Certain Linear Extremal Problems in Bergman Spaces of Analytic Functions

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0. Introduction. The Bergman spaces $A^p := A^p(\mathbf{D})$, $1 \leq p < \infty$, are defined as Banach spaces of functions f analytic in the unit disk $\mathbf{D}$ equipped with the norm

$$\|f\|_p := \left(\int_{\mathbf{D}} |f|^p dA \right)^{1/p} < \infty.$$

(Here, $dA = (1/\pi) dx dy$ is the normalized Lebesgue measure in $\mathbf{D}$.) Let $\omega(z) := \sum_0^N \omega_k z^k$ be a polynomial or, more generally, a rational function. Consider the extremal problem of finding

$$(0.1) \quad \mu := \sup \left\{ \operatorname{Re} \left\{ \int_{\mathbf{D}} f \bar{\omega} dA \right\} : f \in A^p, \|f\|_p \leq 1 \right\}$$

or, equivalently,

$$(0.1') \quad \sup \left\{ \operatorname{Re} \left(\sum_0^N \frac{\bar{\omega}_k}{(k+1)!} f^{(k)}(0) \right) : f \in A^p, \|f\|_p \leq 1 \right\}.$$

An elementary calculation shows that (0.1) is equivalent to the following problem

$$(0.2) \quad \inf \left\{ \|F\|_p : F \in A^p, \int_{\mathbf{D}} F \bar{\omega} dA = 1 \right\},$$

and the extremal functions f^* , F^* in (0.1), (0.2) are related:

$$f^* = F^* \mu.$$

The problem (0.2) is well known to have a unique solution (cf. [DKSS1,2], [V] and references therein). Uniqueness follows, since all spaces A^p are strictly convex. Existence in A^p , $p > 1$, follows at once from the weak $(\star)$ compactness of bounded sets in A^p , $p > 1$. For $p = 1$, it is more delicate and is a corollary of the closedness of A^1 with respect to the weak $(\star)$ convergence of measures $\{g_\alpha dA\}$, $g_\alpha \in A^1$ and continuity of ω . Thus, the problem (0.1) always has a unique solution. The main theme of this paper is to study the qualitative properties of the solutions $f^\star$ of (0.1), more precisely, the smoothness of $f^\star$ near the boundary $\mathbf{T}$ of $\mathbf{D}$, and its structure as to number of zeros in $\mathbf{D}$, its outer factor, singular inner factor, etc. One of our main results is the following:

Theorem A. *For $p > 1$, the solution $f^\star$ of (0.1) is in some Lipschitz class, Lip α in the closed unit disk $\bar{\mathbf{D}}$, $0 < \alpha < 1$ and extends analytically across all points on $\mathbf{T}$ except for at most N points.*

With some extra work we obtain a more refined version of this:

Theorem B. *For $p > 1$, the solution $f^\star$ of (0.1) has the form*

$$(0.3) \quad f^\star(z) = CB(z) \prod_1^N (1 - \bar{\alpha}_j z)^{2/p},$$

where C is a constant, and

$$(0.4) \quad \alpha_j \in \bar{\mathbf{D}}, \quad B(z) = \prod_1^\infty \frac{\beta_j - z}{1 - \bar{\beta}_j z} \frac{|\beta_j|}{\beta_j}$$

is the Blaschke product whose zeros may only accumulate to those points α_j , $j = 1, \dots, N$ that lie on the unit circle.

Let us right away note that for $p = 2$, A^2 becomes a Hilbert space with the inner product

$$\langle f, g \rangle := \int_{\mathbf{D}} f \bar{g} dA,$$

and obviously the necessary and sufficient conditions for having equality in the Cauchy–Schwarz inequality imply that in that case the extremal function $f^\star$ in (0.1) is

$$f^\star = \omega / \|\omega\|_2,$$

and Theorems A and B trivially hold. In the sequel we shall always assume $1 < p < \infty$, $p \neq 2$.

Results similar to (0.3) are well known to hold for solutions of problems similar to (0.1) in the context of Hardy spaces H^p and are results of the elegant functional-analytic approach developed by S.Ya. Khavinson and, independently, by W. Rogosinski and H.S. Shapiro in the early 1950's. It is based on the Hahn–Banach theorem applied together with the well-known characterization of the annihilators of the H^p -spaces in terms of analytic differentials, known as the F. and M. Riesz theorem (see, e.g., [D, Ch.8], [Kh1,2], [RS] for details). The major technical difficulty of the problem (0.1) in the context of A^p -spaces lies in the tremendous size of the annihilator of A^p inside the Lebesgue space $L^q(dA)$, $(1/p) + (1/q) = 1$. Indeed, while $(H^p)^\perp \cong \{f \in H^q : f(0) = 0\}$ (F. and M. Riesz's Theorem), $(A^p)^\perp \cong \left\{ (\partial v / \partial \bar{z}) : v \in W_0^{1,q}(\mathbf{D}) \right\}$, $(1/p) + (1/q) = 1$ (Khavinson's Lemma—cf., e.g., [Sh1], [V]). ($W_0^{1,q}$, as usual, stands for the Sobolev space of functions vanishing on $\mathbf{T} := \partial\mathbf{D}$ with gradients in $L^q := L^q(\mathbf{D})$, $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$.)

In the context of the Hardy spaces H^p , invoking Green's formula for smooth f in (0.1), we can rewrite (0.1) as

$$(0.5) \quad \sup \left\{ \operatorname{Re} \frac{1}{2\pi i} \int_{\mathbf{T}} f \bar{\Omega} dz : f \in H^p, \|f\|_{H^p} \leq 1 \right\},$$

where $\Omega = \int_0^z \omega(\xi) d\xi$ is an antiderivative of ω . Application of the extended version of the Schwarz reflection principle together with the Hahn–Banach duality yields that in the representation (0.3) of the extremal (for (0.5)) function f^* the Blaschke product $B(z)$ consists of at most $(\deg \Omega - 1) = (N + 1) - 1 = N$ factors, i.e., the extremal function f^* has at most N zeros in the unit disk $\mathbf{D}$. We have not been able to close this gap in the A^p -spaces context. So, the following conjecture remains unsolved.

Conjecture 1. *The extremal function f^* for (0.1) has at most N zeros in the closed unit disk $\bar{\mathbf{D}}$. (As usual, zeros on $\mathbf{T}$ are counted with half-multiplicities.)*

Here is what has been known about problem (0.1). In [R1] V.G. Ryabych has solved (0.1) for $\omega = z^N$, i.e.,

$$\sup \left\{ \operatorname{Re} f^{(N)}(0) : \|f\|_p \leq 1 \right\},$$

and showed that the extremal is

$$(0.6) \quad f^*(z) = \left(\frac{np+2}{2} \right)^{1/p} z^N,$$

and for $\omega = (1 - \bar{\beta}z)^{-2}$, $|\beta| < 1$, i.e.,

$$(0.7) \quad \sup \{ \operatorname{Re} f(\beta) : \|f\|_p \leq 1 \},$$

for which

$$(0.7') \quad f^*(z) = (1 - |\beta|^2)^{2/p} (1 - \bar{\beta}z)^{-4/p}.$$

(For further ramifications of (0.7) and a different proof, also cf. [V].) Also, in a later paper [R2], he proved that the extremal function f^* for (0.1) belongs to the Hardy space H^p , in particular it has boundary values on $\mathbf{T}$, even for more general functionals $\omega(z)$ that merely belong to H^q ($(1/p) + (1/q) = 1$). Finally, in an unpublished manuscript [R3] Ryabych tried to prove Theorem B together with Conjecture 1 but the argument, although containing some interesting ideas, is flawed.

K. Osipenko and M. Stessin have in [OS] completely solved the problem with $\omega(z) = \omega_0 + \omega_1 z$, i.e.,

$$\sup \{ \operatorname{Re} (c_1 f(0) + c_1 f'(0)) : \|f\|_{A^p} \leq 1 \},$$

finding

$$(0.8) \quad f^*(z) = \operatorname{const} \frac{\beta - z}{1 - \bar{\alpha}z} (1 - \bar{\alpha}z)^{2/p}, \quad |\beta| \leq 1, |\alpha| \leq 1.$$

In fact, they found α, β explicitly in terms of c_1, c_2 and even solved a more general problem replacing the origin by an arbitrary point $\xi \in \mathbf{D}$. Unfortunately, the methods employed to obtain all the above-mentioned results essentially reduce to a “lucky guess” of the extremal function and then showing it is indeed an extremal, i.e., satisfies the characteristic property of an extremal function (see [V] and the following sections). Needless to say, this approach fails for a general problem (0.1). Finally, in [DKSS1,2] and later on in [MS], the authors studied a minimal interpolation problem of finding

$$(0.9) \quad \inf \left\{ \|f\|_p : f^{(N)}(0) = 1, f(0) = \dots = f^{(N-1)}(0) = f(\beta_1) \right. \\ \left. = \dots = f^{(k_m)}(\beta_m) = 0 \right\},$$

where $|\beta_j| < 1$, $\beta_0 = 0$, $j = 1, \dots, m$, sometimes called the Carathéodory–Fejer problem (cf. [Kh2]). As is well known ([Kh2]), the solution of (0.9) is a constant

multiple of that in the problem of finding

$$(0.10) \quad \sup \left\{ \operatorname{Re} \int_{\mathbf{D}} f \bar{\omega} dA, \quad \omega = \sum_{j=0}^m \sum_{k=0}^{k_j} \frac{\lambda_{jk}}{(1 - \bar{\beta}_j z)^{2+k}}, \quad f \in A^p, \quad \|f\|_p \leq 1 \right\},$$

where λ_{jk} are some constants usually extremely difficult to calculate explicitly. The results from [DKSS1,2] and [MS] combined then yield that the extremal function in (0.9) and hence in (0.10) has the form

$$(0.11) \quad f^*(z) = \text{const } z^N \prod_{j=1}^m \left(\frac{\beta_j - z}{1 - \bar{\beta}_j z} \right)^{k_j} R(z)^{2/p},$$

where $R(z)$ is a rational function of degree less than or equal to $(2N + \sum_{j=1}^m k_j)$. (0.6)–(0.11) are coherent with Theorem B and Conjecture 1. (Remark in passing that the solution of (0.9) plays the major role in the factorization theory in A^p -spaces, similar to that played by Blaschke products in H^p -spaces, and is called a *contractive divisor* associated with the points $0, \beta_1, \dots, \beta_m$ taken with multiplicities $N, k_1, \dots, k_m$, respectively—cf. [He], [DKSS1,2] for details.)

Let us describe the contents of the paper. In Section 1, for the reader's convenience we prove Ryabych's Theorem that the solution f^* of (0.1) is in the Hardy space H^p , thus allowing us to consider boundary values of f^* on $\mathbf{T}$. In Section 2, invoking the “heavy artillery” of regularity theory for solutions of certain nonlinear degenerate elliptic partial differential equations, we show that in fact the extremal function f^* is continuous in the closed disk $\bar{\mathbf{D}}$ and, moreover, belongs to the Lipschitz class $\text{Lip } \gamma$ there, where $\gamma := \gamma(p) > 0$. As a corollary, we show in Section 3 that the outer factor in the canonical H^p -factorization of f^* has form (0.3).

In Section 4, elaborating on an idea of Ryabych, we show that the extremal f^* for (0.1) is always (up to a constant factor) an extremal in a certain problem in the H^p -setting. Then, in Section 5 we show that the kernel in this related H^p problem has the form $H(\xi)/\xi^{N+2}B(\xi)$, $\xi \in \mathbf{T}$ where $H(z) \in H^q(\mathbf{D})$, while $B(z)$ is the Blaschke product constructed out of zeros of f^* . From this we deduce that f^* does not have a singular inner factor, and therefore obtain Theorem B.

In Section 6 we discuss the application of our results to the finite minimal interpolation problem. Finally, in Section 7, we discuss a few possible approaches to Conjecture 1; in particular, a related recent result by J. Lewis concerning the number of critical points for real-valued solutions of the p -Laplace equation [Le2].

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1. Ryabych's Theorem. First, let us state without a proof a simple characterization of solutions to general extremal problems in the Lebesgue spaces L^p , $1 < p < \infty$ which is an easy corollary of the Hahn–Banach theorem (cf. [Kh1,2], [D], [DKSS1,2], [RS], [V] and references cited therein).

Theorem C. Let $p : 1 < p < \infty$, $q : (1/p) + (1/q) = 1$, and $w \in L^q := L^q(\mathbf{D})$. Let X be a closed subspace in $L^p(\mathbf{D})$. The function f^* is a (unique) solution to the problem

$$(1.1) \quad \mu := \sup \left\{ \operatorname{Re} \int_{\mathbf{D}} f w dA : f \in X, \quad \|f\|_p \leq 1 \right\}$$

if and only if

$$(1.2) \quad \|f^*\|_p = 1 \text{ and } \int_{\mathbf{D}} \left(\mu \frac{|f^*|^p}{f^*} - w \right) g dA = 0 \quad \text{for all } g \in X.$$

Theorem 1.1 (Ryabych, [R2]). Let f^* be a unique solution of the problem (0.1). Then f^* belongs to the Hardy space H^p .

