Integration Operators on Bergman Spaces

ALEXANDRU ALEMAN & ARISTOMENIS G. SISKAKIS

ABSTRACT. We study operators of the form

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta,$$

on weighted Bergman spaces $L_a^p(w)$ of the unit disc. For a large class of weights we show that T_g is bounded on $L_a^p(w)$ if and only if g is in the Bloch space, and compact if and only if g is in the little Bloch space. For the weights $w(r) = (1-r)^\alpha$, $\alpha > -1$, we show that T_g is in the Schatten p -class of $L_a^2(w)$ with $p > 1$ if and only if g is in the analytic p -Besov space.

1. Introduction. Let $\mathbb{D}$ be the unit disc in the complex plane and $dm(z) = r dr d\theta/\pi$ the normalized Lebesgue area measure on $\mathbb{D}$. Let $w(r)$, $0 \leq r < 1$, be a positive weight function which is integrable on $(0,1)$. We extend w on $\mathbb{D}$ by setting $w(z) = w(|z|)$. We may assume that our weights are normalized so that $\int_{\mathbb{D}} w(z) dm(z) = 1$.

For $1 \leq p < \infty$ the weighted Bergman space $L_a^p(w)$ is the space of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that

$$(1.1) \quad \|f\|_{p,w}^p = \int_{\mathbb{D}} |f(z)|^p w(z) dm(z) < \infty.$$

Standard estimates show that point evaluations are bounded linear functionals on $L_a^p(w)$ and $L_a^p(w)$ is a Banach space. It is a Hilbert space for $p = 2$.

We are interested in certain operators T_g acting on $L_a^p(w)$. They are defined by

$$(1.2) \quad T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta, \quad f \in L_a^p(w),$$

where g is an analytic function on $\mathbb{D}$. If $g(z) = z$ or $g(z) = \log(1/(1-z))$, then T_g is the integration operator and the Cesàro operator, respectively. We may define the companion operator

$$S_g(f)(z) = \int_0^z f'(\zeta)g(\zeta) d\zeta.$$

Then the relation $T_g(f) = fg - S_g(f)$ gives some sort of cancellation property. Denoting with M_g multiplication by g , this relation is written $T_g = M_g - S_g$ and there are symbols g such that M_g and S_g are not bounded on the Hardy or Bergman space but their difference T_g is bounded. In fact, in some cases it is advantageous to think of T_g and S_g as distant cousins of Hankel and Toeplitz operators, respectively. Interest in T_g arose originally from studying semigroups of analytic composition operators because for certain g , T_g are related to the resolvents of such semigroups—see [11].

Ch. Pommerenke showed [10] that T_g is bounded on the Hardy space H^2 if and only if g is in BMOA. This was extended to other Hardy spaces H^p , $1 \leq p < \infty$, in [1], where also compactness of T_g on H^p , or even Schatten class membership of T_g on H^2 , was completely characterized in terms of the symbol g . In this paper we study similar questions on $L_a^p(w)$. We characterize those g for which T_g is bounded or compact on $L_a^p(w)$, for a large class of weights w that satisfy certain conditions. This class includes the classical weights $w_\alpha(r) = (1-r)^\alpha$, $\alpha > -1$. For the latter weights we also characterize the functions g for which T_g is in the Schatten p -class of $L_a^2(w)$.

As it will turn out, the problem of characterizing the symbols g for which T_g is bounded on $L_a^p(w)$ can be reduced to a question about “Carleson measures” on $L_a^p(w)$. These measures are well understood for a large class of weights (see [6], [7] and [9]), but those results do not cover the whole class of weights considered here and our approach will be different.

While working on this paper we learned from J. Cima, P. Chatage and W. Wogen that they have also considered similar questions for the unweighted Bergman spaces. We would like to thank the referee for useful comments and for pointing out the references [5] and [6].

2. Preliminaries. We collect some facts about Bergman, Bloch and Besov spaces. Additional material for these spaces can be found in references. We use C , $C(p)$, etc. to denote constants in various inequalities. The symbol $\cong$ between two quantities means that each quantity is bounded by a constant multiple of the other as the parameters in the quantities vary.

For f analytic on $\mathbb{D}$ we write, as usual,

$$M_p^p(r, f) = \int_0^{2\pi} |f(re^{i\vartheta})|^p \frac{d\vartheta}{2\pi}, \quad 0 \leq r < 1,$$

for the integral means of f . The norm in $L_a^p(w)$ can then be written

$$\|f\|_{p,w}^p = 2 \int_0^1 M_p^p(r, f) w(r) r dr.$$

The Bloch space $\mathcal{B}$ is the space of all analytic f on $\mathbb{D}$ such that

$$b(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty,$$

and the little Bloch space $\mathcal{B}_0$ consists of all f in $\mathcal{B}$ for which

$$\lim_{|z| \rightarrow 1} \left(\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \right) = 0.$$

$\mathcal{B}$ is a Banach space with norm $\|f\|_{\mathcal{B}} = |F(0)| + b(f)$, and $\mathcal{B}_0$ a closed subspace and coincides with the closure of the polynomials in $\mathcal{B}$.

We shall also use the Besov spaces B_p , $1 < p < \infty$, consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} dm(z) < \infty.$$

There are various characterizations of functions in B_p . We shall need here a characterization that is contained implicitly in our work [1]. To state this result we recall from [8] a decomposition of $\mathbb{D}$ into a disjoint sequence of Carleson half-squares $\{R_j\}$ of hyperbolically equal size. More precisely, let $r_n = 1 - 2^{-n}$, $n = 0, 1, 2, \dots$ and for each n let $A_n = \{z : r_n \leq |z| < r_{n+1}\}$. For fixed n let

$$(2.1) \quad R_{n,k} = \{z = re^{it} : 2\pi k/2^n \leq t < 2\pi(k+1)/2^n\} \cap A_n, \quad k = 0, 1, \dots, 2^n - 1$$

Let $\{R_j\}$ be an enumeration of all sets obtained in this way. The following result is easily seen to follow from [1, Theorem 2].

