Measure Filters: An Extension of Wiener's Theorem

NIKAN B. FIROOZYE & VLADIMÍR ŠVERÁK

ABSTRACT. We present an extension of Wiener's theorem, a nonlinear filter called the *measure filter* which, when applied to a measure, filters out all but the lowest dimensional part of that measure. We also extend the work of Strichartz in the case of codimension one, for functions of bounded variation, by allowing for a wide range of kernels which can be used to extract the jump part of the distributional derivative of a BV function. Our method of proof is different from that of Wiener and Strichartz.

1. Introduction. In this short paper we will present a simple example of a nonlinear filter which, when applied to a measure, filters out the disperse part of the measure leaving only the atomic part with the weights taken to some power, or by choosing a different scaling, filters out the Cantor and absolutely continuous parts of the derivative of a BV function, leaving only the lowest dimensional part behind, the jump part. These results have also been extended by one of the authors (see Firoozye [10]), to filter out the highest dimensional part of a measure, leaving the lowest dimensional part, provided that the lowest dimensional part is concentrated on a countably rectifiable set.

Wiener's original proof in 1933 ([25], Section 20), was merely an extension of his Tauberian theorems. By utilizing his general Tauberian theorem, Wiener was able to show that for a finite measure μ , decomposed into $\mu = \mu_a + \mu_c$, where $\mu_a = \sum_1^\infty a_i \delta_{x_i}$ is the atomic part of the measure and μ_c is the continuous part of the measure, then

$$\lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R |\hat{\mu}(\xi)|^2 d\xi = \sum_1^\infty a_i^2,$$

where $\hat{\mu}$ is the Fourier transform of μ . We note that the above formula can be rewritten in real space, by letting $\lambda = 1/R \rightarrow 0$, as

$$\lim_{\lambda \rightarrow 0} \int \left| \frac{1}{\lambda^{1/2}} \hat{\chi}_{B_1} \left(\frac{x}{\lambda} \right) * \mu \right|^2 dx = \sum_1^\infty a_i^2,$$

where $\hat{\chi}_{B_1}$ is the Fourier transform of the characteristic function of the unit ball. As is, this statement is stronger than the Riemann-Lebesgue theorem, and its applications have included the proof of energy decay for continuous one-parameter unitary groups whose infinitesimal generators have no point spectrum (see Lax-Phillips, [17]) and von Neumann's original proof of the ergodic theorem ([24]).

This result was later extended to n dimensions and commutative groups, (see, e.g., Hewitt-Stromberg [12] or Graham-McGehee [11]). Taylor ([23]), eventually extended it to the consideration of spectral expansions of elliptic operators on compact Riemannian manifolds rather than the Fourier transform, (which is, after all, the spectral expansion for the Laplacian on the torus). Agmon and Hörmander considered basic extensions of Wiener's theorem to the case of μ being surface measure on a C^1 manifold ([1]), for extensions of Rellich's uniqueness theorem to the case of homogeneous pseudo-differential operators, whose symbols had zero sets contained in these C^1 manifolds. Strichartz ([21]) was the first to consider applying these Tauberian theorems to the consideration of more general measures $\mu = \mu_\alpha + \nu$, where μ_α is locally uniformly α -dimensional, i.e., $\mu(B_r) \leq Cr^\alpha$ for $0 < r \leq 1$, and ν is the higher dimensional part. He later extended his results for measures on manifolds, and also developed some basic L^p analogs ([22]). Lau ([16]) and Hudson-Leckband ([13]) expanded on these ideas of Strichartz. The results thusfar quoted rely upon the use of Tauberian theorems for their method of proof.

Our results, which grew from an example in a paper by one of the authors ([8]), do not rely on Tauberian arguments, our method of proof being entirely different from that of the authors quoted above, and, it seems, simpler. The method works for a wide variety of kernels: in the atomic case, including heat kernels and Dirichlet kernels, as well as other, compactly supported kernels; and, in the general case, including a large variety of compactly supported kernels. Also, by considering compactly supported kernels, we allow for potential applications, such as that by one of the authors for the convergence of integration mask algorithms for edge-detection in computer image processing ([9]). We note that similar methods have been used by one of the authors (see Firoozye [10]) in the case where the measure has a density which exists a.e., and thus if the support of the lowest part of the measure is on a rectifiable set.