Proof. Consider problem (0.1) with an additional constraint $f \in \mathcal{P}_m := \{\text{polynomials of degree } \leq m\}$. Let f_m^* be the corresponding extremal polynomial and μ_m be the value of the supremum in $\mathcal{P}_m$. Clearly, $\lim_{m \rightarrow \infty} \mu_m \leq \mu$. Applying (1.2) to f_m^* (with $X := \mathcal{P}_m$) we obtain

$$(1.3) \quad \int_{\mathbf{D}} \left[\mu_m \frac{|f_m^*|^p}{f_m^*} - \bar{\omega} \right] g dA = 0, \quad \text{for all } g \in \mathcal{P}_m.$$

Applying Green's formula, we can compute the H^p -norm of f_m^* by using (1.3) as follows:

$$\begin{aligned} (1.4) \quad \|f_m^*\|_{H^p}^p &:= \frac{1}{2\pi} \int_{\mathbf{T}} |f_m^*|^p d\theta = \frac{i}{2\pi} \int_{\mathbf{T}} |f_m^*|^p z d\bar{z} \\ &= \int_{\mathbf{D}} \frac{\partial}{\partial z} (|f_m^*|^p) z dA + \int_{\mathbf{D}} |f_m^*|^p dA \quad (\|f_m^*\|_p = 1) \\ &= \frac{p}{2} \int_{\mathbf{D}} \frac{|f_m^*|^{p-2}}{f_m^*} z (f_m^*)' dA + 1 \quad (\text{since } z (f_m^*)' \in \mathcal{P}_m) \\ &= \frac{p}{2\mu_m} \int_{\mathbf{D}} z (f_m^*)' \bar{\omega} dA + 1. \end{aligned}$$

Assuming without loss of generality that $m \geq N := \deg \omega$ and writing $f_m^* = \sum_{n=0}^N b_{n,m} z^n + \sum_{m=N+1}^n b_{n,m} z^n =: P_{N,m} + \varphi_m$, $P_{N,m} \in \mathcal{P}_N$ we conclude that the last integral in (1.4) equals

$$(1.5) \quad \frac{p}{2\mu_m} \int_{\mathbf{D}} z P'_{N,m} \bar{\omega} dA.$$

Since $\|f_m^*\|_p = 1$, the first N Taylor coefficients of $f_m^*(b_{n,m}, n = 0, \dots, N)$ are uniformly bounded. Hence, the integral (1.5) (its absolute value) is bounded by a constant independent of m . Taking a subsequence if necessary, we can assume that f_m^* converge to f^* uniformly in $\mathbf{D}$. Indeed, $\|f_m^*\|_p = 1$, so we can choose a subsequence converging weakly in A^p to a function g . We shall still denote that subsequence $\{f_m^*\}$. We have

$$(1.6) \quad \lim_{m \rightarrow \infty} \mu_m = \lim_{m \rightarrow \infty} \int_{\mathbf{D}} f_m \bar{\omega} dA = \int_{\mathbf{D}} g \bar{\omega} dA \leq \mu.$$

Yet, approximating f^* by $g_r := f^*(rz)$, $r < 1$ in A^p -norm and then $g_r(z)$ by its Taylor polynomials in the uniform norm on $\bar{\mathbf{D}}$ we see that equality must hold in (1.6). Hence, $g = f^*$ in view of the uniqueness of the extremal function. Now (1.4) and uniform boundedness of (1.5) imply the theorem. $\square$

2. Lipschitz regularity of the extremal function. Recall Khavin's Lemma (cf., e.g., [Sh], [V]) characterizing the annihilator $(A^p)^\perp$ of A^p in the Lebesgue space $L^q(\mathbf{D})$.

For $p : 1 < p < \infty$

$$(2.1) \quad (A^p)^\perp := \left\{ g \in L^q(\mathbf{D}) : \int_{\mathbf{D}} f g dA = 0 \quad \text{for all } f \in A^p \right\} \\ = \left\{ \frac{\partial u}{\partial \bar{z}}, \quad u \in W_0^{1,q}(\mathbf{D}) \right\},$$

where $W_0^{1,q}$ is defined as the closure of the space $C_0^\infty(\mathbf{D})$ of compactly-supported test functions in $\mathbf{D}$ with respect to the L^q -norm of the gradient or, equivalently, with respect to the L^q -norm of their $\partial/\partial \bar{z}$ derivatives.

(The latter follows because for $\varphi \in C_0^\infty(\mathbf{D})$,

$$\frac{\partial \varphi}{\partial z}(z) = P.V. \int_{\mathbf{D}} \frac{\partial \varphi}{\partial \xi} \frac{1}{(\xi - z)^2} dA(\xi),$$

i.e., $\partial\varphi/\partial z$ is the Hilbert transform of $\partial\varphi/\partial\bar{z}$ which is a bounded operator in L^q .) Invoking (2.1) and Theorem C, we obtain the following characterization of functions f^* extremal for (0.1).

Theorem C'. *A function f^* is the (unique) extremal in the problem (0.1) if and only if $\|f^*\|_p = 1$ and there exists a function $u^* \in W_0^{1,q}$ such that*

$$(2.2) \quad \mu \frac{|f^*|^p}{f} - \bar{\omega} = \frac{\partial u^*}{\partial \bar{z}} \quad \text{a.e. in } \mathbf{D}$$

Moreover, the Hahn–Banach distance formula and necessary and sufficient conditions for equality in Hölder's inequality immediately imply that $\partial u^*/\partial \bar{z}$ is the (unique) solution in the dual to (0.1) problem of finding the infimum

$$(2.3) \quad \inf \left\{ \|\bar{\omega} + g\|_q : g \in (A^p)^\perp \right\}.$$

(cf. [Kh1], [D., Ch.8]; also, cf. [V]).

Denoting the antiderivative of ω by $\Omega(z) := \int_0^z \omega dz$ and letting $v(z) := u^*(z) + \bar{\Omega}(z)$ we obtain from (2.2) and (2.3) that $v \in W^{1,q}(\mathbf{D})$:

$$(2.4) \quad \begin{cases} \frac{\partial v}{\partial \bar{z}} = \mu \frac{|f^*|^p}{f^*} & \text{a. e. in } \mathbf{D} \\ v = \bar{\Omega} & \text{on } \mathbf{T}, \end{cases}$$

and v is the unique solution of the variational problem of finding

$$(2.5) \quad \inf \left\{ \left(\int_{\mathbf{D}} \left| \frac{\partial \varphi}{\partial \bar{z}} \right|^q dA \right)^{1/q} : \varphi \in W^{1,q}(\mathbf{D}) : \varphi = \bar{\Omega} \text{ on } \mathbf{T} \right\}.$$

Either applying a simple variational argument by taking in (2.5) $\varphi = v + \lambda\psi$, $\psi \in C_0^\infty(\mathbf{D})$, $\lambda \in \mathbf{R}$ and using the fact that the minimum occurs for $\lambda = 0$, or directly noticing from (2.4) that $|\partial v/\partial \bar{z}|^{q-2} \partial v/\partial \bar{z} = \mu^{q-1} \overline{f^*(z)}$ in $\mathbf{D}$ ($|\partial v/\partial \bar{z}|^{q-2} \partial v/\partial \bar{z} := 0$ whenever $\partial v/\partial \bar{z} = 0$) where $q = p/(p-1)$, we conclude that v is the solution of the following nonlinear boundary value problem

$$(2.6) \quad \begin{cases} \frac{\partial}{\partial z} \left(\left| \frac{\partial v}{\partial \bar{z}} \right|^{q-2} \frac{\partial v}{\partial \bar{z}} \right) = 0 & \text{in } \mathbf{D}; \\ v = \bar{\Omega} & \text{on } \mathbf{T}. \end{cases}$$

The equation (2.6) is the so-called degenerate (complex) elliptic equation. (It can also be viewed as a system of two real such equations.) Ellipticity fails precisely at the points where $v_{\bar{z}} = 0$, i.e., at zeros of f^* (cf. 2.4). The real-valued analogue of (2.6) is the well-known q -harmonic equation for a real-valued function V

$$(2.7) \quad \begin{cases} \operatorname{div} \left(|\operatorname{grad} V|^{q-2} \operatorname{grad} V \right) = 0 & \text{in } \mathbf{D} \\ V = \operatorname{Re} \Omega & \text{on } \mathbf{T}, \end{cases}$$

whose solution is the minimizer for the variational problem of finding

$$(2.8) \quad \inf \left\{ \left(\int_{\mathbf{D}} |\operatorname{grad} \varphi|^q dA \right)^{1/q} : \varphi \in W^{1,q}(\mathbf{D}), \quad \varphi = \operatorname{Re} \Omega \text{ on } \mathbf{T} \right\}.$$

The problems similar to (2.6), (2.7) or their variational analogues (2.5), (2.8) have been studied in a much more general framework by Ch. Morrey [M1-3], Ladyzhenskaya and Ural'tseva [LU] and many others, cf., e.g., the references in [Ar], [BI], [IM], [Le1], [Lie], [Lin], and see also [CH, v.II], [GT]. The corollary from those investigations that we shall use in the sequel is the following a priori regularity theorem.

Theorem D. *The (unique) solutions of the boundary value problems (2.6) (or, (2.7)) or, equivalently of variational problems (2.5) (or, (2.8)) belong to the class $C^{1,\beta}(\bar{\mathbf{D}})$ of continuously differentiable functions in $\mathbf{D}$ whose first derivatives belong to a Lipschitz class $\operatorname{Lip} \beta$ in the closed disk $\bar{\mathbf{D}}$ with exponent $\beta := \beta(q) > 0$.*

Although including a detailed proof of Theorem D would be beyond the scope of this paper, and it would not be a new, original result anyway, we nevertheless give a sketch of an argument here exclusively for the reader's convenience. We shall only discuss the equation (2.6) that is of the main interest to us.

An outline of the proof of Theorem D. Introducing a regularizing parameter $\varepsilon > 0$, one considers the equation

$$(2.9) \quad \begin{cases} \frac{\partial}{\partial z} \left(\left(\varepsilon + \left| \frac{\partial v}{\partial \bar{z}} \right|^2 \right)^{(q-2)/2} \frac{\partial v}{\partial \bar{z}} \right) = 0, \\ v|_{\mathbf{T}} = \bar{\Omega} \end{cases}$$

or, differentiating and simplifying,

$$(2.10) \quad \left(\varepsilon + \frac{q}{2} |v_{\bar{z}}|^2 \right) \Delta v + \frac{q-2}{2} (v_{\bar{z}})^2 \overline{v_{z\bar{z}}} = 0,$$

where $\Delta v := \partial^2 v / \partial z \partial \bar{z}$ is one-fourth of the regular Laplace operator. (2.10) is a strongly elliptic (complex) equation, and the standard method of a priori estimates due to Schauder shows that the solution v_ε of (2.10) satisfying smooth boundary conditions (2.9) is in $C^\infty(\bar{\mathbf{D}})$. (One must first prove that $v_\varepsilon \in C^{1+\beta}(\bar{\mathbf{D}})$ for some $\beta : 0 < \beta < 1$ and then, using Schauder's estimate it follows that $v_\varepsilon \in C^{2+\beta}(\bar{\mathbf{D}})$, i.e., $\text{grad} v_\varepsilon \in C^{1+\beta}(\bar{\mathbf{D}})$ so again by Schauder's estimates $v_\varepsilon \in C^{3+\beta}$, etc.—cf. [Lie].) In fact, using a stronger result of Morrey, since the boundary value of v_ε on $\mathbf{T}$ is a polynomial, we can claim even that v_ε is real-analytic across $\mathbf{T}$. All the constants in such estimates do depend on ε except for those in the very first one! There a miracle occurs and (see [IM] for terminology and more details) since (2.10) yields that the complex dilatation of the function $u_\varepsilon := (v_\varepsilon)_{\bar{z}}$ satisfies

$$(2.11) \quad \frac{|(u_\varepsilon)_z|}{|(u_\varepsilon)_{\bar{z}}|} \leq \left| 1 - \frac{2}{q} \right| < 1,$$

u_ε is k -quasiregular in $\bar{\mathbf{D}}$ with $k = |1 - 2/q| < 1$, independent of ε . Hence (see, e.g., [IM]),

$$(2.12) \quad \|u_\varepsilon\|_{\text{Lip } \beta(\bar{\mathbf{D}})} \leq C,$$

where $\beta = \beta(q)$ and the constant C depends only on q and the boundary data $\bar{\Omega}$ but not on ε . Letting ε tend to zero, we obtain that $v_\varepsilon, (v_\varepsilon)_{\bar{z}}$ tend to $v, v_{\bar{z}}$ uniformly on compact subsets of $\mathbf{D}$ while (2.12) holds for all ε . Therefore, $v_{\bar{z}} \in \text{Lip } \beta(\bar{\mathbf{D}})$. Now, by the Cauchy–Green formula, we have for $z \in \mathbf{D}$

$$(2.13) \quad \begin{aligned} v(z) &= \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{\bar{\Omega}(\xi) d\xi}{\xi - z} - \int \int_{\mathbf{D}} \frac{\partial v}{\partial \bar{\xi}} \frac{1}{\xi - z} dA(\xi) \\ &= - \int \int_{\mathbf{D}} \frac{\partial v}{\partial \bar{\xi}} \frac{1}{\xi - z} dA(\xi). \end{aligned}$$

$(\int_{\mathbf{T}} \bar{\xi}^n / (\xi - z) d\xi = 0, n \geq 1, z \in \mathbf{D}!)$. Thus, $\partial v / \partial z$ is given by a Hilbert transform of $\partial v / \partial \bar{z}$ and hence is in $\text{Lip } \beta(\bar{\mathbf{D}})$ as well as $\partial v / \partial \bar{z}$. Thus, $v \in C^{1+\beta}(\bar{\mathbf{D}})$. □

For our problem, Theorem D yields an important a priori Lipschitz regularity of the extremal function f^* .

Corollary 2.1. *The solution f^* of the extremal problem (0.1) belongs to a Lipschitz class $\text{Lip } \gamma$ for some $\gamma > 0$ in the closed unit disk $\bar{\mathbf{D}}$.*

Proof. From (2.4) it follows that

$$(2.14) \quad f^* = \mu^{q-1} \frac{|v_{\bar{z}}|^q}{v_{\bar{z}}} \quad \text{a. e. in } \mathbf{D},$$

since $v_{\bar{z}} \in \text{Lip } \beta(\bar{\mathbf{D}})$ and $q > 1$ (2.14) holds pointwise in $\bar{\mathbf{D}}$ and, in particular, f^* is continuous in $\bar{\mathbf{D}}$. The mapping $h : w \mapsto |w|^q/w$, $q > 1$ belongs to the $\text{Lip } \alpha$ class on compact subsets of $\mathbf{C}$. The easiest way to see that is by simply noticing that h maps k -quasiconformally disks of radius R centered at the origin onto the disks of radius R^{q-1} , $k = |1 - 2/q|$ and apply the result of Morrey [Ah]. One can also prove this directly or (cf. [IM]) by noticing that $|\text{grad} h| \leq |w|^{q-2} \in L^1_{\text{loc}}$. (In the latter argument, one obtains $\alpha = q - 1 - [\text{IM}]$.) Since $f^* = h \circ v_{\bar{z}}$ and a composition of $\text{Lip } \alpha$ and $\text{Lip } \beta$ functions belongs to the $\text{Lip } (\alpha\beta)$ class, the corollary follows. $\square$

3. The outer factor of f^* . According to Theorem 1.1, we can factor f^* in the form

$$(3.1) \quad f^*(z) = B(z)S(z)F(z),$$

where, as usual, B stands for a Blaschke product constructed out of zeros of f^* , S is the singular inner factor, and F is the outer part of f^* (cf. [D]).