Theorem 0 [1]. *Suppose $1 < p < \infty$, and g analytic on $\mathbb{D}$. Then $g \in B_p$ if and only if*

$$(2.2) \quad \sum_{j \geq 1} \left(\int_{R_j} |g'(z)|^2 dm(z) \right)^{p/2} < \infty.$$

Condition (2.2) for membership of g in B_p is one of a number of equivalent conditions found in the literature, some of them involving integration (instead of summation) of the averages of $|g'(z)|^2$ over hyperbolic discs [12, Theorem 7(4)] and others involving summation of these averages taken over hyperbolic discs whose centers form a separated sequence (for this terminology see [8]). Here the centers of R_j 's form a separated sequence and the role of hyperbolic discs is played by the R_j 's.

Lemma 1. *Suppose $1 \leq p < \infty$ and $f : \mathbb{D} \rightarrow \mathbb{C}$ is analytic. Then*

$$(2.3) \quad \frac{d}{dr} M_p^p(r, f) \leq p M_p^{p-1}(r, f) M_p(r, f'), \quad 0 < r < 1.$$

Proof. If f is zero, the assertion is obvious. If f is nonzero, at points z where f is not zero we have

$$\begin{aligned} r \frac{\partial |f(z)|^p}{\partial r} &= r p |f(z)|^{p-1} \frac{\partial |f(z)|}{\partial r} = p |f(z)|^p \operatorname{Re} \left(z \frac{f'(z)}{f(z)} \right) \\ &\leq r p |f(z)|^{p-1} |f'(z)|. \end{aligned}$$

By dominated convergence we have

$$\frac{d}{dr} M_p^p(r, f) = \int_0^{2\pi} \frac{\partial |f(re^{i\vartheta})|^p}{\partial r} \frac{d\vartheta}{2\pi} \leq p \int_0^{2\pi} |f(re^{i\vartheta})|^{p-1} |f'(re^{i\vartheta})| \frac{d\vartheta}{2\pi}.$$

If $p = 1$, the assertion is clear. If $p > 1$, apply Hölder's inequality in the last integral and the assertion follows. $\square$

Remark 1.

- (i) For $f \in L_a^p(w)$, $p \geq 1$, using the subharmonicity of $|f(z)|^p$ we get the estimate,

$$(2.4) \quad |f(z)| \leq \frac{2^{1/p}}{\left((1 - |z|) \int_{\xi}^1 w(u) u \, du \right)^{1/p}} \|f\|_{p, w}, \quad \xi = \frac{1 + |z|}{2},$$

for each $z \in \mathbb{D}$.

- (ii) It is easy to see that if w, v are two weights and there is $\rho \in (0, 1)$ such that w/v is bounded and bounded away from zero on $(\rho, 1)$, then $L_a^p(w) = L_a^p(v)$ with equivalence in norms.

3. Boundedness and compactness. In this section we find a large class of Bergman spaces, on which we characterize the bounded or compact operators T_g .

Lemma 2. *Suppose w is a weight and there is a constant c such that*

$$(P1) \quad w(r) \geq \frac{c}{1-r} \int_r^1 w(u) du, \quad 0 < r < 1.$$

Then for $1 \leq p < \infty$ there is a constant $C = C(p)$ such that

$$(3.1) \quad \int_{\mathbb{D}} |f(z)|^p w(z) dm(z) \leq C \left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p w(z) dm(z) \right),$$

for all functions f analytic on $\mathbb{D}$.

Proof. Suppose f is nonzero, $f(0) = 0$, and both integrals in (3.1) are finite. We have

$$\begin{aligned} \|f\|_{p,w}^p &= 2 \int_0^1 M_p^p(r, f) w(r) r dr \\ &= 2 \int_0^1 r w(r) \left(\int_0^r \frac{d}{du} (M_p^p(u, f)) du \right) dr \\ &\leq 2p \int_0^1 w(r) \int_0^r M_p^{p-1}(u, f) M_p(u, f') du dr \quad (\text{Lemma 1}) \\ &= 2p \int_0^1 M_p^{p-1}(u, f) M_p(u, f') \int_u^1 w(r) dr du \quad (\text{Fubini's Theorem}) \\ &\leq \frac{2p}{c} \int_0^1 M_p^{p-1}(u, f) M_p(u, f') (1-u) w(u) du \quad (\text{from P1}). \end{aligned}$$

If $p = 1$, this gives the desired conclusion. For $p > 1$, apply Hölder's inequality in the last integral with exponents $p/p-1$, p to get

$$\begin{aligned} \|f\|_{p,w}^p &\leq \frac{2p}{c} \left(\int_0^1 M_p^p(u, f) w(u) du \right)^{(p-1)/p} \left(\int_0^1 M_p^p(u, f') (1-u)^p w(u) du \right)^{1/p} \\ &\cong \frac{2p}{c} \|f\|_{p,w}^{p-1} \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p w(z) dm(z) \right)^{1/p}, \end{aligned}$$

and this gives the desired result.

We now remove the restriction $f(0) = 0$. Write $f = f(0) + g$. Then $f' = g'$ and we have

$$\begin{aligned} \|f\|_{p,w}^p &\leq (|f(0)| + \|g\|_{p,w})^p \leq 2^p(|f(0)|^p + \|g\|_{p,w}^p) \\ &\leq \max(2^p, 2^p C) \left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^p w(z) dm(z) \right) \end{aligned}$$

as desired.

