The Stability of Small Stationary Solutions in Morrey Spaces of the Navier-Stokes Equation

HIDEO KOZONO & MASAO YAMAZAKI

*Dedicated to Professor Hikosaburo Komatsu
on the occasion of his sixtieth birthday*

ABSTRACT. The unique existence of small stationary solutions of the Navier-Stokes equations in $\mathbb{R}^n$ belonging to suitable Morrey spaces is proved under appropriate assumptions on the external force in the case $n \geq 3$. Also verified is the stability of the stationary solutions above, which are small enough.

1. Introduction. Recently, many authors studied the Cauchy problem for the Navier-Stokes equation in $\mathbb{R}^n$ in the framework of Morrey spaces. For example, Cottet [6], Giga, Miyakawa and Osada [11], Giga and Miyakawa [10] and Kato [15] gave sufficient conditions for the unique existence of time-global solutions. Time-local solutions are discussed in Taylor [26] and Federbush [7] as well as [10] and [15]. Time-global solutions to the Navier-Stokes equations on compact manifolds are also obtained in [26]. For previous papers related to this problem, see the references of Kozono and Yamazaki [22], which studied the Cauchy problem above in new function spaces larger than the corresponding Morrey spaces. These papers treated solutions which do not necessarily tend to 0 as $|x| \rightarrow \infty$. However, these papers treated only the case where the external force vanishes identically or decays as $t \rightarrow \infty$.

The purpose of this paper is to generalize the results on the global unique solvability in Kato [15] and Kozono and Yamazaki [22] to the case with stationary external forces. Namely, we show the unique existence and the stability of small stationary solutions in suitable Morrey spaces under appropriate assumptions on the external forces.

More precisely, we consider the following stationary Navier-Stokes equation with an external force $f(x)$ in $\mathbb{R}^n$ for $n \geq 3$:

$$(1.1) \quad -\Delta_x w(x) + (w(x) \cdot \nabla_x) w(x) + \nabla_x \pi(x) = f(x),$$

$$(1.2) \quad \nabla_x \cdot w(x) = 0,$$

and show the unique existence of a small solution of (1.1)–(1.2) in suitable Morrey spaces under appropriate assumptions on $f(x)$.

We also verify the stability of the stationary solution in the same Morrey space above by showing the time-global unique solvability and the bound of the solution of the following nonstationary Navier-Stokes equation in $(0, \infty) \times \mathbb{R}_x^n$ with the same external force as above:

$$(1.3) \quad \frac{\partial v}{\partial t}(t, x) - \Delta_x v(t, x) + (v(t, x) \cdot \nabla_x) v(t, x) + \nabla_x q(t, x) = f(x),$$

$$(1.4) \quad \nabla_x \cdot v(t, x) = 0,$$

$$(1.5) \quad v(0, x) = a(x) \quad \text{on } \mathbb{R}_x^n,$$

for the Cauchy data $a(x)$ close enough to the stationary solution. We also obtain the decay rate of the solution of (1.3)–(1.5) in related function spaces, which implies the asymptotic stability of the above stationary solutions in other topologies.

Furthermore, we can take initial values in suitable function spaces introduced by [22] as Besov-type spaces based on Morrey spaces, motivated by the works on the Navier-Stokes equations in the framework of Besov spaces by Grubb [12] and Kobayashi-Muramatsu [18]. For other works on the Navier-Stokes equation in function spaces larger than the standard L^p -spaces, see Cannone and Planchon [4], Cannone [3] for the homogeneous Besov spaces, and Kato and Ponce [17] for the Sobolev spaces of negative order. The spaces introduced by [22] are strictly larger than the corresponding Morrey spaces, and contain distributions other than Radon measures. Part of the results of this paper is announced in [23].

In order to state our results more precisely, let us recall the definition of the Morrey spaces. Let p and q be real numbers such that $1 \leq q \leq p$. Then the Morrey space $\mathcal{M}_{p,q}$ on $\mathbb{R}^n$ is defined to be the set of functions $u(x) \in L^q_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{M}_{p,q}} = \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{n/p-n/q} \left(\int_{|x-x_0|<R} |u(x)|^q dx \right)^{1/q} < \infty.$$

Remark 1.1.

- (1) If $r < q \leq p$, then we have $\mathcal{M}_{p,q} \subset \mathcal{M}_{p,r}$.

- (2) In the case $p = q$, the above space coincides with the usual Lebesgue space L^p .
- (3) If $q < p$, then the Lorentz space (weak- L^p space) $L^{p,\infty}$ is contained in $\mathcal{M}_{p,q}$.
- (4) The space above is denoted by $\mathcal{M}_q^p$ in Taylor [26] and Kozono and Yamazaki [22]. Different notations are employed in other papers such as Peetre [25] and Kato [15].
- (5) The space above is the homogeneous version of the Morrey spaces, as opposed to the uniformly local version, which Taylor [26] and Federbush [7] mainly treated. It is impossible to show the time-global solvability on local spaces since the heat semigroup fails to have the decay property on these spaces.

Next, for p and q as above and for every $s \in \mathbb{R}$, we define the space $\mathcal{M}_{p,q}^s$, the Sobolev-type space based on the Morrey space $\mathcal{M}_{p,q}$, by the formula

$$\mathcal{M}_{p,q}^s = \left\{ u(x) \in \mathcal{S}'/\mathcal{P} \mid \|u\|_{\mathcal{M}_{p,q}^s} = \|(-\Delta_x)^{s/2} u\|_{\mathcal{M}_{p,q}} < \infty \right\},$$

where $\mathcal{S}'$ and $\mathcal{P}$ denote the set of tempered distributions on $\mathbb{R}^n$ and the set of polynomials with n variables, respectively.

If $s < n/p$, then [22] showed, by way of the function spaces $\mathcal{N}_{p,q,\infty}^s$ explained later, that every class of $\mathcal{M}_{p,q}^s$ has a canonical representative in $\mathcal{S}'$, in the same way as in Bourdaud [2]. Hence, $\mathcal{M}_{p,q}^s$ can be regarded as a subspace of $\mathcal{S}'$. We hereafter consider these spaces only with $s < n/p$, and regard them as subspaces of $\mathcal{S}'$.

Then we can state our result on the unique existence of small solutions of (1.1)–(1.2), as follows:

Theorem 1. *Suppose that r satisfies $2 < r \leq n$. Then there exist a positive number δ_0 and a continuous, strictly monotone-increasing function $\omega(\delta)$ on $[0, \delta_0]$ satisfying $\omega(0) = 0$ such that the following hold:*

- (1) *For every $f(x) \in (\mathcal{D}')^n$, there exists at most one solution $w(x)$ of (1.1)–(1.2) in $\mathcal{M}_{n,r}$ satisfying the condition $\|w\|_{\mathcal{M}_{n,r}} < \omega(\delta_0)$.*
- (2) *For every $f(x) \in (\mathcal{M}_{n,r}^{-2})^n$ satisfying $\delta = \|f\|_{\mathcal{M}_{n,r}^{-2}} < \delta_0$, there exists a solution $w(x) \in (\mathcal{M}_{n,r})^n$ of (1.1)–(1.2) such that $\|w\|_{\mathcal{M}_{n,r}} \leq \omega(\delta)$.*

Remark 1.2. In particular, we obtain a sufficient condition on the existence of a small stationary solution in the standard space L^n with external force ∇F with F small in $L^{n/2}$, by choosing $r = n$ in the theorem above and applying the Sobolev imbedding theorem $\dot{H}_{n/2}^{-1} \subset \dot{H}_n^{-2}$.

This corresponds to Theorem A of Kozono and Sohr [21]. Although only the case $n = 3$ is treated there, it is easy to see that the argument works for general $n \geq 3$, since the stationary Stokes equation is uniquely solvable in the space $\dot{H}_{n/2}^1(\mathbb{R}^n)$ for $n \geq 3$.

Theorem 9.7 of Galdi [8], Chap. IX asserts that, in the special case $f \equiv 0$, the condition $u \in L^n$ is sufficient to imply $u \equiv 0$, and no smallness condition is necessary.

The stationary problem (1.1)–(1.2) is widely studied in the class of finite Dirichlet integral, which coincides with the class $\dot{H}_2^1$ in the case of the whole space. (See Galdi [8], Chapter IX and the references therein.) The condition $u \in L^n$ has something to do with the uniqueness property in this class. Namely, Theorem 4 of Kozono and Sohr [20] asserts that, if a solution belongs to $\dot{H}_2^1 \cap L^n$ and its L^n -norm is sufficiently small, then every solution in $\dot{H}_2^1$ satisfying the energy inequality must coincide with the solution above.

Remark 1.3. In view of Remark 1.2, (3), we can choose homogeneous functions of degree -3 as f , and then the stationary solution w becomes homogeneous of degree -1 in view of the uniqueness.

We can also treat solutions $w(x)$ which does not necessarily satisfy $w(x) \rightarrow 0$ as $|x| \rightarrow \infty$, as can be seen in the following example.

Example 1.4. Suppose that $n \geq 4$, and let r be an integer satisfying $2 < r < n$. Next, let $\bar{w} = (w_1(x_1, \dots, x_r), \dots, w_r(x_1, \dots, x_r))$ be a sufficiently small function in $L^r(\mathbb{R}^r)$ satisfying (1.2) on $\mathbb{R}^r$, and put $\bar{f} = -\Delta_x \bar{w} + \nabla_x(\bar{w} \otimes \bar{w})$ on $\mathbb{R}^r$. Then $w = (\bar{w}, 0)$ and $f = (\bar{f}, 0)$ satisfies (1.1)–(1.2) on $\mathbb{R}^n$. Moreover, we have $w \in \mathcal{M}_{n,r}$ and $f \in \mathcal{M}_{n,r}^{-2}$ on $\mathbb{R}^n$.

Example 1.5. Let n and r be the same as in the previous example, and let $w(x_1, \dots, x_r)$ be an arbitrary function on $\mathbb{R}^r$, homogeneous of degree -1 , satisfying $\nabla w = 0$ on $\mathbb{R}^r$. Then, in view of Remark 1.1, (3), for every q such that $2 < qr$ we have $w \in \mathcal{M}_{r,q}$ on $\mathbb{R}^r$. Then we have $w \in \mathcal{M}_{n,q}$ on $\mathbb{R}^n$ as in the previous example. In this case the external force $f(x)$ depends only on $x_1, \dots, x_r$, and is homogeneous of degree -3 with respect to these variables.

Our result on the stability of the solution above consists of the following theorem.

Theorem 2. *Let r be the same as in Theorem 1, and suppose that p, q and σ_0 satisfy $n/2 < p < \infty$, $1 < q \leq pr/n$ and $n/2p < \sigma_0 < \min\{1, n/p\}$. Further, let $w(x)$ be the solution of (1.1)–(1.2) given in Theorem 1, (2). Then*

there exists a positive number $\delta_1 \leq \delta_0$ such that, for every $f(x) \in (\mathcal{M}_{n,r}^{-2})^n$ satisfying $\|f\|_{\mathcal{M}_{n,r}^{-2}} < \delta_1$, there exist positive numbers ε_0 and M_0 such that, for every $a(x) \in (\mathcal{M}_{p,q}^{n/p-1})^n$ satisfying $\nabla_x \cdot a(x) = 0$ and $\varepsilon = \|a(x) - w(x)\|_{\mathcal{M}_{p,q}^{n/p-1}} < \varepsilon_0$, there uniquely exists a time-global solution $v(t, x)$ of (1.3)–(1.4) satisfying the conditions

$$(1.6) \quad \sup_{0 < t \leq T'} t^{1/2-n/4p} \|v(t, \cdot) - w\|_{\mathcal{M}_{p,q}^{n/2p}} < \infty$$

for every $T' > 0$,

$$(1.7) \quad \limsup_{t \rightarrow 0} t^{1/2-n/4p} \|v(t, \cdot) - w\|_{\mathcal{M}_{p,q}^{n/2p}} < M_0,$$

and the initial condition (1.5) in the following sense: For every s such that $-1 \leq s \leq n/p - 1$ and for every $T' > 0$, we have

$$(1.8) \quad \sup_{0 < t \leq T'} t^{s/2+1/2-n/2p} \|v(t, \cdot) - a\|_{\mathcal{M}_{p,q}^s} < \infty.$$

Moreover, for every T such that $0 < T \leq \infty$, any solution of (1.3)–(1.4) on $(0, T) \times \mathbb{R}^n$ satisfying (1.6) for every $T' \in (0, T)$, (1.7) and $v(t, \cdot) - a \rightarrow 0$ in $\mathcal{M}_{p,q}^{-1}$ coincides with the restriction on $(0, T) \times \mathbb{R}^n$ of the solution above.

