An L^∞ Bound for a Time-discrete Regularization of a Forward-backward Parabolic Equation

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ABSTRACT. In this paper we consider a time discrete regularization of a backward-forward parabolic equation that is not necessarily well posed. This time-discrete scheme in $\mathbb{R}^n$ can be formulated as

$$v_{m+1} = K_{\Delta t} * v_m + \Delta t K_{\Delta t} * \sum_{i,j=1}^n f_{ij}(v_m, x),$$

with $f_{ij}(v, \cdot)$ constant when v is large, K a positive kernel with compact support,

$$K_s(x) = \frac{1}{s^{n/2}} K\left(\frac{x}{s^{1/2}}\right),$$

$s > 0$. The result we obtain is a uniform bound for $(v_m)_{m \in \mathbb{N}}$ independent of $\Delta t > 0$.

1. Introduction. This paper is mainly concerned with the existence of an L^∞ bound for the following time-discrete scheme in $\mathbb{R}^n$,

$$(1.1) \quad \begin{cases} v_{m+1} = K_{\Delta t} * v_m + \Delta t K_{\Delta t} * \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(v_m, x) \\ v_0. \end{cases}$$

For the kernel $K_s, s > 0$ we assume $K_s(x) = (1/s^{n/2})K(x/s^{1/2})$, where

$$(1.2) \quad K \in C_c^3(\mathbb{R}^n), \quad K \geq 0, \quad \int_{\mathbb{R}^n} K(x) dx = 1$$

and

$$(1.3) \quad \int_{\mathbb{R}^n} x_i K(x) dx = 0, \quad \int_{\mathbb{R}^n} x_i x_j K(x) dx = \delta_{ij}, \quad i, j = 1 \dots n.$$

Note that K can be regarded as a density function of a random vector with all the first moments being zero and the identity as the covariance matrix.

The functions f_{ij} $i, j = 1 \dots n$ are constant outside a bounded interval. To fix the notation we consider this interval as $[-1, 1]$, and we assume that

$$(1.4) \quad f_{ij}: \mathbb{R}^{n+1} \rightarrow \mathbb{R} \quad i, j = 1 \dots n \quad \text{is Lipschitz continuous,}$$

$f_{ij}(\lambda, x) = C_1^{ij}$ where $\lambda \geq 1$, and $f_{ij}(\lambda, x) = C_2^{ij}$ where $\lambda \leq -1$ $i, j = 1 \dots n$, uniformly in $x \in \mathbb{R}^n$.

For the initial condition we assume that

$$(1.5) \quad v_0 \in L^\infty, \quad \nabla v_0 \in (L^\infty)^n.$$

Notice that for a given fixed time, the repeated convolution of the positive kernel $K_{\Delta t}$ with itself tends to the heat kernel as $\Delta t \rightarrow 0$. Therefore, at the limit ($\Delta t \rightarrow 0$) we formally get (1.6) below:

$$(1.6) \quad \begin{cases} v_t = \Delta v + \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(v, x) \\ v(x, 0) = v_0(x) \end{cases} \quad \text{in } x \in \mathbb{R}^n, t > 0.$$

For this reason, (1.1) can be seen as a time-discrete regularization of (1.6), that is, an initial-value problem that could be ill-posed depending on the monotonicity of the second-order operator. There is only one result in the one-dimensional case that ensures the existence of infinitely many solutions with $f = f_{11}$ piecewise linear (see Hollig-Nohel [HN]).

The main result of this paper is as follows:

Theorem 1. *Let $(v_m)_{m \in \mathbb{N}}$ be solutions to (1.1), satisfying (1.2)–(1.4). Then, there is a positive constant c such that*

$$(1.7) \quad \|v_m\|_{L^\infty} \leq c, \quad \Delta t > 0, \quad m \in \mathbb{N}.$$

Although this result does not ensure convergence, it gives a certain notion of stability as $\Delta t \rightarrow 0$. Theorem 1 is analogous to getting a uniform L^∞ bound for the following rescaled problem in $\mathbb{R}^n$:

$$(1.8) \quad \begin{cases} u_{m+1} = K * u_m + K * \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(u_m, x) \\ u_0, \end{cases}$$

where $u_m(x) = v_m(\sqrt{\Delta t}x)$. We can rewrite (1.8) as

$$(1.9) \quad \begin{cases} u_{m+1} = K^{(m+1)} * u_0 + \sum_{\ell=1}^{m+1} K^{(\ell)} * \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(u_{m-\ell+1}, x), \\ u_0, \end{cases}$$

where $K^{(m)}$ is the m -th convolution of K with itself.

