad \int_{\Omega} |\nabla u_i|^t dx \leq C, \quad i = 0, 1, \dots$$

where $C < \infty$ is independent of i .

The function $u_i - \theta$ belongs to $W_0^{1,p_i}(\Omega)$ and hence by the Sobolev imbedding theorem

$$(4.4) \quad \|u_i - \theta\|_{p_i^*} \leq C \|\nabla(u_i - \theta)\|_{p_i}$$

where p_i^* is the Sobolev conjugate exponent $p_i n / (n - p_i)$ if $p_i < n$ and $p_i^* = t$ if $p_i \geq n$. The constant C is again independent of i . Choosing $t \in (p_0, p_0^*)$ we thus obtain from (4.3) and (4.4)

$$(4.5) \quad \|u_i\|_t \leq C \|u_i\|_{p_i^*} \leq C [\|u_i - \theta\|_{p_i^*} + \|\theta\|_{p_i^*}] \leq C [\|\nabla(u_i - \theta)\|_{p_i} + \|\theta\|_{p_i^*}]$$

and this holds for $i = 0$ and for large i . Now (4.3) and (4.4) yield

$$(4.6) \quad \|u_i\|_{1,t} \leq C$$

where $C < \infty$ and $t \in [p_0, s]$ are independent of i ; the estimate (4.6) holds for $i = 0$ and for large i .

Since the Sobolev norms in (4.6) for $i = 1, 2, \dots$ are bounded, for some subsequence of u_i we have that $u_i \rightarrow \bar{u}$ weakly in $W^{1,t}(\Omega)$ and that

$$(4.7) \quad u_i \rightarrow \bar{u} \quad \text{a.e.}$$

$$u_i \rightarrow \bar{u} \quad \text{in } L^t(\Omega).$$

Note that in the sequel we shall prove that $\bar{u} = u_0$ and hence this convergence holds for the whole sequence, not only for some subsequence of the original sequence.

Now $u_i \geq \psi$ and hence by (4.7) $\bar{u} \geq \psi$ as well. On the other hand, $u_i - \theta \in W_0^{1,p_i}(\Omega)$ and Lemma 3.25 together with (4.6) and (4.7) implies that $\bar{u} - \theta \in W_0^{1,p_0}(\Omega)$. Thus we have shown that $\bar{u} \in \mathcal{K}_{\psi,\theta}^{p_0}$.

Next we shall show that

$$(4.8) \quad \nabla u_i \rightarrow \nabla \bar{u} \quad \text{a.e.}$$

Together with the weak convergence and (4.7) this implies that

$$(4.9) \quad u_i \rightarrow \bar{u} \quad \text{in } W^{1,t'}(\Omega)$$

for each $t' < t$. Thus a small adjustment of t gives

$$(4.10) \quad u_i \rightarrow \bar{u} \quad \text{in } W^{1,t}(\Omega)$$

where t depends only on n, p_0, α, β and μ .

To prove (4.8) we use a refinement of the method introduced by Kilpeläinen and Malý [KilM]. Since $u_i, i = 1, 2, \dots$, is the solution to an obstacle problem, it is also an A_i -supersolution, i.e.

$$(4.11) \quad \int_{\Omega} A_i(x, \nabla u_i) \cdot \nabla \varphi \, dx \geq 0$$

for each $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Choose subdomains $G \Subset G' \Subset \Omega$ and let $\varepsilon > 0$. Write

$$E_i^\varepsilon = \{x \in G' : (A_i(x, \nabla u_i) - A_i(x, \nabla \bar{u})) \cdot (\nabla u_i - \nabla \bar{u}) > \varepsilon\}$$

for $i = 1, 2, \dots$. Pick a cut off function $\eta \in C_0^\infty(G')$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on G . Set

$$w_i = \min((\bar{u} + \varepsilon^2 - u_i)^+, 2\varepsilon^2)$$

and note that the function ηw_i is nonnegative and $\eta w_i \in W_0^{1,t}(G') \subset W_0^{1,p_i}(G')$. Hence we can use ηw_i as a test function in (4.11) and this yields