2. Atomic measure filters. As an immediate extension of Wiener's Theorem, we will discuss atomic measure filters, those which filter out everything except

the atomic part of the measure. Although our method of proof will be more general than the discussion immediately following, this discussion gives a general heuristic for the arguments contained in this paper.

Let μ be a finite measure, and let φ be a smooth function of compact support, $\varphi \geq 0$ and $\int |\varphi|^p = 1$. It is known (via the Radon-Nikodým theorem) that μ can be decomposed, albeit nonconstructively, into atomic and continuous parts, $\mu = \mu_a + \mu_c$, where $\mu_a = \sum_1^\infty a_i \delta_{x_i}$. Let $\varphi_\lambda(x) = \lambda^{-n} \varphi(x/\lambda)$. (This notation will be used throughout the rest of the paper.) For p a positive integer greater than one, we will show that

$$\lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^p \xrightarrow{*} \sum |a_i|^p \delta_{x_i},$$

as $\lambda \rightarrow 0_+$, where convergence is weak-* in the sense of measures. First, we show boundedness for $p \in [1, \infty)$:

$$\begin{aligned} & \int \lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^p dx \\ &= \|\lambda^{n-n/p} \varphi_\lambda * \mu\|_{L^p}^p \\ &\leq C \|\lambda^{n-n/p} \varphi_\lambda\|_{L^p}^p \|\mu\|_{\mathcal{M}}^p \\ &= C \|\varphi\|_{L^p}^p \|\mu\|_{\mathcal{M}}^p, \end{aligned}$$

by Young's inequality. Thus, the measure filter $\lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^p$ is bounded in measure and, after extracting a subsequence, it converges weak-* in the sense of measures.

We characterize the limiting measure only for $p = 2$ in our illustration, but the generalization to integer p is obvious. Letting $g \in C_0$ be a test function, and applying Fubini's Theorem,

$$\begin{aligned} & \int g \lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^2 dx \\ &= \iint \left(\int \frac{1}{\lambda^n} g(x) \varphi\left(\frac{x-y}{\lambda}\right) \varphi\left(\frac{x-z}{\lambda}\right) dx \right) d\mu(y) d\mu(z), \end{aligned}$$

we see that this can be rewritten as

$$\int \int \Phi_\lambda(y, z) d\mu(y) d\mu(z),$$

where we define $\Phi_\lambda(y, z)$ by

$$\Phi_\lambda(y, z) = \int g(y + \lambda\zeta) \varphi(\zeta) \varphi\left(\zeta + \frac{y-z}{\lambda}\right) d\zeta.$$

We note that since φ and g both have compact support, $\Phi_\lambda(\cdot, \cdot)$ also has compact support on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover we easily see that $|\Phi_\lambda(y, z)| \leq \|g\|_C$ and that

$\Phi_\lambda(y, z) \rightarrow \chi_\Delta(y, z)g(y)$, pointwise, as $\lambda \rightarrow 0$. where $\Delta = \{(y, z) \in \mathbb{R}^n \times \mathbb{R}^n : y = z\}$ is the diagonal in the product space. Since μ is a finite measure, we can use the dominated convergence theorem. A simple exercise, using, for example, covering arguments, gives us that $(\mu \times \mu)(\Delta) = 0$ unless μ has point masses. Using the dominated convergence theorem, we see that the limit must depend only upon the atomic part of the measure. In fact, we have that

$$\begin{aligned} \int \lambda^{-n} g(x) |\lambda^n \varphi_\lambda * \mu|^2 dx &\rightarrow \int \int \chi_{\{y=z\}}(y, z) \|\varphi\|_{L^2}^2 g(y) d\mu(y) d\mu(z) \\ &= \sum g(x_i) a_i^2, \end{aligned}$$

i.e., that $\lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^2 \xrightarrow{*} \sum a_i^2 \delta_{x_i}$. The heuristic of using product spaces is very powerful and is the primary motivation for the following proofs, although this particular method of proof will not be used below (where $p > 1$ is allowed to be a real number rather than just an integer).