Theorem 3.1. *The outer function $F(z)$ in (3.1) has the form ($N := \deg \omega$):*

$$(3.2) \quad F(z) = C \prod_{j=1}^N (1 - \bar{\alpha}_j z)^{2/p}, \quad |\alpha_j| \leq 1.$$

Proof. By (2.2), Theorem D and Corollary 2.1 we have

$$(3.3) \quad \mu \frac{|f^*|^p}{f^*} - \bar{\omega} = \frac{\partial u^*}{\partial \bar{z}} \quad \text{in } \mathbf{D},$$

where $u^* \in C^{1+\beta}(\bar{\mathbf{D}})$, $\beta > 0$, and $u^* = 0$ on $\mathbf{T}$. Differentiating (3.3) with respect to z , we obtain

$$\left(\frac{p-2}{2} \right) \mu \frac{|f^*|^p}{(f^*)^2} (f^*)' = \frac{\partial}{\partial z} \left(\frac{\partial u^*}{\partial \bar{z}} \right)$$

or,

$$(3.4) \quad \left(\frac{p-2}{2}\right) \mu \frac{|f^*|^p}{f^*} (f^*)' = \frac{\partial}{\partial \bar{z}} \left(f^* \frac{\partial u^*}{\partial z} \right).$$

Note that since $f^* \in \text{Lip } \gamma$, $\gamma > 0$ and $p > 1$, $|f^*|^p / f^* \in L^\infty(\mathbf{D})$, while according to a theorem of Hardy and Littlewood [D, Theorem 5.1],

$$(f^*)' = O\left(\frac{1}{(1-r)^{1-\gamma}}\right) \quad \text{in } \mathbf{D}$$

and hence $(|f^*|^p / f^*) (f^*)' \in L^{p_1}(\mathbf{D})$ for some $p_1 > 1$. Therefore, for any polynomial $g(z)$, we obtain from (3.3)

$$(3.5) \quad \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} (f^*)' z^{N+1} g dA = \frac{1}{\mu} \int_{\mathbf{D}} \bar{\omega} (f^*)' z^{N+1} g dA = 0,$$

and taking a closure, we have (3.5) hold for all $g \in A^{q_1}$, $1/p_1 + 1/q_1 = 1$. Hence, Khavin's Lemma, together with (3.4), (3.5) yield that there exists a function $u_1 \in W_0^{1,p_1}$ so that

$$(3.6) \quad \frac{\partial}{\partial \bar{z}} \left(z^{N+1} f^* \frac{\partial u^*}{\partial z} \right) = \frac{\partial u_1}{\partial \bar{z}} \quad \text{a. e. in } \mathbf{D}.$$

Integrating, we obtain from (3.6)

$$(3.7) \quad z^{N+1} f^* \frac{\partial u^*}{\partial z} = u_1 + h \quad \text{in } \mathbf{D},$$

where h is analytic on $\mathbf{D}$. ((3.7) holds pointwise in $\mathbf{D}$ since the left-hand side and h are continuous.) An easy Sobolev space estimate implies

$$(3.8) \quad \sup_{r < 1} \left\{ \int_0^{2\pi} |u_1(re^{i\theta})| d\theta \right\} \leq \text{const } \|u_1\|_{W^{1,p_1}} < +\infty.$$

(Suffices to write $u_1(re^{i\theta}) = \int_0^r \text{grad } u_1(\rho e^{i\theta}) \cdot d\rho + u_1(0)$, and apply Fubini's theorem and Hölder's inequality.) Thus (3.7), (3.8) and $u^* \in C^{1+\beta}(\bar{\mathbf{D}})$ imply that $h \in H^1(\mathbf{D})$ and, in particular, has boundary values. Since $u_1 = 0$ almost everywhere on $\mathbf{T}$ while $u^* = 0$ on $\mathbf{T}$, we have $\bar{z}^2 u_z^* = u_z^*$ on $\mathbf{T}$ because $0 =$

$du^*/d\theta = izu_z^* - i\bar{z}u_{\bar{z}}^*$ on $\mathbf{T}$, and (3.7), (3.3) imply that almost everywhere on $\mathbf{T}$ we have

$$h(e^{i\theta}) = z^{N+1} f^* \bar{z}^2 \frac{\partial u^*}{\partial \bar{z}} = z^{N+1} f^* \bar{z}^2 \left(\mu \frac{|f^*|^p}{f^*} - \bar{\omega} \right).$$

Equivalently,

$$(3.9) \quad r(z) := \frac{h}{z^{N-1}} + f^* \left(\sum_0^N \frac{\bar{\omega}_k}{z^k} \right) = \mu |f^*|^p \geq 0 \quad \text{a. e. on } \mathbf{T}.$$

The remaining part of the argument is standard. The left-hand side $r(z)$ of (3.9) belongs to H^1 in an annulus near the unit circle $\mathbf{T}$ and applying the reflection principle for H^1 -functions (cf., e.g., [D, Ch. 10], [Kh1], [RS]) we extend $r(z)$ analytically across $\mathbf{T}$ by setting

$$(3.10) \quad R(z) := \overline{r\left(\frac{1}{\bar{z}}\right)}, \text{ for } z : |z| > 1.$$

Since $r(z)$ is in fact analytic in $\mathbf{D} \setminus \{0\}$ and has only a pole of order N at 0 while being non-negative on $\mathbf{T}$, (3.10) implies that $r(z)$ extends to be a rational function on the Riemann sphere $\hat{\mathbf{C}}$ that has a pole of order N at ∞ , zeros $\alpha_1, \dots, \alpha_N$ in $\bar{\mathbf{D}}$ (the zeros on $\mathbf{T}$ are necessarily of an even order and must be counted with half multiplicities!) and zeros at the reflected points $1/\bar{\alpha}_1, \dots, 1/\bar{\alpha}_N$ outside of $\mathbf{D}$, i.e., for some constant C

$$(3.11) \quad r(z) = C \frac{1}{z^N} \prod_{j=1}^N (z - \alpha_j)(1 - \bar{\alpha}_j z).$$

Thus, by (3.9), (3.11) we have on $\mathbf{T}$

$$(3.12) \quad |f^*(z)| = |F(z)| = C \left(\frac{1}{\mu} \right)^{1/p} \left(\prod_{j=1}^N \frac{(z - \alpha_j)(1 - \bar{\alpha}_j z)}{z^N} \right)^{1/p},$$

where C is a constant. Now, the harmonic extension of $\log|z - \alpha_j|$, $|\alpha_j| \leq 1$ from $\mathbf{T}$ is given by $\log|1 - \bar{\alpha}_j z|$. Since $F(z)$ is an outer function, $\log|F(z)|$ is harmonic in $\mathbf{D}$ and hence (3.12) yields

$$(3.13) \quad \log|F(z)| = \frac{2}{p} \sum_{j=1}^N \log|1 - \bar{\alpha}_j z| - \frac{1}{p} \log \mu + \log C$$

in **D**. From (3.13) the theorem follows. $\square$

Corollary 3.1. *The zeros of the Blaschke product $B(z)$ in (3.1) may only accumulate at finitely many points on $\mathbf{T}$, more precisely, only at those points α_j , $1 \leq j \leq k \leq N$ in the representation (3.2) that lie on $\mathbf{T}$. The singular factor $S(z)$ in (3.1) has the form*

$$(3.14) \quad S(z) = \prod_{j=1}^k \exp \left(-\lambda_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z} \right),$$

where $\lambda_j \geq 0$, $e^{i\theta_j} = \alpha_j$, $1 \leq j \leq k \leq N$. (The set of $\alpha_j : |\alpha_j| = 1$ may, of course, be empty.)

The proof follows at once from Corollary 2.1, Theorem 3.1, and a well-known fact that an outer part of a function analytic in $\mathbf{D}$ and continuous in $\bar{\mathbf{D}}$ must vanish on the support of a singular inner part and on the limit set of zeros of its Blaschke product (cf. [Ho]). Thus, we have the following corollary.

Corollary 3.2. (Theorem A, Section 0) *The extremal solution f^* of (0.1) is Lipschitz continuous in the closed unit disk $\bar{\mathbf{D}}$. f^* extends analytically across all points on $\mathbf{T}$ except for at most N branch points of its outer part.*

4. Coupled extremal problems in Hardy and Bergman spaces

I. Recall the basics of the theory of extremal problems in Hardy spaces similar to (0.1)—cf. [Kh1,2], [RS], [D, Ch. 8]. Let $k(z)$ be a meromorphic function in $\mathbf{D}$ with poles $\beta_1, \dots, \beta_n$ repeated according to their multiplicity, $1 < p < \infty$. Consider the extremal problem of finding

$$(4.1) \quad \lambda := \sup \left\{ \operatorname{Re} \int_{\mathbf{T}} f(z) k(z) \frac{dz}{2\pi i} : \|f\|_{H^p} \leq 1 \right\}$$

and its dual problem (in the sense of Theorem C') of finding

$$(4.2) \quad \lambda' := \inf \left\{ \|k(\xi) - \varphi(\xi)\|_{L^q(\mathbf{T})} : \frac{1}{p} + \frac{1}{q} = 1, \varphi \in H^q \right\}.$$

The following theorem combines the results of S.Ya. Khavinson, A. Macintyre, W. Rogosinski, and H.S. Shapiro.

Theorem E. $\lambda = \lambda'$ and there exist unique solutions f^*, φ^* to problems (4.1)–(4.2). The function f^* extremal for (4.1) has the form

$$(4.3) \quad f^*(z) = C \prod_{j=1}^{\sigma} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \prod_{j=1}^{n-1} (1 - \bar{\alpha}_j z)^{2/p} \prod_{j=1}^n (1 - \bar{\beta}_j z)^{-2/p},$$

where $\sigma \leq n-1$, $\alpha_1, \dots, \alpha_{\sigma} \in \mathbf{D}$ and all $\alpha_j \in \bar{\mathbf{D}}$. Moreover, in general, for the functions f^*, φ^* to be extremal in problems (4.1), (4.2) respectively, with $k \in L^q(\mathbf{T})$ it is necessary and sufficient that the following duality relation holds almost everywhere on $\mathbf{T}$:

$$(4.4) \quad \lambda^{q-1} f^*(\xi) [k(\xi) - \varphi^*(\xi)] \frac{d\xi}{i} = |k(\xi) - \varphi^*(\xi)|^q d\theta,$$

or, equivalently,

$$(4.4) \quad \lambda^{-1} f^* [k(\xi) - \varphi^*(\xi)] \frac{d\xi}{i} = |f^*(\xi)|^p d\theta.$$

Remarks.

(i) If we replace the norm in $H^p(\mathbf{T})$ by a weighted norm with weight

$$\rho : \rho \in C(\mathbf{T}), \rho > 0, H^{p,\rho} := \left\{ f \in H^p : \right.$$

$$\left. \|f\|_{H^{p,\rho}} := \left(\frac{1}{2\pi} \int_{\mathbf{T}} |f(e^{i\theta})|^p \rho d\theta \right)^{1/p} < \infty \right\}$$

and pose the problem (4.1) for $f : \|f\|_{H^{p,\rho}} \leq 1$, then in the representation (4.3) of the extremal function f^* we must add an outer factor

$$(4.5) \quad \exp \left(-\frac{1}{2\pi p} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \rho(e^{i\theta}) d\theta \right),$$

while the factors $\rho^{-q/p}, \rho$ must be included in (4.3)–(4.4) in the right-hand side (cf. [Kh1, Section 6]).

(ii) Note that (4.4) implies that a function $f^*, \|f^*\|_{H^p} = 1$ is extremal in the problem (4.1) if and only if there is a function $\varphi^* \in H^q$ so that

$$(4.6) \quad \lambda \frac{|f^*|^p}{f^*} - \xi k(\xi) = -\xi \varphi^*(\xi) \quad \text{a. e. on } \mathbf{T}$$

or equivalently, invoking the F. and M. Riesz Theorem, when

$$(4.7) \quad \int_{\mathbf{T}} \left(\lambda \frac{|f^*|^p}{f^*} - \xi k(\xi) \right) g(\xi) d\theta = 0 \quad \text{for all } g \in H^p,$$

i.e., we have a complete analogy with (1.2).

We need the following.

