Multiple Positive and Nodal Solutions for Semilinear Elliptic Problems with Critical Exponents

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ABSTRACT. We are concerned with the multiplicity of positive and nodal solutions of

$$-\Delta u = Q(x)|u|^{2^*-2}u + \lambda u, \quad x \in \Omega, \quad u \in H_0^1(\Omega),$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $0 \neq Q(x) \geq 0$, $Q \in C(\bar{\Omega})$, and $\lambda \in (0, \lambda_1)$. The existence of k positive solutions (if $N \geq 4$) and k nodal solutions (if $N \geq 7$) is established for $\lambda \in (0, \lambda_0)$, for some $\lambda_0 > 0$, provided that $\max_{\bar{\Omega}} Q$ is achieved as strict local maximums at k interior points in Ω .

1. Introduction. We consider the existence of multiple positive and nodal solutions (solutions which change sign in Ω) of the following problem

$$(1.1) \quad -\Delta u = Q(x)|u|^{2^*-2}u + \lambda u, \quad x \in \Omega, \quad u \in H_0^1(\Omega),$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 4$, $2^* = 2N/(N-2)$, $\lambda \in (0, \lambda_1)$ (λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions) and $Q(x) \in C(\bar{\Omega})$ satisfies the following condition:

Condition (Q). $Q(x) \geq 0$ in Ω and there exist points $a^1, a^2, \dots, a^k \in \Omega$ such that $Q(a^j)$ are strict local maximums satisfying

$$Q(a^j) = Q_M \equiv \max_{\bar{\Omega}} Q(x) > 0,$$

$$Q(x) - Q(a^j) = \begin{cases} O(|x - a^j|^2) & \text{if } N = 4, \\ o(|x - a^j|^2) & \text{if } N \geq 5, \end{cases}$$

for x near a^j , $j = 1, 2, \dots, k$.

Our main results are:

Theorem 1.1. *Suppose condition (Q) holds. Then there exists a positive number $\lambda_0 \in (0, \lambda_1)$ such that, for $\lambda \in (0, \lambda_0]$, problem (1.1) has at least k positive solutions.*

Theorem 1.2. *Suppose condition (Q) holds and $N \geq 7$. Then there exists a positive number $\lambda_0 \in (0, \lambda_1)$ such that, for $\lambda \in (0, \lambda_0]$, problem (1.1) has at least k nodal solutions.*

The existence of at least one positive solution of (1.1) has been established for the special case $Q(x)$ is a constant in a remarkable paper by H. Brezis and L. Nirenberg [7], and for $Q(x)$ satisfying condition (Q) at a point $a \in \bar{\Omega}$, by J. F. Escobar [14].

For constant Q and Ω is a ball, (1.1) has a unique positive solution [20]. The existence of at least one nodal solution of (1.1) was established in [9], [22].

During the past few years the question whether the topology of Ω or its geometry were responsible for the solvability and/or the multiplicity of positive solutions of problems similar to (1.1) has been the subject of several investigations, beginning from the Pohozaev's classical theorem [19], asserting that (1.1) with $\lambda = 0$, $Q(x) \equiv \text{constant}$ has no solution if Ω is a star shaped domain. Recent important results supporting the topology effect were obtained by, among other people, J. Kazdan and F. Warner [15], J. M. Coron [10], A. Bahri and J. M. Coron [2], asserting that (1.1) with $\lambda = 0$, $Q(x) \equiv 1$ has a positive solution provided Ω is topologically nontrivial (some homology groups are nontrivial); and results by E. N. Dancer [11], W. Ding [12], D. Passaseo [18], in which contractible domains for which (1.1) has one or more positive solutions have been exhibited. We also mention a recent paper by M. Marchi and F. Pacella [17] in which contractible domains for which (1.1) has nodal solutions are exhibited.