As our first true generalization of Wiener's theorem we have the following:

Theorem 2.1 *Let μ be a finite measure on $\mathbb{R}^n$. Let $\mu = \mu_a + \mu_c$, with $\mu_a = \sum a_i \delta_{x_i}$, be the decomposition of μ into atomic and continuous parts. Let $p > 1$, $\varphi \in L^p(\mathbb{R}^n)$ with $\int |\varphi|^p = 1$, and let $\varphi_\lambda(x) = \lambda^{-n} \varphi(x/\lambda)$. Then*

$$\lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^p \xrightarrow{*} \sum a_i^p \delta_{x_i},$$

weak- in the sense of measures as $\lambda \rightarrow 0_+$.*

Note that $\hat{\chi}_{B_1}$ and $e^{-|x|}$ and $e^{-|x|^2}$ all satisfy the hypotheses of our theorem for various choices of p and various normalizations, so we reproduce both Wiener's and Strichartz' results and extend them. Wiener ([25], Section 21), essentially noted the same conclusion as in Theorem 2.1 for purely atomic measures with a finite number of point masses, (or its dual for almost periodic functions), for $p = 2$, again using the approximate identity given by $\hat{\chi}_{B_1}$, in his introduction of the function $\varphi(x) = \lim_{T \rightarrow \infty} (1/2T) \int_{-T}^T f(x + \xi) \bar{f}(\xi) d\xi$, one of the basic objects of *Generalized Harmonic Analysis*. He noted that if $f(x) = \sum_1^N a_i e^{ix_i \cdot x}$ then $\varphi(x) = \sum_1^N |a_i|^2 e^{ix_i \cdot x}$. The direction of his study then diverged from ours when he proceeded to study φ in some detail for choices of f in various function spaces, later leading to the Wiener-Wintner theorem ([26], see also Benedetto [3]), an analog of Bochner's theorem characterizing Fourier transforms of positive measures, (i.e., every positive measure on $\mathbb{R}$ is the Fourier transform of a function of the form of φ).

The method of our proof below is not quite as intuitive as the illustration above, although it relies on very basic identities.

Proof. Note that in our discussion above we showed boundedness, thus, that there is a weak-* convergent subsequence, converging to a limiting measure.

First, for any $\varepsilon > 0$ we choose $\rho > 0$ large enough so that

$$\int_{|x|>\rho} |\varphi(x)|^p < \varepsilon.$$

Define $\varphi^1 = \varphi \chi_{B_\rho}$ and $\varphi^2 = \varphi(1 - \chi_{B_\rho})$. We divide μ into two parts, $\mu = \mu_1 + \mu_2$. Find λ_1 such that $|\mu_c|(B_{\rho\lambda_1}(x)) < \varepsilon^{1/(p-1)}/2$, for all x , and then find N large such that $\sum_N^\infty |a_i| < \varepsilon^{1/(p-1)}/2$. Pick $\lambda_2 > 0$ small such that $\text{dist}(x_i, x_j) > \rho\lambda_2$ for $i \neq j$ and $i, j = 1, \dots, N$, then let $\mu_1 = \sum_1^N a_i \delta_{x_i}$ and $\mu_2 = \mu_c + \sum_N^\infty a_i \delta_{x_i}$. Letting $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, we see that $|\mu_2|(B_{\lambda\rho}(x)) < \varepsilon^{1/(p-1)}$ for all $0 < \lambda \leq \lambda_0$ and all x , and that μ_1 has support on a finite number of points no closer than $\lambda_0\rho$.