Finally, we remove the restriction that both integrals in (3.1) are finite. For f analytic on $\mathbb{D}$ let $f_\rho(z) = f(\rho z)$ with $\rho < 1$. These functions f_ρ are bounded on $\mathbb{D}$ so the conclusion holds for each ρ . By the Monotone Convergence Theorem, when $\rho \rightarrow 1$, we obtain the assertion. $\square$

Corollary 1. *Suppose w satisfies (P1). Then:*

- (i) *If $g \in \mathcal{B}$, then T_g is bounded on $L_a^p(w)$, $p \geq 1$, and $\|T_g\|_p \leq C(p)\|g\|_{\mathcal{B}}$.*
- (ii) *If $g \in \mathcal{B}_0$, then T_g is compact on $L_a^p(w)$.*

Proof. (i) Suppose $g \in \mathcal{B}$ and $f \in L_a^p(w)$. Then from Lemma 2,

$$\begin{aligned} \|T_g(f)\|_{p,w}^p &\leq C \int_{\mathbb{D}} |f(z)g'(z)|^p (1 - |z|)^p w(z) dm(z) \\ &\leq C \|g\|_{\mathcal{B}}^p \|f\|_{p,w}^p, \end{aligned}$$

and the conclusion follows.

- (ii) If $g \in \mathcal{B}_0$, there are polynomials P_n such that $\|g - P_n\|_{\mathcal{B}} \rightarrow 0$. Then

$$\|T_g - T_{P_n}\|_p = \|T_{g-P_n}\|_p \leq C(p)\|g - P_n\|_{\mathcal{B}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But for P a polynomial, T_P is compact. To see this observe that T_P is the product of the multiplication operator $f \rightarrow P'f$, which is bounded because P' is bounded on $\mathbb{D}$, with the integration operator $f \rightarrow \int_0^z f(\zeta) d\zeta$, which is easily seen to be compact on $L_a^p(w)$. Thus T_g is approximated by compact operators and is therefore compact. $\square$

The corollary above gives sufficient conditions on g for T_g to be bounded or compact on $L_a^p(w)$. We now look for necessary conditions. It turns out that we can work in a more general setting.

We consider a Banach space X , consisting of analytic functions on $\mathbb{D}$, satisfying the following conditions:

- (C1) For each $\lambda \in \mathbb{D}$, $L_\lambda(f) = f(\lambda)$ is a bounded linear functional on X .
- (C2) For each $\sigma \in \partial\mathbb{D}$, the operator $U_\sigma(f)(z) = f(\sigma z)$ is bounded on X and $\sup_{\sigma \in \partial\mathbb{D}} \|U_\sigma\| < \infty$.
- (C3) For some $s \in (0,1)$ and $\psi_s(z) = sz + 1 - s$, the composition operator $C_s(f) = f \circ \psi_s$ is bounded on X .

Remark 2. If (C2) holds, then $U_{\sigma}^{-1} = U_{\bar{\sigma}}$. Also from (C2), (C3) it follows that if $\sigma, \tau \in \partial\mathbb{D}$, then the composition operator

$$C_{\sigma,\tau}(f)(z) = f \circ \psi_{\sigma,\tau}, \quad \psi_{\sigma,\tau}(z) = s\tau z + \sigma - s\sigma,$$

is bounded on X and

$$\sup_{\sigma,\tau \in \partial\mathbb{D}} \|C_{\sigma,\tau}\| = M < \infty.$$

To see this observe that $C_{\sigma,\tau} = U_{\sigma}^{-1} \circ U_{\tau} \circ C_s \circ U_{\sigma}$.

For X as above denote by $B(X) = \{f \in X : \|f\| \leq 1\}$ the closed unit ball.

Proposition 1. *Suppose X is a Banach space as above, satisfying the conditions (C1), (C2), and (C3). Then there is $C > 0$ such that*

$$(3.2) \quad \sup_{f \in B(X)} |f'(\lambda)| \leq \frac{C}{1-|\lambda|} \sup_{f \in B(X)} |f(\lambda)|$$

for each $\lambda \in \mathbb{D}$.

Proof. Let $f \in X$, $\|f\| \leq 1$, and $\lambda \in \mathbb{D}$. Write $r = |\lambda|$. An application of the Schwarz Lemma gives

$$|f'(\lambda)| \leq \frac{1}{(1-s)(1-r)} \max\{|f(z)| : |z| = sr + 1 - s\}.$$

Let $\sigma \in \partial\mathbb{D}$ such that $|f((sr + 1 - s)\sigma)| = \max\{|f(z)| : |z| = sr + 1 - s\}$. Then

$$\begin{aligned} |f((sr + 1 - s)\sigma)| &\leq \sup_{g \in B(X)} |g((sr + 1 - s)\sigma)| \\ &= \sup_{g \in B(X)} |C_{\sigma,\sigma\bar{\lambda}/r}(g)(\lambda)| \\ &= \sup_{g \in B(X)} |L_{\lambda} \circ C_{\sigma,\sigma\bar{\lambda}/r}(g)| \\ &\leq \sup_{g \in B(X)} \|C_{\sigma,\sigma\bar{\lambda}/r}(g)\| \|L_{\lambda}\| \\ &\leq M \sup_{g \in B(X)} |g(\lambda)|. \end{aligned}$$

Thus we have

$$|f'(\lambda)| \leq \frac{M}{(1-s)(1-|\lambda|)} \sup_{g \in B(X)} |g(\lambda)|$$

for each $f \in X$, $\|f\| \leq 1$, and the conclusion follows. $\square$

Corollary 2. *Let X be a nontrivial Banach space of analytic functions on $\mathbb{D}$ satisfying the properties (C1), (C2), and (C3) above. If T_g is bounded on X , then $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq C\|T_g\|$.*

Proof. Let $\lambda \in \mathbb{D}$, $f \in X$ with $\|f\| \leq 1$. Then from Proposition 1,

$$|T_g(f)'(\lambda)| \leq \|T_g\| \frac{C}{1-|\lambda|} \sup_{h \in B(X)} |h(\lambda)|.$$

Now choose $f_n \in B(X)$ such that $|f_n(\lambda)| \rightarrow \sup_{h \in B(X)} |h(\lambda)|$. Observe that $\sup_{h \in B(X)} |h(\lambda)| \neq 0$ for all λ except possibly countably many, because X is nontrivial. Replacing f by f_n in the inequality above, we obtain

$$(1-|\lambda|)|g'(\lambda)| |f_n(\lambda)| \leq C\|T_g\| \sup_{h \in B(X)} |h(\lambda)|,$$

and letting $n \rightarrow \infty$ gives $(1-|\lambda|)|g'(\lambda)| \leq C\|T_g\|$ for all λ in $\mathbb{D}$ except countably many, and by continuity this holds for all $\lambda \in \mathbb{D}$. Thus we have the desired conclusion. $\square$

We now assume further that X satisfies the additional condition:

(C4) The multiplication operator $M_z(f)(z) = zf(z)$ is bounded on X and for some fixed $t \in (0,1)$ we have $\lim_{n \rightarrow \infty} \|(tz + 1 - t)^n f(z)\| = 0$.