$$\begin{aligned} (4.12) \quad & \int_{G' \cap \{|u_i - \bar{u}| < \varepsilon^2\}} A_i(x, \nabla u_i) \cdot (\nabla u_i - \nabla \bar{u}) \eta \, dx \\ & \leq \int_{G'} A_i(x, \nabla u_i) \cdot \nabla \eta w_i \, dx \leq C\varepsilon^2 \int_{G'} |A_i(x, \nabla u_i)| \, dx \\ & \leq C\varepsilon^2 \int_{G'} |\nabla u_i|^{p_i-1} \, dx \leq C\varepsilon^2 \left(\int_{\Omega} |\nabla u_i|^{p_i} \, dx \right)^{(p_i-1)/p_i} |\Omega|^{1/p_i} \\ & \leq C\varepsilon^2 \end{aligned}$$

where $C < \infty$ is independent of $i = 1, 2, \dots$

On the other hand,

$$\begin{aligned}
 (4.13) \quad & \frac{1}{\varepsilon} \int_{G' \cap \{|u_i - \bar{u}| < \varepsilon^2\}} (A_i(x, \nabla u_i) - A_i(x, \nabla \bar{u})) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \\
 &= \frac{1}{\varepsilon} \int_{G' \cap \{|u_i - \bar{u}| < \varepsilon^2\}} A_i(x, \nabla u_i) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \\
 &\quad - \frac{1}{\varepsilon} \int_{G' \cap \{|u_i - \bar{u}| < \varepsilon^2\}} (A_i(x, \nabla \bar{u}) - A_0(x, \nabla \bar{u})) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \\
 &\quad - \frac{1}{\varepsilon} \int_{G' \cap \{|u_i - \bar{u}| < \varepsilon^2\}} A_0(x, \nabla \bar{u}) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \\
 &= I_1^i + I_2^i + I_3^i
 \end{aligned}$$

and we have from (4.12) that

$$(4.14) \quad I_1^i \leq C\varepsilon.$$

The second integral can be estimated as follows:

$$\begin{aligned}
 (4.15) \quad |I_2^i| &\leq \frac{1}{\varepsilon} \|A_i(x, \nabla \bar{u}) - A_0(x, \nabla \bar{u})\|_{p'_i} \|\nabla \bar{u} - \nabla u_i\|_{p_i} \\
 &\leq \frac{C}{\varepsilon} \|A_i(x, \nabla \bar{u}) - A_0(x, \nabla \bar{u})\|_{p'_i}
 \end{aligned}$$

where p'_i is the conjugate exponent of p_i . We show that

$$(4.16) \quad \int_{\Omega} |A_i(x, \nabla \bar{u}) - A_0(x, \nabla \bar{u})|^{p'_i} dx \rightarrow 0$$

as $i \rightarrow \infty$. Now by the assumption (2.4)

$$A_i(x, \nabla \bar{u}(x)) \rightarrow A_0(x, \nabla \bar{u}(x))$$

a.e. in Ω and for any $E \subset \Omega$

$$\begin{aligned}
 & \int_E |A_i(x, \nabla \bar{u}) - A_0(x, \nabla \bar{u})|^{p'_i} dx \\
 & \leq C \left[\int_E |\nabla \bar{u}|^{p_i} dx + \int_E |\nabla \bar{u}|^{(p_0-1)p'_i} dx \right] \\
 & \leq C \left[|E|^{1-\frac{p_i}{t}} \left(\int_E |\nabla \bar{u}|^t dx \right)^{p_i/t} + |E|^{1-(p_0-1)p'_i/t} \left(\int_E |\nabla \bar{u}|^t dx \right)^{(p_0-1)p'_i/t} \right]
 \end{aligned}$$

where $C < \infty$ is independent of i . The right side can be made less than $\eta > 0$ if $|E| < \delta_\eta$. The Vitali convergence theorem now implies (4.16) and from (4.15) we obtain the estimate

$$(4.17) \quad |I_2^i| < \varepsilon$$

provided that i is large enough.