Let $g \in C_0$ and, to begin with, suppose that $g \geq 0$. Then, simply by using the triangle inequality, we have that

$$\begin{aligned} & \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * (\mu_1 + \mu_2)|^p dx \right)^{1/p} \\ & \leq \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * \mu_1|^p dx \right)^{1/p} + \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * \mu_2|^p dx \right)^{1/p}. \end{aligned}$$

Similarly, the reverse triangle inequality gives us that

$$\begin{aligned} & \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * (\mu_1 + \mu_2)|^p dx \right)^{1/p} \\ & \geq \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * \mu_1|^p dx \right)^{1/p} - \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * \mu_2|^p dx \right)^{1/p}. \end{aligned}$$

Bounding the term involving μ_2 by using the triangle inequality, we see that,

$$\begin{aligned} & \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * \mu_2|^p dx \right)^{1/p} \\ & \leq \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda^1 * \mu_2|^p dx \right)^{1/p} + \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda^2 * \mu_2|^p dx \right)^{1/p} \\ & \leq \left(\int g \frac{1}{\lambda^n} |\lambda^n \int \varphi_\lambda^1(x-y) d\mu_2(y)|^p dx \right)^{1/p} \\ & \quad + \|g\|_C^{1/p} \|\varphi^2\|_{L^p} \|\mu\|_{\mathcal{M}} \\ & \leq \|g\|_C^{1/p} \left(\left(\iint \lambda^{-n} |\lambda^n \varphi_\lambda^1(x-y)|^p d|\mu|(y) dx \cdot |\mu_2|(B_{\rho\lambda})^{p-1} \right)^{1/p} \right. \\ & \quad \left. + \varepsilon^{1/p} \|\mu\|_{\mathcal{M}} \right) \\ & \leq \|g\|_C^{1/p} \left(\varepsilon^{1/p} \|\mu\|_{\mathcal{M}}^{1/p} \|\varphi^1\|_{L^p} + \varepsilon^{1/p} \|\mu\|_{\mathcal{M}} \right), \end{aligned}$$

where we have used Young's inequality in going from the first to the second line and Hölder's inequality in going from the second to the third line and Young's again in going from the third to the fourth.

The term involving μ_1 can be bounded above and below in terms of the triangle and reverse triangle inequality, respectively (where we write only the upper bound):

$$\begin{aligned} \left(\int g \frac{1}{\lambda^n} |\lambda^n \varphi_\lambda * \mu_1|^p dx \right)^{1/p} &\leq \left(\int g \frac{1}{\lambda^n} \left| \lambda^n \varphi_\lambda^1 * \left(\sum_1^N a_i \delta_{x_i} \right) \right|^p dx \right)^{1/p} \\ &\quad + \left(\int g \frac{1}{\lambda^n} \left| \lambda^n \varphi_\lambda^2 * \left(\sum_1^N a_i \delta_{x_i} \right) \right|^p dx \right)^{1/p}. \end{aligned}$$

We obtain a similar lower bound. We note that by the compact support of φ^1 and the separation of the support of μ_1 , the first term can be rewritten for $\lambda \leq \lambda_0$,

$$\int g \frac{1}{\lambda^n} \left| \lambda^n \varphi_\lambda^1 * \sum_1^N a_i \delta_{x_i} \right|^p dx = \int g \frac{1}{\lambda^n} \sum_1^N |\lambda^n \varphi_\lambda^1 * (a_i \delta_{x_i})|^p dx,$$

which is easily seen to converge to

$$\int |\varphi^1|^p dx \sum_1^N g(x_i) a_i^p \geq (1 - \varepsilon) \sum_1^N g(x_i) a_i^p.$$

We note that the second term can be bounded above by Young's inequality,

$$\int g \frac{1}{\lambda^n} \left| \lambda^n \varphi_\lambda^2 * \left(\sum_1^N a_i \delta_{x_i} \right) \right|^p dx \leq \|g\|_{C_0} \|\varphi^2\|_{L^p}^p \|\mu\|_{\mathcal{M}}^p \leq \varepsilon \|g\|_{C_0} \|\mu\|_{\mathcal{M}}^p.$$

Combining these results, we see that for any $\varepsilon > 0$, there exists N such that