Corollary 3. *Suppose X is a nontrivial Banach space of analytic functions on $\mathbb{D}$ satisfying the conditions (C1) through (C4). If T_g is compact on X , then $g \in \mathcal{B}_0$.*

Proof. We will show that for an arbitrary sequence $\{\lambda_n\}$ in $\mathbb{D}$ with $|\lambda_n| \rightarrow 1$ we have

$$(3.3) \quad \liminf_{n \rightarrow \infty} (1-|\lambda_n|)|g'(\lambda_n)| = 0,$$

which implies that $g \in \mathcal{B}_0$. Observe that the set $S = \{\lambda \in \mathbb{D} : f'(\lambda) = 0 \text{ for all } f \in X\}$ is at most countable. If not, then it has a limit point in $\mathbb{D}$ so $f' \equiv 0$ for each $f \in X$, and since $X \neq \{0\}$, the constant functions are in X and by (C4) $cz \in X$, a contradiction. Clearly it suffices to show (3.3) for $\{\lambda_n\} \in \mathbb{D} - S$.

Let Λ_n be the bounded linear functional on X given by $\Lambda_n(f) = f'(\lambda_n)$. By the above all these functionals are nonzero. If T_g is compact on X , then T_g^* is

compact on dual space X^* , so there is a subsequence $\{n_k\}$ and a bounded linear functional $\Lambda \in X^*$ such that in the norm topology of X^* ,

$$\lim_{k \in \infty} T_g^*(\Lambda_{n_k}/\|\Lambda_{n_k}\|) = \Lambda.$$

Claim. $\Lambda = 0$. Assuming for the moment this claim, we proceed to show that $g \in \mathcal{B}_0$. We may write λ_k for the terms of the subsequence (this will not cause any confusion) so that $\Lambda = \lim_{k \rightarrow \infty} T_g^*(\Lambda_k/\|\Lambda_k\|)$. Consider the linear functional $L_k(f) = f(\lambda_k)$. We have

$$\begin{aligned} T_g^*(\Lambda_k)(f) &= \Lambda_k(T_g(f)) = T_g(f)'(\lambda_k) \\ &= g'(\lambda_k)L_k(f). \end{aligned}$$

Proposition 1 gives

$$\|\Lambda_k\| \leq \left(\frac{C}{1 - |\lambda_k|} \right) \|L_k\|, \quad \text{i.e.,} \quad 1 - |\lambda_k| \leq \frac{C\|L_k\|}{\|\Lambda_k\|}.$$

Now choose f such that $\frac{1}{2}\|L_k\|\|f\| \leq |L_k(f)|$. Then:

$$\begin{aligned} (1 - |\lambda_k|)|g'(\lambda_k)| &\leq \frac{C}{\|\Lambda_k\|} \|L_k\| |g'(\lambda_k)| \\ &\leq \frac{2C}{\|\Lambda_k\| \|f\|} |L_k(f)| |g'(\lambda_k)| \\ &= 2C \left| T_g^* \left(\frac{\Lambda_k}{\|\Lambda_k\|} \right) \left(\frac{f}{\|f\|} \right) \right| \\ &\leq 2C \left\| T_g^* \left(\frac{\Lambda_k}{\|\Lambda_k\|} \right) \right\| \end{aligned}$$

and by the claim this goes to 0 as $k \rightarrow \infty$.

To prove the claim, suppose $f \in X$. We have

$$\begin{aligned} \Lambda(f) &= \lim_{k \rightarrow \infty} T_g^*(\Lambda_k/\|\Lambda_k\|)(f) \\ &= \lim_{k \rightarrow \infty} \frac{1}{\|\Lambda_k\|} f(\lambda_k)g'(\lambda_k). \end{aligned}$$

Let ϑ be a limit point of the sequence $\{\arg \lambda_k\}$ and, by passing to a subsequence,

we may assume that $\lambda_k \rightarrow \sigma = e^{i\vartheta}$. For each n we have

$$\Lambda((t\bar{\sigma}z + 1 - t)^n f(z)) = \lim_{k \rightarrow \infty} \frac{1}{\|\Lambda_k\|} (t\bar{\sigma}\lambda_k + 1 - t)^n f(\lambda_k) g'(\lambda_k) = \Lambda(f),$$

hence

$$|\Lambda(f)| = |\Lambda((t\bar{\sigma}z + 1 - t)^n f(z))| \leq \|\Lambda\| \|(t\bar{\sigma}z + 1 - t)^n f(z)\|.$$

But $\lim_{n \rightarrow \infty} \|(t\bar{\sigma}z + 1 - t)^n f(z)\| = 0$ by (C2) and (C4), hence $\Lambda \equiv 0$. $\square$

Remark 3. Condition (C4) may be replaced by other conditions of similar nature. One such condition is:

(C4)' The multiplication operator $M_z(f)(z) = zf(z)$ is bounded on X and $\lim_{n \rightarrow \infty} \|z^n f(z)\| = 0$.