Furthermore, for every σ such that $n/p - 1 \leq \sigma \leq \sigma_0$, there exists a continuous, strictly monotone-increasing function $\psi_\sigma(\varepsilon)$ on $[0, \varepsilon_0]$ satisfying $\psi_\sigma(0) = 0$ such that the estimate

$$(1.9) \quad \sup_{t > 0} t^{\sigma/2+1/2-n/2p} \|v(t, \cdot) - w\|_{\mathcal{M}_{p,q}^\sigma} \leq \psi_\sigma(\varepsilon)$$

holds if $\varepsilon < \varepsilon_0$.

Remark 1.6. If $p \geq n$, Lemmas 2.3 and 2.1 imply the inclusion relation $\mathcal{M}_{n,r} \subset \mathcal{M}_{p,pr/n}^{n/p-1} \subset \mathcal{M}_{p,q}^{n/p-1}$. Hence, the estimate (1.9) with $\sigma = n/p - 1$ in Theorem 2, together with the fact $\lim_{\varepsilon \rightarrow +0} \psi_{n/p-1}(\varepsilon) = 0$, asserts the Lyapunov stability of the stationary solution $w(x)$ in the topology of $\mathcal{M}_{p,q}^{n/p-1}$. In particular, we can put $p = n$ and $q = r$ in Theorem 2, and in this case the fact above implies the Lyapunov stability of $w(x)$ in the space $\mathcal{M}_{n,r}$ itself.

The estimate (1.9) with $\sigma > n/p - 1$ asserts the asymptotic stability of $w(x)$ in different topologies; more precisely, the rate of the convergence in $\mathcal{M}_{p,q}^\sigma$.

Remark 1.7. In the standard L^p -setting with $f \equiv 0$, Kato [14] proved the unique existence of a time-global solution for small initial data. Since the

Laplacian in L^p is densely defined, the upper limit like the left-hand side of (1.7) becomes 0 for the solutions above.

Remark 1.8. In this theorem, the solution $v(t, x)$ may not be strongly continuous at $t = 0$ in $\mathcal{M}_{p,q}^{n/p-1}$. Hence, in order to ensure the uniqueness, we must assume a condition on some sort of smallness near $t = 0$ like (1.7). A condition of this type was employed, in the case $f \equiv 0$, by Giga, Miyakawa and Osada [11] and Kato [16] in order to ensure uniqueness. In Giga and Miyakawa [10], Taylor [26] and Kato [15], uniqueness is proved under a somewhat stronger assumption like

$$\sup_{t>0} t^{1/2-n/4p} \|v(t, \cdot) - w\|_{\mathcal{M}_{p,q}^{n/2p}} < M,$$

although this assumption may not be explicitly stated in some of the papers above.

Even in the case $f \equiv 0$, it is still unknown whether conditions of this kind are necessary to ensure the uniqueness. (See Kato [16].) However, for a class of semilinear heat equations, conditions like (1.7) are really necessary in order to ensure uniqueness. We can see this fact as follows: Haraux and Weissler [13] constructed a forward self-similar solution of a semilinear heat equation in $(0, \infty) \times \mathbb{R}^n$ such that $u(t, \cdot) \rightarrow 0$ holds as $t \rightarrow +0$ strongly in L^p for some values of p . For the critical exponent p_0 such that the smallness of L^{p_0} -norm of the initial value implies the global solvability, the L^{p_0} -norm of $u(t, \cdot)$ is constant, but $u(t, \cdot)$ converges to 0 weakly in L^{p_0} .

Remark 1.9. In the case $f \equiv 0$, Giga and Kambe [9] and Giga and Miyakawa [10] considered self-similar solutions in the case $n = 2$ and $n = 3$ respectively. In particular, they showed that the asymptotic behavior of solutions are described by forward self-similar solutions in some cases. These studies are developed further by Carpio [5].

If the external force $f(x)$ is homogeneous of degree -3 , we can construct self-similar solution as follows: Suppose further that the initial data $a(x)$ is also homogeneous of degree -1 , and put $v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x)$ and $q_\lambda(t, x) = \lambda^2 q(\lambda^2 t, \lambda x)$. Then we have:

$$\begin{aligned} & \frac{\partial v_\lambda}{\partial t}(t, x) - \Delta_x v_\lambda(t, x) + (v_\lambda(t, x) \cdot \nabla_x) v_\lambda(t, x) + \nabla_x q_\lambda(t, x) \\ &= \lambda^3 \left\{ \frac{\partial v}{\partial t}(\lambda^2 t, \lambda x) - \Delta_x v(\lambda^2 t, \lambda x) + (v(\lambda^2 t, \lambda x) \cdot \nabla_x) v(\lambda^2 t, \lambda x) + \nabla_x q(\lambda^2 t, \lambda x) \right\} \\ &= \lambda^3 f(\lambda x) = f(x) \end{aligned}$$

and

$$\lim_{t \rightarrow +0} v_\lambda(t, x) = \lim_{t \rightarrow +0} \lambda v(\lambda^2 t, \lambda x) = \lambda a(\lambda x) = a(x).$$

It follows from the uniqueness that $v_\lambda(t, x) = v(t, x)$; namely, the solution $v(t, x)$ becomes self-similar.

In view of Example 1.5, there also exists some solution independent of some spacial variables and self-similar with respect to other variables.

Example 1.10. Suppose that $1 \leq n_1 < n$, and put $x' = (x_1, \dots, x_{n_1})$ for $x = (x_1, \dots, x_n)$. Then the relation $|x'|^{-a} \in L^{n_1/a, \infty} \subset \mathcal{M}_{n_1/a, q}$ on $\mathbb{R}^{n_1}$ for every $a \in (0, n_1)$ and every $q \in (1, n_1/a)$. It follows that

$$(-\Delta_x)^{(n-n_1p)/2p} \delta(x') = c|x'|^{-n/p} \in \mathcal{M}_{p, q}$$

on $\mathbb{R}^n$ for every p and q such that $1 < q < n_1 p/n$.

In particular, we can take $a(x) = w(x) + \varepsilon(0, \dots, 0, \delta(x_1))$ in Theorem 2 provided the constant ε is sufficiently small, since $\delta(x_1) \in \mathcal{M}_{p, q}^{n/p-1}$ holds for every p and q such that $1 < q < p/n$. This case is treated by Kato [15], and leads to solution self-similar with respect to (t, x_1) if $f(x)$ is homogeneous of degree -3 with respect to x_1 and independent of $x_2, \dots, x_n$.

Example 1.11. In view of the Biot-Savard law, we can take

$$a(x) = w(x) + \varepsilon \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0, \dots, 0 \right)$$

in Theorem 2 provided the constant ε is sufficiently small, since

$$(-\Delta_x)^{-1/2} \delta(x_1, x_2) \in \mathcal{M}_{p, q}^{n/p-1}$$

holds for every p and q such that $1 < q < 2p/n$. This case is treated by Giga and Miyakawa [10], and leads to solution self-similar with respect to (t, x_1, x_2) if $f(x)$ is homogeneous of degree -3 with respect to (x_1, x_2) and independent of $x_3, \dots, x_n$.

Furthermore, we can take initial values in the function spaces $\mathcal{N}_{p, q, \infty}^{n/p-1}$ introduced in [22]. Here, for p, q and s as above and $1 \leq r \leq \infty$, the space $\mathcal{N}_{p, q, r}^s$ is defined as the set of $u(x) \in \mathcal{S}'/\mathcal{P}$ such that

$$\|u\|_{\mathcal{N}_{p, q, r}^s} = \|\{2^{js} \|\mathcal{F}^{-1}[\varphi(2^{-j} \cdot) \mathcal{F}[u]]\|_{\mathcal{M}_{p, q}}\}_{j=-\infty}^{\infty}\|_{\ell^r} < \infty,$$

where $\{\varphi(2^{-j}\xi)\}_{j=-\infty}^{\infty}$ is a homogeneous Littlewood-Paley partition of unity. (See Bergh and Löfström [1], for example.) Then, as is observed in [22], the space $\mathcal{N}_{p,q,r}^s$ coincides with the real interpolation space $(\mathcal{M}_{p,q}^{s_0}, \mathcal{M}_{p,q}^{s_1})_{\vartheta,r}$, where $s_0 \neq s_1$, $0 < \vartheta < 1$ and $s = (1 - \vartheta)s_0 + \vartheta s_1$. (For the definition of the real interpolation space, see [1] for example.) This implies that that this space is independent of the choice of s_0 and s_1 and of the choice of the Littlewood-Paley decomposition, up to equivalent norms. It is shown in [22] that $\mathcal{N}_{p,q,1}^s \subset \mathcal{M}_{p,q}^s \subset \mathcal{N}_{p,q,\infty}^s$, and that the space $\mathcal{N}_{p,q,r}^s$ can be canonically regarded as a subspace of $\mathcal{S}'$ if $s < n/p$, in the same way as in Bourdaud [2]. In this paper we employ only their characterization by real interpolation.

Then our result on the stability under larger perturbation consists of the following theorem.

Theorem 3. *Let $p, q, r, \sigma_0, f(x)$, and $w(x)$ be the same as in Theorem 2. Then there exist positive numbers ε_1 and M_1 such that, for every $a(x) \in \left(\mathcal{N}_{p,q,\infty}^{n/p-1}\right)^n$ satisfying $\nabla_x \cdot a(x) = 0$ and $\varepsilon = \|a(x) - w(x)\|_{\mathcal{N}_{p,q,\infty}^{n/p-1}} < \varepsilon_1$, there uniquely exists a time-global solution $v(t, x)$ of (1.3)–(1.4) satisfying the conditions (1.6) for every $T' \in (0, \infty)$, (1.7), and (1.8) for every s such that $-1 \leq s < n/p - 1$ and every $T' > 0$.*

Moreover, for every T such that $0 < T \leq \infty$, any solution of (1.3)–(1.4) on $(0, T) \times \mathbb{R}^n$ satisfying (1.6) for every $T' \in (0, T)$, (1.7) and $v(t, \cdot) - a \rightarrow 0$ in $\mathcal{M}_{p,q}^{-1}$ coincides with the restriction on $(0, T) \times \mathbb{R}^n$ of the above solution.

Furthermore, there exists a continuous, strictly monotone-increasing function $\bar{\psi}(\varepsilon)$ on $[0, \varepsilon_1]$ satisfying $\bar{\psi}(0) = 0$ such that the estimate

$$(1.10) \quad \sup_{t>0} \|v(t, \cdot) - w\|_{\mathcal{N}_{p,q,\infty}^{n/p-1}} \leq \bar{\psi}(\varepsilon)$$

holds if $\varepsilon < \varepsilon_1$, and for every σ such that $n/p - 1 < \sigma < \sigma_0$, there exists a continuous, strictly monotone-increasing function $\psi'_\sigma(\varepsilon)$ on $[0, \varepsilon_1]$ satisfying $\psi'_\sigma(0) = 0$ such that the estimate

$$(1.11) \quad \sup_{t>0} t^{\sigma/2+1/2-n/2p} \|v(t, \cdot) - w\|_{\mathcal{M}_{p,q}^\sigma} \leq \psi'_\sigma(\varepsilon)$$

holds if $\varepsilon < \varepsilon_1$.

