Estimates in L^p for Magnetic Schrödinger Operators

ZHONGWEI SHEN

ABSTRACT. We study the magnetic Schrödinger operator $H(\mathbf{a}, V)$ in $\mathbb{R}^n$, $n \geq 3$. The L^p ($1 < p < \infty$) and weak-type (1,1) estimates are obtained under certain conditions, given in terms of the reverse Hölder inequality, on the magnetic field $\mathbf{B} = \text{curl } \mathbf{a}$ and the electrical potential V . In particular, we show that the L^p and weak-type (1,1) estimates hold if the components of $\mathbf{a}$ are polynomials, and V is a nonnegative polynomial.

Introduction. This paper concerns the Schrödinger operator with magnetic field

$$(0.1) \quad H(\mathbf{a}, V) = \sum_{j=1}^n \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j(x) \right)^2 + V(x) \quad \text{in } \mathbb{R}^n, n \geq 3,$$

where $i = \sqrt{-1}$, $\mathbf{a} = (a_1(x), a_2(x), \dots, a_n(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the magnetic potential and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is the electrical potential. Let

$$(0.2) \quad L_j = \frac{1}{i} \frac{\partial}{\partial x_j} - a_j(x) \quad \text{for } 1 \leq j \leq n.$$

We shall be interested in the L^p estimates:

$$(0.3) \quad \sum_{1 \leq j, k \leq n} \|L_j L_k(f)\|_p \leq C_p \|H(\mathbf{a}, V)f\|_p$$

for $1 < p < \infty$ and $f \in C_0^\infty(\mathbb{R}^n)$. Note that, in the classical case $\mathbf{a}(x) \equiv \mathbf{0}$ and $V(x) \equiv 0$, estimate (0.3), which plays a fundamental role in the elliptic theory, is a consequence of the well-known Calderón-Zygmund inequality.

Concerning (0.3), much has been done in the case when the magnetic potential $\mathbf{a}$ is not present, i.e., $H(\mathbf{a}, V) = -\Delta + V(x)$. Estimates like (0.3) were

studied first by Helffer and Nourrigat [HN1][HN2]. They proved that, if V is a nonnegative polynomial and $p = 2$, then (0.3) holds with some constant C depending only on n and the *degree* of V . This result was recently generalized to the case $1 < p < \infty$ by Guibourg [Gu1][Gu2], and independently by Zhong [Z]. In particular, Zhong showed that $\nabla^2(-\Delta + V)^{-1}$ is in fact a Calderón–Zygmund operator if V is a nonnegative polynomial. In [Sh2], the author considered the Schrödinger operator $-\Delta + V$ with V in the reverse Hölder class. We show, among other things, that $\nabla^2(-\Delta + V)^{-1}$ is bounded on $L^p(\mathbb{R}^n)$, if $1 < p \leq q$ and $V \in (RH)_q$ (See Definition 0.4) for some $q \geq n/2$. We remark that the nonnegative polynomials belong to $(RH)_q$ for any $q > 1$.

The magnetic Schrödinger operator has received a considerable amount of attention in recent years. See e.g. [AHS], [LS], [Si], [H], [I]. Estimates like (0.3) are of great interest in the study of spectral theory of $H(\mathbf{a}, V)$. Nevertheless, the only known result is an L^2 -estimate given by Guibourg. It is shown in [Gu1] that, if $a_j(x)$ and $V(x)$ are polynomials and $V \geq 0$, then (0.3) holds for $p = 2$.

In this paper we shall establish the L^p and weak-type (1,1) estimates for a broad class of potentials with minimal smooth conditions. This class, which contains all polynomials, is given in terms of the reverse Hölder inequality.

Definition 0.4. Suppose $W \in L^p_{\text{loc}}(\mathbb{R}^n)$ ($1 < p \leq \infty$) and $W \geq 0$ a.e. on $\mathbb{R}^n$. We say $W \in (RH)_{p,\text{loc}}$ if there exists $C_0 \geq 1$ such that

$$(0.5) \quad \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} W^p(y) dy \right)^{1/p} \leq C_0 \cdot \frac{1}{|Q(x,r)|} \int_{Q(x,r)} W(y) dy$$

for every $x \in \mathbb{R}^n$ and $0 < r \leq 1$, where $Q(x,r)$ denotes the ball centered at x with radius r . If (0.5) holds for $0 < r < \infty$, we say $W \in (RH)_p$.

The reverse Hölder class $(RH)_p$, which was introduced independently by Gehring [Ge] and Muchenhaupt [Mu], has been studied extensively in harmonic analysis. See [St] for references.

Definition 0.6. For $x \in \mathbb{R}^n$, the function $m(x, W)$ is defined by

$$\frac{1}{m(x, W)} = \sup \left\{ r > 0 : \frac{r^2}{|Q(x,r)|} \int_{Q(x,r)} W(y) dy \leq 1 \right\}.$$

The function $m(x, W)$, which is closely related to the uncertainty principle, plays a very important role in the paper. The definition of $m(x, W)$ was given first in [Sh1] for potentials in $(RH)_\infty$. It generalizes the early version of a very useful auxiliary function for polynomial potentials (see (0.14)).

Let

$$(0.7) \quad \mathbf{B}(x) = \operatorname{curl} \mathbf{a}(x) = (b_{jk}(x))_{1 \leq j, k \leq n}$$

be the magnetic field generated by $\mathbf{a}(x)$ where

$$(0.8) \quad b_{jk}(x) = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}.$$

Note that, because of the gauge invariance of the operator $H(\mathbf{a}, V)$ and the nature of the L^p -estimates, any quantitative condition should be imposed on the magnetic field $\mathbf{B}(x)$, not directly on $\mathbf{a}(x)$.

We are now in a position to state the main results of the paper.

Theorem 0.9. *Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$ and $V \geq 0$ a.e. on $\mathbb{R}^n$, $n \geq 3$. Also assume that:*

$$(0.10) \quad \begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2, \text{loc}}, \\ V(x) \leq C_1 \{m(x, |\mathbf{B}| + V + 1)\}^2, \\ |\nabla \mathbf{B}(x)| \leq C_2 \{m(x, |\mathbf{B}| + V + 1)\}^3 \end{cases}$$

where $|\mathbf{B}| = |\mathbf{B}(x)| = \sum_{j,k} |b_{jk}(x)|$. Then, for $1 < p < \infty$, there exists a constant $C > 0$, which depends only on n , p , and the constants C_0 , C_1 , C_2 in (0.5) and (0.10), such that

$$(0.11) \quad \sum_{1 \leq j, k \leq n} \|L_j L_k(f)\|_p \leq C \{\|H(\mathbf{a}, V)f\|_p + \|f\|_p\}$$

for any $f \in C_0^\infty(\mathbb{R}^n)$.

Theorem 0.12. *Under the same assumption as in Theorem 0.9, we have the weak-type (1,1) estimate*

$$\left| \left\{ x \in \mathbb{R}^n : \sum_{1 \leq j, k \leq n} |L_j L_k(f)(x)| > \lambda \right\} \right| \leq \frac{C}{\lambda} \{\|H(\mathbf{a}, V)f\|_1 + \|f\|_1\}$$

for any $\lambda > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$.