For the last integral we set

$$M_i = \{x \in G' : |u_i(x) - \bar{u}(x)| < \varepsilon^2\}.$$

Let χ_i and $\chi_{G'}$ be the characteristic functions of M_i and G' , respectively. Now

$$\begin{aligned} I_3^i &= \frac{1}{\varepsilon} \int_{G'} \chi_i A_0(x, \nabla \bar{u}) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \\ &= \frac{1}{\varepsilon} \int_{G'} (\chi_i - \chi_{G'}) A_0(x, \nabla \bar{u}) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \\ &\quad + \frac{1}{\varepsilon} \int_{\Omega} \chi_{G'} A_0(x, \nabla \bar{u}) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \end{aligned}$$

and since $\nabla u_i \rightarrow \nabla u$ weakly in $L^t(\Omega)$, and hence weakly in $L^{p_0}(\Omega)$, we have

$$\int_{\Omega} \chi_{G'} A_0(x, \nabla \bar{u}) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \rightarrow 0$$

as $i \rightarrow \infty$. For the first integral observe that

$$|(\chi_i - \chi_{G'}) A_0(x, \nabla \bar{u})| \leq 2|A_0(x, \nabla \bar{u})| \in L^{p'_0}(G')$$

and since $u_i \rightarrow u$ a.e. in G' ,

$$(\chi_i - \chi_{G'}) A_0(x, \nabla \bar{u}) \rightarrow 0 \quad \text{a.e. in } G'.$$

Thus the Lebesgue dominated convergence theorem and the Hölder inequality imply that the first integral converges to zero as $i \rightarrow \infty$. Consequently we have

$$(4.18) \quad |I_3^i| < \varepsilon$$

for large i .

Combining (4.14), (4.17), and (4.18) we obtain for large i

$$\frac{1}{\varepsilon} \int_{G' \cap \{|u_i - \bar{u}| < \varepsilon^2\}} (A_i(x, \nabla u_i) - A_i(x, \nabla \bar{u})) \cdot (\nabla u_i - \nabla \bar{u}) \eta dx \leq C\varepsilon$$

where C is independent of i and ε . Since $E_i^\varepsilon \subset G'$, the ellipticity assumption (d) yields

$$\begin{aligned}
 (4.19) \quad |E_i^\varepsilon| &\leq |E_i^\varepsilon \cap \{|u_i - \bar{u}| \geq \varepsilon^2\}| \\
 &\quad + \frac{1}{\varepsilon} \int_{E_i^\varepsilon \cap \{|u_i - \bar{u}| < \varepsilon^2\}} (A_i(x, \nabla u_i) - A_i(x, \nabla \bar{u})) \cdot (\nabla u_i - \nabla \bar{u}) dx \\
 &\leq |E_i^\varepsilon \cap \{|u_i - \bar{u}| \geq \varepsilon^2\}| + C\varepsilon \\
 &\leq C\varepsilon
 \end{aligned}$$

for large i , say $i \geq i_0$, because $u_i \rightarrow \bar{u}$ a.e. Now the ellipticity assumption, together with (4.19), easily shows that

$$(4.20) \quad \lim_{i \rightarrow \infty} I_i(x) = \lim_{i \rightarrow \infty} (A_i(x, \nabla u_i) - A_i(x, \nabla \bar{u})) \cdot (\nabla u_i - \nabla \bar{u}) = 0$$

for a.e. $x \in G'$. This is the main step for (4.8).

It suffices to show that (4.8) holds a.e. in G' . To this end let $x_0 \in G'$ with

- (i) $I_i(x_0) \rightarrow 0$ as $i \rightarrow \infty$,
- (ii) $|\nabla \bar{u}(x_0)| < \infty$,
- (iii) (b)–(d) hold for $x = x_0$ and $A = A_i$, $i = 0, 1, \dots$, and
- (iv) $A_i(x_0, \xi) \rightarrow A_0(x_0, \xi)$ locally uniformly in $\mathbb{R}^n$, see (2.4).