The aim of this paper is to show the effect of the shape of the graph of $Q(x)$ on the existence and multiplicity of both positive and nodal solutions. Our solutions are obtained as local minimum points of the functional

$$(1.2) \quad I_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \lambda u^2 - \frac{1}{2^*} \int_{\Omega} Q(x) |u|^{2^*},$$

for $\lambda \in (0, \lambda_1)$, constrained to suitably constructed closed subsets of $H_0^1(\Omega)$. Our positive solutions will correspond to critical values in $(0, S^{N/2}/NQ_M^{(N-2)/2})$, and our nodal solutions correspond to critical values in $(0, 2S^{N/2}/NQ_M^{(N-2)/2})$. Our results seem to suggest that the geometry of the graph of $Q(x)$ has a similar effect to the geometry of Ω , on the existence and multiplicity of both kinds of solutions.

Remark 1.3. The hypothesis $N \geq 7$ in Theorem 1.2 is necessary for the existence of radial nodal solutions by a result of F. V. Atkinson, H. Brezis and L. A. Peletier [1].

Remark 1.4. The more general problem

$$-\Delta u = Q(x)|u|^{2^*-2}u + q(x)|u|^{\gamma-1}u$$

could be considered in a general domain $\Omega \subseteq \mathbb{R}^N$, under suitable assumptions on Q and q , for $1 < \gamma < 2^* - 1$, by similar arguments.

2. Notations and Preliminary Results. Let $\|\cdot\|$ denote the norm of $H_0^1(\Omega)$, $\|u\| = \int |\nabla u|^2$ for all $u \in H_0^1(\Omega)$. Here all integrals are Lebesgue integrals over Ω unless otherwise stated.

Let $g : H_0^1(\Omega) \rightarrow \mathbb{R}^N$ be defined by

$$(2.1) \quad g(u) = \frac{\int x|u|^{2^*}}{\int |u|^{2^*}}.$$

For $r > 0$, $y \in \mathbb{R}^N$, set $B_r(y) = \{x \in \mathbb{R}^N, |x - y| < r\}$ and let $\bar{B}_r(y)$, $S_r(y)$ denote the closure and the boundary of $B_r(y)$, respectively. By condition (Q) we may choose $\ell > 0$ small enough so that $B_{2\ell}(a^j) \subset \Omega$ are disjoint and $Q(x) < Q(a^j)$ for $x \in \bar{B}_{2\ell}(a^j)$, $x \neq a^j$, $j = 1, 2, \dots, k$.

For $\lambda > 0$, $j = 1, 2, \dots, k$, define

$$(2.2) \quad \Sigma_\lambda = \{u \in H_0^1(\Omega) : u \not\equiv 0, \langle I'_\lambda(u), u \rangle = 0\},$$

where $I'_\lambda(u)$ denotes the Fréchet derivative of $I_\lambda(u)$.

For $\lambda > 0$ and $j = 1, 2, \dots, k$ define

$$(2.3) \quad \begin{aligned} O_\lambda^j &= \{u \in \Sigma_\lambda : g(u) \in B_\ell(a^j)\}, \\ U_\lambda^j &= \{u \in \Sigma_\lambda : g(u) \in S_\ell(a^j)\}, \\ V_\lambda^j &= \{u : u^\pm \in O_\lambda^j\}, \\ W_\lambda^j &= \{u : u^\pm \in \Sigma_\lambda, \quad g(u^\pm) \in \bar{B}_\ell(a^j) \text{ and either } g(u^+) \in S_\ell(a^j) \\ &\quad \text{or } g(u^-) \in S_\ell(a^j)\}. \end{aligned}$$

None of O_λ^j , U_λ^j , V_λ^j , W_λ^j are empty, as can be easily verified.