A rescaling argument shows that, if

$$(0.13) \quad \begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ V(x) \leq C_1 \{m(x, |\mathbf{B}| + V)\}^2, \\ |\nabla \mathbf{B}(x)| \leq C_2 \{m(x, |\mathbf{B}| + V)\}^3, \end{cases}$$

we have

$$\sum_{1 \leq j, k \leq n} \|L_j L_k(f)\|_p \leq C \|H(\mathbf{a}, V)f\|_p$$

for $1 < p < \infty$, and

$$\left| \left\{ x \in \mathbb{R}^n : \sum_{1 \leq j, k \leq n} |L_j L_k(f)(x)| > \lambda \right\} \right| \leq \frac{C}{\lambda} \|H(\mathbf{a}, V)f\|_1.$$

See Theorem 4.7. In particular, the condition (0.13), hence the L^p and weak-type (1,1) estimates, hold if $a_j(x)$ and $V(x) \geq 0$ are polynomials. This follows easily from the fact that, if $W = |P(x)|^\alpha$ where $\alpha > 0$ and $P(x)$ is a polynomial of degree k , then $W \in (RH)_\infty$ and

$$(0.14) \quad m(x, W) \approx \sum_{|\beta| \leq k} |\partial_x^\beta P(x)|^{\alpha/(\alpha|\beta|+2)}.$$

See [Sh3].

It was pointed out in [Sh4] that, if $\mathbf{a} \in C^{k+2}(\mathbb{R}^n)$, $V \in C^{k+2}(\mathbb{R}^n)$, $V \geq 0$ and

$$(0.15) \quad \sum_{|\beta|=k+1} |\partial_x^\beta \mathbf{B}(x)| + \sum_{|\beta|=k+2} |\partial_x^\beta V(x)| \leq C \{1 + M(x)\}$$

where

$$M(x) = \sum_{|\beta| \leq k} |\partial_x^\beta \mathbf{B}(x)| + \sum_{|\beta| \leq k+1} |\partial_x^\beta V(x)|,$$

then $(\mathbf{B}, V)$ satisfies the condition (0.10) in Theorem 0.9. Thus our results contain that in [Gu1], where the L^2 -estimate is proved under a condition slightly stronger than (0.15). As we mentioned before, the L^p -estimates for $p \neq 2$ and the weak (1,1) estimate are new even for polynomial potentials.

We briefly sketch the argument we will use in this paper. Our method is different from the one used in [Gu1], which relies heavily on the technique of higher order commutators. First, in the local scale where $|x - y| \leq \{m(x, |\mathbf{B}| + V + 1)\}^{-1}$, we construct the potential function $\mathbf{h}(x)$ from the magnetic field

$\mathbf{B}(x)$, so that, $\mathbf{h} = \mathbf{a} - \nabla\Phi$ and $\mathbf{h}$ can be estimated by $\mathbf{B}$. This enables us to establish an L^p -estimate in the local scale with the help of the classical estimates for $-\Delta$ and the gauge invariant property of the magnetic Schrödinger operator. It then follows from an integration argument that

$$(0.16) \quad \int_{\mathbb{R}^n} |L_j L_k(f)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V) f|^p dx + \int_{\mathbb{R}^n} |m(x) |L(f)||^p dx + \int_{\mathbb{R}^n} |\{m(x)\}^2 f|^p dx \right\}$$

where $1 < p < \infty$, $m(x) = m(x, |\mathbf{B}| + V + 1)$ and $|L(f)| = (\sum_{j=1}^n |L_j(f)|^2)^{1/2}$.

To estimate the last two terms in (0.16), we study the kernel function $\Gamma(x, y)$ of the operator $(H(\mathbf{a}, V) + 1)^{-1}$. We will show that, if $|x - y| \leq 1$,

$$(0.17) \quad |\Gamma(x, y)| \leq \frac{C_k}{\{1 + |x - y| m(x)\}^k} \cdot \frac{1}{|x - y|^{n-2}}$$

and

$$(0.18) \quad \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j(x) \right) \Gamma(x, y) \right| \leq \frac{C_k}{\{1 + |x - y| m(x)\}^k} \cdot \frac{1}{|x - y|^{n-1}}$$

for any $k > 0$. From (0.17) and (0.18), through a localization argument, we establish the desired estimate:

$$(0.19) \quad \int_{\mathbb{R}^n} |m(x) |L(f)||^p dx + \int_{\mathbb{R}^n} |\{m(x)\}^2 f|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V) f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}$$

for $f \in C_0^2(\mathbb{R}^n)$ and $1 \leq p \leq \infty$. Estimate (0.11) in Theorem 0.8 follows immediately from (0.16) and (0.19).

Finally, we remark that the proof of the estimates (0.17) and (0.18) relies on the Kato-Simon inequality and the following Fefferman-Phong type estimate:

$$(0.20) \quad \int_{\mathbb{R}^n} |m(x) u(x)|^2 dx \leq C \left\{ \int_{\mathbb{R}^n} |Lu|^2 dx + \int_{\mathbb{R}^n} (V + 1) |u|^2 dx \right\}.$$

(0.20) was established in [Sh4] to study the eigenvalue asymptotics and exponential decay of eigenfunctions of the operator $H(\mathbf{a}, V)$. See [F] [Sm] [Z] [Sh2] for the case $H(\mathbf{0}, V) = -\Delta + V(x)$.

The paper is organized as follows. In Section 1 we derive the size estimate (0.17) of the kernel function $\Gamma(x, y)$ from (0.20). Section 2 contains the proof of the derivative estimate (0.18). Estimate (0.19) is established in Section 3. Our main results, Theorem 0.9 and 0.12, are proved in Section 4.

Throughout this paper, unless otherwise indicated, we will use C and c to denote positive constants, which depend at most on the dimension n , p (if applicable) and the constants C_0 , C_1 , C_2 in (0.5) and (0.10). To avoid the confusion with the magnetic field $\mathbf{B}(x)$, we will use $Q(x, r)$ to denote the ball centered at x with radius r .

1. Estimates of the kernel function. Recall that

$$L_j = \frac{1}{i} \frac{\partial}{\partial x_j} - a_j(x), \quad 1 \leq j \leq n.$$

We let

$$(1.1) \quad |Lu(x)| = \left(\sum_{j=1}^n |L_j u(x)|^2 \right)^{1/2}.$$

The following is a Caccioppoli-type inequality.

Lemma 1.2. *Suppose $H(\mathbf{a}, V)u + u = f$ in $Q(x_0, 2R)$. Then*

$$\int_{Q(x_0, R)} |Lu|^2 dx \leq C \left\{ \int_{Q(x_0, 2R)} |f| |u| dx + \frac{1}{R^2} \int_{Q(x_0, 2R)} |u|^2 dx \right\}.$$

The proof of the lemma is identical to that of the usual Caccioppoli inequality, hence omitted.

Lemma 1.3. *Suppose $H(\mathbf{a}, V)u + u = f$ in $Q(x_0, 2R)$. Then*

$$\begin{aligned} \left(\frac{1}{|Q(x_0, R/4)|} \int_{Q(x_0, R/4)} |u|^q dx \right)^{1/q} &\leq C \left\{ \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |u|^2 dx \right)^{1/2} \right. \\ &\quad \left. + R^2 \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |f|^p dx \right)^{1/p} \right\} \end{aligned}$$

where $2 \leq p < q \leq \infty$ and $(1/q) - (1/p) > (-2/n)$.