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \left(\int \lambda^{-n} |\lambda^n \varphi_\lambda * \mu|^p g dx \right)^{1/p} &\leq \left(\sum_1^N a_i^p g(x_i) \right)^{1/p} + C\varepsilon^{1/p} \\ &\geq \left((1 - \varepsilon) \sum_1^N a_i^p g(x_i) \right)^{1/p} - C\varepsilon^{1/p} \end{aligned}$$

and thus we have shown the result. $\square$

3. BV Measure Filters. In our extension of Wiener's results to measures which have a lower bound on the dimension of their support, we will consider only kernels of compact support. Strichartz considered only the heat kernel, but had to make extra assumptions on the measure, such as that the support of the lowest dimensional part should be a C^2 embedded manifold. In considering only compactly supported kernels, we are given the flexibility of dealing with measures whose supports are merely countably rectifiable. In particular we will consider (possibly vector-valued) $BV \cap L^\infty$ functions. This the most relevant form of extensions of Wiener's theorem for applications to image processing ([9]). We note that the distributional derivative of a BV function u , is a measure Du which can be divided using Radon-Nikodým-Lebesgue decomposition into three mutually singular parts, (see, e.g., Ambrosio [2]),

$$Du = [u] \otimes \nu \mathcal{H}^{n-1} \llcorner S_u + C_u + \nabla u,$$

where $[u]$ is the jump in u , ν is the generalized unit normal to the jump set, which is defined $\mathcal{H}^{n-1}$ a.e., and S_u is the jump set, C_u is the Cantor part of the measure, and ∇u is the absolutely continuous part of the measure. By choosing the appropriate scaling, we will have a constructive means of finding the jump part of the measure.

Recall the BV "blowup" results (see, for instance Ziemer [27] Section 5.14.1 or Federer [6] Section 4.5.9):

$$\frac{1}{\lambda^n} \int_{B_\lambda(y)} |u(x) - u(y)|^{n/(n-1)} dx \rightarrow 0,$$

for $\mathcal{H}^{n-1}$ a.e. $y \notin S_u$, and, for $\mathcal{H}^{n-1}$ a.e. $y \in S_u$,

$$\frac{1}{\lambda^n} \int_{B_\lambda^\pm(y)} |u(x) - u^\pm(y)|^{n/(n-1)} dx \rightarrow 0,$$

where $u^\pm(y)$ are the approximate limsup and approximate liminf at y respectively, and $B_\lambda^\pm(y) = B_\lambda(y) \cap N^\pm(y)$ where the half-spaces, $N^+(y) = \{x : (x - y) \cdot \nu > 0\}$ and where $N^-(y) = \{x : (x - y) \cdot \nu < 0\}$, with ν the approximate unit normal to S_u at y .

We shall prove the following theorem:

Theorem 3.1 *Let $\varphi \in C_0^1(B_1)$ and let u be a (possibly vector-valued) function belonging to $BV \cap L^\infty(\Omega)$ and let $p > 1$. Then, the BV measure filter,*

$$\lambda^{-1} |\lambda \varphi_\lambda * Du|^p \xrightarrow{*} G(\nu) |[u]|^p d\mathcal{H}^{n-1} \llcorner S_u,$$

converging weak- in the sense of measures as $\lambda \rightarrow 0_+$, where $[u]$ is the jump in u , and S_u is the jump set of u , ν is the generalized unit normal, and $G \in C(S^{n-1})$ is an odd function ($G(\nu) = -G(-\nu)$). If, in addition, φ is radial,*

(i.e., $\varphi(x) = \varphi(|x|)$), then $G \equiv \text{const}$. Conversely, if the measure filter of a BV function u converges weak-* in the sense of measures to some limiting measure, and φ is a positive radial function, then $[u] \in L^p(d\mathcal{H}^{n-1}|_{S_u})$.