The organization of the paper is as follows.

In Section 2, we prove that the gradient of the solutions of (1.8) has an L^∞ bound, uniformly in $m \in \mathbb{N}$. For the proof we use the fact that $K^{(m)}$ approaches the heat kernel as $m \rightarrow +\infty$ (see for instance Durrett [D], Feller [Fe], and many others). Roughly speaking, this means that $K^{(m)}$ behaves as the heat kernel for m large. In order to get the result, one must assume that f_{ij} is bounded.

In Section 3, we prove first that $(u_{m+1} - u_m)$, namely the difference between two consecutive solutions of (1.8), has a bound in L^∞ uniformly in $m \in \mathbb{N}$. We then obtain that the solutions of (1.8) have a bound in L^∞ uniformly in $m \in \mathbb{N}$ and, as a consequence, we get the result of Theorem 1. In this section we use for the first time the compact support of K (in the previous section an exponential decay at infinity would be enough).

Note: The Cahn-Hilliard equation (see for instance Cahn [C], Pego [P], Slemrod [S]) can be seen as a fourth-order regularization of (1.6). In a previous paper (see Caffarelli-Muler [CM]) we obtained an L^∞ bound for the solutions of Cahn-Hilliard. Notice that in [CM] and in this paper we succeed in proving the existence of a uniform bound for regularizations of (1.6), even though we cannot demonstrate the existence of solutions in the limit.

Notation.

- (1) $L^p = L^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$.
- (2) $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
- (3) The one star (*) denotes the space convolution in $\mathbb{R}^n$.
- (4) $B_a(x_0) = \{x/|x - x_0| < a\}$
- (5) In Section 2, given a function $g: \mathbb{R}^n \rightarrow \mathbb{R}$, we denote

$$D^\alpha g(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} g(x),$$

where $\alpha \in \mathbb{N}_0^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$

2. Bounds for the gradient. Let u_m be the solution to (1.8). In this section we shall prove that ∇u_m have an L^∞ bound uniformly in $m \in \mathbb{N}$.

The characteristic function φ of the distribution function with density K can be written as

$$(2.1) \quad \varphi(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} K(\xi) d\xi.$$

Hence,

$$(2.2) \quad K^{(m)}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi^m(\xi) d\xi.$$

First, let us state some elementary results.

Lemma 1. *For $x \in \mathbb{R}^n$ the following holds:*

- (i) $|\varphi(x)| < 1$ $x \neq 0$ and $\varphi(0) = 1$;
- (ii) $\lim_{\ell \rightarrow +\infty} |\varphi^{\ell-k}(x/\sqrt{\ell})| = e^{-|x|^2/2}$, $k \in \mathbb{N}$;
- (iii) *Given $\alpha \in \mathbb{N}_0^n$, there is a positive constant c such that*

$$|D_x^\alpha \varphi(x)| \leq c;$$

- (iv) *There is a positive constant c such that*

$$|\nabla \varphi(x)| \leq c|x|;$$

- (v) *There is a positive constant c such that*

$$|\varphi(x)| \leq c \prod_{i=1}^n \frac{1}{|x_i|^3}.$$

Proof.

- (i) From the definition of φ it follows that $|\varphi(x)| \leq 1$ in $\mathbb{R}^n$ and $\varphi(0) = 1$.

We should now see that the strict inequality holds for $x \neq 0$. Let us assume that there exists $\bar{x} \in \mathbb{R}^n$ such that $\varphi(\bar{x}) = 1$. Hence,

$$(2.3) \quad \int_{\mathbb{R}^n} e^{i\bar{x} \cdot \xi} (K(\xi) - 1) d\xi = \varphi(\bar{x}) - 1 = 0.$$

In particular,

$$\bar{x} \cdot \xi = k_\xi \pi, \quad \forall \xi \in B_\delta(a) \subset \text{supp}(K), \quad k_\xi \in \mathbb{Z}$$

and this implies $\bar{x} = 0$.