Proof. Let $\varphi \in C_0^\infty(Q(x_0, R/2))$ such that $\varphi \equiv 1$ on $Q(x_0, R/3)$ and $|\nabla^2 \varphi| \leq C/R^2$. Note that

$$(1.4) \quad H(\mathbf{a}, V)(u\varphi) = \varphi H(\mathbf{a}, V)u + \frac{2}{i} L_j(u) \cdot \frac{\partial \varphi}{\partial x_j} - u \Delta \varphi.$$

It follows from the Kato-Simon inequality [LS, Lemma 6, p. 7] that

$$\begin{aligned} |u(x)\varphi(x)| &= \left| (H(\mathbf{a}, V) + 1)^{-1} \left(f\varphi + \frac{2}{i} L_j(u) \cdot \frac{\partial \varphi}{\partial x_j} - u\Delta\varphi \right) \right| \\ &\leq (-\Delta + 1)^{-1} \left| f\varphi + \frac{2}{i} L_j(u) \cdot \frac{\partial \varphi}{\partial x_j} - u\Delta\varphi \right| \\ &\leq c_n \int_{\mathbb{R}^n} \frac{|f\varphi| + |L(u)| |\nabla\varphi| + |u| |\Delta\varphi|}{|x - y|^{n-2}} dy. \end{aligned}$$

Thus, for $x \in Q(x_0, R/4)$,

$$\begin{aligned} |u(x)| &\leq C \left\{ \int_{Q(x, R)} \frac{|f(y)| dy}{|y - x|^{n-2}} + \frac{1}{R^{n-1}} \int_{Q(x_0, R)} |Lu| dy + \frac{1}{R^n} \int_{Q(x_0, R)} |u| dy \right\} \\ &\leq C \left\{ \int_{Q(x, R)} \frac{|f(y)| dy}{|y - x|^{n-2}} + R^2 \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |f|^2 dy \right)^{1/2} \right. \\ &\quad \left. + \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |u|^2 dy \right)^{1/2} \right\} \end{aligned}$$

where we have used Lemma 1.2 in the second inequality.

Now, by Young's inequality, if $2 \leq p < q \leq \infty$,

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1 \quad \text{and} \quad (n-2)r < n,$$

$$\begin{aligned} \|u\|_{L^q(Q(x_0, R/4))} &\leq CR^{-n(1-(1/r))+2} \|f\|_{L^p(Q(x_0, 2R))} \\ &\quad + CR^{n/q+2} \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |f|^2 dx \right)^{1/2} \\ &\quad + CR^{n/q} \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |u|^2 dy \right)^{1/2}. \end{aligned}$$

The lemma then follows since $p \geq 2$. □

Lemma 1.5. *Suppose $H(\mathbf{a}, V)u + u = 0$ in $Q(x_0, R)$. Then*

$$|u(x_0)| \leq C \left(\frac{1}{|Q(x_0, R)|} \int_{Q(x_0, R)} |u(x)|^2 dx \right)^{1/2}.$$

Lemma 1.5 follows immediately by taking $q = \infty$ in Lemma 1.3.

Recall that, for a nonnegative function W , the auxiliary function $m(x, W)$ is defined by

$$\frac{1}{m(x, W)} = \sup \left\{ r > 0 : \frac{r^2}{|Q(x, r)|} \int_{Q(x, r)} W(y) dy \leq 1 \right\}.$$

We state the basic properties of $m(x, W)$ in the following lemma which was proved in [Sh4].

Lemma 1.6. *Suppose $W \in (RH)_{n/2, \text{loc}}$ and $W(x) \geq 1$ a.e. on $\mathbb{R}^n$. Then:*

- (a) $1 \leq m(x, W) < \infty$ for every $x \in \mathbb{R}^n$,
- (b) if $|x - y| \leq \{m(x, W)\}^{-1}$, then $m(x, W) \approx m(y, W)$,
- (c) if $|x - y| \leq 1$, then

$$m(y, W) \leq C \{1 + |x - y| m(x, W)\}^{k_0} m(x, W),$$

$$m(y, W) \geq \frac{cm(x, W)}{\{1 + |x - y| m(x, W)\}^{k_0/(k_0+1)}}$$

for some $k_0 = k_0(n, C_0) > 0$.

The following is a Fefferman-Phong type estimate whose proof may also be found in [Sh4].

Theorem 1.7. *Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L_{\text{loc}}^{n/2}(\mathbb{R}^n)$ and $V(x) \geq 0$. Also assume that*

$$\begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2, \text{loc}}, \\ |\nabla \mathbf{B}(x)| \leq C \{m(x, |\mathbf{B}| + V + 1)\}^3 \end{cases}$$

where $\mathbf{B}(x) = \text{curl } \mathbf{a}(x)$. Then, for $u \in C_0^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(x, |\mathbf{B}| + V + 1)u|^2 dx \leq C \left\{ \int_{\mathbb{R}^n} |Lu(x)|^2 dx + \int_{\mathbb{R}^n} (V(x) + 1)|u|^2 dx \right\}.$$

Remark 1.8. In [Sh4, Theorem 2.7], there is an extra condition:

$$(1.9) \quad |\mathbf{B}(x)| \leq C \{m(x, |\mathbf{B}| + V + 1)\}^2.$$

We claim that (1.9) in fact follows from

$$|\nabla \mathbf{B}(x)| \leq C \{m(x, |\mathbf{B}| + V + 1)\}^3.$$

Indeed, let $R = \{m(x, |\mathbf{B}| + V + 1)\}^{-1}$. Then, for $y \in Q(x, R)$, by part (b) of Lemma 1.6,

$$\begin{aligned} |\mathbf{B}(x) - \mathbf{B}(y)| &\leq C\{m(x, |\mathbf{B}| + V + 1)\}^3|x - y| \\ &\leq C\{m(x, |\mathbf{B}| + V + 1)\}^2. \end{aligned}$$

Thus,

$$\begin{aligned} |\mathbf{B}(x)| &\leq \frac{1}{|Q(x, R)|} \int_{Q(x, R)} |\mathbf{B}(y)| dy + C\{m(x, |\mathbf{B}| + V + 1)\}^2 \\ &\leq C\{m(x, |\mathbf{B}| + V + 1)\}^2 \end{aligned}$$

by the definition of $m(x, |\mathbf{B}| + V + 1)$.

For the rest of the paper, we will let

$$(1.10) \quad m(x) = m(x, |\mathbf{B}| + V + 1).$$

Lemma 1.11. Suppose $H(\mathbf{a}, V)u + u = 0$ in $Q(x_0, R)$ for some $x_0 \in \mathbb{R}^n$ and $0 < R \leq 1$. Then

$$|u(x_0)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \left(\frac{1}{|Q(x_0, R)|} \int_{Q(x_0, R)} |u(x)|^2 dx \right)^{1/2}$$

for any integer $k > 0$.

Proof. Let $\varphi \in C_0^\infty(Q(x_0, R/2))$ such that $\varphi \equiv 1$ on $Q(x_0, R/4)$ and $|\nabla^2 \varphi| \leq C/R^2$. Applying Theorem 1.7 to the function $u\varphi$, we obtain

$$\begin{aligned} \int_{Q(x_0, R/4)} |m(x)u|^2 dx &\leq C | \langle (H(\mathbf{a}, V) + 1)(u\varphi), u\varphi \rangle | \\ &\leq \frac{C}{R} \int_{Q(x_0, R/2)} |Lu| |u| dx + \frac{C}{R^2} \int_{Q(x_0, R/2)} |u|^2 dx \\ &\leq \frac{C}{R^2} \int_{Q(x_0, R)} |u|^2 dx \end{aligned}$$

where we have used Lemma 1.2.