This condition (C4)', however, is more restrictive than (C4). Indeed (C4)' implies (C4) for all $t \in (0, 1)$ because,

$$\|(tz + 1 - t)^n f(z)\| \leq \sum_{k=0}^n \binom{n}{k} t^k (1 - t)^{n-k} \|z^k f(z)\|.$$

Now if $\{a_n\}$ is a zero sequence, then so is the sequence

$$A_n = \sum_{k=0}^n \binom{n}{k} t^k (1 - t)^{n-k} a_k,$$

because for fixed N , $\lim_{n \rightarrow \infty} \sum_{k=0}^N \binom{n}{k} t^k (1 - t)^{n-k} = 0$.

We observe that (C4)' is actually (C4) with $t = 1$ and the argument above can be used to show that if (C4) holds for some t , then it holds for all $t' < t$.

Proposition 2. For each $p \geq 1$, $L_a^p(w)$ satisfies (C1), (C2), and (C4). If in addition there is $s \in (0, 1)$ and $d > 0$ such that

$$(P2) \quad w(sr + 1 - s) \geq dw(r), \quad 0 < r < 1,$$

then $L_a^p(w)$ satisfies conditions (C3) for this value of s .

Proof. That point evaluations are bounded linear functionals follows from (2.4). Condition (C2) is obvious since $w(z) = w(|z|)$. Condition (C4) follows from the Dominated Convergence Theorem. It remains to show that if (P2) is satisfied, then the composition operator $C_s(f)(z) = f(sz + 1 - s)$ is bounded on $L_a^p(w)$. The proof of this is essentially due to Kriete and MacCluer [5, Theorem 1.2]. We include a proof here with the necessary modifications for our hypothesis and for the sake of completeness. We have:

$$\begin{aligned}
 \|C_s(f)\|_{p,w}^p &= \int_{\mathbb{D}} |f(sz + 1 - s)|^p w(z) dm(z) \\
 &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |f(sre^{i\vartheta} + 1 - s)|^p d\vartheta w(r)r dr \\
 &\leq \frac{C}{\pi} \int_0^1 \int_0^{2\pi} |f((sr + 1 - s)e^{i\vartheta})|^p d\vartheta w(r)r dr \quad (\text{by subordination}) \\
 &= 2C \int_0^1 M_p^p(sr + 1 - s, f) w(r)r dr \\
 &= \frac{2C}{s} \int_{1-s}^1 M_p^p(u, f) w\left(\frac{u+s-1}{s}\right) \left(\frac{u+s-1}{s}\right) du \\
 &\leq \frac{2C}{sd} \int_0^1 M_p^p(u, f) w(u)u du \quad (\text{using (P2)}) \\
 &= \frac{2C}{sd} \|f\|_{p,w}^p,
 \end{aligned}$$

and the conclusion follows. $\square$

From Corollaries 1, 2, 3 and Proposition 2 we have the following,

Theorem 1. *Suppose $1 \leq p < \infty$ and w is a weight satisfying the conditions (P1) and (P2). Then:*

- (i) T_g is bounded on $L_a^p(w)$ if and only if $g \in \mathcal{B}$. In this case $\|T_g\| \cong \|g\|_{\mathcal{B}}$.
- (ii) T_g is compact on $L_a^p(w)$ if and only if $g \in \mathcal{B}_0$.

To end this section we state a corollary which is the analogue for Bergman spaces of [10, (3.4) and (3.5)] or [2, Theorem 1] concerning Hardy spaces and BMOA.

Corollary 4. *Suppose w satisfies (P1) and g is analytic on $\mathbb{D}$. Then:*

- (i) If $g \in \mathcal{B}$, then the function $e^g \in L_a^p(w)$ for some $p > 0$.
- (ii) If $g \in \mathcal{B}_0$, then $e^g \in L_a^p(w)$ for all $p < \infty$.

Proof. (i) Let $g \in \mathcal{B}$, then by Corollary 1 T_g is bounded on $L_a^2(w)$. Without loss of generality assume $g(0) = 0$. Applying T_g to the function 1, repeatedly we find $T_g(1)(z) = g(z)$ and

$$T_g^n(1)(z) = \frac{1}{n!} g(z)^n, \quad n = 2, 3, \dots$$

Denote by $r(T_g)$ the spectral radius of T_g and let s be a real number such that $0 < s < 1/r(T_g)$. Then the series $\sum_{n \geq 0} s^n T_g^n$ converges in the operator norm topology [3, Lemma VII.8.4], hence the series

$$\sum_{n \geq 0} \frac{(sg(z))^n}{n!} = \sum_{n \geq 0} s^n T_g^n(1)(z)$$

converges in $L_a^2(w)$. Since point evaluation are bounded linear functionals on $L_a^2(w)$, the sum of this series coincides with the pointwise sum which is e^{sg} . Thus $e^{sg} \in L_a^2(w)$ hence $e^g \in L_a^{2s}(w)$.

(ii) If $g \in \mathcal{B}_0$, then T_g is compact on $L_a^2(w)$. We show that T_g has no nonzero eigenvalues. Indeed if f is analytic on $\mathbb{D}$ with $T_g(f) = \lambda f$ for some $\lambda \neq 0$, then from (1.2) it follows that $f(0) = 0$. Differentiating (1.2) and setting $z = 0$, we obtain $f'(0) = 0$. Repeating this, we find inductively that $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \dots$ thus $f = 0$. Thus T_g has no eigenvalues $\lambda \neq 0$ and since it is compact, it follows that $r(T_g) = 0$. From (i) we then find $g \in L_a^{2s}(w)$ for each $s > 0$, and the proof is complete. $\square$

4. Examples and discussion of the weights considered. We are going to give examples of weights that satisfy the stated conditions,

$$(P1) \quad w(r) \geq \frac{c}{1-r} \int_r^1 w(u) du, \quad 0 < r < 1,$$

$$(P2) \quad w(sr + 1 - s) \geq dw(r), \quad 0 < r < 1.$$

and hence for such weights Theorem 1 holds. As a general remark we see that because of Remark 1 we may assume that conditions (P1) and (P2) hold in a small interval $r \in (\rho, 1)$ for some $\rho < 1$, without affecting the conclusions.