(ii) From the assumptions on K and the definition of φ , we have

$$(2.4) \quad \varphi(0) = 1, \quad \nabla \varphi(0) = 0 \quad \text{and} \quad \frac{\partial^2 \varphi(0)}{\partial x_i \partial x_j} = -\delta_{ij}.$$

Therefore,

$$(2.5) \quad \varphi\left(\frac{x}{\sqrt{\ell}}\right)^{\ell-k} = \left(1 - \frac{|x|^2}{2\ell} + o\left(\frac{|x|^2}{\ell}\right)\right)^\ell \varphi\left(\frac{x}{\sqrt{\ell}}\right)^{-k} \xrightarrow{\ell \rightarrow +\infty} e^{-|x|^2/2}.$$

(iii) and (iv) follow from the compact support of K and $\nabla \varphi(0) = 0$. In fact, it would be enough to assume K with an exponential decay at infinity.

(v) Using $K \in C_c^3(\mathbb{R}^n)$, we integrate (2.1) three times by parts and we get the result. $\square$

Now, we need some technical lemmas.

Lemma 2. *There is a constant $c > 0$ such that*

$$(2.6) \quad \left| D_x^\alpha \varphi^\ell\left(\frac{x}{\sqrt{\ell}}\right) \right| \leq c \sum_{k=1}^s \left| \varphi^{\ell-k}\left(\frac{x}{\sqrt{\ell}}\right) \right| (1 + |x|^k), \quad \forall \ell \geq |\alpha|,$$

where $s = \min(|\alpha|, \ell)$, $\alpha_i \in \{0, 1, 2\}$, $i = 1 \dots n$.

Proof. Opening $D_x^\alpha \varphi^\ell(x/\sqrt{\ell}) = D_x^\alpha \prod_{j=1}^\ell \varphi(x/\sqrt{\ell})$, for each $1 \leq k \leq s$ we have a summation such that the typical term becomes

$$(2.7) \quad \ell^{-|\alpha|/2} \binom{\ell}{k} \binom{k}{k_1} (k - k_1)! k! \varphi(y)^{\ell-k} \prod_{j=1}^k D_y^{\alpha(j)} \varphi(y),$$

where $y = x/\sqrt{\ell}$, $0 \leq k_1 \leq k$, $\sum_{j=1}^k \alpha(j)_i = \alpha_i$, $i = 1 \dots n$, $|\alpha(j)| \geq 2$, $j = 1 \dots k_1$, $|\alpha(j)| = 1$, $j = k_1 + 1 \dots k$.

Using Lemma 1 (iii) and (iv) and taking into account that $|\alpha| \geq k_1 + k$, we can obtain that (2.7) is bounded by

$$\begin{aligned} & \ell^{-(|\alpha|/2 + k - k_1)} \left| \varphi\left(\frac{x}{\sqrt{\ell}}\right)^{\ell-k} \right| \binom{\ell}{k} \binom{k}{k_1} (k - k_1)! k! |x|^{k-k_1} \\ & \leq d \left| \varphi\left(\frac{x}{\sqrt{\ell}}\right)^{\ell-k} \right| (|x|^k + 1), \end{aligned}$$

where d is a positive constant.

Hence, we get the result. $\square$

Lemma 3. *There is a constant $c > 0$ that does not depend on $\ell \in \mathbb{N}$ such that*

$$(2.8) \quad \int_{\mathbb{R}^n} \left| P(x) \varphi \left(\frac{x}{\sqrt{\ell}} \right)^{\ell-k} \right| dx \leq c,$$

where $0 \leq k \leq 2n$, $k < \ell$ and $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial of degree at most one in each variable x_i .