Remark 1.12. Cannone and Planchon [4] and Cannone [3] considered the Cauchy problem in the case $n = 3$ with $f \equiv 0$, and applied the result to obtain self-similar solutions in the homogeneous Besov space $\dot{B}_{3,\infty}^0$ and $\dot{B}_{p,\infty}^{3/p-1}$, respectively. Here, in view of the fact $\mathcal{M}_{p,p} = L^p$, we have $\mathcal{M}_{p,p,r}^s = \dot{B}_{p,r}^s$. Namely,

our spaces coincide with homogeneous Besov spaces as particular examples, and are larger than them in general.

Example 1.13. It was shown in [22] that the distribution $\text{p.v.}(1/x_1)$ belongs to the space $\mathcal{N}_{p,p/n,\infty}^{n/p-1}$ if $p > n$, and $q = p/n$ enjoys the condition of Theorem 3 for every r . Hence, we can take

$$a = w + \varepsilon \left(0, \dots, 0, \text{p.v.} \frac{1}{x_1} \right)$$

in Theorem 3, provided the constant ε is sufficiently small. This example leads to the solution self-similar with respect to (t, x_1) if $f(x)$ is homogeneous of degree -3 with respect to x_1 and independent of $x_2, \dots, x_n$.

Remark 1.14. If $p \geq n$, Remark 1.6 and the property of the space $\mathcal{N}_{p,q,r}^s$ imply the inclusion relation $\mathcal{M}_{n,r} \subset \mathcal{M}_{p,q}^{n/p-1} \subset \mathcal{N}_{p,q,\infty}^{n/p-1}$. Hence the estimate (1.10) and the fact $\lim_{\varepsilon \rightarrow +0} \bar{\psi}(\varepsilon) = 0$ assert the Lyapunov stability of the stationary solution $w(x)$ in the topology of $\mathcal{N}_{p,q,\infty}^{n/p-1}$.

The estimate (1.11) asserts the asymptotic stability of $w(x)$ in different topologies; more precisely, the rate of the convergence in $\mathcal{M}_{p,q}^\sigma$.

Remark 1.15. The main result of [22] on the Navier-Stokes equation can be regarded as the stability of the stationary solution 0 of the equation (1.1)–(1.2) with the external force $f \equiv 0$. Hence Theorem 3 can be regarded as a generalization of the results above to the case with more general stationary solutions. As described in this paper, our function spaces are strictly larger than the ones considered in [10] and [15].

Theorem 1 is proved by virtue of the inverse function theorem. Contrary to the study in the usual L^p -space, we cannot exclude so far the possibility of the existence of a large solution of (1.1)–(1.2) other than the one given in Theorem 1, even in the case $f = 0$. Hence we can show only the uniqueness of small solutions.

In the proof of Theorem 2, we cannot put the external force on the right-hand side and rely on the estimate of the Stokes semigroup, since its decay property is not good enough to ensure the time-global solvability with stationary external forces. Instead, we make use of the perturbation of the Stokes semigroup on the spaces $\mathcal{M}_{p,q}^s$. For this purpose, we make use of the fractional power of the resolvent of the Laplacian. However, the fact that the Stokes operator is not densely defined in $\mathcal{M}_{p,q}^s$ makes it necessary to develop the theory of the perturbation of analytic semigroups once again. Moreover, the fact that the stationary solution $w(x)$ is not very smooth calls for a new method to estimate

the Neumann series representation of the resolvent. Once Theorem 2 is proved, we can prove Theorem 3 essentially in the same way, replacing the necessary estimates by new ones obtained by way of the real interpolation. See Komatsu [19] for the relation between semigroup theory and interpolation.

This paper is organized as follows. In Section 2 we recall some properties of the Morrey spaces, by quoting some results of Taylor [26], and we prove Theorem 1. We next consider a perturbation of the Stokes operator, and obtain its resolvent estimate in Section 3. The semigroup generated by the perturbed operator above is studied in Section 4. In Section 5 we formally rewrite the equation (1.3)–(1.5) into an integral equation, and justify the formal procedure. Then Theorem 2 is proved in Section 6. Finally, in Section 7 we prove Theorem 3, together with necessary estimates.

2. Proof of Theorem 1. We first recall some of the properties of the spaces $\mathcal{M}_{p,q}^s$ needed in this paper. For more detailed properties, see Peetre [25], Taylor [26] and Kozono and Yamazaki [22]. The following two lemmas immediately follow from Hölder's inequality.

Lemma 2.1. *If $1 \leq r < q \leq p < \infty$, we have $\mathcal{M}_{p,q} \subset \mathcal{M}_{p,r}$.*

Lemma 2.2. *Suppose that $q_1 \leq p_1 < \infty$, $q_2 \leq p_2 < \infty$ and $q = q_1 q_2 / (q_1 + q_2) \geq 1$. Then we have $p = p_1 p_2 / (p_1 + p_2) \geq q$, and the estimate $\|u_1 u_2 | \mathcal{M}_{p,q}\| \leq \|u_1 | \mathcal{M}_{p_1,q_1}\| \cdot \|u_2 | \mathcal{M}_{p_2,q_2}\|$ holds for every $u_1 \in \mathcal{M}_{p_1,q_1}$ and $u_2 \in \mathcal{M}_{p_2,q_2}$.*

We next have an imbedding theorem of Sobolev type, which immediately follows from Proposition 3.7 of Taylor [26].

Lemma 2.3. *If $s \in \mathbb{R}$, $1 < q \leq p < \infty$ and $0 < \vartheta < 1$, we have the inclusion relation $\mathcal{M}_{p,q}^s \subset \mathcal{M}_{p/\vartheta,q/\vartheta}^{s-(1-\vartheta)n/p}$.*

Further, we have the following mapping property of singular integrals, which is proved by Peetre [24]. See Proposition 3.3 of Taylor [26].

Lemma 2.4. *Suppose that $1 < q \leq p < \infty$ and $s, \sigma \in \mathbb{R}$. Let $P(\xi)$ be a $C^{[n/2]+1}$ -function on $\mathbb{R}^n \setminus \{0\}$ satisfying the estimate $|(\partial^{|\alpha|} P / \partial \xi^\alpha)(\xi)| \leq C |\xi|^{\sigma-|\alpha|}$ for every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq [n/2] + 1$ and every $\xi \in \mathbb{R}^n$ such that $\xi \neq 0$. Then the operator $P(D)$ is bounded from $\mathcal{M}_{p,q}^s$ to $\mathcal{M}_{p,q}^{s-\sigma}$.*

As an application of this lemma we have the following result on complex interpolation. For complex interpolation, see Bergh and Löfström [1], for example.

Lemma 2.5. Suppose that $1 < q \leq p < \infty$, $s_1, s_2 \in \mathbb{R}$ and $0 < \vartheta < 1$, and put $s = (1 - \vartheta)s_0 + \vartheta s_1$. Then the space $\mathcal{M}_{p,q}^s$ coincides with the complex interpolation space $[\mathcal{M}_{p,q}^{s_0}, \mathcal{M}_{p,q}^{s_1}]_{\vartheta}$.

This lemma immediately follows from the estimate

$$\|(-\Delta_x)^{\alpha+i\beta} |L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^{s-2\alpha})\| \leq C_s(1 + |\beta|)^{n/2+1},$$

which is a direct consequence of Lemma 2.4.

We now prove Theorem 1 by the inverse mapping theorem. For $w(x) \in \mathcal{M}_{n,r}$, put

$$F(w) = -P\Delta_x w(x) + P\nabla_x(w(x) \otimes w(x)),$$

where P denotes the projection onto the solenoidal vector fields. Then P is represented by an $n \times n$ -matrix of singular integrals. It follows that P is a projection in $(\mathcal{M}_{n,r}^s)^n$ for every $s \in \mathbb{R}$. Let $(\mathcal{M}_{n,r}^s)_\sigma$ denote the range of P on $(\mathcal{M}_{n,r}^s)^n$.

Then Lemmas 2.2 and 2.3 imply that

$$w(x) \otimes w(x) \in (\mathcal{M}_{n/2,r/2})^{n^2} \subset (\mathcal{M}_{n,r}^{-1})^{n^2}$$

and that there exists a constant C such that the estimate

$$\|w(x) \otimes w(x) | \mathcal{M}_{n,r}^{-1}\| \leq C \|w(x) | \mathcal{M}_{n,r}\|^2$$

holds. Hence, in view of Lemma 2.4, we see that the mapping F is continuous from $(\mathcal{M}_{n,r})_\sigma$ to $(\mathcal{M}_{n,r}^{-2})_\sigma$, where $(\mathcal{M}_{n,r})_\sigma = (\mathcal{M}_{n,r}^0)_\sigma$. Moreover, it is Fréchet differentiable at every point of $(\mathcal{M}_{n,r})_\sigma$, and its Fréchet derivative at w_0 is of the form

$$-P\Delta_x + P\nabla_x(w_0(x) \otimes \cdot) + P\nabla_x(\cdot \otimes w_0(x)).$$

Hence the Fréchet derivative at 0 coincides with the Stokes operator $A = -P\Delta_x$, which is an isomorphism from $(\mathcal{M}_{n,r})_\sigma$ to $(\mathcal{M}_{n,r}^{-2})_\sigma$. Hence the Banach inverse function theorem implies that, for sufficiently small positive constants γ_0 and δ_0 , the mapping F is injective on the set

$$U = \{w(x) \in (\mathcal{M}_{n,r})_\sigma \mid \|w | \mathcal{M}_{n,r}\| < \gamma_0\},$$

and that the image of U by F contains the set

$$V = \{f(x) \in (\mathcal{M}_{n,r}^{-2})_\sigma \mid \|f(x) | \mathcal{M}_{n,r}^{-2}\| < \delta_0\}.$$

Moreover, the mapping F^{-1} is continuous from V to U . This completes the proof of Theorem 1. $\square$

3. The resolvent of the perturbed operator. Let $w(x)$ be the solution of (1.1)–(1.2) given in Theorem 1, and put $u(x) = v(x) - w(x)$ and $b(x) = a(x) - w(x)$. Then the system (1.3)–(1.5) is transformed into the following system:

$$(3.1) \quad \begin{aligned} \frac{\partial u}{\partial t}(t, x) - \Delta_x u(t, x) + (w(x) \cdot \nabla_x)u(t, x) \\ + (u(t, x) \cdot \nabla_x)(w(x) + u(t, x)) + \nabla_x(q(t, x) - \pi(x)) = 0, \end{aligned}$$

$$(3.2) \quad \nabla_x u(t, x) = 0,$$

$$(3.3) \quad u(0, x) = b(x).$$

Applying P to (3.1) and observing (3.2), we obtain

$$(3.4) \quad \frac{\partial u}{\partial t}(t, x) + \mathcal{A}[u(t, \cdot)](x) + P\nabla_x(u(t, x) \otimes u(t, x)) = 0,$$

where

$$\begin{aligned} \mathcal{B}[f](x) &= P\nabla_x(w(x) \otimes f(x) + f(x) \otimes w(x)), \\ \mathcal{A}[f](x) &= -\Delta_x f(x) + \mathcal{B}[f](x). \end{aligned}$$

Further, the conditions (1.6)–(1.9) can be rewritten as

$$(3.5) \quad \sup_{0 < t \leq T'} t^{1/2-n/4p} \left\| u(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} < \infty,$$

$$(3.6) \quad \limsup_{t \rightarrow 0} t^{1/2-n/4p} \left\| u(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} < M_0,$$

$$(3.7) \quad \sup_{0 < t \leq T'} t^{s/2+1/2-n/2p} \left\| u(t, \cdot) - b \right\|_{\mathcal{M}_{p,q}^s} < \infty, \quad \text{and}$$

$$(3.8) \quad \sup_{t > 0} t^{\sigma/2+1/2-n/2p} \left\| u(t, \cdot) \right\|_{\mathcal{M}_{p,q}^\sigma} \leq \psi_\sigma(\varepsilon)$$

respectively. Hence it suffices to find a solution of (3.3)–(3.4) satisfying the conditions (3.5)–(3.8) for every $T' > 0$, and to show its uniqueness. In this section we show some estimates of the resolvent of the linear operator $\mathcal{A}$.