Now, by part (c) of Lemma 1.6 and the assumption $R \leq 1$, we have

$$\int_{Q(x_0, R/4)} |u|^2 dx \leq \frac{C\{1 + Rm(x_0)\}^{2k_0/(k_0+1)}}{\{Rm(x_0)\}^2} \int_{Q(x_0, R)} |u|^2 dx.$$

Hence,

$$\int_{Q(x_0, R/4)} |u|^2 dx \leq \frac{C}{\{1 + Rm(x_0)\}^{2/(k_0+1)}} \int_{Q(x_0, R)} |u|^2 dx.$$

It follows that

$$\begin{aligned} |u(x_0)| &\leq C \left(\frac{1}{|Q(x_0, R/4^k)|} \int_{Q(x_0, R/4^k)} |u|^2 dx \right)^{1/2} \\ &\leq \frac{C_k}{\{1 + Rm(x_0)\}^{k/(k_0+1)}} \left(\frac{1}{|Q(x_0, R)|} \int_{Q(x_0, R)} |u|^2 dx \right)^{1/2}. \end{aligned}$$

The lemma then follows since $k > 0$ is arbitrary. $\square$

We are now ready to estimate the kernel function $\Gamma(x, y)$ of the operator $(H(\mathbf{a}, V) + 1)^{-1}$.

First, observe that, by the Kato–Simon inequality,

$$|(H(\mathbf{a}, V) + 1)^{-1} f(x)| \leq (-\Delta + 1)^{-1} |f|(x)$$

for $V \geq 0$ [LS], we have

$$(1.12) \quad |\Gamma(x, y)| \leq \frac{c_n}{|x - y|^{n-2}}.$$

Theorem 1.13. *Under the same assumption as in Theorem 1.7, we have*

$$|\Gamma(x, y)| \leq \frac{C_k}{\{1 + |x - y|m(x)\}^k} \cdot \frac{1}{|x - y|^{n-2}}$$

for $x, y \in \mathbb{R}^n$ and $|x - y| \leq 1$.

Proof. Fix $x_0, y_0 \in \mathbb{R}^n$ such that $|x_0 - y_0| \leq 1$. Let $u(x) = \Gamma(x, y_0)$ and $R = |x_0 - y_0|/2$. Then

$$H(\mathbf{a}, V)u + u = 0 \quad \text{in } Q(x_0, R).$$

The theorem then follows easily from Lemma 1.11 and (1.12). $\square$

Remark 1.14. By part (c) of Lemma 1.6, for $|x - y| \leq 1$,

$$(1.15) \quad 1 + |x - y|m(y) \leq C \{1 + |x - y|m(x)\}^{k_0+1}.$$

Hence, Theorem 1.13 also gives

$$(1.16) \quad |\Gamma(x, y)| \leq \frac{C_k}{\{1 + |x - y|m(y)\}^k} \cdot \frac{1}{|x - y|^{n-2}}$$

for $|x - y| \leq 1$ and any integer $k > 0$.

Remark 1.17. It follows from Theorem 1.13 and (1.16) that

$$(1.18) \quad \int_{|y-x| \leq 1} |\Gamma(x, y)| dy \leq \frac{C}{\{m(x)\}^2}$$

and

$$(1.19) \quad \int_{|x-y| \leq 1} |\Gamma(x, y)| \{m(x)\}^2 dx \leq C.$$

Thus, if $f \in C_0^\infty(Q(x_0, 1/2))$ for some $x_0 \in \mathbb{R}^n$ and $u = (H(\mathbf{a}, V) + 1)f$,

$$\begin{aligned} |f(x)| &\leq \int_{\mathbb{R}^n} |\Gamma(x, y)| |u(y)| dy \\ &\leq \frac{C}{\{m(x)\}^{2/p'}} \left\{ \int_{|y-x| \leq 1} |\Gamma(x, y)| |u(y)|^p dy \right\}^{1/p} \end{aligned}$$

for $1 \leq p \leq \infty$, by Hölder inequality. It follows from (1.19) that

$$(1.20) \quad \int_{\mathbb{R}^n} |\{m(x)\}^2 f(x)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}.$$

2. Derivative estimates. Throughout this section we will assume that

$$(2.1) \quad \begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2, \text{loc}}, \\ V(x) \leq C\{m(x)\}^2, \\ |\nabla \mathbf{B}(x)| \leq C\{m(x)\}^3 \end{cases}$$

where $m(x) = m(x, |\mathbf{B}| + V + 1)$. By Remark 1.8, (2.1) implies that

$$(2.2) \quad |\mathbf{B}(x)| \leq C\{m(x)\}^2.$$

Lemma 2.3. Suppose $H(\mathbf{a}, 0)u + u = 0$ in $Q(x_0, 2R)$ and $0 < R \leq 1$. Then

$$|Lu(x_0)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \cdot \frac{1}{R} \cdot \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |u|^2 dx \right)^{1/2}.$$

for any $k > 0$.

Proof. Note that, for $1 \leq j, k \leq n$,

$$(2.4) \quad [L_j, L_k] = L_j L_k - L_k L_j = \frac{1}{i} \left\{ \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j} \right\} = \frac{1}{i} b_{jk}$$

and

$$\begin{aligned} [L_k, H(\mathbf{a}, 0) + 1]u &= [L_k, H(\mathbf{a}, 0)]u = \sum_{j=1}^n [L_k, L_j^2]u \\ &= \sum_{j=1}^n \{L_j [L_k, L_j]u + [L_k, L_j]L_j u\} \\ &= \sum_{j=1}^n \left\{ L_j \left\{ \frac{1}{i} b_{kj} u \right\} + \frac{1}{i} b_{kj} L_j u \right\} \\ &= \sum_{j=1}^n \left\{ \frac{2}{i} b_{kj} L_j u - \frac{\partial b_{kj}}{\partial x_j} u \right\}. \end{aligned}$$

Hence,

$$(H(\mathbf{a}, V) + 1)L_k u = \sum_{j=1}^n \left\{ \frac{\partial b_{kj}}{\partial x_j} \cdot u - \frac{2}{i} b_{kj} L_j u \right\}.$$

It then follows from Lemma 1.3 that, if $2 \leq p < q \leq \infty$ and $(1/q) - (1/p) > (-2/n)$,

$$\begin{aligned} &\left(\frac{1}{|Q(x_0, R/32)|} \int_{Q(x_0, R/32)} |Lu|^q dx \right)^{1/q} \\ &\leq C \left(\frac{1}{|Q(x_0, R/4)|} \int_{Q(x_0, R/4)} |Lu|^2 dx \right)^{1/2} \\ &\quad + CR^2 \left(\frac{1}{|Q(x_0, R/4)|} \int_{Q(x_0, R/4)} (|\nabla \mathbf{B}(x)| |u|)^p dx \right)^{1/p} \\ &\quad + CR^2 \left(\frac{1}{|Q(x_0, R/4)|} \int_{Q(x_0, R/4)} (|\mathbf{B}(x)| |Lu|)^p dx \right)^{1/p} \\ &\leq \frac{C}{R} \{1 + Rm(x_0)\}^{k_0} \left(\frac{1}{|Q(x_0, R/2)|} \int_{Q(x_0, R/2)} |u|^2 dx \right)^{1/2} \\ &\quad + C \{1 + Rm(x_0)\}^{k_0} \left(\frac{1}{|Q(x_0, R/2)|} \int_{Q(x_0, R/2)} |Lu|^p dx \right)^{1/p} \end{aligned}$$

where we have used Lemma 1.2, assumption (2.1), (2.2) and part (c) of Lemma 1.6.