Proof. We can split the integral as

$$(2.9) \quad \int_{\mathbb{R}^n} \left| P(x) \varphi \left(\frac{x}{\sqrt{\ell}} \right)^{\ell-k} \right| dx = \sum_{i=1}^3 \int_{I_i} \left| P(x) \varphi \left(\frac{x}{\sqrt{\ell}} \right)^{\ell-k} \right| dx,$$

where

$$I_1 = \left\{ x/|x_i| \leq \sqrt{\ell}\varepsilon, i = 1 \dots n \right\},$$

$$I_2 = \left\{ x/\sqrt{\ell}\varepsilon \leq |x_i| \leq \ell^2, i = 1 \dots n \right\}$$

$$I_3 = \left\{ x/|x_i| \geq \ell^2, i = 1 \dots n \right\}.$$

The proof consists in estimating each term in the previous expression separately. From Lemma 1, for all $x \in I_1$ and ε small enough,

$$(2.10) \quad \left| \varphi \left(\frac{x}{\sqrt{\ell}} \right)^{\ell-k} \right| \leq e^{-|x|^2/4}.$$

In view of Lemma 1 (i), for all $x \in I_2$ there is a constant $0 < q < 1$ such that

$$(2.11) \quad \left| \varphi \left(\frac{x}{\sqrt{\ell}} \right) \right| \leq q.$$

Therefore,

$$(2.12) \quad \int_{I_2} \left| P(x) \varphi \left(\frac{x}{\sqrt{\ell}} \right) \right|^{\ell-k} dx \leq q^{\ell-k} \int_{I_2} |P(x)| dx.$$

and this last integral can be bounded by a polynomial with variable ℓ .

Finally, from Lemma 1 (v) and the definition of I_3 , we have that there is a positive constant c such that

$$(2.13) \quad \begin{aligned} \int_{I_3} \left| P(x) \varphi \left(\frac{x}{\sqrt{\ell}} \right)^{\ell-k} \right| dx &\leq c \int_{I_3} |P(x)| \prod_{i=1}^n \left(\frac{\sqrt{\ell}}{|x_i|} \right)^{3(\ell-k)} dx \\ &\leq c \int_{I_3} |P(x)| \prod_{i=1}^n \frac{1}{|x_i|^{(9/4)(\ell-k)}} dx, \end{aligned}$$

that is bounded for ℓ large enough. Collecting the bounds (2.10)–(2.13), we obtain the result of the lemma. $\square$

Lemma 4. *There is a constant $c > 0$ that does not depend on ℓ such that*

$$(2.14) \quad \left\| \int_{\mathbb{R}^n} |P(\eta) \varphi^\ell\left(\frac{\eta}{\sqrt{\ell}}\right) e^{i\eta \cdot x} d\eta| \right\|_{L^1} \leq c$$

where $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is a polynomial of degree at most one in each variable η_i .

Proof. For $\ell \leq 2n$, using the decay of φ , we can easily see that the result holds. Now, let us consider the case $\ell > 2n$.

Choose a function $\vartheta \in C_c^1$ such that

$$(2.15) \quad \vartheta(\lambda) \leq 1 \quad \text{and} \quad \vartheta(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \leq 1 \\ 0 & \text{if } |\lambda| \geq 2, \end{cases}$$

we have

$$(2.16) \quad 1 = \prod_{j=1}^n (\vartheta(x_j) + \bar{\vartheta}(x_j)), \quad \text{where} \quad \bar{\vartheta} = 1 - \vartheta.$$

Therefore, we can write the integral in (2.14) as

$$(2.17) \quad \int_{\mathbb{R}^n} \prod_{j=1}^n (\vartheta(x_j) + \bar{\vartheta}(x_j)) \left| \int_{\mathbb{R}^n} |P(\eta) \varphi^\ell\left(\frac{\eta}{\sqrt{\ell}}\right)| e^{i\eta \cdot x} d\eta \right| dx.$$

We open all the sums in products (after reordering $(\eta_1, \dots, \eta_n), (x_1, \dots, x_n)$) and the typical term becomes

$$(2.18) \quad \int_{\mathbb{R}^n} \prod_{j=1}^k \bar{\vartheta}(x_j) \prod_{j=k+1}^n \vartheta(x_j) \left| \int_{\mathbb{R}^n} |P(\eta) \varphi^\ell\left(\frac{\eta}{\sqrt{\ell}}\right)| e^{i\eta \cdot x} d\eta \right| dx,$$

where $0 \leq k \leq n$. We integrate by parts two times with respect to the variables η_j , $1 \leq j \leq k$ (if there are any, that is, when $k \neq 0$) and the integral can be bounded by a sum of terms that can be written as

$$(2.19) \quad \int_{\mathbb{R}^n} \prod_{j=1}^k \frac{1}{x_j^2} \bar{\vartheta}(x_j) \prod_{j=k+1}^n \vartheta(x_j) \int_{\mathbb{R}^n} |\bar{P}(\eta) D_\eta^\alpha \varphi^\ell\left(\frac{\eta}{\sqrt{\ell}}\right)| d\eta dx,$$

where $\bar{P} : \mathbb{R}^n \rightarrow \mathbb{R}$ are polynomials of degree at most one in each variable η_i , $\alpha_i \in \{0, 1, 2\}$ $i = 1 \dots n$, $|\alpha| \leq 2k$.