Lemma 3.1. *Suppose that the real numbers p, q and r satisfy the conditions $2 < r \leq n$ and $1 < q \leq pr/n$. Then, for every s such that $s \leq 1$ and that $n/pr < s < n/p$, the operators $\mathcal{A}$ and $\mathcal{B}$ are bounded from $\mathcal{M}_{p,q}^s$ to $\mathcal{M}_{p,q}^{s-2}$.*

Proof. In view of Lemma 2.4, it suffices to prove the conclusion for $\mathcal{B}$. Choose p_1, q_1, p_2 , and q_2 such that $n/p_1 = n/p - s$, $q_1 = p_1 q/p$, $p_2 = np_1/(n + p_1)$ and $q_2 = rq_1/(r + q_1)$. Since $0 \leq s < n/p$ and $q \leq p$, we have $q_1 \leq p_1 < \infty$. Further, since $r \leq n$, it follows that $q_2 \leq p_2 < n$. Finally, since $ps > n/r$, it follows that

$$\frac{1}{q_2} = \frac{1}{r} + \frac{1}{q_1} = \frac{1}{r} + \frac{p}{nq} \left(\frac{n}{p} - s \right) < \frac{1}{r} + \frac{1}{q} - \frac{1}{qr} = 1 - \left(1 - \frac{1}{r} \right) \left(1 - \frac{1}{q} \right) < 1,$$

which implies that $q_2 > 1$.

Let f be an element of $\mathcal{M}_{p,q}^s$. Then Lemma 2.3 implies that $f \in \mathcal{M}_{p_1,q_1}$. Next, Lemma 2.2 implies that $f \otimes w, w \otimes f \in \mathcal{M}_{p_2,q_2}$. Now Lemma 2.4 implies that $\mathcal{B}[f] \in \mathcal{M}_{p_2,q_2}^{-1}$. Since $s \leq 1$ and $n/p_2 - (1 - s) = n/p$, we can apply Lemma 2.3 again to conclude that $\mathcal{B}[f] \in \mathcal{M}_{p,q_3}^{s-2}$, where

$$\frac{1}{q_3} = \frac{p_2}{pq_2} = \frac{np_1}{p(n + p_1)} \left(\frac{1}{r} + \frac{p}{p_1 q} \right) = \frac{n}{n + p_1} \left(\frac{p_1}{pr} + \frac{1}{q} \right) \leq \frac{n}{n + p_1} \left(\frac{p_1}{qn} + \frac{1}{q} \right) = \frac{1}{q}$$

in view of the inequality $n/r \leq p/q$. Hence Lemma 2.1 implies that $\mathcal{B}[f] \in \mathcal{M}_{p,q}^{s-2}$. $\square$

We next verify some estimates for the resolvent of $\mathcal{A}$.

Proposition 3.2. *Let p, q , and r be the same as in Proposition 3.1, and let s and σ be real numbers such that*

$$(3.9) \quad s, \sigma \in \left(\frac{n}{pr} - 2, \frac{n}{p} \right) \quad \text{and} \quad |s - \sigma| \leq 2.$$

Then, for every $\varepsilon > 0$, we can take a positive number $\gamma_1 = \gamma_1(s, \sigma)$ such that, if $w(x) \in (\mathcal{M}_{n,r})_\sigma$ satisfies $\|w|_{\mathcal{M}_{n,r}}\| < \gamma_1$, then every $\lambda \in \mathbb{C} \setminus [0, +\infty)$ such that $|\arg \lambda| \geq \varepsilon$ belongs to the resolvent of the operator $\mathcal{A}$ in $\mathcal{M}_{p,q}^s$, and the estimate

$$(3.10) \quad \left\| (-\Delta_x)^{\sigma/2} (\lambda + \Delta_x)^{-1} \mathcal{B}(\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2} \Big| L(\mathcal{M}_{p,q}) \right\| \leq \frac{C|\lambda|^{(\sigma-s)/2-1}}{\gamma_1 - \|w|_{\mathcal{M}_{n,r}}\|}$$

holds with a positive constant C . Furthermore, if $\sigma \geq s$, the operator $(-\Delta_x)^{\sigma/2} (\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2}$ is extended to a bounded operator on $\mathcal{M}_{p,q}$, and enjoys the estimate

$$(3.11) \quad \left\| (-\Delta_x)^{\sigma/2} (\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2} \Big| L(\mathcal{M}_{p,q}) \right\| \leq \frac{C|\lambda|^{(\sigma-s)/2-1}}{\gamma_1 - \|w|_{\mathcal{M}_{n,r}}\|}$$

with a positive constant C .

For the proof of this proposition we make use of the next two lemmas.

Lemma 3.3. *Let p and q be real numbers such that $1 < q \leq p$. Then, for every positive number ε such that $\varepsilon < \pi/2$ and nonnegative numbers a and b such that $a \leq b$, there exists a positive constant $C = C_{n,\varepsilon,a,b}$ such that the estimate $\|(-\Delta_x)^a(\lambda + \Delta_x)^{-b}|L(\mathcal{M}_{p,q})\| \leq C|\lambda|^{a-b}$ holds for every $\lambda \in \mathbb{C} \setminus [0, +\infty)$ such that $|\arg \lambda| \geq \varepsilon$.*

This lemma follows immediately from Lemma 2.4.

Lemma 3.4. *Let p, q and r be the same as in Proposition 3.1, and suppose that $\vartheta \in (0, 1)$ satisfies $n/2pr < \vartheta < n/2p$. Then the operator $(-\Delta_x)^{\vartheta-1} \cdot \mathcal{B}(-\Delta_x)^{-\vartheta}$ is bounded on $\mathcal{M}_{p,q}$.*

Proof. In view of Lemma 2.4, there exists a constant C such that, for every $f(x) \in (\mathcal{M}_{p,q})^n$, the function $g = (-\Delta_x)^{-\vartheta} f$ satisfies the estimate $\|g|\mathcal{M}_{p,q}^{2\vartheta}\| \leq C\|f|\mathcal{M}_{p,q}\|$. We also have the estimate

$$\begin{aligned} & \|(-\Delta_x)^{\vartheta-1} \mathcal{B}(-\Delta_x)^{-\vartheta} f | \mathcal{M}_{p,q}\| \\ & \leq C \left\| (-\Delta_x)^{\vartheta-1} P \nabla_x (-\Delta_x)^{1/2-\vartheta} \right\| L(\mathcal{M}_{p,q}) \left\| w \otimes g + g \otimes w | \mathcal{M}_{p,q}^{2\vartheta-1} \right\| \\ & \leq C \left\| w \otimes g + g \otimes w | \mathcal{M}_{p,q}^{2\vartheta-1} \right\|. \end{aligned}$$

Hence it suffices to show

$$(3.12) \quad \left\| w \otimes g + g \otimes w | \mathcal{M}_{p,q}^{2\vartheta-1} \right\| \leq \|w|\mathcal{M}_{n,r}\| \cdot \|g|\mathcal{M}_{p',q'}^{2\vartheta}\|.$$

The assumption $\vartheta < n/2p$ implies $n/p - 2\vartheta > 0$. Define p' and q' by the formulas $n/p - 2\vartheta = n/p'$ and $q' = p'q/p$. Then Lemma 2.3 implies that $g \in \mathcal{M}_{p',q'}$ and

$$(3.13) \quad \|g|\mathcal{M}_{p',q'}\| \leq C\|g|\mathcal{M}_{p,q}^{2\vartheta}\|.$$

Next we choose p'' and q'' by the formula $1/p'' = 1/n + 1/p'$ and $1/q'' = 1/r + 1/q'$. Then, the assumption on ϑ implies

$$\frac{1}{q''} = \frac{1}{r} + \frac{p}{qp'} = \frac{1}{r} + \frac{p}{nq} \left(\frac{n}{p} - 2\vartheta \right) < \frac{1}{r} + \frac{1}{q} - \frac{1}{qr} = 1 - \left(1 - \frac{1}{r} \right) \left(1 - \frac{1}{q} \right) < 1.$$

Next, since $r \leq n$ and $q' \leq p'$, it follows that $q'' \leq p''$. These facts and Lemma 2.2 imply

$$(3.14) \quad \|w \otimes g + g \otimes w | \mathcal{M}_{p'',q''}\| \leq C\|w|\mathcal{M}_{n,r}\| \cdot \|g|\mathcal{M}_{p',q'}\|.$$

Finally, it follows from the inequalities $2\vartheta \leq 1$ and $q \leq pr/n$ that

$$\frac{n}{p''} - (1 - 2\vartheta) = \frac{n}{p'} + 2\vartheta = \frac{n}{p}$$

and

$$\frac{p''}{pq''} = \frac{p'n}{p' + n} \left(\frac{1}{pq'} + \frac{1}{pr} \right) \leq \frac{p'n}{p' + n} \left(\frac{1}{p'q} + \frac{1}{nq} \right) = \frac{1}{q}.$$

It follows from Lemmas 2.1 and 2.3 that

$$\begin{aligned} (3.15) \quad \|w \otimes g + g \otimes w\|_{\mathcal{M}_{p,q}^{2\vartheta-1}} &\leq C \|w \otimes g + g \otimes w\|_{\mathcal{M}_{p,pq'',p''}^{2\vartheta-1}} \\ &\leq C \|w \otimes g + g \otimes w\|_{\mathcal{M}_{p'',q'',q''}}. \end{aligned}$$

Now (3.12) is an immediate consequence of (3.13)–(3.15). $\square$

Proof of Proposition 3.2. In view of the inequalities $\max\{s, \sigma\}/2 < \min\{1, n/2p\}$, $\max\{s, \sigma\}/2 \leq 1 + \min\{s, \sigma\}/2$, and $n/2pr < 1 + \min\{s, \sigma\}/2$, we can take a number $\vartheta \in (0, 1)$ such that $\max\{s, \sigma\}/2 \leq \vartheta \leq 1 + \min\{s, \sigma\}/2$ and that $n/2pr < \vartheta < n/2p$. Then Lemma 3.4 implies that the operator

$$\mathcal{C} = (-\Delta_x)^{\vartheta-1} \mathcal{B} (-\Delta_x)^{-\vartheta}$$

is bounded on $\mathcal{M}_{p,q}$, and the estimate $\|\mathcal{C}\|_{L(\mathcal{M}_{p,q})} \leq C_1 \|w\|_{\mathcal{M}_{n,r}}$ holds with some constant C_1 .

On the other hand, the operator $(-\Delta_x)(\lambda + \Delta_x)^{-1}$ is bounded on $\mathcal{M}_{p,q}$ for every $\lambda \in \mathbb{C} \setminus [0, +\infty)$, and there exists a constant C_2 such that the estimate $\|(-\Delta_x)(\lambda + \Delta_x)^{-1}\|_{L(\mathcal{M}_{p,q})} \leq C_2$ holds for every $\lambda \in \mathbb{C} \setminus [0, +\infty)$ such that $|\arg \lambda| \geq \varepsilon$. In the same way, since $0 \leq \sigma/2 + 1 - \vartheta \leq 1$ and $0 \leq \vartheta - s/2 \leq 1$, the operators $(-\Delta_x)^{\sigma/2+1-\vartheta}(\lambda + \Delta_x)^{-1}$ and $(-\Delta_x)^{\vartheta-s/2}(\lambda + \Delta_x)^{-1}$ are bounded on $\mathcal{M}_{p,q}$ for every $\lambda \in \mathbb{C} \setminus [0, +\infty)$, and there exists a constant C_3 such that the estimates

$$\begin{aligned} \|(-\Delta_x)^{\sigma/2+1-\vartheta}(\lambda + \Delta_x)^{-1}\|_{L(\mathcal{M}_{p,q})} &\leq C_3 |\lambda|^{\sigma/2-\vartheta}, \\ \|(-\Delta_x)^{\vartheta-s/2}(\lambda + \Delta_x)^{-1}\|_{L(\mathcal{M}_{p,q})} &\leq C_3 |\lambda|^{\vartheta-1-s/2} \end{aligned}$$

hold for every $\lambda \in \mathbb{C} \setminus [0, +\infty)$ such that $|\arg \lambda| \geq \varepsilon$.