Lemma 4.1. *Let $f \in H^p$ and define the function*

$$(4.8) \quad r(z) := \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{|f|^p}{f} \frac{1}{\xi - z} d\bar{\xi}$$

analytic in $\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}}$. Then, $r(z) \in H^q(\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$, in particular $r(z)$ has boundary values $r(\xi)$ almost everywhere on $\mathbf{T}$ and $r(\xi) \in L^q(\mathbf{T})$. A constant multiple of $f(z)$ is the extremal function in the problem (4.1) with kernel $k_1(z) \in L^q$ equal to

$$(4.9) \quad k_1(z) := zr(z).$$

Also,

$$(4.10) \quad \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{\xi k_1(\xi) d\xi}{\xi^2(\xi - z)} = -r(z), \quad |z| > 1.$$

Proof. Since $f \in H^p$, $|f|^p/f \in L^q(\mathbf{T})$ and therefore, since the analytic projection is bounded in $L^r(\mathbf{T})$, $1 < r < \infty$ ([D]), $r \in H^q(\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$. Then, (4.10) is simply Cauchy's Representation Formula for functions in $H^q(\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$ that vanish at ∞ in view of (4.9). Also, for all $z : |z| > 1$ we have by (4.10)

$$\frac{1}{2\pi i} \int_{\mathbf{T}} \frac{|f|^p}{f} \frac{d\xi}{\xi^2(\xi - z)} = -\frac{1}{2\pi i} \int_{\mathbf{T}} \frac{|f|^p}{f} \frac{d\bar{\xi}}{\xi - z} = -r(z) = \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{\xi k_1(\xi) d\xi}{\xi^2(\xi - z)},$$

i.e.,

$$\left(\frac{|f|^p}{\xi^2 f} - \frac{k_1(\xi)}{\xi} \right) d\xi$$

annihilate all functions analytic in $\mathbf{D}$. The F. and M. Riesz Theorem then implies that

$$(4.11) \quad \frac{|f|^p}{\xi^2 f} - \frac{k_1(\xi)}{\xi} = h(\xi) \quad \text{a. e. on } \mathbf{T}$$

with $h(\xi) \in H^1(\mathbf{D})$. Since the left-hand side of (4.11) is in $L^q(\mathbf{T})$, $h \in H^q(\mathbf{T})$. Letting $f^* = f/\|f\|_{H^p}$, we obtain from (4.11)

$$\|f\|_{H^p}^{p-1} \frac{|f^*|^p}{f^*} - \xi k_1(\xi) = \xi^2 h(\xi).$$

Setting $\|f\|_{H^p}^{p-1} = \lambda$, $\xi h(\xi) = -\varphi^*(\xi) \in H^q$ and comparing with (4.6) we obtain the lemma. $\square$

Recall (cf. [DSS]) that a function $F(z) \in H^q(\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$, $q \geq 1$, is said to have a *meromorphic pseudocontinuation* $m(z)$ into $\mathbf{D}$ if $m(z)$ is a meromorphic function of bounded characteristic in $\mathbf{D}$ such that $F = m$ almost everywhere on $\mathbf{T}$.

Corollary 4.1. *If $r(z)$ admits a meromorphic pseudocontinuation into $\mathbf{D}$ with $m(z) = H(z)/(B_n(z)B(z))$, where $H \in H^q(\mathbf{D})$, $B_n(z)$ is a finite Blaschke product and $B(z)$ is a Blaschke product built with zeros of f , then f has no singular inner factor.*

Proof. Invoking (4.9), (4.11) we obtain that the equality

$$(4.12) \quad f(\xi^2 r(\xi) + \xi^2 h(\xi)) = |f|^p \geq 0$$

holds almost everywhere on $\mathbf{T}$. Our hypothesis on r now implies ($h \in H^q$) that the left-hand side in (4.12) represents boundary values of an H^1 -function in the annulus $\{z : r < |z| < 1\}$. By the Schwarz Reflection Principle it extends analytically across $\mathbf{T}$. Thus, writing $f = BSF$, where S is a singular inner factor and F is outer, we obtain that

$$\frac{SF}{B_n}(\xi^2 H + \xi^2 B_n B h)$$

is analytic across $\mathbf{T}$. Since all factors except $1/B_n$ belong to the Smirnov class N^+ while $1/B_n$ is analytic across $\mathbf{T}$, S must be a constant. This proves the corollary. $\square$

5. Coupled extremal problems in Hardy and Bergman spaces

II. The idea of the following result goes back to the Ryabych manuscript [R3], although there the author heavily relied on the a priori finiteness of zeros of an extremal function, i.e., on Conjecture 1.

Theorem 5.1. *Let f^* be the extremal function in the problem (0.1) and let $r(z)$, $|z| > 1$ be defined by (4.8). Then, $r(z)$ admits a meromorphic pseudocontinuation $M(z) = H(z)/(z^{N+2}B(z))$ into $\mathbf{D}$, where $H \in H^q(\mathbf{D})$ and $B(z)$ is the Blaschke product built with the zeros of f^* .*

Proof. Theorem 3.1 and Corollary 3.1 yield that $f^* = BSF$, where F is given by (3.2), S by (3.14), and the zeros of B satisfy Corollary 3.1. In order not to overwhelm the reader with technicalities, we study first the technically more transparent case $p \geq 2$. An easy modification of this argument allowing to extend it to all $p > 1$ will be outlined at the end of the proof. By Corollary 2.1 and its proof, f^* and $\frac{|f^*|^p}{f^*}$ belong to some class $\text{Lip } \gamma$, $\gamma > 0$ in $\bar{\mathbf{D}}$. Then,

$$\frac{\partial}{\partial z} \left(\frac{|f^*|^p}{f^*} \right) = \left(\frac{p}{2} - 1 \right) \frac{|f^*|^p}{f^{*2}} (f^*)' \in L^1(\mathbf{D}).$$

Indeed, $|f^*|^p (f^*)^{-2}$ is bounded ($p \geq 2$) and

$$|(f^*)'(z)| = O\left(\frac{1}{(1-|z|)^{1-\gamma}}\right)$$

near $\mathbf{T}$ and hence integrable. Thus, we can apply Green's formula to (4.8) and obtain, for any $z \in \mathbf{C} \setminus \mathbf{D}$,

$$(5.1) \quad r(z) = -\left(\frac{p}{2} - 1\right) \iint_{\mathbf{D}} \frac{|f^*|^p}{(f^*)^2} (f^*)' \frac{1}{\xi - z} dA(\xi) + \iint_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{(\xi - z)^2} dA.$$

The second integral in (5.1) can be calculated easily. In view of the extremality of f^* using, e.g., (2.2) we obtain, invoking Green's formula ($\Omega := \int_0^z \omega$):

$$(5.2) \quad \begin{aligned} \iint_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{(\xi - z)^2} dA(\xi) &= \frac{1}{\mu} \int_{\mathbf{D}} \bar{\omega} \frac{1}{(\xi - z)^2} dA(\xi) \\ &= \frac{1}{2\pi i \mu} \int_{\mathbf{T}} \overline{\Omega(\xi)} \frac{1}{(\xi - z)^2} d\xi \quad (\omega = \sum_0^N \omega_n z^N) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\mu} \sum_0^N \frac{\bar{\omega}_n}{n+1} \int_{\mathbf{T}} \frac{d\xi}{\xi^{n+1}(\xi-z)^2} \\
&= \frac{1}{\mu} \sum_0^N \frac{\bar{\omega}_n}{n+1} \left[\left(\frac{d}{d\xi} \right)^n \left[\frac{(-1)^n}{n!} \frac{1}{(\xi-z)^2} \right] \right]_{\xi=0} \\
&= \frac{1}{\mu} \sum_0^N \frac{\bar{\omega}_n}{z^{n+2}} = \frac{1}{\mu} \frac{1}{z^2} \omega^\# \left(\frac{1}{z} \right) \\
&= -\frac{1}{\mu} \left(\Omega^\# \left(\frac{1}{z} \right) \right)'
\end{aligned}$$

(Here, we used the notation $\omega^\#$ for $\omega^\#(z) := \sum_0^N \bar{\omega}_k z^k$). The second integral in (5.1) can be rewritten in the form

$$\begin{aligned}
(5.3) \quad & \frac{2-p}{2} \int \int_{\mathbf{D}} \frac{|f^\star|^p}{f^\star} (\log f^\star)' \frac{1}{\xi-z} dA(\xi) \\
&= \frac{2-p}{2} \int \int_{\mathbf{D}} \frac{|f^\star|^p}{f^\star} \frac{1}{\xi-z} [\log B + \log S + \log F]' dA(\xi) =: I_1 + I_2 + I_3.
\end{aligned}$$

Let us calculate each integral separately. By (3.2) in Theorem 3.1 $(\log F)' \in A^1$, so using again extremality of $f^\star$, we find:

$$\begin{aligned}
(5.4) \quad I_3 &= \frac{2-p}{2\mu} \int_{\mathbf{D}} \bar{\omega} \frac{1}{\xi-z} (\log F)' dA(\xi) \\
&= \frac{2-p}{2\mu} \int_{\mathbf{D}} \bar{\omega} \frac{1}{\xi-z} \left(\sum_{j=1}^N \frac{-\frac{2}{p} \bar{\alpha}_j}{1 - \bar{\alpha}_j \xi} \right) dA(\xi) = (|z| > 1!) \\
&= \frac{p-2}{p\mu} \sum_{j=1}^N \bar{\alpha}_j \int_{\mathbf{D}} \bar{\omega} \left(-\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{\xi}{z} \right)^n \right) \left(\sum_{k=0}^{\infty} (\bar{\alpha}_j \xi)^k \right) dA(\xi) \\
&= \frac{p-2}{p\mu} \sum_{j=1}^N \bar{\alpha}_j \int_{\mathbf{D}} \left(\sum_{\ell=0}^N \bar{\omega}_\ell \bar{\xi}^\ell \right) \left(\sum_{m=0}^{\infty} \xi^m \left(-\sum_{k=0}^m z^{-(k+1)} \bar{\alpha}_j^{m-k} \right) \right) dA(\xi) \\
&= \frac{p-2}{p\mu} \sum_{j=1}^N \bar{\alpha}_j \sum_{\ell=0}^N \frac{\bar{\omega}_\ell}{\ell+1} \left(-\sum_{k=0}^{\ell} z^{-(k+1)} \bar{\alpha}_j^{\ell-k} \right) \\
&= \frac{2-p}{p\mu} \sum_{k=0}^N \frac{B_k}{z^{k+1}},
\end{aligned}$$

where

$$B_k = \sum_{j=1}^N \bar{\alpha}_j \left(\sum_{\ell=k}^N \bar{\alpha}_j^{\ell-k} \frac{\bar{\omega}_\ell}{\ell+1} \right) = \sum_{\ell=k}^N \frac{\bar{\omega}_\ell}{\ell+1} \left(\sum_{j=1}^N \bar{\alpha}_j^{\ell-k+1} \right).$$

(Note that in the calculation above we have tacitly assumed that $\sum_0^\infty (\bar{\alpha}_j \xi)^k$ converges, i.e., all $\alpha_j \in \mathbf{D}$, $j = 1, \dots, N$. However, if some $\alpha_j \in \mathbf{T}$, we can replace the second integral in (5.4) by $\lim_{r \rightarrow 1} \int_{r\mathbf{D}}(\dots)$, perform the next step, obtain still a finite sum as above with factors r^ℓ and let $r \rightarrow 1$. Of course, the final result is the same.) To calculate I_2 in (5.3) we use (2.2) and (3.14). By (3.14)

$$(5.5) \quad (\log S)' = -2 \sum_{j=1}^k \lambda_j \alpha_j (\alpha_j - \xi)^{-2}, \quad k \leq N, \quad |\alpha_j| = 1.$$

Since f^* is Lip γ , $\gamma > 0$ in $\bar{\mathbf{D}}$ and $f^*(\alpha_j) = 0$ for all $1 \leq j \leq k$ (cf. (3.2)) while $p > 1$, the integrand in I_2 is $O(|\alpha_j - \xi|^{-2+\varepsilon})$, $\varepsilon > 0$ near each point α_j , $j = 1, \dots, k$. Therefore, I_2 is an absolutely convergent integral. It is clear from (5.5) that $(\log S)'$ is a sum of outer functions in H^r , for any $r < \frac{1}{2}$ and hence is in the Smirnov class N^+ . According to a theorem of V.I. Smirnov ([Sm], also, cf. [KS]), there exists a sequence of bounded analytic functions φ_n , such that $\varphi_n \rightarrow (\log S)'$ uniformly on compact subsets of $\mathbf{D}$ and $|\varphi_n(z)| \uparrow |(\log S(z))'|$ monotonically at all $z \in \mathbf{D}$. Then, applying Lebesgue's Dominated Convergence Theorem and (2.2), we obtain $(z \in \widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$:

$$(5.6) \quad \begin{aligned} I_2 &= \frac{2-p}{2} \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{\xi-z} (\log S)' dA(\xi) \\ &= \frac{2-p}{2} \lim_{n \rightarrow \infty} \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{\xi-z} \varphi_n(\xi) dA(\xi) \\ &= \frac{2-p}{2\mu} \lim_{n \rightarrow \infty} \int_{\mathbf{D}} \bar{\omega}(\xi) \frac{\varphi_n(\xi)}{\xi-z} dA(\xi) \end{aligned}$$

(applying Green's formula)

$$\begin{aligned} &= \frac{2-p}{2\mu} \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\mathbf{T}} \bar{\Omega}(\xi) \frac{\varphi_n(\xi)}{\xi-z} d\xi \\ &= \frac{2-p}{2\mu} \lim_{n \rightarrow \infty} \sum_{\ell=0}^N \frac{\bar{\omega}_\ell}{(\ell+1)!} \left[\left(\frac{d}{d\xi} \right)^\ell \left(\frac{\varphi_n(\xi)}{\xi-z} \right) \right]_{\xi=0} \end{aligned}$$

$$\begin{aligned}
 &= \frac{p-2}{2\mu} \sum_{\ell=0}^N \frac{\bar{\omega}_\ell}{(\ell+1)!} \left[\left(\frac{d}{d\xi} \right)^\ell \left(\sum_{j=1}^k 2\lambda_j \alpha_j (\alpha_j - \xi)^{-2} (\xi - z)^{-1} \right) \right] \Big|_{\xi=0} \\
 &= \sum_{\ell=0}^N C_\ell z^{-(\ell+1)}.
 \end{aligned}$$

(We do not bother to calculate constants C_ℓ since our ultimate goal, as the reader will see shortly, is to eliminate the singular factor $S(z)$ altogether; i.e., to show that all $\lambda_j = 0$, $j = 1, \dots, k$ and hence $I_2 \equiv 0$.) Finally, let us calculate I_1 . We write

$$B(\xi) = \xi^m \prod_{j=1}^{\infty} \frac{|\beta_j|}{\beta_j} \frac{\beta_j - \xi}{1 - \bar{\beta}_j \xi},$$

where $\{\beta_j\}_1^\infty$ are zeros of f^* that may accumulate only to those points of α_j , $j = 1, \dots, k \leq N$ from (3.2) that lie on $\mathbf{T}$. We have

$$\begin{aligned}
 (5.7) \quad [\log B(\xi)]' &= \left(\sum_{j=1}^{\infty} \{ \log(\beta_j - \xi) - \log(1 - \bar{\beta}_j \xi) - i \arg \beta_j \} + m \log \xi \right)' \\
 &= \sum_{j=1}^{\infty} \left(\frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi} \right) + \frac{m}{\xi}.
 \end{aligned}$$