Define

$$\begin{aligned}
 m_\lambda^j &= \inf\{I_\lambda(u) : u \in O_\lambda^j\}, \\
 \bar{m}_\lambda^j &= \inf\{I_\lambda(u) : u \in U_\lambda^j\}, \\
 M_\lambda^j &= \inf\{I_\lambda(u) : u \in V_\lambda^j\}, \\
 \bar{M}_\lambda^j &= \inf\{I_\lambda(u) : u \in W_\lambda^j\},
 \end{aligned}
 \tag{2.4}$$

$$S = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 : u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{2^*} = 1 \right\}.
 \tag{2.5}$$

In the following lemmas we establish estimates on m_λ^j , M_λ^j , $\bar{M}_\lambda^j$, which are crucial to our construction of P.S. sequences in the subsets O_λ^j , V_λ^j . (The estimate for $\bar{m}_\lambda^j$ will be given in Section 3.)

Lemma 2.1. *For $j = 1, \dots, k$, and $\lambda > 0$ we have*

$$m_\lambda^j < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.
 \tag{2.6}$$

Proof. Let $x_0 \in \mathbb{R}^N$, $\varepsilon > 0$, and

$$U_{\varepsilon, x_0} = \frac{C_N \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x - x_0|^2)^{(N-2)/2}}$$

with $C_N = (N(N-2)/Q_M)^{(N-2)/4}$.

It is easy to verify that U_{ε, x_0} satisfies

$$-\Delta u = Q_M |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N.$$

Furthermore, it is well known [7] that the infimum in (2.5) is achieved by the functions $U_\varepsilon, x_0 / \|U_\varepsilon, x_0\|_{L^{2^*}(\mathbb{R}^N)}$.

Let $0 < \rho < \ell/2$, ρ fixed. Define a radial nonnegative function $\varphi \in C_0^2(\mathbb{R}^N)$ by

$$\varphi(x) = \begin{cases} 1 & x \in B_\rho(0), \\ 0 & x \notin B_{2\rho}(0). \end{cases}$$

Set $u_{\varepsilon, a^j} = U_{\varepsilon, a^j} \varphi(x - a^j)$, and we shall simply write u_ε for u_{ε, a^j} when there is no confusion.

The following estimates can be easily established from similar estimates in [7]:

$$(2.7) \quad \int |u_\varepsilon|^{2^*} = K'_2 + O(\varepsilon^N),$$

$$(2.8) \quad \int |\nabla u_\varepsilon|^2 = K_1 + O(\varepsilon^{N-2}),$$

$$K'_2 = C_N^{2^*} \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^N} = \|U_{1,0}\|_{L^{2^*}(\mathbb{R}^N)}^{2^*},$$

$$K_1 = (N-2)^2 C_N^2 \int_{\mathbb{R}^N} \frac{|x|^2}{(1+|x|^2)^N} dx = \|\nabla U_{1,0}\|_{L^2(\mathbb{R}^N)}^2,$$

$$K_1/K_2 = S, \quad K'_2 = K_2^{N/(N-2)},$$

$$(2.9) \quad \int |u_\varepsilon|^2 = \begin{cases} K_3 \varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \\ K_3 \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \end{cases}$$

$$(2.10) \quad \left(\int |u_\varepsilon|^{2^*} \right)^{2/2^*} = K_2 + O(\varepsilon^{N-2}),$$

where K_3 is a positive constant.

$$(2.11) \quad \int |u_\varepsilon| \leq C_1 \varepsilon^{(N-2)/2},$$

$$(2.12) \quad \int |u_\varepsilon|^{2^*-1} \leq C_2 \varepsilon^{(N-2)/2},$$

for some positive constants C_1, C_2 , independent of ε .