We utilize the following convention: if f and g are two (integrable) vector valued functions defined on $\mathbb{R}^n$, we denote by $f * g(x)$ the matrix-valued function on $\mathbb{R}^n$ which is defined by

$$f * g(x) = \int f(x-y) \otimes g(y) dy.$$

Let us prove boundedness of the measure filter:

$$\begin{aligned} \int \lambda^{-1} |\lambda \varphi_\lambda * Du|^p dx &= \int |\lambda \varphi_\lambda * Du|^{p-1} |\varphi_\lambda * Du| dx \\ &= \int |u * (D\varphi)_\lambda|^{p-1} |\varphi_\lambda * Du| dx \\ &\leq \|u * (D\varphi)_\lambda\|_{L^\infty}^{p-1} \int |\varphi_\lambda * Du| dx \\ &\leq \|\varphi\|_{L^1} \|d\varphi\|_{L^1}^{p-1} \|u\|_{L^\infty}^{p-1} \|Du\|_{\mathcal{M}}, \end{aligned}$$

where $(D\varphi)_\lambda(x) = \lambda^{-n}(D\varphi)(x/\lambda)$. Thus, it makes sense to speak of the weak-* limit in the sense of measures, after passing to a subsequence.

We will again provide an illustration for $p = 2$. In what follows, we will use the following convention: if Ω is an open set, $\Omega \subset \mathbb{R}^n$, $u \in BV(\Omega; \mathbb{R}^m)$, and $F = (F_{ij})$ is a function on Ω with values in $m \times n$ matrices, we use the coordinate-free notation $\int_\Omega F(x) d(Du(x))$ to mean $\int_\Omega F_{ij}(x) d\mu_{ij}(x)$, where μ_{ij} is the measure given by $\mu_{ij}(x) = D_j u_i(x)$. We also use the convention of summing over repeated indices. For $p = 2$, we use Fubini's theorem, and integrate by parts once to get,

$$\begin{aligned} &\int \frac{1}{\lambda} |\lambda \varphi_\lambda * Du|^2 g dx \\ &= \iint u(y) \otimes \left(\int \frac{1}{\lambda^n} (D\varphi) \left(\frac{x-y}{\lambda} \right) \frac{1}{\lambda^n} \varphi \left(\frac{x-z}{\lambda} \right) g(x) dx \right) dy d(Du)(z). \end{aligned}$$

Then, changing variables in the inner integral, we see that the right-hand side becomes

$$(1) \quad \iint u(y) \otimes \left(\int \frac{1}{\lambda^n} (D\varphi) \left(\zeta + \frac{z-y}{\lambda} \right) \varphi(\zeta) g(z + \lambda\zeta) d\zeta \right) d(Du)(z).$$

Define $u^{(z,\lambda)}(y) = u(z + \lambda y)$, and

$$(2) \quad u^{(z,0)}(y) = \begin{cases} u(z) & \text{for } z \notin S_u \\ u^+(z)\chi_+^z(y) + u^-(z)\chi_-^z(y) & \text{for } z \in S_u, \end{cases}$$

(the notation will become more apparent later in the proof), where $\chi_+^z = \chi_{N^+(z)}$ and $\chi_-^z = \chi_{N^-(z)}$ are characteristic functions of the half-planes defined in the blowup results. With this definition, we have a convenient way of encoding the three separate blowup results at once, i.e.,

$$\begin{aligned} & \frac{1}{\lambda^n} \int_{B_\lambda(z)} |u(y) - u^{(z,0)}(y)|^{n/(n-1)} dy \\ &= \int_{B_1} |u^{(z,\lambda)}(y) - u^{(z,0)}(y)|^{n/(n-1)} dy \rightarrow 0, \end{aligned}$$

for $\mathcal{H}^{n-1}$ a.e. y . We rewrite (1) as

$$(3) \quad \iint \frac{1}{\lambda^n} u(y) \otimes \Psi_\lambda\left(\frac{z-y}{\lambda}, z\right) dy \, dDu(z),$$

where

$$\Psi_\lambda(r, z) = \int D\varphi(\zeta + r) \varphi(\zeta) g(z + \lambda\zeta) d\zeta.$$

Changing variables in (3), we see that it is equal to

$$\begin{aligned} (4) \quad & \iint u(z - \lambda r) \otimes \Psi_\lambda(r, z) dr \, dDu(z) \\ &= \iint u^{(z,\lambda)}(-r) \otimes \Psi_\lambda(r, z) dr \, dDu(z). \end{aligned}$$