Note also that since $(I_1 + I_2 + I_3)$ and I_2, I_3 are absolutely convergent integrals ($p \geq 2$), so is I_1 . First, using (2.2) and calculations similar to those in (5.2), we obtain

$$\begin{aligned}
 (5.8) \quad &\frac{2-p}{2} \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{m}{\xi(\xi-z)} dA(\xi) \\
 &= \frac{m(p-2)}{2z} \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \left(\frac{1}{\xi} - \frac{1}{\xi-z} \right) dA(\xi) \\
 &= \frac{m(p-2)}{2z} \left[\int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{\xi} dA(\xi) - \frac{1}{\mu} \int_{\mathbf{D}} \bar{\omega}(\xi) \frac{1}{\xi-z} dA(\xi) \right] \\
 &= \frac{m(p-2)}{2z} \left[D_0 - \frac{1}{\mu} \Omega^\# \left(\frac{1}{z} \right) \right] = \sum_1^{N+2} D_j z^{-j},
 \end{aligned}$$

where D_j are constants. For a given β_j , $|\beta_j| < 1$ in (5.7) we have:

$$(5.9) \quad \frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi} = \left(\beta_j - \frac{1}{\bar{\beta}_j} \right) \frac{1}{\left(\frac{1}{\bar{\beta}_j} - \xi \right) (\beta_j - \xi)}.$$

The density $\rho := (|f^*|^p / f^*)(1/(\xi - z))$ in the integral I_1 satisfies for all j

$$(5.10) \quad |\rho(\xi)| \leq C |\xi - \beta_j|^\delta$$

for some $\delta > 0$ and $C = C(f^*)$ independent of β_j . Indeed, $f^* \in \text{Lip } \gamma$ in $\bar{\mathbf{D}}$ and $f^*(\beta_j) = 0$, so for all $\xi \in \mathbf{D}$ and all j , we have

$$|f^*(\xi)| = |f^*(\xi) - f^*(\beta_j)| \leq C(f^*) |\xi - \beta_j|^\gamma,$$

and so ($z : |z| > 1$ and fixed):

$$|\rho(\xi)| \leq \left(\max_{\xi \in \mathbf{D}} \frac{1}{|\xi - z|} \right) |f^*(\xi)|^{p-1} \leq C(f^*) |\xi - \beta_j|^{\gamma(p-1)},$$

and (5.10) follows since $p > 1$. Thus, for a fixed j we have from (5.9)–(5.10)

$$(5.11) \quad \begin{aligned} & \int_{\mathbf{D}} |\rho(\xi)| \left| \frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi} \right| dA(\xi) \\ & \leq C_{f^*} \left| \beta_j - \frac{1}{\bar{\beta}_j} \right| \int_{\mathbf{D}} \frac{dA(\xi)}{|\xi - \beta_j|^{1-\delta} \left| \frac{1}{\bar{\beta}_j} - \xi \right|} \\ & \leq C \left| \beta_j - \frac{1}{\bar{\beta}_j} \right| \int_{|\xi - \beta_j| \leq 2} \frac{dA(\xi)}{|\xi - \beta_j|^{2-\delta}} \\ & \leq C \left| \beta_j - \frac{1}{\bar{\beta}_j} \right|, \end{aligned}$$

where we also used $|\xi - \beta_j| \leq |(\xi - 1/\bar{\beta}_j)|$ for all $\xi \in \mathbf{D}$. Since $\{\beta_j\}$ is a Blaschke sequence, we conclude that

$$(5.12) \quad \int_{\mathbf{D}} \frac{|f^*|^{p-1}}{|\xi - z|} \sum_1^\infty \left| \frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi} \right| dA(\xi) < +\infty,$$

and hence (by Fubini's theorem) we can indeed interchange summation and integration while calculating I_1 using (5.7).

Fix β_j , $j = 1, \dots$. We have

$$(5.13) \quad \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \left(\frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi} \right) \frac{1}{\xi - z} dA(\xi) =: J_1^{(j)} - J_2^{(j)}.$$

Calculating $J_1^{(j)}$ in the same fashion as the integral in (5.4), we obtain

$$(5.14) \quad J_1^{(j)} = \sum_{k=0}^N \frac{E_k^{(j)}}{z^{k+1}},$$

a rational function. To calculate $J_2^{(j)}$, we write

$$\frac{1}{(\beta_j - \xi)(\xi - z)} = \frac{1}{\beta_j - z} \left[\frac{1}{\beta_j - \xi} + \frac{1}{\xi - z} \right],$$

and using (2.2), we obtain, similarly to (5.8):

$$(5.15) \quad \begin{aligned} J_2^{(j)} &= \frac{1}{\beta_j - z} \left[\int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{\beta_j - \xi} dA(\xi) + \frac{1}{\mu} \int_{\mathbf{D}} \bar{\omega}(\xi) \frac{1}{\xi - z} dA(\xi) \right] \\ &= \frac{1}{\beta_j - z} \left[F_0^{(j)} - F_1^{(j)} \Omega^{\#} \left(\frac{1}{z} \right) \right], \end{aligned}$$

where $F_0^{(j)}, F_1^{(j)}$ are some constants. Combining (5.14), (5.15) and using partial fractions decomposition, we obtain that the integral (5.13) gives for all $z \in \mathbf{C} \setminus \bar{\mathbf{D}}$

$$(5.16) \quad J_1^{(j)} - J_2^{(j)} = \sum_{k=0}^N \frac{H_k^{(j)}}{z^{k+1}} + \frac{K^{(j)}}{\beta_j - z},$$

where $H_k^{(j)}, K^{(j)}$ are some constants. Thus, from (5.7), (5.8), (5.13), (5.16) and

using the remark following (5.12) we obtain that for $z : |z| > 1$,

$$(5.17) \quad I_1 := I_1(z) = \sum_{j=1}^{\infty} r_j(z),$$

where r_j is a rational function for $j = 1, \dots$ with a simple pole at β_j and a pole of order $\leq N+2$ at the origin.

Moreover, the series in (5.17) converges uniformly on compact subsets of $\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}}$. Thus, to complete the proof of Theorem 5.1, it suffices in view of (5.1)–(5.4), (5.7)–(5.8), (5.13), (5.16) to establish the following.

Proposition 5.1. *The function $I_1(z)$ admits a meromorphic pseudocontinuation $M_1(z)$ into $\mathbf{D}$ so that $z^{N+2}B(z)M_1 \in H^q(\mathbf{D})$.*

Proof. Note that in view of (4.8), (5.1)–(5.4), (5.7)–(5.8) I_1 differs from the Cauchy projection (in $\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}}$) of

$$\left(-\frac{|f^*|^p}{\xi^2 f^*} \right) \in L^\infty(\mathbf{T})$$

by a rational function with the only pole at 0 of order $\leq N+2$, and hence $I_1(z) \in \bigcap_{r < \infty} H^r(\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$, in particular $I_1(z) \in H^q(\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$. According to (5.3), (5.7) and (5.8), if we set $R_n(z) := \sum_{j=1}^n r_j(z)$ to be partial sums of the series (5.17), we have

$$(5.18) \quad R_n(z) = \sum_{j=1}^n \left\{ \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} \frac{1}{\xi - z} \left(\frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi} \right) dA(\xi) \right\} + \sum_1^{N+2} D_j z^{-j},$$

where D_j are fixed constants. By (5.17), $R_n(z)$ also have simple poles at β_j , $j = 1, \dots, n$.

Let us separate the following.

Lemma 5.1. $\inf \left\{ \|R_n(\xi) - h(\xi)\|_{L^1(\mathbf{T})} : h \in H^1(\mathbf{D}) \right\} \leq C < \infty$, where the constant C does not depend on n .

Proof. The duality of H^p extremal problems described in (4.1)–(4.2) and Theorem E extend to $p = \infty$, $q = 1$ (also, cf. [Kh1], [D, Ch.8], [RS]). Thus, to prove the lemma it suffices to show that

$$\sup\left\{\left|\frac{1}{2\pi i}\int_{\mathbf{T}} R_n(z)g(z)dz\right| : g \in H^\infty(\mathbf{D}), \|g\|_\infty = 1\right\} \leq C < +\infty$$

or, in view of (5.18), that

$$(5.19) \quad \sup\left\{\left|\frac{1}{2\pi i}\int_{\mathbf{T}} \left\{\sum_{j=1}^n \int_{\mathbf{D}} \frac{|f^\star|^p}{f^\star} \frac{1}{\xi - z} \left(\frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi}\right) dA(\xi)\right\} g(z) dz\right| : \right. \\ \left. g \in H^\infty(\mathbf{D}), \|g\|_\infty = 1\right\} \leq C < +\infty.$$

Applying Fubini's theorem and using Cauchy's representation formula for g , we can see that (5.19) is precisely

$$(5.20) \quad \sup\left\{\left|\sum_{j=1}^n \int_{\mathbf{D}} \frac{|f^\star|^p}{f^\star} g(\xi) \left(\frac{\bar{\beta}_j}{1 - \bar{\beta}_j \xi} - \frac{1}{\beta_j - \xi}\right) dA(\xi)\right| : \|g\|_\infty = 1\right\}.$$

Since g is bounded, the estimates (5.10), (5.11) hold for the integrals in (5.20) for each j and hence the lemma follows with

$$C := C(f^\star) \sum_{j=1}^{\infty} \left|\beta_j - \frac{1}{\beta_j}\right| < +\infty.$$

Now we are ready to finish the proof of the proposition. For each n find $h_n \in H^1(\mathbf{D})$ so that $\|R_n - h_n\|_{L^1(\mathbf{T})} \leq C < +\infty$, i.e., the best H^1 -approximation to R_n . It exists and is, in fact, analytic across $\mathbf{T}$ (cf. [Kh1], [D, Ch.8], [RS]). $R_n - h_n$ has poles only at the origin (of order $\leq N + 2$) and at $\beta_1, \dots, \beta_n$, so we can rewrite it in the form (in $\mathbf{D}$):

$$(5.21) \quad R_n - h_n = \frac{H_n}{z^{N+2} B_n}, \text{ where } H_n \in H^1(\mathbf{D}), \\ \|H_n\|_{L^1(\mathbf{T})} = \|R_n - h_n\|_{L^1(\mathbf{T})} \leq C < +\infty,$$

$$B_n := \prod_{j=1}^n \frac{|\beta_j|}{\beta_j} \frac{\beta_j - z}{1 - \bar{\beta}_j z}.$$

Moreover, for $z : |z| > 1$, since R_n is analytic in $\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}}$, we have ($R_n(\infty) = 0$):

$$(5.22) \quad -\frac{1}{2\pi i} \int_{\mathbf{T}} \frac{R_n(\xi) - h_n(\xi)}{\xi - z} d\xi = R_n(z).$$

According to the F. and M. Riesz Theorem, bounded closed sets in $H^1(\mathbf{D})$ are weak $(\star)$ compact (when imbedded into the space of measures on $\mathbf{T}$), so taking a subsequence if necessary, we can assume that functions H_n in (5.21) converge weak $(\star)$ to $H_0 \in H^1(\mathbf{D})$, i.e., the measures $\{H_n d\xi\}$ converge weak $(\star)$ to $H_0 d\xi$ on $\mathbf{T}$. This and Cauchy's formula imply that $R_n - h_n$ converge uniformly on compact subsets of $\mathbf{D} \setminus \{0, \{\beta_j\}_1^\infty\}$ to the function $H_0/(z^{N+2}B)$. Taking another subsequence if necessary, we can assume that the measures $(h_n - R_n) d\xi|_{\mathbf{T}}$ weak $(\star)$ converge to a finite measure ν on $\mathbf{T}$. Since for all $z : |z| < 1$ we have

$$(5.23) \quad \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{R_n(\xi) d\xi}{\xi - z} = 0, \quad |z| < 1$$

(all poles of R_n are inside of $\mathbf{T}$ and $R_n(\infty) = 0$!), it follows that $(h_n \in H^1(\mathbf{D}))$:

$$(5.24) \quad h_n(z) \rightarrow \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{d\nu(\xi)}{\xi - z} =: \tilde{\nu}(z) \in \bigcap_{0 < r < 1} H^r(\mathbf{D})$$

uniformly on compact subsets of $\mathbf{D}$. Therefore, the sequence of rational functions $R_n(z)$ converges uniformly on compact subsets of $\mathbf{D} \setminus \{0, \{\beta_j\}_1^\infty\}$ to $H_0/(z^{N+2}B) + \tilde{\nu}(z)$. Summarizing, we have the sequence of rational functions $R_n(z)$ with poles at 0 (of order $\leq N+2$) and $\{\beta_j\}_1^\infty$ converge uniformly on compact subsets of $\widehat{\mathbf{C}} \setminus \mathbf{T} \setminus \{0\} \setminus \{\beta_j\}_1^\infty$ to

$$(5.25) \quad R_n(z) \rightarrow \begin{cases} I_1(z), & |z| > 1, \\ \frac{H_0}{z^{N+2}B(z)} + \tilde{\nu}(z), & |z| < 1. \end{cases}$$

Moreover, near the arcs γ on $\mathbf{T}$ free of zeros α_j , $j = 1, \dots, k \leq N$ of the outer part F of f^* and hence (Cor. 3.1) away from zeros and poles of the Blaschke product $B(z)\{R_n(z)\}$ clearly satisfy the hypothesis of the celebrated Beurling theorem (cf., e.g., [Do]):

If we have a family of functions holomorphic in a domain $\mathcal{O}$ and there exists a function $J(z) : J(z) \geq |f(z)|$ for all f in

the family and for some $\varepsilon > 0$

$$(5.26) \quad \int \int_{\mathcal{O}} (\log^+ \log^+ J)^{1+\varepsilon} dA < +\infty,$$

then the family of functions is normal in $\mathcal{O}$.