It follows that, if $\|w\|_{\mathcal{M}_{n,r}} < \gamma = 1/C_1 C_2$, the series

$$\begin{aligned} &(-\Delta_x)^{\sigma/2+1-\vartheta}(\lambda + \Delta_x)^{-1} \mathcal{C} \sum_{j=0}^{\infty} \{(\lambda + \Delta_x)^{-1} (-\Delta_x) \mathcal{C}\}^j (-\Delta_x)^{\vartheta-s/2}(\lambda + \Delta_x)^{-1} \\ &= (-\Delta_x)^{\sigma/2}(\lambda + \Delta_x)^{-1} \mathcal{B} \sum_{j=0}^{\infty} \{(\lambda + \Delta_x)^{-1} \mathcal{B}\}^j (-\Delta_x)^{-s/2}(\lambda + \Delta_x)^{-1} \end{aligned}$$

converges in $L(\mathcal{M}_{p,q})$, and the operator norm of the limit is dominated by $C_1^2 C_2^2 C_3^2 / (\gamma - \|w| \mathcal{M}_{n,r}\|)$. Moreover, the limit is

$$\begin{aligned} & (-\Delta_x)^{\sigma/2} (\lambda + \Delta_x)^{-1} \mathcal{B} \{1 - (\lambda + \Delta_x)^{-1} \mathcal{B}\}^{-1} (\lambda + \Delta_x)^{-1} (-\Delta_x)^{-s/2} \\ &= (-\Delta_x)^{\sigma/2} (\lambda + \Delta_x)^{-1} \mathcal{B} (\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2}. \end{aligned}$$

This completes the proof of (3.10).

In order to prove (3.11), observe the equality

$$\begin{aligned} & (-\Delta_x)^{\sigma/2} (\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2} \\ &= (-\Delta_x)^{\sigma/2} (\lambda + \Delta_x)^{-1} (-\Delta_x)^{-s/2} \\ &\quad + (-\Delta_x)^{\sigma/2} (\lambda + \Delta_x)^{-1} \mathcal{B} (\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2} \\ &= (-\Delta_x)^{(\sigma-s)/2} (\lambda + \Delta_x)^{-1} + (-\Delta_x)^{\sigma/2} (\lambda + \Delta_x)^{-1} \mathcal{B} (\lambda - \mathcal{A})^{-1} (-\Delta_x)^{-s/2}. \end{aligned}$$

The estimate (3.10) implies that the second term in the right-hand side enjoys the required estimate. Since $0 \leq \sigma - s \leq 2$, Lemma 3.3 implies that the first term also enjoys the estimate. These facts yield (3.11). $\square$

4. The semigroup generated by the perturbed operator. In this section we show some estimates of the semigroup $\exp(-t\mathcal{A})$ defined by the formula

$$\exp(-t\mathcal{A}) = \int_{\Gamma} \exp(-t\lambda) (\lambda - \mathcal{A})^{-1} d\lambda,$$

for $t \in \mathbb{C}$ such that $\Re t > 0$, where the contour Γ satisfies the condition

$$(4.1) \quad \Gamma \subset \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| > \varepsilon\} \text{ connects } \exp(-i\omega)\infty \text{ to } \exp(i\omega)\infty,$$

where $0 < \varepsilon < \omega < \pi/2 - |\arg t|$.

We start with the following estimate for this semigroup, which provides the key estimate of the linearized equation.

Theorem 4.1. *Let p , q , and r be the same as in Proposition 3.1, and suppose that s and σ satisfy (3.9). Suppose moreover that $w(x) \in (\mathcal{M}_{n,r})_{\sigma}$ satisfies $\|w| \mathcal{M}_{n,r}\| < \gamma_1(s, \sigma)$. Then there exists a positive constant C such that the following estimates hold for every $t > 0$:*

$$(4.2) \quad \left\| \exp(-t\mathcal{A}) \Big| L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^{\sigma}) \right\| \leq \frac{C t^{(s-\sigma)/2}}{\gamma_1 - \|w| \mathcal{M}_{n,r}\|} \quad \text{if } s \leq \sigma,$$

$$(4.3) \quad \left\| \exp(-t\mathcal{A}) - 1 \Big| L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^{\sigma}) \right\| \leq \frac{C t^{(s-\sigma)/2}}{\gamma_1 - \|w| \mathcal{M}_{n,r}\|} \quad \text{if } \sigma \leq s.$$

Proof. (4.2) immediately follows from (3.11) since

$$\|\exp(-t\mathcal{A})|L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^\sigma)\| = \|(-\Delta_x)^{\sigma/2} \exp(-t\mathcal{A})(-\Delta_x)^{-s/2}|L(\mathcal{M}_{p,q})\|.$$

We next prove (4.3). For this purpose we write

$$\begin{aligned} \exp(-t\mathcal{A}) - 1 &= \int_{\Gamma} \exp(-t\lambda)(\lambda - \mathcal{A})^{-1} d\lambda - 1 \\ &= \int_{\Gamma} \exp(-t\lambda)(\lambda + \Delta_x)^{-1} d\lambda - 1 \\ &\quad + \int_{\Gamma} \exp(-t\lambda)(\lambda + \Delta_x)^{-1} \mathcal{B}(\lambda - \mathcal{A})^{-1} d\lambda. \end{aligned}$$

It follows that

$$\begin{aligned} &\|\exp(-t\mathcal{A}) - 1|L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^\sigma)\| \\ &= \|(-\Delta_x)^{\sigma/2} \{\exp(-t\mathcal{A}) - 1\}(-\Delta_x)^{-s/2}|L(\mathcal{M}_{p,q})\| \\ &\leq \|(-\Delta_x)^{\sigma/2} \{\exp(t\Delta_x) - 1\}(-\Delta_x)^{-s/2}|L(\mathcal{M}_{p,q})\| \\ &\quad + \int_{\Gamma} \exp(t\lambda) \|(-\Delta_x)^{\sigma/2}(\lambda + \Delta_x)^{-1} \mathcal{B}(\lambda - \mathcal{A})^{-1}(-\Delta_x)^{-s/2}|L(\mathcal{M}_{p,q})\| d\lambda. \end{aligned}$$

The inequality $s - 2 \leq \sigma \leq s$ implies that the first term, which denotes the operator norm in $L(\mathcal{M}_{p,q})$ of the Fourier multiplier with symbol $|\xi|^{\sigma-s}(\exp(-t|\xi|^2) - 1)$, is dominated by $Ct^{(s-\sigma)/2}$ in view of Lemma 2.4. The estimate (3.10) shows that the second term is also dominated by $Ct^{(s-\sigma)/2}$. This completes the proof of (4.3). $\square$

We see from the theorem above, exactly in the same way as in the usual analytical semigroup theory, that $\exp(-t\mathcal{A})$ is independent of the choice of ω , and that the semigroup property $\exp(-(t+s)\mathcal{A}) = \exp(-t\mathcal{A}) \cdot \exp(-s\mathcal{A})$ holds for every t and s such that $\Re t, \Re s > 0$.

However, we can prove the strong continuity of $\exp(-t\mathcal{A})f$ at $t = 0$ and the strong differentiability of $\exp(-t\mathcal{A})f$ only for f with some restriction, since the domain of $\mathcal{A}$ considered in $\mathcal{M}_{p,q}^s$ is not dense in $\mathcal{M}_{p,q}^s$. Hence, we must prove these properties under appropriate assumption, as in the following proposition.

Proposition 4.2. *Let p , q , r , and γ be the same as in Theorem 4.1, and suppose that $n/pr < \sigma + 2 < s < \min\{1, n/p\}$. Suppose moreover that $w(x) \in (\mathcal{M}_{n,r})_\sigma$ satisfies the estimate*

$$\|w|_{\mathcal{M}_{n,r}}\| < \min\{\gamma_1(s, s-2), \gamma_1(s-2, \sigma+2)\}.$$

Then, for every $f(x) \in \mathcal{M}_{p,q}^s$, we have

$$\begin{aligned} \frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f &\in \mathcal{M}_{p,q}^\sigma \quad \text{for every } t > 0, \\ \lim_{t \rightarrow +0} \frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f &= 0 \quad \text{in the topology of } \mathcal{M}_{p,q}^\sigma. \end{aligned}$$

Proof. Let Γ be a contour satisfying (4.1). Then, putting $\mu = t\lambda$, we have

$$\begin{aligned} \frac{\exp(-t\mathcal{A})f - f}{t} &= \int_{\Gamma} \frac{\exp(-t\lambda)}{t} \{(\lambda - \mathcal{A})^{-1} - \lambda^{-1}\} f d\lambda \\ &= \int_{\Gamma} \frac{\exp(-t\lambda)}{t\lambda} (\lambda - \mathcal{A})^{-1} \mathcal{A}f d\lambda \\ &= \int_{\Gamma'} \frac{\exp(-\mu)}{\mu^2} \left(1 - \frac{t}{\mu} \mathcal{A}\right)^{-1} \mathcal{A}f d\mu, \end{aligned}$$

where the contour $\Gamma' = \{t\lambda \mid \lambda \in \Gamma\}$ also satisfies (4.1). Since

$$\frac{n}{pr} < \sigma + 2 < s < \frac{n}{p} \quad \text{and} \quad \|w|_{\mathcal{M}_{n,r}}\| < \gamma_1(s-2, \sigma+2),$$

it follows from Lemma 3.1 and Proposition 3.2 that

$$\begin{aligned} &\left\| \frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f \right\|_{\mathcal{M}_{p,q}^\sigma} \\ &\leq \int_{\Gamma'} \left\| \frac{t\mathcal{A} \exp(-\mu)}{\mu} \frac{1}{\mu^2} \left(1 - \frac{t}{\mu} \mathcal{A}\right)^{-1} \mathcal{A}f \right\|_{\mathcal{M}_{p,q}^\sigma} d\mu \\ &\leq \int_{\Gamma'} \|t\mathcal{A}\| L(\mathcal{M}_{p,q}^{\sigma+2}, \mathcal{M}_{p,q}^\sigma) \left\| \frac{\exp(-\mu)}{\mu^3} \right\| \left\| \left(1 - \frac{t}{\mu} \mathcal{A}\right)^{-1} \right\| L(\mathcal{M}_{p,q}^{s-2}, \mathcal{M}_{p,q}^{\sigma+2}) \\ &\quad \cdot \|\mathcal{A}f\|_{\mathcal{M}_{p,q}^{s-2}} d\mu \\ &\leq C \int_{\Gamma'} \frac{t \exp(-\mu)}{\mu^3} \left(\frac{t}{\mu}\right)^{(s-\sigma)/2-2} d\mu \|f\|_{\mathcal{M}_{p,q}^s} \\ &\leq C t^{(s-\sigma)/2-1} \|f\|_{\mathcal{M}_{p,q}^s} \int_{\Gamma'} \frac{\exp(-\mu)}{\mu^{(s-\sigma)/2+1}} d\mu. \end{aligned}$$

Moreover, the right-hand side of this formula tends to 0 as $t \rightarrow +0$. This completes the proof. $\square$

5. Equivalence between differential equations and integral equations. In this section we prove, by virtue of the estimates of the semigroup $\exp(-t\mathcal{A})$ established in the previous section, the equivalence of the original differential equation and the associated integral equation.