We note that, by the continuity of g , $\Psi_\lambda(r, z) \rightarrow g(z) \int D\varphi(\zeta + r) \varphi(\zeta) d\zeta$, pointwise. Also, by the blowup results, $u^{(z,\lambda)} \rightarrow u^{(z,0)}$ strongly in $L_{\text{loc}}^{n/(n-1)}$. Incorporating these into (4), and using the dominated convergence theorem (since $u \in L^\infty$), we see that

$$\begin{aligned} & \int \frac{1}{\lambda} |\lambda \varphi_\lambda * Du|^2 g dx \\ & \rightarrow \int g(z) \left(\int u^{(z,0)}(-r) \otimes D\varphi(\zeta + r) \varphi(\zeta) d\zeta \right) dr \, dDu(z). \end{aligned}$$

A more careful analysis of the definition of $u^{(z,0)}$ completes the illustration.

We now turn to the proof of the general case:

Proof. First, we note that since $u \in L^\infty$ and there is a representation for u defined $\mathcal{H}^{n-1}$ a.e., we may as well assume that u is bounded $\mathcal{H}^{n-1}$ a.e.

We make several preliminary definitions. Let $A(f) = |f|^{p-2}f$ for vector or matrix valued functions f . Define $\psi = D\varphi$, $v_\lambda = u * \psi_\lambda$ and $w_\lambda = |v_\lambda|^{p-2}v_\lambda = A(v_\lambda)$.

Let $g \in C_0$ be a test function. Then, noting that $|a| = \langle a, a \rangle / |a|$, integrating by parts, and using Fubini's theorem,

$$\begin{aligned}
 (5) \quad & \int g(x) \frac{1}{\lambda^n} |(\lambda \varphi_\lambda * Du)(x)|^p dx \\
 &= \int g(x) \langle w_\lambda(x), \varphi_\lambda * Du(x) \rangle dx \\
 &= \int \left\langle \int \varphi_\lambda(x-y) g(x) w_\lambda(x) dx, \theta(y) \right\rangle d|Du(y)| \\
 &= \int \langle \tilde{\varphi}_\lambda * (g w_\lambda)(y), \theta(y) \rangle d|Du(y)|,
 \end{aligned}$$

where $\theta(y)$ is the Radon-Nikodým derivative, $\theta = d(Du)/d|Du|$, and $\tilde{\varphi}(t) = \varphi(-t)$.

We see that in order to find the limit of (5), we need only characterize the limit of $\tilde{\varphi}_\lambda * (g w_\lambda)$ a.e. with respect to $|Du|$, and then we will use the dominated convergence theorem.

By changing variables several times, we see that

$$\begin{aligned}
 (6) \quad & \tilde{\varphi}_\lambda * (g w_\lambda)(y) \\
 &= \int \frac{1}{\lambda^n} \tilde{\varphi}\left(\frac{y-x}{\lambda}\right) g(x) w_\lambda(x) dx \\
 &= \int \varphi(\zeta) g(y + \lambda \zeta) w_\lambda(y + \lambda \zeta) d\zeta \\
 &= \int \varphi(\zeta) A\left(\int u(y + \lambda(\zeta - \eta)) \otimes \psi(\eta) d\eta\right) g(y + \lambda \zeta) d\zeta \\
 &= \int \varphi(\zeta) A\left(\int u^{(y, \lambda)}(\zeta - \eta) \otimes \psi(\eta) d\eta\right) g(y + \lambda \zeta) d\zeta
 \end{aligned}$$

where $u^{(y, \lambda)}(\zeta) = u(y + \lambda \zeta)$. As we noted above, by the BV blowup results, $u^{(y, \lambda)}(\cdot) \rightarrow u^{(y, 0)}(\cdot)$ strongly in $L^{n/(n-1)}$ for $\mathcal{H}^{n-1}$ a.e. y , where $u^{(y, 0)}$ is defined as in equation (2). Thus, also using the continuity of g , we can incorporate this into (6) to get that

$$(7) \quad \tilde{\varphi}_\lambda * (g w_\lambda)(y) \rightarrow \int \varphi(\zeta) A\left(\int u^{(y, 0)}(\zeta + \eta) \otimes \psi(\eta) d\eta\right) d\zeta g(y),$$

for $\mathcal{H}^{n-1}$ a.e. point y (and thus for $|Du|$ a.e. y).