In our situation we take $\mathcal{O}$ to be any sufficiently small domain away from all the poles of R_n 's and containing the arc γ . In view of (5.21)–(5.22) we have for all $z : |z| > 1$ in H

$$(5.27) \quad |R_n(z)| \leq \frac{C}{|z| - 1},$$

while in view of (5.21), (5.23) and the fact that $\mathcal{O}$ stays away from 0 and $\{\beta_j\}_1^\infty$, we have for $z \in \mathcal{O} \cap \mathbf{D}$ by using Cauchy's formula:

$$(5.28) \quad \begin{aligned} |R_n(z)| &\leq \frac{|H_n(z)|}{|z^{N+1}B(z)|} + |h_n(z)| \\ &\leq \frac{C_1 \|H_n\|_{L^1(\mathbf{T})}}{1 - |z|} + \frac{\|R_n - h_n\|_{L^1(\mathbf{T})}}{1 - |z|} \leq \frac{C'}{1 - |z|}, \end{aligned}$$

where C_1 (and, hence, C') depend only on the distance from $\mathcal{O}$ to $\{0\} \cup \{\beta_j\}_1^\infty$. Letting $J(z)$ to be the right-hand side of (5.27) for $z \in \mathcal{O} \cap (\widehat{\mathbf{C}} \setminus \bar{\mathbf{D}})$ and that of (5.28) for $z \in \mathcal{O} \cap \mathbf{D}$, we see that (5.26) is satisfied (with a vengeance) and therefore a subsequence of functions $\{R_n(z)\}$ converges uniformly in $\mathcal{O}$ to an analytic function. Hence (cf. 5.25) $I_1(z)$ coincides in fact everywhere on $\mathbf{T} \setminus \{\alpha_j\}_1^k$, $k \leq N$ with boundary values of the second function in (5.25). The latter can be written in the form

$$(5.29) \quad \frac{\tilde{H}(z)}{z^{N+2}B(z)}, \quad \tilde{H} \in \bigcap_{0 < r < 1} H^r(\mathbf{D}).$$

Now if we recall that (see the beginning of the proof of Proposition 5.1) the boundary values of the integral $I_1(z)$ from outside of $\mathbf{D}$ belong to $L^q(\mathbf{T})$ (in fact, to $\bigcap_{1 < r < \infty} L^r(\mathbf{T})$!), we obtain that $\tilde{H} \in L^q(\mathbf{T})$, and since $\tilde{H} \in N^+$, $\tilde{H} \in H^q(\mathbf{T})$.

The proof of Proposition 5.1 and that of Theorem 5.1 is now complete for all $p \geq 2$. However, the assumption $p > 2$ was in fact almost totally superficial, needed only to justify the application of Green's formula in the beginning. If,

instead, we had carried out the calculations (5.17)–(5.20) first (and those rely only on $p > 1$) and then went on calculating I_2, I_3 and eventually applied Green's formula, we could have easily extended our argument to all $p > 1$. $\square$

Now from Theorem 5.1 and Corollary 4.1 the main result of our paper, Theorem B, follows immediately.

Corollary 5.1. (cf. Theorem B in Section 0) *The singular inner factor $S(z)$ in the representation (3.1) of the function $f^*(z)$ extremal for the problem (0.1) is trivial ($S \equiv 1$). Thus, the solution f^* of (0.1) has the form*

$$(5.30) \quad f^*(z) = CB(z) \prod_1^N (1 - \bar{\alpha}_j z)^{2/p},$$

where C is a constant, $\alpha_j \in \bar{\mathbf{D}}$, B is a Blaschke product whose zeros may only accumulate at those points α_j , $j = 1, \dots, k \leq N$ that lie on the unit circle. So, in particular, if all $\alpha_j \in \mathbf{D}$, $B(z)$ is a finite Blaschke product.

Remark. The techniques used in proving Proposition 5.1 is based on ideas developed in the works of H.S. Shapiro and G.C. Tumarkin (cf., e.g., [Sh2], [Tu1–3]).

6. Extremal problems with rational kernels: Minimal interpolation problems. All the results from the previous section can be readily extended to more general extremal problems similar to (0.1) with a rational kernel ω instead of a polynomial (cf. [Kh1, 2]). For the sake of brevity we shall consider here the case of problem (0.1) with

$$(6.1) \quad \omega(z) := \sum_{k=1}^N \omega_k (1 - \bar{w}_k z)^{-2}, \quad |w_k| < 1,$$

i.e., $\overline{\omega(z)}$ being a linear combination of the Bergman reproducing kernels for the spaces A^p at w_j . Then (0.1) reduces to finding

$$(6.2) \quad \mu := \sup \left\{ \operatorname{Re} \left(\sum_0^N \bar{\omega}_k f(w_k) \right) : f \in A^p, \|f\|_p \leq 1 \right\}$$

(cf. (0.1')). We have then the following analogue of Corollary 5.1.

Theorem 6.1. *The (unique) solution f^* of the problem (6.2) has the form*

$$(6.3) \quad f^*(z) = CB(z) \prod_1^{2N-2} (1 - \bar{\alpha}_j z)^{2/p} \prod_1^N (1 - \bar{w}_j z)^{-4/p},$$

where C is a constant, $\alpha_j \in \bar{\mathbf{D}}$, B is a Blaschke product whose zeros may only accumulate at those points of α_j , $j = 1, \dots, k \leq 2N - 2$ that lie on the unit circle. So, in particular, if all $\alpha_j \in \mathbf{D}$, $B(z)$ is a finite Blaschke product.

Proof. The proof of (6.3) follows precisely the same steps as that of Corollary 5.1. Indeed, the proof of Ryabych in Section 1 must be modified only slightly: Following (1.4) one has to write $f_m^* = P_{N,m} + \varphi_m$ where $P_{N,m} \in \mathcal{P}_N$ is the Lagrange polynomial that interpolates the values of f_m^* at $\{w_j\}_1^N$, e.g., and proceed as before. The results in Section 2 only rely on the fact that $\omega(z)$ is analytic in the closed disk $\mathbf{D}$, while all the calculations in the proof of Theorem 5.1 leading to Corollary 5.1, an analogue of Theorem 6.1, are only based on Theorem 3.1 and the fact that the function $r(z)$ defined for $z := |z| > 1$ by

$$(6.4) \quad r(z) := \int_{\mathbf{D}} \bar{\omega}(\xi) \frac{1}{\xi - z} dA(\xi) = \frac{1}{2\pi i} \int_{\mathbf{T}} \bar{\Omega}(\xi) \frac{1}{\xi - z} d\xi, \quad \Omega' = \omega,$$

is a rational function with poles inside $\mathbf{D}$ and vanishing at infinity. This is the case in our situation as well, if we set

$$\Omega(\xi) = \sum_1^N \frac{\omega_k}{\bar{w}_k(1 - \bar{w}_k \xi)} - C,$$

C is a constant, so,

$$(6.5) \quad \overline{\Omega(\xi)} \Big|_{\mathbf{T}} = \sum_1^N \frac{\bar{\omega}_k \xi}{w_k(\xi - w_i)} - \bar{C} = \Omega^\#(\xi),$$

a rational function with all poles $\{w_k\}_1^N$ inside $\mathbf{D}$, and if we choose $C := \sum_0^N \omega_k / \bar{w}_k$, $\Omega^\#(\infty) = 0$, and hence the integral in (6.4) equals $\Omega^\#(z)$ as desired. Thus, to finish the proof of Theorem 6.1 we only need to introduce appropriate changes in Theorem 3.1 when calculating the outer factor of f^* . Using (2.2) and keeping in mind that ω is a linear combination of reproducing kernels, we arrive

at the following analogue of (3.5):

$$(6.6) \quad \int_{\mathbf{D}} \frac{|f^*|^p}{f^*} (f^*)' \prod_{j=1}^N (z - w_j) g dA = 0,$$

for all polynomials g . Repeating the argument following (3.5), we arrive (using (6.6)) at the equality

$$(6.7) \quad \prod_1^N (z - w_j) f^* z^2 \left(\mu \frac{|f^*|^p}{f^*} - \bar{\omega} \right) = h(z),$$

with h analytic in $\mathbf{D}$ that holds on $\mathbf{T}$. Equivalently,

$$(6.8) \quad q(z) = \frac{h \cdot z^2}{\prod_1^N (z - w_j)} + f^* \left(\sum_1^N \frac{\bar{\omega}_k z^2}{(z - w_k)^2} \right) = \mu |f^*|^p \geq 0 \quad \text{on } \mathbf{T}.$$

By the same token as in Section 3, $q(z)$ extends as a rational function to the Riemann sphere $\widehat{C}$ that has a double zero at ∞ and 0, double poles at the points $w_1, \dots, w_N$ and at the reflected points $(1/\bar{w}_1), \dots, (1/\bar{w}_N)$, and $(2N-2)$ zeros $\alpha_1, \dots, \alpha_{2N-2}$ inside $\bar{\mathbf{D}}$ and $(2N-2)$ zeros at $(1/\bar{\alpha}_1), \dots, (1/\bar{\alpha}_{2N-2})$ outside of $\mathbf{D}$. (As always, zeros on $\mathbf{T}$ are of even order and must be counted with half-multiplicities.) Thus, we have

$$(6.9) \quad |f^*|^p = C z^2 \frac{\prod_1^{2N-2} (z - \alpha_j) (1 - \bar{\alpha}_j z)}{\prod_1^N (z - w_j)^2 (1 - \bar{w}_j z)^2} \quad \text{on } \mathbf{T}.$$

From (6.9), since $|f^*|^p = |F|^p$ on $\mathbf{T}$, where F is the outer part of f^* , we conclude that the harmonic extension of $\log |F(z)|$ into $\mathbf{D}$ is given by

$$(6.10) \quad \log |F(z)| = \frac{2}{p} \sum_{j=1}^{2N-2} \log |1 - \bar{\alpha}_j z| - \frac{4}{p} \sum_1^N \log |1 - \bar{w}_j z| + \log C.$$

Exponentiating (6.10), we obtain the desired formula for the outer factor of f^* and hence the theorem. $\square$

Let $w_1, \dots, w_N$ be as above, fixed points in the unit disk and $c_1, \dots, c_N$ are given scalars, and consider the following minimal interpolation problem, sometimes called a general Carathéodory–Fejér problem, of finding

$$(6.11) \quad \min \{ \|f\|_p : f \in A^p, f(w_k) = c_k, \quad k = 1, \dots, N \}.$$

From the general theory of such problems (cf. [Kh2]), it follows that the solution f^* exists and is unique and, moreover, is a constant multiple of the extremal function in a problem (6.2) with some (genuinely impossible to calculate explicitly) constants ω_k . Thus, we have the following important corollary.

Corollary 6.1. *Solutions to all problems (6.11) have form (6.3).*

Remarks.

- (i) A particular problem (6.11) with $w_1 = 0$, $c_1 = 1$, $c_2 = \cdots = c_N = 0$ gives rise to so-called *contractive divisors*, analogues of Blaschke products in Bergman spaces. It was treated in [DKSS1,2] and later in [MS] by totally different methods. (Earlier, Hedenmalm solved a similar problem in A^2 —[He].) It was found that in that case the solution has the form (cf. (0.11))

$$(6.12) \quad f^*(z) = C \prod_2^{N-1} \frac{z - w_j}{1 - \bar{w}_j z} \prod_1^{2N-2} (1 - \bar{\alpha}_j z)^{2/p} \prod_2^N (1 - \bar{w}_j z)^{-4/p},$$

$|\alpha_j| < 1$ for all j .

—cf. [MS].

- (ii) A very important and interesting problem would be to study qualitative properties of solutions to problems (6.11) with infinitely many interpolating conditions. The general feeling is that if the solutions exists, it must extend analytically across any arc on $\mathbf{T}$ free of cluster points of the sequence $\{w_k\}$. This is indeed the case when $p = 2$ [AC]. For $w_1 = 0$, $c_1 = 1$, $c_2 = \cdots = 0$ (again a contractive divisor!) and arbitrary $0 < p < \infty$, it was established very recently by C. Sundberg—[Su]. An easier case when $\{\omega_j\}_1^\infty$ is a Blaschke sequence was treated in [DKS]. (It can also be obtained by letting $N \rightarrow \infty$ in (6.12).)

7. Conjecture 1: Some final remarks.

I. All the proofs, especially that of Theorem 5.1 and hence Corollary 5.1 could be drastically shortened and simplified if we could gain an a priori control over the number of zeros of the external function in (0.1), or (6.2), i.e., if we could establish Conjecture 1. This is precisely what makes the H^p -theory so much easier, since from (4.4) it readily follows from the argument principle that $\#$ (zeros of f^*) $\leq \#$ (poles of k in $\mathbf{D}$) $- 1$. Moreover, even an easier result, Theorem A (Cor. 3.2) can be obtained almost at once without appealing to the deep and tedious theory of the nonlinear partial differential equation by only using regularity of the reproducing kernels in weighted spaces of analytic functions that rests on a much more transparent regularity theory of *linear* elliptic partial differential equations (cf. [Ga], [DKSS1,2]). To illustrate this latter point let us prove the following.

Proposition 7.1. *Assume that Conjecture 1 holds. Then $f^* = B\mathcal{F}^{2/p}$, where $\mathcal{F} \neq 0$ in $\mathbf{D}$ and $(N = \deg \omega)$*

$$(7.1) \quad \mathcal{F}(z) = \sum_0^N \lambda_j \left(\frac{\partial}{\partial \xi} \right)^j J(z, \xi)|_{\xi=0},$$

where $B(z) = \prod_1^N (z - \beta_j)/(1 - \bar{\beta}_j z)$, λ_j are some constants and $J(z, \xi)$ is the reproducing kernel in the weighted space $A^2(|B|^p dA) := \{ \text{closure of polynomials in } L^2(|B|^p dA) \}$. In particular, since $J(z, 0)$ is analytic across $\mathbf{T}$ (cf. [Ga], [DKSS1, 2]), (7.1) immediately implies Theorem A.