If $|\nabla u_i(x_0)| \rightarrow \infty$, then

$$\begin{aligned}
 &\alpha |\nabla u_i(x_0)|^{p_i} \\
 &\leq A_i(x_0, \nabla u_i(x_0)) \cdot \nabla u_i(x_0) \\
 &= (A_i(x_0, \nabla u_i(x_0)) - A_i(x_0, \nabla \bar{u}(x_0))) \cdot (\nabla u_i(x_0) - \nabla \bar{u}(x_0)) \\
 &\quad + A_i(x_0, \nabla \bar{u}(x_0)) \cdot (\nabla u_i(x_0) - \nabla \bar{u}(x_0)) + A_i(x_0, \nabla u_i(x_0)) \cdot \nabla \bar{u}(x_0) \\
 &\leq I_i(x_0) + \beta |\nabla \bar{u}(x_0)|^{p_i-1} (|\nabla u_i(x_0)| + |\nabla \bar{u}(x_0)|) + \beta |\nabla u_i(x_0)|^{p_i-1} |\nabla \bar{u}(x_0)|,
 \end{aligned}$$

and this is a contradiction as $i \rightarrow \infty$. Thus we may assume, passing to a subsequence if necessary, that

$$\nabla u_i(x_0) \rightarrow \xi \in \mathbb{R}^n.$$

If now $\xi \neq \nabla \bar{u}(x_0)$, then

$$0 = \lim_{i \rightarrow \infty} I_i(x_0) = (A_0(x_0, \xi) - A_0(x_0, \nabla \bar{u}(x_0))) \cdot (\xi - \nabla \bar{u}(x_0)) > 0$$

provides a contradiction. Since a.e. point $x_0 \in G'$ satisfies (i)–(iv), we have proved that $\nabla u_i \rightarrow \nabla u_0$ a.e. in G' and hence a.e. in Ω . This completes the proof for (4.8) and (4.9) also follows.

At the final step of the proof we will show that

$$(4.20) \quad u_0 = \bar{u}.$$

We know that $\bar{u} \in \mathcal{K}_{\psi, \theta}^{p_0}$ and hence

$$(4.21) \quad \int_{\Omega} A_0(x, \nabla u_0) \cdot (\nabla \bar{u} - \nabla u_0) dx \geq 0.$$

In order to obtain the inequality

$$(4.22) \quad \int_{\Omega} A_0(x, \nabla \bar{u}) \cdot (\nabla u_0 - \nabla \bar{u}) dx \geq 0$$

we use an approximation method; the difficulty is that we do not know that $u_0 - \theta \in W_0^{1,t}(\Omega)$ although $u_0, \theta \in W^{1,t}(\Omega)$.

For (4.22) we may again assume that $\theta \geq \psi$. Now $u_0 - \theta \in W_0^{1,p_0}(\Omega)$ and hence there is a sequence $\varphi_i \in C_0^\infty(\Omega)$ such that $\varphi_i \rightarrow u_0 - \theta$ in $W^{1,p_0}(\Omega)$. Since $\psi - \theta \leq 0$ and since φ_i has compact support,

$$\max(\varphi_i, \psi - \theta) \in W_0^{1,t}(\Omega)$$

and then $v_i - \theta \in W_0^{1,t}(\Omega)$ where $v_i = \max(\varphi_i, \psi - \theta) + \theta$. On the other hand

$$v_i \geq \psi - \theta + \theta = \psi \quad \text{a.e.}$$

and thus $v_i \in \mathcal{K}_{\psi, \theta}^t$. If $v \in \mathcal{K}_{\psi, \theta}^t$, then we may assume that $v \in \mathcal{K}_{\psi, \theta}^{p_i}$ for all $i = 0, 1, \dots$. Hence

$$\int_{\Omega} A_i(x, \nabla u_i) \cdot (\nabla v - \nabla u_i) dx \geq 0$$

and letting $i \rightarrow \infty$ we obtain from (4.10)

$$(4.23) \quad \int_{\Omega} A_0(x, \nabla \bar{u}) \cdot (\nabla v - \nabla \bar{u}) dx \geq 0.$$

Set $v = v_i$ in (4.23). Now $u_0 \geq \psi$ a.e. and hence $v_i \rightarrow u_0$ in $W^{1,p_0}(\Omega)$. Thus (4.23) implies (4.22).

It is now easy to complete the proof for (4.20). From (d) and from (4.21) and (4.22) we obtain

$$0 \leq \int_{\Omega} (A_0(x, \nabla \bar{u}) - A_0(x, \nabla u_0)) \cdot (\nabla \bar{u} - \nabla u_0) dx \leq 0$$

and this is possible only if (4.20) holds.

The proof for Theorem 2.5 is complete. $\square$

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