We note easily that

$$\int u^{(y, 0)}(\zeta + \eta) \otimes \psi(\eta) d\eta = 0,$$

for $y \notin S_u$, since ψ has zero average. If we restrict y to S_u , then we must study

$$\begin{aligned} \int \psi(\eta) \chi_+^y(\zeta + \eta) d\eta &= \int_{\eta \cdot \nu(y) > 0} D\varphi(\zeta + \eta) d\eta \\ &= \int_{\eta \in T_y} \varphi(\zeta + \eta) d\mathcal{H}^{n-1}(\eta) \nu(y), \end{aligned}$$

where $T_y = \{\eta : \eta \cdot \nu(y) = 0\}$ is the approximate tangent to $y \in S_u$.

Noting that $\int \psi(\eta)(\chi^+(\zeta + \eta) + \chi^-(\zeta + \eta)) d\eta = 0$, we see that

$$(8) \quad \int u^{(y,0)}(\zeta + \eta) \otimes \psi(\eta) d\eta = \chi_{S_u}(y) \int_{T_y} \varphi(\zeta + \eta) d\mathcal{H}^{n-1}(\eta) [u] \otimes \nu(y).$$

Incorporating (*) into (7), we see that

$$\tilde{\varphi}_\lambda * (gw_\lambda)(y) \rightarrow \int \varphi(\zeta) \left| \int_{T_y} \varphi(\zeta + \eta) d\mathcal{H}^{n-1}(\eta) \right|^{p-1} d\zeta [|u|]^{p-2} [u] \otimes \nu(y) g(y),$$

for $|du|$ a.e. y . Using the fact that $u \in L^\infty$ to dominate, we see that (5) can be seen to converge,

$$\begin{aligned} &\int \langle \tilde{\varphi}_\lambda * (gw_\lambda)(y), \theta(y) \rangle d|Du(y)| \\ &\rightarrow \int \left\langle \int \varphi(\zeta) \left| \int_{T_y} \varphi(\zeta + \eta) d\mathcal{H}^{n-1}(\eta) \right|^{p-1} d\zeta \right. \\ &\quad \left. \times g(y) [|u|]^{p-2} [u] \otimes \nu(y), \theta(y) \right\rangle d|Du(y)| \\ &= \int [|u|]^p G(\nu) g(y) d\mathcal{H}^{n-1}|S_u|, \end{aligned}$$

where we have used the fact that $Du|_{S_u} = [u] \otimes \nu \mathcal{H}^{n-1}|_{S_u}$, where

$$G(\nu) = \int \varphi(\zeta) \left| \int_{T_y} \varphi(\zeta + \eta) d\mathcal{H}^{n-1}(\eta) \right|^{p-1} \nu d\zeta.$$

Finally, we claim that $G(\nu) = \text{const}$ in the case where $\varphi(\xi) = \varphi(|\xi|)$. In this case, $D\varphi(\xi) = (\xi/|\xi|)\varphi'(|\xi|)$. Just by changing variables in both integrals, we can see that $G(R\nu) = G(\nu)$ for $R \in SO(n)$. Thus, we have completed the proof. $\square$

Remark 3.2 We note that the above proof still works if we assume that

$$|M_{n-1}(|Du|)|^{p-1} \in L^1(|Du|),$$

where M_{n-1} is the $n - 1$ dimensional maximal function, i.e.,

$$M_{n-1}[f](x) = \sup_{r>0} r^{-(n-1)} \int_{B_r(x)} |f| dy.$$

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NIKAN B. FIROOZY
Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801

VLADIMÍR ŠVERÁK
School of Mathematics
University of Minnesota
Vincent Hall
206 Church St. SE
Minneapolis, Minnesota 55455

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