A bootstrap argument then yields that

$$\begin{aligned}
 |Lu(x_0)| &\leq \frac{C}{R} \cdot \{1 + Rm(x_0)\}^{k_1} \left(\frac{1}{|Q(x_0, R/2)|} \int_{Q(x_0, R/2)} |u|^2 dx \right)^{1/2} \\
 &\quad + C \{1 + Rm(x_0)\}^{k_1} \left(\frac{1}{|Q(x_0, R/2)|} \int_{Q(x_0, R/2)} |Lu|^2 dx \right)^{1/2} \\
 &\leq \frac{C}{R} \cdot \{1 + Rm(x_0)\}^{k_1} \left(\frac{1}{|Q(x_0, R)|} \int_{Q(x_0, R)} |u|^2 dx \right)^{1/2} \\
 &\leq \frac{C_k}{R} \cdot \frac{1}{\{1 + Rm(x_0)\}^k} \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |u|^2 dx \right)^{1/2}
 \end{aligned}$$

where we have used Lemma 1.11 in the last inequality. $\square$

Remark 2.5. Let $\Gamma_0(x, y)$ denote the kernel function of the operator $H(\mathbf{a}, 0) + 1$. It follows easily from Lemma 2.3 that

$$(2.6) \quad |L_j^x \Gamma_0(x, y)| \leq \frac{C_k}{\{1 + |x - y|m(x)\}^k} \cdot \frac{1}{|x - y|^{n-1}}$$

for $x, y \in \mathbb{R}^n$ and $|x - y| \leq 1$, where $L_j^x = (1/i)(\partial/\partial x_j) - a_j(x)$.

Lemma 2.7. Suppose $H(\mathbf{a}, V)u + u = 0$ in $Q(x_0, 2R)$ and $R \leq 1$. Then

$$|Lu(x_0)| \leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \cdot \frac{1}{R} \cdot \left(\frac{1}{|Q(x_0, 2R)|} \int_{Q(x_0, 2R)} |u|^2 dx \right)^{1/2}.$$

Proof. Let $\varphi \in C_0^\infty(Q(x_0, R/2))$ such that $\varphi \equiv 1$ in $Q(x_0, R/8)$ and $|\nabla^2 \varphi| \leq C/R^2$. Note that

$$(H(\mathbf{a}, 0) + 1)(u\varphi) = \frac{2}{i} L_j(u) \cdot \frac{\partial \varphi}{\partial x_j} - u \Delta \varphi - V u \varphi.$$

It follows that

$$u(x)\varphi(x) = \int_{\mathbb{R}^n} \Gamma_0(x, y) \left\{ \frac{2}{i} L_j(u) \frac{\partial \varphi}{\partial x_j} - u \Delta \varphi - V u \varphi \right\} dy.$$

Now, using (2.6), we get

$$\begin{aligned}
 |Lu(x_0)| &\leq \frac{C}{R^n} \int_{|y-x_0| \leq R/2} |Lu| dy + \frac{C}{R^{n+1}} \int_{|y-x_0| \leq R/2} |u| dy \\
 &\quad + C \int_{|y-x_0| \leq R/2} \frac{V(y)|u(y)| dy}{|x_0 - y|^{n-1}} \\
 &\leq \frac{C}{R} \left(\frac{1}{R^n} \int_{Q(x_0, R)} |u|^2 dy \right)^{1/2} \\
 &\quad + CR \{1 + Rm(x_0)\}^{k_0} \{m(x_0)\}^2 \left(\frac{1}{R^n} \int_{Q(x_0, R)} |u|^2 dy \right)^{1/2} \\
 &\leq \frac{C_k}{\{1 + Rm(x_0)\}^k} \cdot \frac{1}{R} \cdot \left(\frac{1}{R^n} \int_{Q(x_0, 2R)} |u(y)|^2 dy \right)^{1/2},
 \end{aligned}$$

where we have used Lemma 1.2, assumption (2.1), Lemma 1.6 in the second inequality and Lemma 1.11 in the third. $\square$

Let

$$|L^x f| = \left(\sum_{j=1}^n \left| \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j(x) \right) f \right|^2 \right)^{1/2}$$

and

$$|\overline{L^y} f| = \left(\sum_{j=1}^n \left| \left(\frac{1}{i} \frac{\partial}{\partial y_j} + a_j(y) \right) f \right|^2 \right)^{1/2}.$$

Theorem 2.8. Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$ and $V \geq 0$ a.e. on $\mathbb{R}^n$. Also assume that

$$\begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2, \text{loc}}, \\ V(x) \leq C\{m(x)\}^2, \\ |\nabla \mathbf{B}(x)| \leq C\{m(x)\}^3. \end{cases}$$

Then, for $x, y \in \mathbb{R}^n$, $|x - y| \leq 1$,

$$|L^x \Gamma(x, y)| + |\overline{L^y} \Gamma(x, y)| \leq \frac{C_k}{\{1 + |x - y|m(x)\}^k} \cdot \frac{1}{|x - y|^{n-1}}.$$

Proof. The estimate of $|L^x\Gamma(x, y)|$ follows immediately from Lemma 2.7. For $|\overline{L^y\Gamma}(x, y)|$, it suffices to notice that

$$\Gamma(x, y) = \overline{\Gamma(y, x)}$$

which follows from the fact that the operator $H(\mathbf{a}, V) + 1$ is self-adjoint. $\square$

Remark 2.9. Using the method in this section, one may obtain higher order estimates on $\Gamma(x, y)$ under more conditions on $(\mathbf{B}, V)$. For example, if, in addition, we assume that

$$(2.10) \quad |\nabla^2 \mathbf{B}(x)| \leq C\{m(x, |\mathbf{B}| + 1)\}^4,$$

then

$$(2.11) \quad |L_j^x L_k^x \Gamma_0(x, y)| \leq \frac{C_k}{\{1 + |x - y|m(x)\}^k} \cdot \frac{1}{|x - y|^n}$$

for $x, y \in \mathbb{R}^n$ and $|x - y| \leq 1$.

Estimates like (2.11) are very interesting in their own right. However, we will not pursue them in the present paper, as they are not needed for the L^p boundedness of the operator $L_j L_k (H(\mathbf{a}, V) + 1)^{-1}$.

3. Lower order estimates. This section is devoted to the proof of the following theorem.

Theorem 3.1. Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$ and $V \geq 0$ a.e. on $\mathbb{R}^n$. Also assume that

$$\begin{cases} |\mathbf{B}| + V + 1 \in (RH)_{n/2, \text{loc}}, \\ V(x) \leq C\{m(x)\}^2, \\ |\nabla \mathbf{B}(x)| \leq C\{m(x)\}^3. \end{cases}$$

Then, for $f \in C_0^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(x)Lf|^p dx + \int_{\mathbb{R}^n} |\{m(x)\}^2 f|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}$$

where $1 \leq p \leq \infty$.

We will need a couple of lemmas before we carry out the proof of Theorem 3.1.