We first determine the positive constant δ_1 . Let σ_0 be a constant in Theorem 2. Next, since $n/p(3/2 - q) < n/2p$, there exist real numbers σ_1 and σ_2 such that

$$\max \left\{ \frac{n}{p} \left(\frac{3}{2} - q \right), \frac{n}{pr} \right\} < \sigma_1 < \sigma_2 < \frac{n}{2p}.$$

Then we can take a positive constant $\delta_1 \leq \delta_0$ sufficiently small so that, for every $f \in (\mathcal{M}_{n,r}^{-2})$ such that $\|f\|_{\mathcal{M}_{n,r}^{-2}} < \delta_1$, the solution $w(x)$ of (1.1)–(1.2) given in Theorem 1 enjoys the estimate

$$\begin{aligned} \|w\|_{\mathcal{M}_{n,r}} &< \min \left\{ \gamma_1(\sigma_0 - 2, \sigma_0 - 2), \gamma_1(\sigma_0, \sigma_0 - 2), \gamma_1(\sigma_0 - 2, \sigma_0), \gamma_1(\sigma_0, \sigma_0), \right. \\ &\quad \gamma_1(\sigma_1 - 2, \sigma_1 - 2), \gamma_1(\sigma_1, \sigma_1 - 2), \gamma_1(\sigma_1 - 2, \sigma_1), \gamma_1(\sigma_1, \sigma_1), \\ &\quad \left. \gamma_1 \left(\frac{2n}{p}, \frac{2n}{p} - 2 \right), \gamma_1 \left(\frac{2n}{p} - 2, \sigma_2 \right) \right\}. \end{aligned}$$

In the sequel we assume that $f(x)$ and $w(x)$ are as above.

Then Theorem 4.1 and Lemma 2.5 imply the following proposition.

Proposition 5.1. *The estimate*

$$\|\exp(-t\mathcal{A})\|_{L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^\sigma)} \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ such that $\sigma_1 - 2 \leq s \leq \sigma \leq \sigma_0$ and that $\sigma \leq s + 2$, and the estimate

$$\|\exp(-t\mathcal{A}) - 1\|_{L(\mathcal{M}_{p,q}^s, \mathcal{M}_{p,q}^\sigma)} \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ such that $\sigma_1 - 2 \leq \sigma \leq s \leq \sigma_0$ and that $s \leq \sigma + 2$.

Then we prove the following theorem concerning the equivalence.

Theorem 5.2. *Let T be either a positive number or $+\infty$. Suppose that $b(x) \in \mathcal{M}_{p,q}^{n/p-1}$ and that $u(t, x)$ is a function on $(0, T) \times \mathbb{R}^n$ satisfying (3.5) for every $T' \in (0, T)$. Then the following three conditions on the function $u(t, x)$ are equivalent:*

- (1) *$u(t, x)$ satisfies the differential equation (3.4) on $(0, T)$, and the condition (3.7) for every $s \in [-1, p/n - 1]$ and every $T' \in (0, T)$.*

- (2) $u(t, x)$ satisfies the differential equation (3.4) on $(0, T)$, and we have $u(t, \cdot) - b \rightarrow 0$ in the topology of $\mathcal{M}_{p,q}^{-1}$ as $t \rightarrow +0$.
- (3) $u(t, x)$ satisfies the integral equation

$$(5.1) \quad u(t, \cdot) = \exp(-t\mathcal{A})b + \int_0^t \exp(-(t-\tau)\mathcal{A})P\nabla_x(u(\tau, \cdot) \otimes u(\tau, \cdot)) d\tau$$

on $(0, T)$.

For the proof of this theorem as well as Theorem 2, we first show the following lemma.

Lemma 5.3. *Let $u(t, x)$ be a function on $(0, T) \times \mathbb{R}^n$ satisfying (3.5) for every $T' \in (0, T)$ and (5.1) on $(0, T)$, and suppose that $b(x) \in \mathcal{M}_{p,q}^{n/p-1}$. Then, for every $s \in [-1, \sigma_0]$, there exists a constant C_s such that the inequalities*

$$(5.2) \quad \|u(t, \cdot) - \exp(-t\mathcal{A})b\|_{\mathcal{M}_{p,q}^s} \leq C_s M^2 t^{n/2p-s/2-1/2},$$

$$(5.3) \quad \|u(t, \cdot) - b\|_{\mathcal{M}_{p,q}^s} \leq C_s \left(M^2 + \|b\|_{\mathcal{M}_{p,q}^{n/p-1}} \right) t^{n/2p-s/2-1/2}$$

if $s \leq n/p - 1$,

$$(5.4) \quad \|u(t, \cdot)\|_{\mathcal{M}_{p,q}^s} \leq C_s \left(M^2 + \|b\|_{\mathcal{M}_{p,q}^{n/p-1}} \right) t^{n/2p-s/2-1/2}$$

if $s \geq n/p - 1$

hold for every $t \in (0, T)$, where $M = \sup_{0 < t < T} t^{1/2-n/4p} \|u(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}}$.

Proof. We first prove (5.2). Since $\sigma_1 < -1 \leq s \leq \sigma_0$, Proposition 5.1 implies that

$$(5.5) \quad \begin{aligned} & \|u(t, \cdot) - \exp(-t\mathcal{A})b\|_{\mathcal{M}_{p,q}^s} \\ & \leq \int_0^t \|\exp(-(t-\tau)\mathcal{A})P\nabla_x(u(\tau, \cdot) \otimes u(\tau, \cdot))\|_{\mathcal{M}_{p,q}^s} d\tau \\ & \leq C \int_0^t (t-\tau)^{(-s-1)/2} \|P\nabla_x(u(\tau, \cdot) \otimes u(\tau, \cdot))\|_{\mathcal{M}_{p,q}^{-1}} d\tau \\ & \leq C \int_0^t (t-\tau)^{-s/2-1/2} \|u(\tau, \cdot) \otimes u(\tau, \cdot)\|_{\mathcal{M}_{p,q}} d\tau \\ & \leq C \int_0^t (t-\tau)^{-s/2-1/2} \|u(\tau, \cdot)\|_{\mathcal{M}_{2p,2q}}^2 d\tau \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^t (t-\tau)^{-s/2-1/2} \left\| u(\tau, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}}^2 d\tau \\
&\leq C \int_0^t (t-\tau)^{-s/2-1/2} M^2 \tau^{n/2p-1} d\tau \\
&\leq t^{n/2p-s/2-1/2} C_s M^2,
\end{aligned}$$

from which we conclude (5.2).

If $s \leq n/p - 1$, we have $n/p - 1 < \sigma_0$ and $s - (n/p - 1) \geq -n/p > -2$. Hence Proposition 5.1 yields the inequality

$$\left\| (\exp(-t\mathcal{A}) - 1)b \right\|_{\mathcal{M}_{p,q}^s} \leq C t^{n/2p-s/2-1/2} \left\| b \right\|_{\mathcal{M}_{p,q}^{n/p-1}},$$

which together with (5.2) implies (5.3).

Finally, if $s \geq n/p - 1$, we have $\sigma_1 < n/p - 1$ and $s - (n/p - 1) < 1$. Hence Proposition 5.1 implies the inequality

$$\left\| \exp(-t\mathcal{A})b \right\|_{\mathcal{M}_{p,q}^s} \leq C t^{n/2p-s/2-1/2} \left\| b \right\|_{\mathcal{M}_{p,q}^{n/p-1}},$$

which together with (5.2) implies (5.4). $\square$

Proof of Theorem 5.2. The implication (1) $\Rightarrow$ (2) is trivial.

We next show the implication (3) $\Rightarrow$ (1). Suppose that $u(t, x)$ is a solution of (5.1). Then the estimate (3.7) for every $T' \in (0, T)$ follows from (5.3) for every $s \in [-1, n/p - 1]$.

We next verify that the function $u(t, \cdot)$ is Hölder continuous from $[\varepsilon, T']$ to $\mathcal{M}_{p,q}^{\sigma_2}$ for every ε and T' such that $0 < \varepsilon < T' < T$. Suppose that $\varepsilon \leq \tau < t \leq T'$. Then we have

$$\begin{aligned}
u(t, \cdot) - u(\tau, \cdot) &= \exp(-(t-\tau)\mathcal{A})u(\tau, \cdot) \\
&\quad + \int_{\tau}^t \exp(-(s-\tau)\mathcal{A}) P \nabla_x (u(s, \cdot) \otimes u(s, \cdot)) ds.
\end{aligned}$$

It follows from this equality and Proposition 5.1 that

$$\begin{aligned}
&\left\| u(t, \cdot) - u(\tau, \cdot) \right\|_{\mathcal{M}_{p,q}^{\sigma_2}} \\
&\leq \left\| (\exp(-(t-\tau)\mathcal{A}) - 1) \right\| L \left(\mathcal{M}_{p,q}^{n/2p}, \mathcal{M}_{p,q}^{\sigma_2} \right) \left\| u(\tau, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} \\
&\quad + \int_{\tau}^t \left\| \exp(-(s-\tau)\mathcal{A}) \right\| L \left(\mathcal{M}_{p,q}^{-1}, \mathcal{M}_{p,q}^{\sigma_2} \right) \left\| P \nabla_x (u(s, \cdot) \otimes u(s, \cdot)) \right\|_{\mathcal{M}_{p,q}^{-1}} ds
\end{aligned}$$

$$\begin{aligned}
&\leq C(t-\tau)^{n/4p-\sigma_2/2} M \tau^{n/4p-1/2} + C \int_{\tau}^t (s-\tau)^{-\sigma_2/2-1/2} \|u(s, \cdot) | \mathcal{M}_{2p,2q}\|^2 ds \\
&\leq CM(t-\tau)^{n/4p-\sigma_2/2} \tau^{n/4p-1/2} + C \int_{\tau}^t (s-\tau)^{-\sigma_2/2-1/2} M^2 s^{n/2p-1} ds \\
&\leq CM(t-\tau)^{n/4p-\sigma_2/2} \tau^{n/4p-1/2} + CM^2(t-\tau)^{(1-\sigma_2)/2} (t+\tau)^{n/2p-1},
\end{aligned}$$

which implies that the function $u(t, \cdot)$ is Hölder continuous from $[\varepsilon, T']$ to $\mathcal{M}_{p,q}^{\sigma_2}$.

Putting

$$\begin{aligned}
p' &= \frac{np}{n - \sigma_2 p}, \\
q' &= \frac{nq}{n - \sigma_2 p}, \\
p'' &= \frac{2pp'}{2p + p'}, \quad \text{and} \\
q'' &= \frac{2qq'}{2q + q'},
\end{aligned}$$

we have $q' \leq p'$, $q'' \leq p''$ and

$$\frac{1}{q''} = \frac{1}{2q} + \frac{1}{q'} = \frac{3}{2q} - \frac{\sigma_2 p}{qn} < 1.$$

It follows from the estimate

$$\begin{aligned}
&\|P\nabla_x(u(t, \cdot) \otimes u(t, \cdot) - u(\tau, \cdot) \otimes u(\tau, \cdot)) | \mathcal{M}_{p,q}^{\sigma_2-1}\| \\
&\leq \|u(t, \cdot) \otimes u(t, \cdot) - u(\tau, \cdot) \otimes u(\tau, \cdot) | \mathcal{M}_{p,q}^{\sigma_2}\| \\
&\leq \|u(t, \cdot) \otimes u(t, \cdot) - u(\tau, \cdot) \otimes u(\tau, \cdot) | \mathcal{M}_{p'',q''}\| \\
&\leq \|u(t, \cdot) - u(\tau, \cdot) | \mathcal{M}_{p',q'}\| (\|u(t, \cdot) | \mathcal{M}_{2p,2q}\| + \|u(\tau, \cdot) | \mathcal{M}_{2p,2q}\|) \\
&\leq \|u(t, \cdot) - u(\tau, \cdot) | \mathcal{M}_{p,q}^{\sigma_2}\| (\|u(t, \cdot) | \mathcal{M}_{p,q}^{n/2p}\| + \|u(\tau, \cdot) | \mathcal{M}_{p,q}^{n/2p}\|),
\end{aligned}$$

that the function $P\nabla_x(u(t, \cdot) \otimes u(t, \cdot))$ is Hölder continuous on $[\varepsilon, T']$ with values in $\mathcal{M}_{p,q}^{\sigma_2-1}$.