We also remark that if w is nonincreasing near 1, then (P1) holds for all r near 1 because in this case,

$$\int_r^1 w(u) du \leq w(r)(1-r),$$

while if w is nondecreasing near 1, then condition (P2) is trivially satisfied. Also if w is a decreasing concave function on $[0,1]$ with $w(1) = 0$ then (P1) and (P2) hold.

Example 1. The weights

$$(4.1) \quad w(R) = (1-r)^\alpha, \quad \alpha > -1$$

satisfy (P1) and (P2) since

$$w(r) = \frac{\alpha+1}{1-r} \int_r^1 w(u) du, \quad w(rs+1-s) = w(r)s^\alpha$$

for each $r \in (0,1)$.

The following simple observation allows us to construct many other examples.

Proposition 3. *Let w be a positive function on $(0,1)$.*

(i) *If there exist $\rho < 1$ and a function h integrable on $(0,1)$ such that*

$$(4.2) \quad w(tr+1-t) \leq h(t)w(r), \quad r \in (\rho,1),$$

for almost every $t \in (0,1)$, then w satisfies (P1).

(ii) *If for some $s \in (0,1)$ the limit*

$$(4.3) \quad \lim_{r \rightarrow 1} \frac{w(rs+1-s)}{w(r)}$$

exists and is nonzero, then w satisfies (P2) for this value of s .

Proof. (i) We have

$$\int_r^1 w(u) du = (1-r) \int_0^1 w(tr+1-t) dt \leq (1-r) \left(\int_0^1 h(t) dt \right) w(r),$$

and the assertion follows.

(ii) The assumption gives $w(rs+1-s)/w(r) \geq \delta > 0$ for $r \in (\rho,1)$, and this is a reformulation of (P2). $\square$

We can now list some more weights that satisfy the assumptions of Proposition 3 (by straightforward estimation) and hence satisfy (P1) and (P2).

Example 2. For $\alpha > -1$ and $\beta \in \mathbb{R}$, let

$$(4.4) \quad w(r) = (1-r)^\alpha \left(\log \frac{e}{1-r} \right)^\beta.$$

Example 3. For $\beta > 0$, $0 < \alpha \leq 1$, let

$$(4.5) \quad w(r) = \exp \left(-\beta \left(\log \frac{e}{1-r} \right)^\alpha \right).$$

Example 4. For $\alpha > 0$, let

$$(4.6) \quad w(r) = \left(\log \log \frac{e}{1-r} \right)^\alpha.$$

Remark 4. Suppose w is a continuous weight satisfying (P1) and set $W(r) = \int_r^1 w(u) du$. Then (P1) can be written

$$\frac{W'(r)}{W(r)} \leq -\frac{c}{1-r},$$

which says that (P1) is a regularity (or distortion) condition. The following proposition says that (P2) is really a growth condition.

Proposition 4. Suppose that $\inf_{u < r} w(u) > 0$ for each $r \in (0,1)$ and w satisfies (P2). Then there exist $\alpha > -1$, $C > 0$ and $\rho \in (0,1)$ such that

$$w(r) \geq C(1-r)^\alpha, \quad r \in (\rho, 1).$$

In particular $L_a^p(w) \subset L_a^p((1-r)^\alpha)$.

Proof. We are assuming $w(sr+1-s) \geq dw(r)$ for $r \in (\rho_0, 1)$. Let $\rho = s\rho_0 + 1 - s$. For $r > \rho$ we have $1-r < s(1-\rho_0)$. Hence there exists $n = n(r)$ with

$$s^{n+1}(1-\rho_0) \leq 1-r < s^n(1-\rho_0).$$

This gives

$$(4.7) \quad (i) \quad n \geq \frac{1}{\log s} \log(1-r) + B - 1, \quad (ii) \quad n < \frac{1}{\log s} \log(1-r) + B,$$

with $B = -\log(1-\rho_0)/\log s$. Now set $r = 1 + s^n(\rho_r - 1)$. Then $\rho_r \in [\rho_0, \rho)$, $1 + s^2(\rho_r - 1) > 1 + s(\rho_r - 1) > \rho_r > \rho_0$ and by repeated application $1 + s^k(\rho_r - 1) > \rho_0$ for each k . By applying (P2) repeatedly, we have

$$w(r) = w(1 + s^n(\rho_r - 1)) \geq dw(1 + s^{n-1}(\rho_r - 1)) \geq \cdots \geq d^n w(\rho_r) \geq d^n \gamma,$$

where $\gamma = \inf_{u < \rho} w(u)$. Then according to $d \geq 1$ or $d < 1$, we obtain the desired inequality from (4.7) (i) or (ii). $\square$

Remark 5. There is no real restriction in assuming $\inf_{u < r} w(u) > 0$ for all $r \in (0, 1)$. If w is arbitrary, let

$$\omega(r) = w(r) + \left(\sup\{|f(z)| : |z| \leq r, \|f\|_{p,w} \leq 1\} \right)^{-p}.$$

Then ω satisfies the inf condition and $L_a^p(\omega) = L_a^p(w)$.

A careful reading of the results and proofs in this paper reveals that the notion of Carleson measures arises behind the scenes in many places. We will be brief, referring the reader to [7], [13] for information on Carleson measures. The following proposition shows that if (P1) and (P2) hold for w , then T_g is bounded on $L_a^p(w)$ if and only if $d\mu(z) = |g'(z)|^p(1-|z|)^p dm(z)$ is a Carleson measure for $L_a^p(w)$.