Lemma 3.2. *Under the same assumption as in Theorem 3.1, we have, for $f \in C_0^2(\mathbb{R}^n)$ and $1 \leq p \leq \infty$,*

$$\begin{aligned} & \int_{\mathbb{R}^n} |m(x)L(f)|^p dx + \int_{\mathbb{R}^n} |\{m(x)\}^2 f|^p dx \\ & \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |L(f)|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}. \end{aligned}$$

Proof. Suppose $g \in C_0^2(Q(x_0, 1/2))$ for some $x_0 \in \mathbb{R}^n$. By Remark 1.17, we have

$$(3.3) \quad \int_{\mathbb{R}^n} |\{m(x)\}^2 g|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)g|^p dx + \int_{\mathbb{R}^n} |g|^p dx \right\}.$$

Also note that, Theorem 2.8, together with the argument in Remark 1.17, yields

$$(3.4) \quad \int_{\mathbb{R}^n} |m(x)L(g)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)g|^p dx + \int_{\mathbb{R}^n} |g|^p dx \right\}.$$

Now, let $\varphi \in C_0^\infty(Q(x_0, 1/2))$ such that $\varphi \equiv 1$ on $Q(x_0, 1/4)$. Then, for any $f \in C_0^2(\mathbb{R}^n)$,

$$(3.5) \quad |L_j(f\varphi) - \varphi L_j(f)| \leq C|f|$$

and

$$(3.6) \quad |H(\mathbf{a}, V)(f\varphi) - \varphi H(\mathbf{a}, V)(f)| \leq \{|L(f)| + |f|\}.$$

Finally, we apply (3.3) and (3.4) to the function $g = f\varphi$. Using (3.5) and (3.6), we obtain

$$\begin{aligned} & \int_{Q(x_0, 1/4)} |m(x)L(f)|^p dx + \int_{Q(x_0, 1/4)} |\{m(x)\}^2 f|^p dx \\ & \leq C \left\{ \int_{Q(x_0, 1/2)} |H(\mathbf{a}, V)f|^p dx + \int_{Q(x_0, 1/2)} |L(f)|^p dx + \int_{Q(x_0, 1/2)} |f|^p dx \right\}. \end{aligned}$$

The lemma then follows by integrating above inequality with respect to x_0 over $\mathbb{R}^n$ and using Fubini's Theorem. $\square$

Next, we will construct on each ball a vector potential $\mathbf{h}$, such that, $\mathbf{h} = \mathbf{a} - \nabla\Phi$ and $\mathbf{h}$ can be estimated by the magnetic field $\mathbf{B}(x)$.

Lemma 3.7. Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$ and $Q = Q(x_0, R)$. Then there exist a real vector-valued function $\mathbf{h} \in C^1(Q)$ and a real function $\Phi \in C^2(Q)$, such that

$$(3.8) \quad h_j = a_j - \frac{\partial \Phi}{\partial x_j}, \quad j = 1, 2, \dots, n,$$

$$(3.9) \quad \max_Q |\mathbf{h}| \leq c_n R \max_Q |\mathbf{B}|$$

and

$$(3.10) \quad \max_Q |\nabla \mathbf{h}| \leq c_n \{ \max_Q |\mathbf{B}| + R \max_Q |\nabla \mathbf{B}| \}$$

where c_n depends only on n .

Proof. We use the formula found in [I, p.365]. For $x, y \in Q$, we define

$$(3.11) \quad g_j(x, y) = \sum_{k=1}^n (x_k - y_k) \int_0^1 b_{jk}(y + t(x - y)) t dt$$

where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Let

$$h_j(x) = \frac{1}{|Q|} \int_Q g_j(x, y) dy, \quad j = 1, 2, \dots, n.$$

It is not hard to see that

$$g_j(x, y) = a_j(x) - \frac{\partial}{\partial x_j} \left\{ \sum_{k=1}^n (x_k - y_k) \int_0^1 a_k(y + t(x - y)) dt \right\}.$$

Hence,

$$h_j(x) = a_j(x) - \frac{\partial \Phi}{\partial x_j}(x)$$

where

$$\Phi(x) = \frac{1}{|Q|} \int_Q \left\{ \sum_{k=1}^n (x_k - y_k) \int_0^1 a_k(y + t(x - y)) dt \right\} dy.$$

The estimates (3.9) and (3.10) follow easily from (3.11). $\square$

Remark 3.12. Lemma 3.7 also holds when the ball $Q(x_0, R)$ is replaced by a cube $S(x_0, R)$ centered at x_0 with side length R .

We are now in a position to give the

Proof of Theorem 3.1. By Lemma 3.2, it suffices to show that

$$(3.13) \quad \int_{\mathbb{R}^n} |L(f)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}.$$

Let

$$(3.14) \quad E_T = \{x \in \mathbb{R}^n : m(x) \leq T\}$$

for $T > 1$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} |L(f)|^p dx &\leq \int_{E_T} |L(f)|^p dx + \frac{1}{T^p} \int_{\mathbb{R}^n} |m(x)L(f)|^p dx \\ &\leq \int_{E_T} |L(f)|^p dx + C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\} \\ &\quad + \frac{C}{T^p} \int_{\mathbb{R}^n} |L(f)|^p dx. \end{aligned}$$

Thus, it is enough to show

$$(3.15) \quad \int_{E_T} |L(f)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}$$

for a large but fixed T .

To this end, we divide $\mathbb{R}^n$ into a grid of cubes $\{S_\ell\}$ with side length ε/T where $0 < \varepsilon < 1/(2n)$. Then

$$E_T \subset \bigcup_{\ell} S(x_\ell, \frac{\varepsilon}{T})$$

where $S(x_\ell, \varepsilon/T) \cap E_T \neq \emptyset$.

For each $S = S(x_\ell, \varepsilon/T)$, we construct $\mathbf{h}$ as in Lemma 3.7, such that

$$(3.16) \quad a_j = h_j + \frac{\partial \Phi}{\partial x_j}.$$

Note that, if $y \in S$, then for $x \in S \cap E_T$,

$$|y - x| \leq \frac{\sqrt{n}\varepsilon}{T} \leq \frac{1}{T} \leq \frac{1}{m(x)}.$$

It follows from part (b) of Lemma 1.6 that

$$\max_{y \in S} m(y) \leq CT.$$

Thus, by Lemma 3.7,

$$(3.17) \quad \max_S |\mathbf{h}| \leq C\varepsilon \quad \text{and} \quad \max_S |\nabla \mathbf{h}| \leq C.$$

Now let

$$(3.18) \quad \tilde{L}_j = \frac{1}{i} \frac{\partial}{\partial x_j} - h_j(x) \quad \text{and} \quad |\tilde{L}(f)| = \left(\sum_{j=1}^n |\tilde{L}_j(f)|^2 \right)^{1/2}$$

and $R = \varepsilon/T$. We have

$$\begin{aligned} \int_{S(x_\ell, R)} |\tilde{L}(f)|^p dx &\leq \int_{S(x_\ell, R)} |\nabla f|^p dx + C\varepsilon^p \int_{S(x_\ell, R)} |f|^p dx \\ &\leq CR^p \int_{S(x_\ell, 2R)} |\Delta f|^p dx + C_\varepsilon \int_{S(x_\ell, 2R)} |f|^p dx, \end{aligned}$$

where we have used the classical estimates for $-\Delta$, which follow easily from the representation by the Newtonian potential.