Hence, from Lemma 2 and Lemma 3 we get the desired result. $\square$

Now, let us state the main result of this section.

Proposition 1. *There is a constant $c > 0$ that does not depend on $m \in \mathbb{N}$ such that*

$$(2.20) \quad \|\nabla u_m\|_{(L^\infty)^n} \leq c$$

where u_m $m \in \mathbb{N}$ are solutions of (1.8).

Proof. From (1.9) we can write

$$(2.21) \quad \frac{\partial}{\partial x_k} u_m = K^{(m)} * \frac{\partial}{\partial x_k} u_0 + \sum_{\ell=1}^m K^{(\ell)} * \sum_{i,j=1}^n \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} f_{ij}(u_{m-\ell}, \cdot),$$

$k = 1 \dots n.$

Using Hölder's inequality and standard properties of the convolutions, one has

$$\left\| \frac{\partial}{\partial x_k} u_m \right\|_{L^\infty} \leq \|K^{(m)}\|_{L^1} \left\| \frac{\partial}{\partial x_k} u_0 \right\|_{L^\infty} + \sum_{\ell=1}^m \sum_{i,j=1}^n \left\| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} K^{(\ell)} \right\|_{L^1} \cdot \max_{i,j=1 \dots n} \|f_{ij}\|_{L^\infty}.$$

By (2.2) we obtain

$$(2.23) \quad \|K^{(m)}\|_{L^\infty} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi^m(\xi) d\xi \right| dx.$$

Setting $\eta = \sqrt{\ell} \xi$, $y = \frac{x}{\sqrt{\ell}}$, we have that (2.23) becomes

$$(2.24) \quad \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-iy \cdot \eta} \varphi^m\left(\frac{\eta}{\sqrt{\ell}}\right) d\eta \right| dy.$$

By Lemma 4, this expression is uniformly bounded. Similarly, we can write

$$(2.25) \quad \left\| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} K^{(m)} \right\|_{L^1} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \xi_i \xi_j \xi_k \varphi^m(\xi) d\xi \right| dx.$$

Setting $\eta = \sqrt{\ell} \xi$, $y = \frac{x}{\sqrt{\ell}}$, we have that (2.25) becomes

$$(2.26) \quad \frac{1}{(2\pi)^n \ell^{3/2}} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-iy \cdot \eta} \eta_i \eta_j \eta_k \varphi^m\left(\frac{\eta}{\sqrt{\ell}}\right) d\eta \right| dy.$$

Therefore, according to Lemma 4, there is a positive constant such that

$$(2.27) \quad \sum_{\ell=1}^m \left\| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} K^{(\ell)} \right\| \leq c \sum_{\ell=1}^m \frac{1}{\frac{3}{\ell^2}} < +\infty.$$

By hypothesis, the functions f_{ij} $i, j = 1 \dots n$ are bounded, hence it yields the result of the proposition. $\square$

3. Bounds for the solution of the scheme. In this section we shall prove that u_m , solutions of (1.8), have an L^∞ bound uniformly in $m \in \mathbb{N}$.

As a first step let us state the following proposition.