Proposition 5. Suppose w is a weight satisfying the conditions (P1) and (P2). Then for every f analytic on $\mathbb{D}$ we have,

$$(4.8) \quad \|f\|_{p,w}^p \cong |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1-|z|)^p w(z) dm(z).$$

Proof. In view of (3.1), which holds if (P1) holds, it suffices to show the reverse inequality. This comes from [4, page 80] (proof of Theorem 5.5). Write Cauchy's formula for $f'(re^{i\theta})$ integrating on the circle $|z| = \rho$, $\rho = sr + 1 - s$, and then integrate $|f'(re^{i\theta})|^p$ and apply the continuous form of Minkowski's inequality to get

$$M_p(r, f') \leq \frac{M_p(\rho, f)}{\rho^2 - r^2} \leq C(s) \frac{M_p(\rho, f)}{1 - r},$$

which gives

$$\begin{aligned}
 \int_0^1 M_p^p(r, f')(1-r)^p w(r) r dr &\leq C(s) \int_0^1 M_p^p(sr+1-s, f) w(r) r dr \\
 &= \frac{C(s)}{s} \int_{1-s}^1 M_p^p(u, f) w\left(\frac{u+s-1}{s}\right) \frac{u+s-1}{s} du \\
 &\leq \frac{C(s)}{sd} \int_0^1 M_p^p(u, f) w(u) du \quad (\text{using (P2)}) \\
 &\leq C \|f\|_{p,w}^p.
 \end{aligned}$$

Also from (2.4) there is a constant C' such that $|f(0)| \leq C' \|f\|_{p,w}$. Combining the above, we have the assertion. $\square$

We end this section by discussing a non-example, drawn from the class of exponential weights. These are typical examples of weights that do not satisfy condition (P2).

Example 5. For $\alpha > 0$, $\beta > 0$, let

$$(4.9) \quad w_{\alpha,\beta}(r) = \exp\left(\frac{-\beta}{(1-r)^\alpha}\right).$$

With the change of variable $t = (1-u)^{-\alpha} - (1-r)^{-\alpha}$, we obtain

$$\begin{aligned}
 &\int_r^1 w_{\alpha,\beta}(u) du \\
 &= \int_r^1 \exp(-\beta(1-u)^{-\alpha}) du \\
 &= (1-r)^{\alpha+1} \exp(-\beta(1-r)^{-\alpha}) \frac{1}{\alpha} \int_0^\infty \left(\frac{(1-r)^{-\alpha}}{t + (1-r)^{-\alpha}}\right)^{(\alpha+1)/\alpha} \exp(-\beta t) dt \\
 &\leq C(1-r)^{\alpha+1} w_{\alpha,\beta}(r),
 \end{aligned}$$

so $w_{\alpha,\beta}$ satisfies the following condition, which is slightly stronger than (P1):

$$(P1)' \quad \int_r^1 w_{\alpha,\beta}(u) du \leq C(1-r)^{\alpha+1} w_{\alpha,\beta}(r), \quad 0 < r < 1.$$

It is easy to see that $w_{\alpha,\beta}$ does not satisfy (P2). Now a straight-forward modification of the argument in Lemma 2 shows that for any weight w that satisfies (P1)' we have

$$\|f\|_{p,w}^p \leq C \left(|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p(\alpha+1)} w(z) dm(z) \right).$$

The argument in the proof of Corollary 1 then shows that if w satisfies (P1)', then T_g is bounded on $L_a^p(w)$ whenever

$$(4.10) \quad (1-|z|)^{\alpha+1} |g'(z)| = o(1), \quad z \in \mathbb{D}, |z| \rightarrow 1$$

and T_g is compact on $L_a^p(w)$ whenever the corresponding “little oh” condition holds. We have not been able to answer the following question.

Question. Is boundedness (compactness) of T_g on $L_a^p(w_{\alpha,\beta})$ equivalent to $(1-|z|)^{\alpha+1} |g'(z)| = o(1)$ (resp. $= o(1)$) as $|z| \rightarrow 1$?

5. Schatten classes on $L_a^2(w)$. For the special case of the normalized classical weights

$$w(r) = (\alpha+1)(1-r^2)^\alpha, \quad \alpha \geq -1,$$

the inner product on the Hilbert space $L_a^2(w)$, denoted now by $L_a^2(\alpha)$, is given by

$$(5.1) \quad \langle f, h \rangle = (\alpha+1) \int_{\mathbb{D}} f(z) \overline{h(z)} (1-|z|^2)^\alpha dm(z),$$

and a computation shows

$$(5.2) \quad \langle f, h \rangle = f(0) \overline{h(0)} + \int_{\mathbb{D}} f'(z) \overline{h'(z)} v(|z|) dm(z),$$

where

$$v(r) = 2 \int_r^1 \frac{(1-u^2)^{\alpha+1}}{u} du.$$

The reproducing kernel of $L_a^2(\alpha)$ has the form [13, page 120]

$$k_w(z) = \frac{1}{(1-z\bar{w})^{\alpha+2}},$$

that is, $\langle f, k_w \rangle = f(w)$ for $f \in L_a^2(\alpha)$. Using Stirling's formula in (5.1), we see that an equivalent form of the norm on $L_a^2(\alpha)$ is

$$\|f\|_{2,\alpha}^2 \cong \sum_{n \geq 0} (n+1)^{-(\alpha+1)} |a_n|^2.$$

We denote by $\mathcal{S}_p$, $0 < p < \infty$, the Schatten classes of operators on $L_a^2(\alpha)$. $\mathcal{S}_p$ contains those compact operators whose sequence of characteristic numbers is p -summable. Finite rank operators belong to every $\mathcal{S}_p$, and membership of an operator in $\mathcal{S}_p$ measures in some sense the size of the operator. General information on the $\mathcal{S}_p$ classes of operators on a separable Hilbert space, including the fact that for $p \geq 1$, $\mathcal{S}_p$ is a Banach space, can be found in the references. Here we are going to use the fact that an operator $T \in \mathcal{S}_p$ if and only if $T^*T \in \mathcal{S}_{p/2}$ where T^* is the Hilbert space adjoint of T .