To continue, let

$$\begin{aligned} (3.19) \quad H(\mathbf{h}, V) &= \sum_{j=1}^n \tilde{L}_j^2 + V \\ &= -\Delta + V - 2 \sum_{j=1}^n h_j \tilde{L}_j - \frac{1}{i} \operatorname{div} \mathbf{h} - |\mathbf{h}|^2. \end{aligned}$$

Then, for $f \in C_0^2(\mathbb{R}^n)$,

$$\begin{aligned} (3.20) \quad \int_{S(x_\ell, R)} |\tilde{L}(f)|^p dx &\leq CR^p \int_{S(x_\ell, 2R)} |H(\mathbf{h}, V)f|^p dx \\ &\quad + CR^p \int_{S(x_\ell, 2R)} |\tilde{L}(f)|^p dx + C_\varepsilon \int_{S(x_\ell, 2R)} |f|^p dx, \end{aligned}$$

where we have used the assumption $V(x) \leq C\{m(x)\}^2$.

Finally, using the gauge invariance of the magnetic Schrödinger operator, i.e.,

$$L_j(f) = e^{-i\Phi} \tilde{L}_j(e^{i\Phi} f) \quad \text{and} \quad H(\mathbf{a}, V)(f) = e^{-i\Phi} H(\mathbf{h}, V)(e^{i\Phi} f),$$

we see that

$$\begin{aligned} \int_{S(x_\ell, R)} |L(f)|^p dx &\leq CR^p \int_{S(x_\ell, 2R)} |H(\mathbf{a}, V)f|^p dx \\ &\quad + CR^p \int_{S(x_\ell, 2R)} |L(f)|^p dx + C_\varepsilon \int_{S(x_\ell, 2R)} |f|^p dx. \end{aligned}$$

Summing over ℓ , we obtain

$$\begin{aligned} \int_{E_T} |L(f)|^p dx &\leq C\varepsilon^p \int_{\mathbb{R}^n} |L(f)|^p dx + C \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + C_\varepsilon \int_{\mathbb{R}^n} |f|^p dx \\ &\leq C\varepsilon^p \int_{E_T} |L(f)|^p dx + C_\varepsilon \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \int_{\mathbb{R}^n} |f|^p dx \right\}. \end{aligned}$$

(3.15) then follows by choosing ε so small that $C\varepsilon^p \leq \frac{1}{2}$. $\square$

4. Proof of the main results. In this section we give the proof of Theorems 0.9 and 0.12.

We will need the following classical estimates for $-\Delta$:

$$(4.1) \quad \int_{Q(x_0, R/2)} |\nabla^2 f|^p \leq C \int_{Q(x_0, R/2)} |\Delta f|^p dx + \frac{C}{R^{2p}} \int_{Q(x_0, R)} |f|^p dx$$

for $1 < p < \infty$, and

$$(4.2) \quad \begin{aligned} &|\{x \in Q(x_0, R/2) : |\nabla^2 f(x)| > \lambda\}| \\ &\leq \frac{C}{\lambda} \int_{Q(x_0, R)} |\Delta f| dx + \frac{C}{\lambda R^2} \int_{Q(x_0, R)} |f| dx. \end{aligned}$$

We remark that (4.1) and (4.2) are easy consequences of the Calderón–Zygmund inequality and a cut-off function argument.

Proof of Theorem 0.9. Let $1 < p < \infty$. We need to show that

$$(4.3) \quad \sum_{j,k} \|L_j L_k(f)\|_p \leq C \{ \|H(\mathbf{a}, V)f\|_p + \|f\|_p \}.$$

To this end, we fix $x_0 \in \mathbb{R}^n$. Let $R = \{m(x_0)\}^{-1}$ and $Q(R) = Q(x_0, R)$. By Lemma 3.7, we may construct $\mathbf{h}$ and Φ such that $\mathbf{h} = \mathbf{a} - \nabla \Phi$,

$$(4.4) \quad \max_{Q(R)} |\mathbf{h}| \leq c_n R \max_{Q(R)} |\mathbf{B}| \leq C m(x_0)$$

and

$$(4.5) \quad \max_{Q(R)} |\nabla \mathbf{h}| \leq c_n \left\{ \max_{Q(R)} |\mathbf{B}| + R \max_{Q(R)} |\nabla \mathbf{B}| \right\} \leq C \{m(x_0)\}^2,$$

where we have used Assumptions (0.10), (1.9), and Lemma 1.6.

Now, let

$$\tilde{L}_j = \frac{1}{i} \frac{\partial}{\partial x_j} - h_j(x).$$

Then

$$\tilde{L}_j \tilde{L}_k(f) = -\frac{\partial^2 f}{\partial x_j \partial x_k} - h_j \tilde{L}_k(f) - h_k \tilde{L}_j(f) - \frac{1}{i} \frac{\partial h_k}{\partial x_j} \cdot f - h_j h_k f.$$

Hence, by (4.4) and (4.5),

$$(4.6) \quad \left| \tilde{L}_j \tilde{L}_k(f) + \frac{\partial^2 f}{\partial x_j \partial x_k} \right| \leq C \left\{ m(x_0) |\tilde{L}(f)| + \{m(x_0)\}^2 |f| \right\}.$$

It follows that

$$\begin{aligned} \int_{Q(R/2)} |\tilde{L}_j \tilde{L}_k(f)|^p dx &\leq \int_{Q(R/2)} |\nabla^2 f|^p dx + \frac{C}{R^p} \int_{Q(R/2)} |\tilde{L}(f)|^p dx \\ &\quad + \frac{C}{R^{2p}} \int_{Q(R/2)} |f|^p dx \\ &\leq C \int_{Q(R)} |\Delta f|^p dx + \frac{C}{R^p} \int_{Q(R)} |\tilde{L}(f)|^p dx + \frac{C}{R^{2p}} \int_{Q(R)} |f|^p dx, \end{aligned}$$

where we have used (4.1) in the second inequality.

Recall that

$$H(\mathbf{h}, V) = -\Delta + V - 2 \sum_{j=1}^n h_j \tilde{L}_j - \frac{1}{i} \operatorname{div} \mathbf{h} - |\mathbf{h}|^2.$$

(See (3.19).) We have, on $Q(R)$,

$$|H(\mathbf{h}, V)f + \Delta f| \leq C \left\{ m(x_0) |\tilde{L}(f)| + \{m(x_0)\}^2 |f| \right\}.$$

Thus,

$$\begin{aligned} \int_{Q(R/2)} |\tilde{L}_j \tilde{L}_k(f)|^p dx &\leq C \int_{Q(R)} |H(\mathbf{h}, V)f|^p dx + \frac{C}{R^p} \int_{Q(R)} |\tilde{L}(f)|^p dx \\ &\quad + \frac{C}{R^{2p}} \int_{Q(R)} |f|^p dx. \end{aligned}$$

By the gauge invariance, we obtain, for any $f \in C_0^2(\mathbb{R}^n)$,

$$\begin{aligned} & \int_{Q(R/2)} |L_j L_k(f)|^p dx \\ & \leq C \int_{Q(R)} |H(\mathbf{a}, V)f|^p dx + \frac{C}{R^p} \int_{Q(R)} |L(f)|^p dx + \frac{C}{R^{2p}} \int_{Q(R)} |f|^p dx \\ & \leq C \int_{Q(R)} |H(\mathbf{a}, V)f|^p dx + C \int_{Q(R)} |m(x)L(f)|^p dx + C \int_{Q(R)} |\{m(x)\}^2 f|^p dx. \end{aligned}$$

Recall that $R = \{m(x_0)\}^{-1}$ and $Q(R) = Q(x_0, R)$. To finish the proof, we multiply $\{m(x_0)\}^n$ to the both sides of the inequality above, and then integrate in x_0 over $\mathbb{R}^n$ and use Fubini's Theorem. Since

$$\int_{|x-x_0| < (1/m(x_0))} \{m(x_0)\}^n dx_0 \approx 1$$

by Lemma 1.6, we get

$$\begin{aligned} & \int_{\mathbb{R}^n} |L_j L_k(f)|^p dx \\ & \leq C \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + C \int_{\mathbb{R}^n} |m(x)L(f)|^p dx + C \int_{\mathbb{R}^n} |\{m(x)\}^2 f|^p dx \\ & \leq C \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + C \int_{\mathbb{R}^n} |f|^p dx \end{aligned}$$

where the last inequality follows from Theorem 3.1. The proof is complete. $\square$

Finally we give the proof of the weak-type (1,1) estimate.