We now show that $u(t, x)$ satisfies (3.4). Let t_0 be an arbitrary point of (ε, T') , and let t_1, t_2 be points of (ε, T') such that $t_1 < t_2$. Then we have

$$\begin{aligned}
& \frac{u(t_2) - u(t_1)}{t_2 - t_1} \\
&= \frac{\exp(-(t_2 - t_1)\mathcal{A}) - 1}{t_2 - t_1} u(t_1) \\
&\quad + \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \exp(-(\tau - t_1)\mathcal{A}) P\nabla_x (u(\tau, \cdot) \otimes u(\tau, \cdot)) d\tau \\
&= \frac{\exp(-(t_2 - t_1)\mathcal{A}) - 1}{t_2 - t_1} u(t_0) + \frac{\exp(-(t_2 - t_1)\mathcal{A}) - 1}{t_2 - t_1} (u(t_1) - u(t_0)) \\
&\quad + \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \exp(-(\tau - t_1)\mathcal{A}) P\nabla_x (u(\tau, \cdot) \otimes u(\tau, \cdot) - u(t_0, \cdot) \otimes u(t_0, \cdot)) d\tau \\
&\quad + \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\exp(-(\tau - t_1)\mathcal{A}) - 1) P\nabla_x (u(t_0, \cdot) \otimes u(t_0, \cdot)) d\tau \\
&\quad + P\nabla_x (u(t_0, \cdot) \otimes u(t_0, \cdot)) \\
&= I_1 + I_2 + I_3 + I_4 + P\nabla_x (u(t_0, \cdot) \otimes u(t_0, \cdot)).
\end{aligned}$$

Now let $t_1, t_2 \rightarrow t_0$. Then Proposition 4.2 and the fact

$$\|w\|_{\mathcal{M}_{n,r}} < \min \left\{ \gamma_1 \left(\frac{2n}{p}, \frac{2n}{p} - 2 \right), \gamma_1 \left(\frac{2n}{p} - 2, \sigma_2 \right) \right\}$$

imply that I_1 tends to $-\mathcal{A}u(t_0)$ in $\mathcal{M}_{p,q}^{\sigma_2-2}$. Next, Proposition 5.1 implies that the operator

$$\frac{\exp(-t\mathcal{A}) - 1}{t}$$

is uniformly bounded in $L(\mathcal{M}_{p,q}^{\sigma_2}, \mathcal{M}_{p,q}^{\sigma_2-2})$. It follows from this fact and the Hölder continuity of $u(t, \cdot)$ in $\mathcal{M}_{p,q}^{\sigma_2}$ that I_2 tends to 0 in $\mathcal{M}_{p,q}^{\sigma_2-2}$. Further, Proposition 5.1 implies that the operator $\exp(-t\mathcal{A})$ is uniformly bounded in $L(\mathcal{M}_{p,q}^{\sigma_2-1}, \mathcal{M}_{p,q}^{\sigma_2-1})$. It follows from this fact and from the Hölder continuity of $P\nabla_x(u(t, \cdot) \otimes u(t, \cdot))$ in $\mathcal{M}_{p,q}^{\sigma_2-1}$ that I_3 tends to 0 in $\mathcal{M}_{p,q}^{\sigma_2-1}$. Finally, (4.4) implies that the operator

$$\frac{\exp(-t\mathcal{A}) - 1}{\sqrt{t}}$$

is uniformly bounded in $L(\mathcal{M}_{p,q}^{\sigma_2-1}, \mathcal{M}_{p,q}^{\sigma_2-2})$. Hence I_4 tends to 0 in $\mathcal{M}_{p,q}^{\sigma_2-2}$.

From these facts we see that the equality

$$\lim_{\substack{t_1, t_2 \rightarrow t_0 \\ t_1 < t_2}} \frac{u(t_2) - u(t_1)}{t_2 - t_1} = -\mathcal{A}u(t_0) + P\nabla_x(u(t_0, \cdot) \otimes u(t_0, \cdot))$$

holds in $\mathcal{S}'$. This implies that $u(t, x)$ enjoys (3.4) on (ε, T') . Since ε and T' are arbitrary, we see that $u(t, x)$ enjoys (3.4) on $(0, T)$.

We finally prove the implication (2) $\Rightarrow$ (3). Assume that $u(t, x)$ satisfies (3.5) for every $T' \in (0, T)$, (3.4) and $u(t, \cdot) - b \rightarrow 0$ in the topology of $\mathcal{M}_{p,q}^{-1}$ as $t \rightarrow +0$, and put

$$v(t, \cdot) = \int_0^t \exp(-(t-\tau)\mathcal{A}) P\nabla_x(u(\tau, \cdot) \otimes u(\tau, \cdot)) d\tau.$$

Then we see that $v(t, \cdot) \in L^\infty((0, T']; \mathcal{M}_{p,q}^{-1})$ holds for every $T' \in (0, T)$. Moreover, we see exactly in the same way as before that $v(t, x)$ enjoys the differential equation $dv/dt = -\mathcal{A}v + P\nabla_x(u(t, \cdot) \otimes u(t, \cdot))$ on $(0, T)$, and the condition $v(t, \cdot) \rightarrow 0$ in $\mathcal{M}_{p,q}^{-1}$ as $t \rightarrow +0$. Hence the function $w(t, x) = u(t, x) - v(t, x)$ enjoys the linear differential equation $dw/dt = -\mathcal{A}w$ on $(0, T)$ and the condition $w(t, \cdot) - b \rightarrow 0$ in $\mathcal{M}_{p,q}^{-1}$ as $t \rightarrow +0$. It follows that $w(t, \cdot) = \exp(-t\mathcal{A})b$, and hence $u(t, x) = w(t, x) + v(t, x)$ enjoys (5.1). This completes the proof of the implication (2) $\Rightarrow$ (3). $\square$

6. Proof of Theorem 2. In this section we construct the solution of the integral equation (5.1) on $(0, \infty)$ by successive approximation, by virtue of the estimates of the semigroup $\exp(-t\mathcal{A})$ established in Proposition 5.1. Then we show that this solution satisfies the required properties. We finally prove the uniqueness.

Define the sequence of functions $\{u_j(t, x)\}_{j=0}^\infty$ inductively by the formulas $u_0(t, \cdot) = \exp(-t\mathcal{A})b$ and

$$u_{j+1}(t, \cdot) = \exp(-t\mathcal{A})b + \int_0^t \exp(-(t-\tau)\mathcal{A}) P\nabla_x(u_j(\tau, \cdot) \otimes u_j(\tau, \cdot)) d\tau,$$

and put $X_j = \sup_{t>0} t^{1/2-n/4p} \|u_j(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}}$. Since $\sigma_1 < n/p - 1 < n/2p < \sigma_0$, Proposition 5.1 implies that the inequality $X_0 \leq C_1 \|b\|_{\mathcal{M}_{p,q}^{n/p-1}} = C_1 \varepsilon$ holds with a positive number C_1 . Next, since $n/pr - 2 < -1 < n/2p < n/p$, Proposition 5.1 implies the estimate

$$\begin{aligned} (6.1) \quad & \left\| u_{j+1}(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} \\ & \leq \left\| u_0(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} \\ & \quad + \int_0^t \left\| \exp(-(t-\tau)\mathcal{A}) P\nabla_x(u_j(\tau, \cdot) \otimes u_j(\tau, \cdot)) \right\|_{\mathcal{M}_{p,q}^{n/2p}} d\tau \end{aligned}$$

$$\begin{aligned}
&\leq C_1 \varepsilon t^{n/4p-1/2} + C \int_0^t (t-\tau)^{-n/4p-1/2} X_j^2 \tau^{n/2p-1} d\tau \\
&\leq t^{n/4p-1/2} (C_1 \varepsilon + C_2 X_j^2)
\end{aligned}$$

in the same way as in (5.5) with $s = n/2p$, where C_2 is another positive constant. This estimate implies the inequality $X_{j+1} \leq C_2 X_j^2 + C_1 \varepsilon$. From this inequality we see that, if $\varepsilon < \varepsilon_0 = 1/4C_1C_2$, we have

$$X_j \leq \psi(\varepsilon) = \frac{1 - \sqrt{1 - 4C_1C_2\varepsilon}}{2C_2} < M_0 = \psi(\varepsilon_0) = \frac{1}{2C_2}$$

for every $j \in \mathbb{N}$, by induction on j . Putting

$$Z_j = \sup_{t>0} t^{1/2-n/4p} \left\| u_{j+1}(t, \cdot) - u_j(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}},$$

we obtain exactly in the same way as in (6.1) that

$$\begin{aligned}
(6.2) \quad &\left\| u_{j+2}(t, \cdot) - u_{j+1}(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} \\
&\leq C \int_0^t (t-\tau)^{-n/4p-1/2} \left\| u_{j+1}(\tau, \cdot) \otimes (u_{j+1}(\tau, \cdot) - u_j(\tau, \cdot)) \right. \\
&\quad \left. + (u_{j+1}(\tau, \cdot) - u_j(\tau, \cdot)) \otimes u_j(\tau, \cdot) \right\|_{\mathcal{M}_{p,q}} d\tau \\
&\leq C_2 t^{n/4p-1/2} (X_{j+1} + X_j) Z_j.
\end{aligned}$$

It follows that $Z_{j+1} \leq C_2 (X_{j+1} + X_j) Z_j \leq 2C_2 \psi(\varepsilon) Z_j$. Since $2RC_2 < 1$, we conclude that $\sum_{j=0}^{\infty} Z_j < \infty$, which implies that the sequence $\{u_j(t, x)\}$ converges to a function $u(t, x)$ which satisfies

$$\sup_{t>0} t^{1/2-n/4p} \left\| u(t, \cdot) \right\|_{\mathcal{M}_{p,q}^{n/2p}} \leq \psi(\varepsilon) < M_0.$$

This implies that $u(t, x)$ enjoys (3.5) for every $T' \in (0, \infty)$ and (3.6). It is also easy to see that $u(t, x)$ enjoys the integral equation (5.1) on $(0, \infty)$. Hence Theorem 5.2 implies that $u(t, x)$ enjoys (3.4).

Further, the estimate (5.3) in Lemma 5.3 implies that $u(t, x)$ enjoys the estimate (3.7) for every $T' \in (0, \infty)$, and the estimate (5.4) implies that $u(t, x)$ enjoys the estimate (3.8) with $\psi_s(\varepsilon) = C_s(\varepsilon + \psi(\varepsilon)^2)$. Hence the function $u(t, x)$ enjoys all the required properties.

We next show the uniqueness. Let $\tilde{u}(t, x)$ be another solution of (3.4) on $(0, T) \times \mathbb{R}^n$ satisfying (3.5) for every $T' \in (0, T)$ and (3.6) such that $\tilde{u}(t, \cdot) - b \rightarrow 0$ in the topology of $\mathcal{M}_{p,q}^{-1}$ as $t \rightarrow +0$, and let T_0 be a positive number less than T such that $M = \sup_{0 < t \leq T_0} t^{1/2-n/4p} \|\tilde{u}(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}} < M_0$. Then Theorem 5.2 implies that $\tilde{u}(t, x)$ solves the integral equation (5.1) on $(0, T)$. Hence, putting $\bar{u}(t, x) = u(t, x) - \tilde{u}(t, x)$ and $Z = \sup_{0 < t \leq T_0} t^{1/2-n/4p} \|\bar{u}(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}}$, we obtain $Z \leq C_2(\psi(\varepsilon) + M)Z$ in the same way as in (6.2). Since $\psi(\varepsilon)$, $M < M_0 = 1/2C_2$, it follows that $Z = 0$. Hence we must have $\tilde{u}(t, x) \equiv u(t, x)$ on $(0, T_0] \times \mathbb{R}^n$.