Proof. For the sake of brevity, we shall only consider the case $1 < p < 2$. In that case, $\mathcal{F}^{2/p-1} \in A^p$ ($f^* \in A^p$ and B is finite!) and so are the functions $g = B\mathcal{F}^{\frac{2}{p}-1} z^{N+k} \varphi(z)$, $k = 1, 2, \dots$, where φ is an arbitrary polynomial. Then (2.2) right away implies that

$$(7.2) \quad \int_{\mathbf{D}} |B|^p \bar{\mathcal{F}} z^{N+k} \varphi dA = 0, \quad k = 1, 2, \dots$$

Since $\mathcal{F}^{2/p} \in H^p$ (by Theorem 1.1), $\mathcal{F} \in H^2$ and hence $\mathcal{F} \in A^2(|B|^p dA)$. Then (7.2) simply says that $\mathcal{F}$ is orthogonal in $A^2(|B|^p dA)$ to the subspace of functions in that space vanishing with the order $\geq N+1$ at the origin. Since the functions $\left\{ \left(\partial/\partial \xi \right)^j J(z, \xi)|_{\xi=0} \right\}_{j=0}^N$ are also orthogonal to the same subspace and are obviously linearly independent (otherwise a nontrivial linear combination of first N derivatives of all functions in $A^2(|B|^p dA)$ would always vanish at 0), (7.1) follows. $\square$

II. The most promising approach to Conjecture 1 lies, we think, in reformulating it in terms of nonlinear partial differential equations. In view of (2.2)–(2.7), Conjecture 1 is equivalent to the following.

Conjecture 2. *The number of critical points (i.e., the points where $v_{\bar{z}} = 0$) in $\mathbf{D}$ of the solution to the boundary value problem (2.6) with the degree of the polynomial $\Omega \leq N+1$ does not exceed N .*

Recall that the points where $v_{\bar{z}}$ vanishes are precisely the points where the ellipticity of the equation (2.6) fails, so Conjecture 2 would establish a bound for the number of degeneracy points for the operator

$$\frac{\partial}{\partial z} \left(\left| \frac{\partial v}{\partial \bar{z}} \right|^{q-2} \frac{\partial v}{\partial \bar{z}} \right)$$

that depends only on the “rotation” on the boundary and is independent of q and the structural constants. Since for any solution v of the equation, (2.6) $v +$ (analytic function) is another solution (with a different boundary data, of course). We can replace the boundary data $\bar{\Omega}$ in (2.6) by a real-valued harmonic polynomial $\operatorname{Re}\Omega$ of degree $(N+1)$. In this form, Conjecture 2 has been recently established by J. Lewis [Le2] under the additional assumption that v is *real-valued* in $\mathbf{D}$. In fact, he has proven Conjecture 2 for real-valued solutions of the p -Laplace equation (2.7). For the reader’s convenience, we shall briefly sketch his argument here, adapted to our situation.

Proposition 7.2. *Let v_ε be a real-valued solution of the regularized equation (2.9)–(2.10) with data P , a real-valued trigonometric polynomial of order $\leq N+1$. Then, the number of critical points of v_ε in $\bar{\mathbf{D}}$ does not exceed N .*

(As usual, the points on $\mathbf{T}$ are counted with half-multiplicity.)

Sketch of the proof (cf. [Le2]). As in Section 2, v_ε is real-analytic across $\mathbf{T}$ and the complex dilatation of $u_\varepsilon = (v_\varepsilon)_{\bar{z}}$ satisfies the estimate (2.11). Hence u_ε is a quasi-regular map and (cf. [LV]) can be factored (u_ε reverses the orientation!) as

$$(7.3) \quad u_\varepsilon = \bar{g} \circ \chi,$$

where g is analytic in $\bar{\mathbf{D}}$ and χ is a $k = |1 - 2/q|$ q.c. automorphism of $\mathbf{C}$. So, the critical points of u_ε are simply zeros of g and the argument principle applies. Now, since v_ε is real-valued, we have on $\mathbf{T}$

$$(7.4) \quad \frac{1}{2} \frac{\partial}{\partial \theta} (v_\varepsilon) := \frac{1}{2} (v_\varepsilon)_\theta = \frac{1}{2} \{iz(v_\varepsilon)_z - i\bar{z}(v_\varepsilon)_{\bar{z}}\} = \operatorname{Re}(izu_\varepsilon).$$

Thus, on each subarc of $\mathbf{T}$ free of zeros of $(v_\varepsilon)_\theta$, $\operatorname{Re}(izu_\varepsilon)$ has a constant sign. So, the increment of the argument of zu_ε along such arc is $\geq -\pi$. Since $(v_\varepsilon)_\theta$ is a *trigonometric* polynomial of degree $\leq N+1$ on $\mathbf{D}$, the number of zeros of $(v_\varepsilon)_\theta$ = the number of arcs on $\mathbf{T}$ where $(v_\varepsilon)_\theta$ has constant sign is $\leq 2(N+1)$, while $\Delta_{\mathbf{T}} \arg z = 2\pi$. Thus, the total increment of the argument of u_ε along all such arcs is at least

$$(7.5) \quad -\pi 2(N+1) + 2\pi = -2\pi N.$$

To study the situation near each zero $z^0 \in \mathbf{T}$ of $(v_\varepsilon)_\theta$, replace the contour $\mathbf{T}$ near z^0 by a small semicircle $\gamma := \{|z - z^0| = \delta, |z| < 1\}$. As long as $u_\varepsilon \neq 0$ at

z^0 , it is obvious that

$$(7.6) \quad \lim_{\delta \rightarrow 0} \Delta_\gamma \arg u_\varepsilon = 0.$$

If $u_\varepsilon|_{z^0} = 0$, using regularity of v_ε across $\mathbf{T}$ and expanding u_ε near z^0 by Taylor's formula,

$$(7.7) \quad u_\varepsilon := (v_\varepsilon)_{\bar{z}} = Q_\ell(z - z^0) + o(|z - z^0|^\ell), \quad 1 \leq \ell \leq 2(N+1),$$

where Q_ℓ is a homogeneous polynomial of degree ℓ . Substituting (7.7) into the equation (2.9)–(2.10) satisfied by u_ε , we find $\varepsilon(Q_\ell)_z + O(|z - z^0|^{3\ell-1}) = 0$, so $(Q_\ell)_z = 0$, i.e., $Q_\ell = c(\bar{z} - \bar{z}^0)^\ell$ and, taking into account the orientation of γ with respect to z^0 , we obtain

$$(7.8) \quad \lim_{\delta \rightarrow 0} \Delta_\gamma \arg u_\varepsilon = -\lim_{\delta \rightarrow 0} \Delta_\gamma \arg(z - z^0)^\ell = \pi\ell \geq 0.$$

From (7.5)–(7.8) it follows that (cf. (7.3))

$$\Delta_{\mathbf{T}} \arg u_\varepsilon = -\Delta_{\mathbf{T}} \arg g \geq -2\pi N,$$

and the proposition is proved. $\square$

The difficulty in extending Lewis' argument to complex-valued solutions of (2.6) lies, of course, in the fact that (7.4) is no longer valid and there is no obvious correlation between v_θ and $v_{\bar{z}}$ on $\mathbf{T}$.

III. A different approach to Conjecture 1 rests on the following considerations. For the motivation consider first the H^p -setting with $k(z) := \sum_1^{N+1} a_k/z^k$ (cf. (4.1)–(4.4)) and let f^* be the extremal function in (4.1). Then, (4.4) implies that

$$(7.9) \quad \int_{\mathbf{T}} \frac{|f^*|^p}{f^*} \xi^{N+k} g d\theta = 0, \quad k = 1, 2, \dots, g \in H^p.$$

For the sake of brevity, as in Proposition 7.1, assume $1 < p \leq 2$ and let $f^* = B\mathcal{F}^{2/p}$, where B is a Blaschke product and $\mathcal{F} \neq 0$ in $\mathbf{D}$, $\mathcal{F} \in H^2$. Suppose, contradictory to Theorem E ((4.3)), that B contains more than N -factors. Fix

$m > N$ and write $B = B_m \tilde{B}_m$, where B_m is a Blaschke product with m factors. Taking in (7.9) $g = \tilde{B}_m \mathcal{F}^{2/p-1} z^{N+k} \varphi$, where $k \geq 1$ and φ is an arbitrary polynomial, we obtain ($g \in H^p$, $|B| = 1$ on $\mathbf{T}$):

$$(7.10) \quad \int_{\mathbf{T}} \bar{\mathcal{F}} \xi^{N+k} \frac{\varphi(\xi)}{B_m(\xi)} d\theta = 0, \quad k = 1, \dots$$

In other words, $\bar{\mathcal{F}}$ annihilates all meromorphic functions with poles at the zeros of B_m , $\beta_1, \dots, \beta_m$. Without loss of generality, to fix the ideas, assume that all $\beta_j \neq 0$, $j = 1, \dots, m$. Define the space

$$H_m^2(\mathbf{D}) = \left\{ H^2(\mathbf{D}) + \text{linear span of } \left\{ \frac{1}{\xi - \beta_j} \right\}_1^m \right\} = \left\{ \frac{g}{B_m}, \quad g \in H^2 \right\},$$

and consider it as a closed subspace in $L^2(\mathbf{T})$. Obviously, for any $z^0 \in \mathbf{D} \setminus \{\beta_j\}_1^m$, the functional $\Lambda_{z^0} : z^0 \mapsto g(z^0)$, $g \in H_m^2$ is a bounded linear functional and so H_m^2 has a reproducing kernel $K_m(z, \xi)$ reproducing the H_m^2 -functions at all points in $\mathbf{D} \setminus \{\beta_j\}_1^m$. Of course, $K_m(z, \xi)(z \in \mathbf{D}, \xi \in \mathbf{T})$ has poles at $\{\beta_j\}_1^m$ and, in fact, it is rather easy to guess that

$$(7.11) \quad K_m(z, \xi) = \frac{1}{2\pi} \frac{1}{(1 - z\bar{\xi})} \frac{1}{B_m(z) \overline{B_m(\xi)}}.$$

Now we can interpret (7.10) by saying that the function $\mathcal{F} \in H^2 \subset H_m^2$ is orthogonal to all functions $g \in H_m^2$ such that $g(0) = \dots = g^{(N)}(0) = 0$. But then, as in Proposition 7.1, it yields that

$$(7.12) \quad \mathcal{F}(z)|_{\mathbf{T}} = \sum_0^N \lambda_j \left(\frac{\partial}{\partial \bar{\xi}} \right)^j \Big|_{\xi=0} K_m(z, \xi).$$

From (7.11)–(7.12) it follows then that

$$\mathcal{F}(z)|_{\mathbf{T}} = \frac{P_N(z)}{B_m(z)},$$

where P_N is a polynomial of degree N . On the other hand, $\mathcal{F}$ is analytic in $\mathbf{D}$, and so the zeros of P_N must cancel all of those of B_m , which is impossible because of our assumption that $m > N$. This eventually gives us a (new, to the best of our knowledge) proof of (4.3) that does not appeal to the Schwarz

Reflection Principle and Argument Principle, but is based entirely on (4.4) and a Hilbert space argument.

Applying these ideas to the Bergman spaces situation, assuming once again $1 < p \leq 2$, $f^* = B\mathcal{F}^{2/p}$, $B = B_m\tilde{B}_m$, $m > N$ and using (2.2) as in (7.2) with $g = \tilde{B}_m\mathcal{F}^{2/p-1}z^{N+k}\varphi$, $k \geq 1$, $\varphi \in H^\infty$, we obtain similarly to (7.10)

$$(7.13) \quad \int_{\mathbf{D}} w \bar{\mathcal{F}}\xi^{N+k} \frac{\varphi(\xi)}{B_m(\xi)} dA(\xi) = 0, \quad k = 1, \dots,$$

where we set $w := |B|^p$. So, similarly to (7.11) we have

$$(7.14) \quad \mathcal{F}(z) = \sum_0^N \lambda_j \left(\frac{\partial}{\partial \xi} \right)^j \bigg|_{\xi=0} J_m(z, \xi),$$

where $J_m(z, \xi)$ is the representing kernel in the space $A_m^2 := \{\text{closure of rational functions } g/B_m \text{ with fixed poles in } \mathbf{D} \text{ in } L^2(w dA)\}$. Once again (cf. (7.11)), it is not hard to see that

$$(7.15) \quad J_m(z, \xi) = k_m(z, \xi) \frac{1}{B_m(z)\overline{B_m(\xi)}},$$

where $k_m(z, \xi)$ is the reproducing kernel in the space of analytic functions $\mathcal{A}_{w_1}^2$ obtained by taking the $L^2(w_1 dA)$ closure of polynomials where the weight $w_1 := |B_m|^{p-2} |\tilde{B}_m|^p$. Thus, to obtain Conjecture 1 in a slightly weaker form would be equivalent to proving:

Conjecture 3. *Any linear combination*

$$(7.16) \quad \sum_0^N \lambda_j \left(\frac{\partial}{\partial \xi} \right)^j k_m(z, \xi) \bigg|_{\xi=0},$$

where k_m is the reproducing kernel in the weighted Bergman space $\mathcal{A}_{w_1}^2$ with the weight $w_1 = |B_m|^{p-2} |\tilde{B}_m|^p$ has at most $2N+1$ zeros in $\mathbf{D}$.

Placing Conjecture 3 in the context of $p \geq 2$, it is very tempting in light of the results in, e.g., [DKS] to propose:

Conjecture 3'. *The conclusion of Conjecture 3 holds for any weight $w > 0$, say bounded in $\mathbf{D}$, such that $\log w$ is subharmonic.*

A similar statement holds even in greater generality in the H^p -setting.

Proposition 7.3. *Let $\rho > 0$ be, for the sake of simplicity, a real-analytic function on the unit circle $\mathbf{T}$. Let $H^{2,\rho}$ (cf. Theorem E in Section 4 and the following remarks) be the closure of polynomials in the weighted space $L^2(\rho d\theta)$ and let $K_\rho(z, \xi)$ be the reproducing kernel in $H^{2,\rho}$. Then any nontrivial linear combination*

$$(7.16) \quad f(z) \Big|_{\mathbf{T}} := \sum_0^N \lambda_j \left(\frac{\partial}{\partial \xi} \right)^j K_\rho(z, \xi) \Big|_{\xi=0}$$

extended as an analytic function into the unit disk has at most N zeros in $\mathbf{D}$.