Proof of Theorem 0.12. The proof of the weak-type estimate is similar to that of the L^p estimates.

Let $Q(R)$, $\tilde{L}_j$ be the same as in the proof of Theorem 0.9. It follows from (4.6) and (4.2) that, for $\lambda > 0$,

$$\begin{aligned} & \left| \left\{ x \in Q\left(\frac{R}{2}\right) : |\tilde{L}_j \tilde{L}_k(f)(x)| > \lambda \right\} \right| \\ & \leq \left| \left\{ x \in Q\left(\frac{R}{2}\right) : |\nabla^2 f(x)| > \frac{\lambda}{2} \right\} \right| \\ & \quad + \left| \left\{ x \in Q\left(\frac{R}{2}\right) : C m(x) |\tilde{L}(f)(x)| + C \{m(x)\}^2 |f(x)| > \frac{\lambda}{2} \right\} \right| \leq \end{aligned}$$

$$\leq \frac{C}{\lambda} \int_{Q(R)} |H(\mathbf{h}, V)f| dx + \frac{C}{\lambda R} \int_{Q(R)} |\tilde{L}(f)| dx + \frac{C}{\lambda R^2} \int_{Q(R)} |f| dx.$$

As before, by the gauge invariance, this implies that

$$\begin{aligned} & |\{x \in Q(R/2) : |L_j L_k(f)(x)| > \lambda\}| \\ & \leq \frac{C}{\lambda} \left\{ \int_{Q(R)} |H(\mathbf{a}, V)f| dx + \int_{Q(R)} |m(x)L(f)| dx + \int_{Q(R)} |\{m(x)\}^2 f| dx \right\}. \end{aligned}$$

The rest of the proof is the same as in the proof of Theorem 0.9. We omit the details. $\square$

We end this paper with a scale invariant version of Theorem 0.9 and 0.12.

Theorem 4.7. Suppose $\mathbf{a} \in C^2(\mathbb{R}^n)$, $V \in L_{\text{loc}}^\infty(\mathbb{R}^n)$ and $V \geq 0$ a.e. on $\mathbb{R}^n$. Also assume that

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ V(x) \leq C_1 \{m(x), |\mathbf{B}| + V\}^2, \\ |\nabla \mathbf{B}(x)| \leq C_2 \{m(x), |\mathbf{B}| + V\}^3. \end{cases}$$

Then

$$\int_{\mathbb{R}^n} |L_j L_k(f)|^p dx \leq C \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx, \quad \text{for } 1 < p < \infty,$$

$$|\{x \in \mathbb{R}^n : |L_j L_k(f)(x)| > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f| dx,$$

and for $1 \leq p \leq \infty$,

$$\begin{aligned} & \int_{\mathbb{R}^n} |m(x), |\mathbf{B}| + V| L_j(f)|^p dx + \int_{\mathbb{R}^n} |\{m(x), |\mathbf{B}| + V\}^2 f|^p dx \\ & \leq C \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx. \end{aligned}$$

Proof. For $\varepsilon > 0$, let

$$a_j^\varepsilon(x) = \frac{1}{\varepsilon} a_j\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad V^\varepsilon(x) = \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right).$$

Then

$$b_{jk}^\varepsilon(x) = \frac{\partial a_j^\varepsilon}{\partial x_k} - \frac{\partial a_k^\varepsilon}{\partial x_j} = \frac{1}{\varepsilon^2} b_{jk}\left(\frac{x}{\varepsilon}\right).$$

Also, it is not hard to see that

$$m(x, |\mathbf{B}^\varepsilon| + V^\varepsilon) = \frac{1}{\varepsilon} m\left(\frac{x}{\varepsilon}, |\mathbf{B}| + V\right)$$

where $\mathbf{B}^\varepsilon(x) = (b_{jk}^\varepsilon(x))_{n \times n}$.

It follows that

$$\begin{aligned} V^\varepsilon(x) &= \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right) \leq \frac{C_1}{\varepsilon^2} \left\{ m\left(\frac{x}{\varepsilon}, |\mathbf{B}| + V\right) \right\}^2 \\ &= C_1 \{m(x, |\mathbf{B}^\varepsilon| + V^\varepsilon)\}^2 \end{aligned}$$

and

$$\begin{aligned} |\nabla \mathbf{B}^\varepsilon(x)| &= \frac{1}{\varepsilon^3} \left| \nabla \mathbf{B}\left(\frac{x}{\varepsilon}\right) \right| \leq \frac{C_2}{\varepsilon^3} \left\{ m\left(\frac{x}{\varepsilon}, |\mathbf{B}| + V\right) \right\}^3 \\ &= C_2 \{m(x, |\mathbf{B}^\varepsilon| + V^\varepsilon)\}^3. \end{aligned}$$

Clearly, $|\mathbf{B}^\varepsilon| + V^\varepsilon \in (RH)_{n/2}$ with a constant independent of ε . Thus, by Theorem 0.9,

$$(4.8) \quad \int_{\mathbb{R}^n} |L_j^\varepsilon L_k^\varepsilon(g)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}^\varepsilon, V^\varepsilon)g|^p dx + \int_{\mathbb{R}^n} |g|^p dx \right\}$$

for $1 < p < \infty$ and $g \in C_0^2(\mathbb{R}^n)$, where

$$L_j^\varepsilon = \frac{1}{i} \frac{\partial}{\partial x_j} - a_j^\varepsilon(x) \quad \text{and} \quad H(\mathbf{a}^\varepsilon, V^\varepsilon) = \sum_{j=1}^n L_j^\varepsilon L_j^\varepsilon + V^\varepsilon.$$

Now, given any $f \in C_0^2(\mathbb{R}^n)$, let $g(x) = f(x/\varepsilon)$ in (4.8). Note that

$$L_j^\varepsilon(g)(x) = \frac{1}{\varepsilon} L_j(f)\left(\frac{x}{\varepsilon}\right)$$

and

$$H(\mathbf{a}^\varepsilon, V^\varepsilon)(g)(x) = \frac{1}{\varepsilon^2} H(\mathbf{a}, V)(f)\left(\frac{x}{\varepsilon}\right).$$

It follows that

$$\int_{\mathbb{R}^n} |L_j L_k(f)|^p dx \leq C \left\{ \int_{\mathbb{R}^n} |H(\mathbf{a}, V)f|^p dx + \varepsilon^2 \int_{\mathbb{R}^n} |f|^p dx \right\}.$$

The desired estimate follows by letting $\varepsilon \rightarrow 0$. The other estimates in Theorem 4.7 follow in the same manner. $\square$

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Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506
E-MAIL: shenz@ms.uky.edu

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