Next, take $T' \in (0, T)$ arbitrarily, and put

$$M' = \sup_{0 < t \leq T'} t^{1/2-n/4p} \|\tilde{u}(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}}$$

and

$$Z(\tau) = \sup_{0 < t \leq \tau} t^{1/2-n/4p} \|\bar{u}(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}}$$

for every $\tau \in (0, T']$. Then we have

$$t^{n/4p-1/2} Z(t) \leq C \int_0^t (t-\tau)^{-n/4p-1/2} \tau^{n/2p-1} (\psi(\varepsilon) + M') Z(\tau) d\tau$$

in the same way as in (6.2). Now suppose that $Z(T_1) = 0$ holds with some $T_1 \geq T_0$. Then we have

$$(6.3) \quad Z(t) \leq C Z(t) t^{1/2-n/4p} \int_{T_1}^t (t-\tau)^{-n/4p-1/2} \tau^{n/2p-1} (\psi(\varepsilon) + M') d\tau.$$

Choose a positive number δ such that

$$0 < \delta < \frac{T_0}{2} \min \left\{ 1, \left(\frac{2p-n}{4pC(\psi(\varepsilon) + M')} \right)^{4p/(2p-n)} \right\}.$$

Then $t = \min\{T_1 + \delta, T'\}$ satisfies $t \leq 2T_1$. Hence the inequality $n/2p < 1$ implies $\tau^{n/2p-1} \leq T_1^{n/2p-1}$ for $\tau \in [T_1, t]$, and $t^{1/2-n/4p} \leq (2T_1)^{1/2-n/4p}$. It follows that

$$\begin{aligned} t^{1/2-n/4p} \int_{T_1}^t (t-\tau)^{-n/4p-1/2} \tau^{n/2p-1} d\tau &\leq \frac{2^{1/2-n/4p} T_1^{n/4p-1/2} (t-T_1)^{1/2-n/4p}}{1/2-n/4p} \\ &= \frac{4p}{2p-n} \left(\frac{2\delta}{T_1} \right)^{(2p-n)/4p} \\ &< \frac{1}{C(\psi(\varepsilon) + M')}. \end{aligned}$$

Substituting this estimate into (6.3), we see that $Z(T_1) = 0$ implies $Z(t) = 0$. Starting at $T_1 = T_0$ and repeating this argument, we arrive at $t = T'$ after finite steps, and we have $u(t, x) \equiv \tilde{u}(t, x)$ on $(0, T'] \times \mathbb{R}^n$. Since $T' \in (0, T)$ is arbitrary, we conclude that $u(t, x) \equiv \tilde{u}(t, x)$ on $(0, T) \times \mathbb{R}^n$. $\square$

7. Proof of Theorem 3. In this section we prove Theorem 3, by modifying the argument given in the previous sections.

We first obtain necessary estimates from previous results.

Proposition 7.1. *The estimate*

$$\|\exp(-t\mathcal{A}) \mid L(\mathcal{N}_{p,q,\infty}^s, \mathcal{N}_{p,q,\infty}^\sigma)\| \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ such that $\sigma_1 - 2 < s \leq \sigma < \sigma_0$ and that $\sigma \leq s + 2$, and the estimate

$$\|\exp(-t\mathcal{A}) - 1 \mid L(\mathcal{N}_{p,q,\infty}^s, \mathcal{N}_{p,q,\infty}^\sigma)\| \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ such that $\sigma_1 - 2 < \sigma \leq s < \sigma_0$ and that $s \leq \sigma + 2$.

Moreover, the estimate

$$\|\exp(-t\mathcal{A}) \mid L(\mathcal{N}_{p,q,\infty}^s, \mathcal{M}_{p,q}^\sigma)\| \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ such that $\sigma_1 - 2 < s < \sigma < \sigma_0$ and that $\sigma < s + 2$, and the estimate

$$\|\exp(-t\mathcal{A}) - 1 \mid L(\mathcal{N}_{p,q,\infty}^s, \mathcal{M}_{p,q}^\sigma)\| \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ such that $\sigma_1 - 2 < \sigma < s < \sigma_0$ and that $s < \sigma + 2$.

Proof. We obtain the first two estimates from Proposition 5.1 by real interpolation, by changing the values of s and σ simultaneously. The last two estimates are also obtained similarly, by fixing σ and changing the value of s . $\square$

Theorem 7.2. *Let T be either a positive number or $+\infty$. Suppose that $b(x) \in \mathcal{N}_{p,q,\infty}^{n/p-1}$ and that $u(t, x)$ is a function on $(0, T) \times \mathbb{R}^n$ satisfying (3.5) for every $T' \in (0, T)$. Then the following three conditions on the function $u(t, x)$ are equivalent:*

- (1) $u(t, x)$ satisfies the differential equation (3.4) on $(0, T)$, and the condition (3.7) for every $T' \in (0, T)$ and every $s \in [-1, p/n - 1)$.
- (2) $u(t, x)$ satisfies the differential equation (3.4) on $(0, T)$, and we have $u(t, \cdot) - b \rightarrow 0$ in the topology of $\mathcal{M}_{p,q}^{-1}$ as $t \rightarrow +0$.
- (3) $u(t, x)$ satisfies the integral equation (5.1) on $(0, T)$.

For the proof of this theorem as well as Theorem 3, we use the following lemma.

Lemma 7.3. *Let $u(t, x)$ be a function on $(0, T) \times \mathbb{R}^n$ satisfying (3.5) for every $T' \in (0, T)$ and (5.1) on $(0, T)$, and suppose that $b(x) \in \mathcal{N}_{p,q,\infty}^{n/p-1}$. Then, for every $s \in [-1, \sigma_0)$, there exists a constant C_s such that the inequalities*

$$\begin{aligned} \|u(t, \cdot) - \exp(-t\mathcal{A})b\|_{\mathcal{M}_{p,q}^s} &\leq C_s M^2 t^{n/2p-s/2-1/2}, \\ \|u(t, \cdot) - b\|_{\mathcal{M}_{p,q}^s} &\leq C_s \left(M^2 + \|b\|_{\mathcal{N}_{p,q,\infty}^{n/p-1}} \right) t^{n/2p-s/2-1/2} \\ &\quad \text{if } s < n/p - 1, \\ \|u(t, \cdot)\|_{\mathcal{M}_{p,q}^s} &\leq C_s \left(M^2 + \|b\|_{\mathcal{N}_{p,q,\infty}^{n/p-1}} \right) t^{n/2p-s/2-1/2} \\ &\quad \text{if } s > n/p - 1 \end{aligned}$$

hold for every $t \in (0, T)$, where $M = \sup_{0 < t < T} t^{1/2-n/4p} \|u(t, \cdot)\|_{\mathcal{M}_{p,q}^{n/2p}}$.

We can prove this lemma in the same way as Lemma 5.3, by using Proposition 7.1 in place of Proposition 5.1.

Once this lemma is established, we can prove Theorem 7.2 in the same way as Theorem 5.2, by using Propositions 7.1 and 7.3 in place of Propositions 5.1 and 5.3 respectively.

Hence we can prove Theorem 3 in the same way as in Theorem 2. In fact, we first rewrite the original equation into the integral equation (5.1) by virtue of Theorem 7.2. Then we solve (5.1) by successive approximation, using Proposition 7.1 in place of Proposition 5.1. We next verify the required properties of this solution, by using Lemma 7.3 in place of Lemma 5.3. The uniqueness is proved exactly in the same way as in the proof of the uniqueness in Theorem 2.

REFERENCES

- [1] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces*, Springer, Berlin, 1976.
- [2] G. BOURDAUD, *Réalisations des espaces de Besov homogènes*, Ark. Mat. **26** (1988), 41–54.
- [3] M. CANNONE, *A generalization of a theorem by Kato on Navier-Stokes equations*, (Preprint).
- [4] M. CANNONE AND F. PLANCHON, *Self-similar solutions for Navier-Stokes equations in $\mathbb{R}^3$* , Comm. Partial Differential Equations **21** (1996), 179–193.
- [5] A. CARPIO, *Comportement asymptotique des solutions des équations de tourbillon en dimensions 2 et 3*, C. R. Acad. Sci. Paris, Sér. I **316** (1993), 1289–1294.
- [6] G.-H. COTTET, *Équations de Navier-Stokes dans le plan avec tourbillon initial mesure*, C. R. Acad. Sci. Paris, Sér. I **303** (1986), 105–108.
- [7] P. FEDERBUSH, *Navier and Stokes meet the wavelet*, Comm. Math. Phys. **155** (1993), 219–248.

- [8] G. P. GALDI, *An introduction to the mathematical theory of the Navier-Stokes equations, Vol. II Nonlinear steady problems*, Springer, New York, 1994.
- [9] Y. GIGA AND T. KAMBE, *Large time behaviour of the vorticity of two-dimensional viscous flow and its application to vortex formation*, Comm. Math. Phys. **117** (1988), 549–568.
- [10] Y. GIGA AND T. MIYAKAWA, *Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces*, Comm. Partial Differential Equations **14** (1989), 577–618.
- [11] Y. GIGA, T. MIYAKAWA AND H. OSADA, *Two-dimensional Navier-Stokes flow with measures as initial vorticity*, Arch. Rat. Mech. Anal. **104** (1988), 223–250.
- [12] G. GRUBB, *Initial value problems for the Navier-Stokes equations with Neumann boundary conditions*, The Navier-Stokes Equations II (J. G. Heywood, K. Masuda, R. Rautmann and S. A. Solonnikov, eds.), Proc. Conf. Oberwolfach 1991, Lecture Notes in Math., **1530**, Springer, Berlin, 1992, pp. 262–283.
- [13] A. HARAUX AND F. B. WEISSLER, *Non-uniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. **31** (1982), 167–189.
- [14] T. KATO, *Strong L^p -solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), 471–480.
- [15] T. KATO, *Strong solutions of the Navier-Stokes equations in Morrey spaces*, Boll. Soc. Brasil. Mat. (N. S.) **22-2** (1992), 127–155.
- [16] T. KATO, *The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as initial vorticity*, Differential and Integral Equations **7** (1994), 949–966.
- [17] T. KATO AND G. PONCE, *The Navier-Stokes equation with weak initial data*, Int. Math. Res. Notices **1994-10**, 435–444.
- [18] T. KOBAYASHI AND T. MURAMATU, *Abstract Besov space approach to the non-stationary Navier-Stokes equations*, Math. Methods in the Appl. Sci. **15** (1992), 599–620.
- [19] H. KOMATSU, *Fractional powers of operators, VI Interpolation of non-negative operators and imbedding theorems*, J. Fac. Sci. Univ. Tokyo, Sect. IA **19** (1972), 1–63.
- [20] H. KOZONO AND H. SOHR, *Density properties for solenoidal vector fields, with applications to the Navier-Stokes equations in exterior domains*, J. Math. Soc. Japan **44** (1992), 307–330.
- [21] H. KOZONO AND H. SOHR, *On stationary Navier-Stokes equations in unbounded domains*, Ricerche Mat. **42** (1993), 69–86.
- [22] H. KOZONO AND M. YAMAZAKI, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations **19** (1994), 959–1014.
- [23] H. KOZONO AND M. YAMAZAKI, *Small stable stationary solutions in Morrey spaces of the Navier-Stokes equation*, Proc. Japan Acad., Ser. A **71** (1995), 199–201.
- [24] J. PEETRE, *On convolution operators leaving $L^{p,\lambda}$ spaces invariant*, Ann. Mat. Pura Appl. **72** (1966), 295–304.
- [25] J. PEETRE, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87.
- [26] M. E. TAYLOR, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations **17** (1992), 1407–1456.

HIDEO KOZONO
 Department of Applied Physics
 Nagoya University
 Chikusa-ku, Nagoya 464-01 Japan

MASAO YAMAZAKI
 Department of Mathematics
 Hitotsubashi University
 Kunitachi, Tokyo 186 Japan

Received: February 10th, 1995.