Proposition 2. *There is a constant $c > 0$ that does not depend on $m \in \mathbb{N}$ such that*

$$(3.1) \quad \|u_{m+1} - u_m\|_{L^\infty} \leq c.$$

where u_m , $m \in \mathbb{N}$ are solutions of (1.8)

Proof. In order to prove the result, notice that it holds

$$(3.2) \quad u_{m+1}(x) - u_m(x) = \int_{\mathbb{R}^n} K(y)(u_m(x-y) - u_m(x)) dy \\ + \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2}{\partial y_i \partial y_j} K(y) f_{ij}(u_m(x-y), x-y) dy$$

From Proposition 1, there is a constant $c_1 > 0$ such that

$$(3.3) \quad |u_m(x-y) - u_m(x)| \leq c_1 |y|.$$

Using Hölder's inequality, we can bound (3.2) as

$$(3.4) \quad |u_{m+1}(x) - u_m(x)| \\ \leq c_1 \int_{\mathbb{R}^n} |K(y)| dy + \max_{i,j=1 \dots n} \|f_{ij}\|_{L^\infty} \sum_{i,j=1}^n \left\| \frac{\partial^2}{\partial y_i \partial y_j} K \right\|_{L^1}.$$

By hypothesis f_{ij} $i, j = 1 \dots n$ are bounded and $K \in C_c^3(\mathbb{R}^n)$. Hence, we have obtained the result of the proposition. $\square$

Notice that for the proof of the previous result it would be enough to have K with exponential decay at infinity.

Now, we can prove the main result of the paper.

Proposition 3. *There is a constant $c > 0$ that does not depend on $m \in \mathbb{N}$ such that*

$$(3.5) \quad \|u_m\|_{L^\infty} \leq c$$

where u_m , $m \in \mathbb{N}$ are solutions of (1.8)

Proof. Let us consider M such that

$$(3.6) \quad M > 1 + c_1 a + c_2 + \|u_0\|_{L^\infty},$$

where c_1 is the L^∞ bound of ∇u_m obtained in Proposition 1, c_2 is the L^∞ bound of $(u_{m+1}(x) - u_m(x))$ obtained in Proposition 2 and a is a constant such that $\text{supp } K \subset B_a(0)$.

Now, suppose that u_m , $m \in \mathbb{N}$ does not have an L^∞ bound, uniformly in $m \in \mathbb{N}$. Therefore, there exists $m_0 \in \mathbb{N}$ and $\bar{x} \in \mathbb{R}^n$ such that

$$(3.7) \quad |u_{m_0}(\bar{x})| \geq M \quad \text{and} \quad \sup_{m \leq m_0-1} \|u_m\|_{L^\infty} < M.$$

From the definition of u_{m_0} we have

$$(3.8) \quad |u_{m_0}(\bar{x})| \leq |K_* u_{m_0-1}(\bar{x})| + \left| \left(K_* \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(u_{m_0-1}, \cdot) \right)_{(\bar{x})} \right|.$$

From (1.2), (3.7) and using Hölder's inequality, it follows that

$$(3.9) \quad |K_* u_{m_0-1}(\bar{x})| \leq \|K\|_{L^1} \|u_{m_0-1}\|_{L^\infty} < M.$$

From Proposition 1 we obtain

$$(3.10) \quad |u_{m_0-1}(\bar{x}) - u_{m_0-1}(x)| \leq c_1 |x - \bar{x}|, \quad \forall x \in \mathbb{R}^n.$$

From Proposition 2 we have

$$(3.11) \quad |u_{m_0}(\bar{x}) - u_{m_0-1}(\bar{x})| \leq c_2, \quad \forall x \in \mathbb{R}^n.$$

Hence, we get

$$(3.12) \quad |u_{m_0-1}(x)| \geq |u_{m_0-1}(\bar{x})| - c_2 - c_1 |x - \bar{x}| \geq |u_{m_0}(\bar{x})| |x - \bar{x}|, \quad \forall x \in \mathbb{R}^n.$$

From (3.6) and (3.7) we have

$$(3.13) \quad |u_{m_0-1}(\bar{x})| - c_2 - c_1 |x - \bar{x}| \geq |u_{m_0}(\bar{x})| |x - \bar{x}| > 1, \quad \forall x \in B_a(\bar{x}).$$

This combined with the assumption (1.4) for f_{ij} yields that

$$(3.14) \quad \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(u_{m_0-1}(x), x) \equiv 0, \quad \forall x \in B_a(\bar{x}), \quad i, j = 1 \dots n.$$

Now, since $\text{supp } K \subset B_a(0)$ we have that

$$(3.15) \quad \left(K_* \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(u_{m_0-1}, \cdot) \right)(\bar{x}) = 0$$

In view of (3.9) and (3.15), we have obtained $|u_{m_0}(\bar{x})| < M$, which is a contradiction with (3.7). $\square$

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