In this section we consider the operators T_g on $L_a^2(\alpha)$ and characterize the functions g for which $T_g \in \mathcal{S}_p$. It turns out that the method used in our work [1, Section 4], dealing with the same question on the Hardy space H^2 , can be used here with only minor changes. For this reason we will be brief, describing only those parts of the arguments that differ from the Hardy space case. As in [1], we are using the result of D. Luecking [8] concerning the Schatten classes of operators of the form

$$(5.3) \quad Q_\mu(f)(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \bar{z}w)^{\alpha+2}} d\mu(z), \quad f \in L_a^2(\alpha),$$

where μ is a positive Borel measure on $\mathbb{D}$. We can do this because the operators $T_g^*T_g$ are of the form (5.3). Indeed, using (5.2), we have

$$\langle T_g^*T_g(f), h \rangle = \langle T_g(f), T_g(h) \rangle = \int_{\mathbb{D}} f(z) \overline{h(z)} |g'(z)|^2 v(|z|) dm(z).$$

for $f, h \in L_a^2(\alpha)$. In particular if $h(z) = k_w(z)$, then

$$T_g^*T_g(f)(w) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \bar{z}w)^{\alpha+2}} |g'(z)|^2 v(|z|) dm(z).$$

Luecking's result [8, page 347], stated conveniently for our needs, says that for an operator Q_μ defined by (5.3) we have:

$$(5.4) \quad Q_\mu \in \mathcal{S}_p \Leftrightarrow \sum_j \left(\frac{\mu(R_j)}{1(R_j)^{\alpha+2}} \right)^p < \infty,$$

and the Schatten norm $|Q_\mu|_p$ is comparable to this sum. Here $\{R_j\}$ is the decomposition of $\mathbb{D}$ described in (2.1), and $1(R_j)$ is the side-length of R_j . More details on this decomposition of $\mathbb{D}$ can be found in [8] (see also [1]). For each $z \in R_j$ the estimates $1(R_j) \cong 1 - |z|^2$ and $m(R_j) \cong (1 - |z|^2)^2$ hold.

In our case the measure μ is $|g'(z)|^2 v(|z|) dm(z)$. For z near the boundary of $\mathbb{D}$ we have $v(|z|) \cong (1 - |z|^2)^{\alpha+2}$, and from the remarks above we easily see that

$$\begin{aligned} \mu(R_j) &= \int_{R_j} |g'(z)|^2 v(|z|) dm(z) \\ &\cong \int_{R_j} |g'(z)|^2 (1 - |z|^2)^{\alpha+2} dm(z) \\ &\cong (1 - |z|^2)^{\alpha+2} \int_{R_j} |g'(z)|^2 dm(z), \quad z \in R_j. \end{aligned}$$

Therefore,

$$(5.5) \quad \frac{\mu(R_j)}{1(R_j)^{\alpha+2}} \cong \int_{R_j} |g'(z)|^2 dm(z).$$

Theorem 2. For $1 < p < \infty$ we have $T_g \in \mathcal{S}_p$ if and only if $g \in B_p$. For $0 < p \leq 1$ the operator T_g does not belong to $\mathcal{S}_p$ unless g is constant.

Proof. As pointed out, $T_g \in \mathcal{S}_p$ if and only if $T_g^* T_g \in \mathcal{S}_{p/2}$ and by Luecking's Theorem and (5.5), this is equivalent to

$$(5.6) \quad \sum_{j \geq 0} \left(\int_{R_j} |g'(z)|^2 dm(z) \right)^{p/2} < \infty.$$

From Theorem 0 then we find $T_g \in \mathcal{S}_p$ if and only if $g \in B_p$, $1 < p < \infty$. For $p = 1$ we can repeat the argument for the corresponding case in [1, Theorem 2] to conclude that if the sum (5.6) with $p = 1$ is finite, then g is constant. The proof is complete. $\square$

REFERENCES

- [1] A. ALEMAN AND A. G. SISKAKIS, *An integral operator on H^p* , Complex Variables **28** (1995), 149–158.
- [2] J. A. CIMA AND G. SCHÖBER, *Analytic functions with bounded mean oscillation and logarithms of H^p functions*, Math. Z. **151** (1976), 295–300.

- [3] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators I*, Wiley, New York, 1988.
- [4] P. L. DUREN, *Theory of H^p spaces*, Academic Press, New York and London, 1970.
- [5] T. L. KRIETE AND B. D. MACCLUER, *Composition operators on large weighted Bergman spaces*, Indiana Univ. Math. J. **41** (1992), 755–788.
- [6] P. LIN AND R. ROCHBERG, *Hankel operators on the weighted Bergman spaces with exponential weights*, Integr. Equat. Oper. Th. **21** (1995), 460–483.
- [7] D. H. LUECKING, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. **107** (1985), 85–111.
- [8] D. H. LUECKING, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal. **73** (1987), 345–368.
- [9] V. L. OLEINIK, *Embedding theorems for weighted classes of harmonic and analytic functions*, J. Soviet Math. **9** (1978), 228–243.
- [10] CH. POMMERENKE, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, Comment. Math. Helv. **52** (1977), 122–129.
- [11] A. G. SISKAKIS, *Semigroups of composition operators on spaces of analytic functions*, (A Review), Studies in Composition Operators, Contemporary Mathematics, Volume 213, Amer. Math. Soc., Providence, Rhode Island, 1997.
- [12] K. ZHU, *Analytic Besov Spaces*, J. Math. Anal. Appl. **157** (1991), 318–336.
- [13] K. ZHU, *Operator Theory in Function Spaces*, Marcel Dekker Inc., New York and Basel, 1990.

ALEXANDRU ALEMAN
Fachbereich Mathematik
Fern Universität Hagen
Postfach 940
D-58084 Hagen, Germany

ARISTOMENIS G. SISKAKIS
Department of Mathematics
University of Thessaloniki
54006 Thessaloniki, Greece

Received: December 3rd, 1996; revised: April 7th, 1997.