Proof. Since for any $g \in H^{2,\rho}$, $\|g\|_{H^{2,\rho}} \leq 1$ the Cauchy–Schwarz inequality yields

$$|\langle f, g \rangle_{H^{2,\rho}}| = \left| \sum_0^N \lambda_j g^{(j)}(0) \right| \leq \|f\|_{H^{2,\rho}} \|g\|_{H^{2,\rho}} \leq \|f\|_{H^{2,\rho}}$$

with equality attained if and only if

$$g = \frac{e^{i\gamma} f}{\|f\|_{H^{2,\rho}}},$$

it follows that a scalar multiple of f is the extremal function in the problem (4.1) posed in the weighted space $H^{2,\rho}$ (with $k := \sum_1^{N+1} a_k / \xi^k$). In view of the Remark (i) following Theorem E, there exists $\varphi^* \in H^{2,\rho}$ such that

$$(7.17) \quad (f(\xi) [k(\xi) - \varphi^*(\xi)] d\xi = \text{const} |k - \varphi^*|^2 \rho d\theta \quad \text{a.e. on } \mathbf{T}.$$

Then, $f(k - \varphi^*)$ extends analytically across $\mathbf{T}$, and since k has $\leq (N+1)$ poles in $\mathbf{D}$, (7.17) implies that f has at most N zeros in $\mathbf{D}$. $\square$

Remark. Everywhere in the discussion above we could consider linear combinations

$$\sum_0^N \lambda_j k(z, \xi_j), \quad \xi_j \in \mathbf{D}$$

instead, and formulate the corresponding Conjectures and prove a related version of Proposition 7.3. We leave this to the reader.

Let us finish this discussion with two simple cases for which Conjecture 3 is easily checked.

Proposition 7.4. *Conjecture 3 holds for*

- (i) $w = |z|^\alpha$, $\alpha > -2$;
- (ii) $w = |z - \beta/1 - \bar{\beta}z|^\alpha$, $\alpha > -2$, $|\beta| < 1$.

Proof. (i) Note that $\{(n + \alpha/2 + 1)^{1/2} z^n\}$ is the orthonormal basis in $\mathcal{A}_w^2 := \{L^2(w dA)\text{-closure of polynomials}\}$. Thus, the reproducing kernel $k_w(z, \xi)$ in $\mathcal{A}_w^2$ is easily calculated:

$$(7.18) \quad k_w(z, \xi) = \sum_0^\infty \left(n + \frac{\alpha}{2} + 1\right) z^n \bar{\xi}^n = (1 - z\bar{\xi})^{-2} + \frac{\alpha}{2} (1 - z\bar{\xi})^{-1}.$$

Hence, $\sum_0^N \lambda_j \left(\frac{\partial}{\partial \xi}\right)^j k_w(z, \xi)|_{\xi=0}$ is a polynomial of degree $\leq N$ and the statement follows.

(ii) follows from (i) by a simple change of variable. Indeed, let $K(z, \xi)$ be the reproducing kernel in $\mathcal{A}_w^2$. So,

$$(7.19) \quad \int_{\mathbf{D}} \left| \frac{z - \beta}{1 - \bar{\beta}z} \right|^\alpha K(\xi, z) f(z) dA(z) = f(\xi),$$

for all $f \in H^\infty$. Setting $z = \tau(t) = (t + \beta)/(1 + \bar{\beta}t)$, $t = \tau^{-1}(z) = (z - \beta)/(1 - \bar{\beta}z)$, $\xi = \tau(\eta)$,

$$dA(z) = \frac{(1 - |\beta|^2)^2}{|1 + \bar{\beta}t|^4} dA(t),$$

we obtain from (7.19)

$$(7.20) \quad \begin{aligned} & \int_{\mathbf{D}} |t|^\alpha K(\tau(\eta), \tau(t)) f(\tau(t)) \frac{(1 - |\beta|^2)^2}{(1 + \bar{\beta}t)^2 (1 + \beta\bar{t})^2} dA(t) \\ &= f(\tau(t)) = \frac{f(\tau(\eta))}{(1 + \bar{\beta}\eta)^2} (1 + \bar{\beta}\eta)^2. \end{aligned}$$

Since

$$g(t) := \frac{f(\tau(t))}{(1 + \bar{\beta}t)^2}$$

is an arbitrary H^∞ -function, we obtain from (7.20) that

$$(7.21) \quad (1 + \bar{\beta}\eta)^{-2} K(\tau(\eta), \tau(t)) \frac{(1 - |\beta|^2)}{(1 + \beta\bar{t})^2} = k_w(\eta, t),$$

where k_w is defined by (7.18). Hence,

$$\begin{aligned} K(\xi, z) &= k_w \left(\frac{\xi - \beta}{1 - \bar{\beta}\xi}, \frac{z - \beta}{1 - \bar{\beta}z} \right) \left(\frac{1 - |\beta|^2}{(1 - \bar{\beta}\xi)(1 - \beta\bar{z})} \right)^2 \\ &= \frac{(1 - |\beta|^2)^2}{(1 - \bar{\beta}\xi)^2} \left\{ (1 - \beta\bar{z})^{-2} \left[\left(1 - \frac{(\xi - \beta)(\bar{z} - \bar{\beta})}{(1 - \bar{\beta}\xi)(1 - \beta\bar{z})} \right)^{-2} \right. \right. \\ &\quad \left. \left. + \frac{\alpha}{2} \left(1 - \frac{(\xi - \beta)(\bar{z} - \bar{\beta})}{(1 - \bar{\beta}\xi)(1 - \beta\bar{z})} \right)^{-1} \right] \right\}. \end{aligned}$$

Thus,

$$\sum_0^N \lambda_j \left(\frac{\partial}{\partial \bar{z}} \right)^j K(\xi, z)|_{z=0} = \frac{(1 - |\beta|^2)^2}{(1 - \bar{\beta}\xi)^2} r \left(\frac{\xi - \beta}{1 - \bar{\beta}\xi} \right),$$

where $r(\eta)$ is a rational function with the numerator equal to a polynomial of degree $\leq N + 1$.

(IV) In almost all of the arguments carried out in this paper, we relied quite heavily on the assumption $1 < p < \infty$. The case $p = \infty$ is included into the H^p -theory (cf. [Kh1], [D], [RS]) and hence is not very interesting from the Bergman spaces standpoint. The case $p = 1$ is another matter. In general, when the “kernel w ” in (1.1) is not continuous in $\bar{\mathbf{D}}$, even the existence for the extremal functions may fail (cf., e.g., [V]). However, this is certainly not an obstacle for the extremal problems we have studied here. So, the “goodness” of the problems we have considered strongly suggests that all results in this paper concerning the qualitative structure of extremal functions should extend mutatis mutandis to $p = 1$. However, we have not been able to find an argument providing such a smooth transition by simply applying our results while letting $p \rightarrow 1$.

Finally, the qualitative properties of solutions of the best approximation problems in the L^p -metric like that of finding

$$\inf \{ \|k - \varphi\|_p : \varphi \in A_p \}$$

for say, a real analytic in $\bar{\mathbf{D}}$, function k have not been studied at all to the best of our knowledge. We think that these problems, and especially their relations

with nonlinear partial differential equations, deserve some spotlight in future investigations.

REFERENCES

- [Ah] L. V. AHLFORS, *Lectures on Quasiconformal Mappings*, Von Nostrand, Princeton, New Jersey, 1966.
- [Ar] G. A. ARONSSON, *Construction of singular solutions to the p -harmonic equation and its limit equation for $p = \infty$* , Manuscripta Math. **56** (1986), 135–138.
- [AC] E. J. AKUTOWICZ AND L. CARLESON, *The analytic continuation of interpolatory functions*, J. Analyse Math. **7** (1959–1960), 223–247.
- [BI] B. BOJARSKI AND T. IWANIEC, *The p -harmonic equation and quasiregular mappings*, Report No. 617, Bonn **SFB 72** (1983).
- [CH] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, Vol. II, Wiley, 1989.
- [Do] Y. DOMAR, *On the existence of a largest subharmonic minorant of a given function*, Arkiv för Math. **3** (1957), 429–440.
- [DSS] R. DOUGLAS, H. S. SHAPIRO, AND A. L. SHIELDS, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier **20**(1) (1970), 37–76.
- [D] P. DUREN, *The Theory of H^p -Spaces*, Academic Press, New York, 1980.
- [DKS] P. DUREN, D. KHAVINSON, AND H. S. SHAPIRO, *Extremal functions in invariant subspaces of Bergman spaces*, Illinois Math. J. **40** (1992), 202–210.
- [DKSS1] P. DUREN, D. KHAVINSON, H. S. SHAPIRO, AND C. SUNDBERG, *Contractive zero divisors in Bergman spaces*, Pacific J. Math. **157** (1993), 37–56.
- [DKSS2] P. DUREN, D. KHAVINSON, H. S. SHAPIRO, AND C. SUNDBERG, *Invariant subspaces in Bergman spaces and the biharmonic equation*, Mich. Math. J. **41** (1994), 247–259.
- [Ga] P. GARABEDIAN, *A partial differential equation arising in conformal mapping*, Pacific J. Math. **1** (1951), 485–524.
- [GT] D. GILBARG AND N. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag, New York, 1977.
- [He] H. HEDENMALM, *A factorization theorem for square-integrable analytic functions*, J. Reine Angew. Math. **422** (1991), 45–68.
- [Ho] K. HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice–Hall, Englewood Cliffs New Jersey, 1962.
- [IM] T. IWANIEC AND J. MANFREDI, *Regularity of p -harmonic functions on the plane*, Revista Math. Iber. **5** (1989), 1–19.
- [KS] D. KHAVINSON AND H. S. SHAPIRO, *Invariant subspaces in Bergman spaces and Hedenmalm’s boundary value problem*, Arkiv för Math. **32** (1994), 309–321.
- [Kh1] S. YA. KHAVINSON, *Foundations of the theory of extremal problems for bounded analytic functions and various generalizations of them*, Amer. Math. Soc. Transl. **129** (1986), 1–56.
- [Kh2] S. YA. KHAVINSON, *Foundations of the theory of extremal problems for bounded analytic functions with additional conditions*, Amer. Math. Soc. Transl. **129** (1986), 64–114.
- [LU] O. A. LADYZHENSKAYA AND N. M. URAL’TSEVA, *Linear and Quasi-linear Elliptic Equations*, (translated from Russian), Academic Press, 1968.
- [LV] O. LEHTO AND K. VIRTANEN, *Quasiconformal Mappings in the Plane*, Springer-Verlag, New York, 1973.

- [Le1] J. LEWIS, *Smoothness of certain degenerate elliptic equations*, Proc. Amer. Math. Soc. **80** (1980), 259–265.
- [Le2] J. LEWIS, *On critical points of the p -Laplacian*, published July 6, 1994, Electronic J. on Diff. Equations **3** (1994), 1–4.
- [Lie] G. M. LIEBERMAN, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.
- [Lin] P. LINDQUIST, *On p -harmonic functions in the complex plane and curvature*, Israel J. Math. **64** (1988), 257–269.
- [MS] T. H. MACGREGOR AND M. I. STESSIN, *Weighted reproducing kernels in Bergman spaces*, Mich. Math. J. **41** (1994), 523–533.
- [M1] C. B. MORREY, JR., *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
- [M2] C. B. MORREY, JR., *Second-order elliptic equations in several variables and Hölder continuity*, Math. Z. **72** (1959), 146–164.
- [M3] C. B. MORREY, JR., *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York, 1966.
- [OS] K. U. OSIPENKO AND M. I. STESSIN, *On recovery problems in Hardy and Bergman spaces*, (in Russian), Mat. Zametki **49** (1991), 95–104.
- [RS] W. W. ROGOSINSKI AND H. S. SHAPIRO, *On certain extremum problems for analytic functions*, Acta Math. **90** (1953), 287–318.
- [R1] V. G. RYABYCH, *Certain extremal problems*, (in Russian), Nauchnye soobscheniya R.G.U. (1965), 33–34.
- [R2] V. G. RYABYCH, *Extremal problems for summable analytic functions*, (in Russian), Siberian Math. J. **XXVIII** (1986), 212–217.
- [R3] V. G. RYABYCH, *The simplest extremal problems for functions of A^p -classes*, Unpublished Manuscript (1988).
- [Sh1] H. S. SHAPIRO, *The Schwarz Function and its Generalization to Higher Dimensions*, John Wiley, New York, 1991.
- [Sh2] H. S. SHAPIRO, *Overconvergence of sequences of rational functions with sparse poles*, Arkiv för Math. **7** (1968), 343–349.
- [Sm] V. I. SMIRNOV, *Sur les formules de Cauchy et de Green et quelques problèmes qui s'y rattachent*, Izv. Akad. Nauk SSSR (Ser. Mat.) **VIII** (1932), 331–372.
- [Su] C. SUNDBERG, *Analytic continuability of Bergman inner functions*, Mich. Math. J. **44** (1997), 399–407.
- [Tu1] G. TS. TUMARKIN, *Conditions for the convergence of the boundary values of analytic functions and approximation on rectifiable curves*, (in Russian), ‘Contemporary Problems of the Theory of Analytic Functions’, Nauka, Moscow, 1966, pp. 283–295.
- [Tu2] G. TS. TUMARKIN, *Description of a class of functions approximable by rational functions with fixed poles*, (in Russian), Izv. Akad. Nauk Arm. SSR **I** (1966), 89–105.
- [Tu3] G. TS. TUMARKIN, *Conditions for the uniform convergence and convergence of the boundary values of analytic and meromorphic functions of uniformly bounded characteristic*, (in Russian), Mat. Zametki **V** (1964), 387–417.
- [V] D. VUKOTIĆ, *Linear extremal problems for Bergman spaces*, Expositiones Math. (to appear).

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