For $\varepsilon > 0$, let $t_\varepsilon > 0$ be selected such that $v_\varepsilon \equiv t_\varepsilon u_\varepsilon \in \Sigma_\lambda$. That is

$$(2.13) \quad t_\varepsilon^{2^*-2} = \frac{\int (|\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2)}{\int Q(x) |u_\varepsilon|^{2^*}}.$$

Notice that $v_\varepsilon \in O_\lambda^j$. This follows easily from the symmetry of u_ε about a^j , which implies that $g(v_\varepsilon) \in B_\ell(a^j)$. Hence (2.6) will follow if we show that

$$(2.14) \quad \sup_{t>0} I_\lambda(tu_\varepsilon) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

To establish (2.14) we set

$$h(t) = \frac{t^2}{2} \int (|\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2) - \frac{t^{2^*}}{2^*} \int Q(x) u_\varepsilon^{2^*},$$

$h'(t) > 0$ for $t \in (0, t_\varepsilon)$, $h'(t) < 0$ for $t > t_\varepsilon$, and $h'(t_\varepsilon) = 0$. Therefore,

$$\begin{aligned} \sup_{t>0} I_\lambda(tu_\varepsilon) &= I_\lambda(t_\varepsilon u_\varepsilon) \\ &= \frac{t_\varepsilon^2}{N} \int |\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2 \\ &= \frac{1}{N} [(K_1 + O(\varepsilon^{N-2})) - K_3 \eta(\varepsilon) \lambda] t_\varepsilon^2, \end{aligned}$$

where

$$\eta(\varepsilon) = \begin{cases} \varepsilon^2 & \text{if } N \geq 5, \\ \varepsilon^2 |\ln \varepsilon| & \text{if } N = 4. \end{cases}$$

Using condition (Q) we have

$$\begin{aligned} (2.15) \quad \int Q(x) u_\varepsilon^{2^*} &= \int Q_M |u_\varepsilon|^{2^*} + \int (Q(x) - Q_M) |u_\varepsilon|^{2^*} \\ &= \begin{cases} Q_M K_2' + O(\varepsilon^N) + o(\varepsilon^2) & \text{if } N \geq 5, \\ Q_M K_2' + O(\varepsilon^N) + O(\varepsilon^2) & \text{if } N = 4. \end{cases} \end{aligned}$$

Thus

$$(2.16) \quad t_\varepsilon^2 = \left[\frac{K_1 + O(\varepsilon^{N-2}) - \lambda K_3 \eta(\varepsilon)}{Q_M K_2' + O(\varepsilon^N) + \alpha(\varepsilon)} \right]^{(N-2)/2},$$

where

$$\alpha(\varepsilon) = \begin{cases} o(\varepsilon^2) & \text{if } N \geq 5, \\ O(\varepsilon^2) & \text{if } N = 4. \end{cases}$$

Using (2.16) we have

$$\sup_{t>0} I_\lambda(tu_\varepsilon) = \frac{K_1^{N/2}}{NQ_M^{(N-2)/2} (K_2')^{(N-2)/2}} \beta(\varepsilon) = \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \beta(\varepsilon),$$

where

$$\beta(\varepsilon) = \begin{cases} 1 - \frac{N}{2} \frac{K_3}{K_1} \lambda \varepsilon^2 + o(\varepsilon^2) & \text{if } N \geq 5, \\ 1 - 2 \frac{K_3}{K_1} \lambda \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4. \end{cases}$$

It then follows that $\beta(\varepsilon) < 1$ for sufficiently small ε , which implies (2.14), and Lemma 2.1 follows. $\square$

Lemma 2.2. Assume condition (Q) holds. Then there exist $\varepsilon > 0$ and $\lambda_\varepsilon > 0$ such that

$$(2.17) \quad \begin{aligned} \text{(i)} \quad \bar{m}_\ell^j &> \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + \varepsilon, \\ \text{(ii)} \quad \bar{M}_\lambda^j &> \frac{2S^{N/2}}{NQ_M^{(N-2)/2}} + \varepsilon, \end{aligned}$$

for $j = 1, 2, \dots, k$, and $\lambda \in (0, \lambda_\varepsilon)$.

Proof. To prove (i) suppose to the contrary that we could find a sequence $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, such that $\bar{m}_{\lambda_n}^j \rightarrow c \leq S^{N/2}/NQ_M^{(N-2)/2}$. Consequently, there exists $u_n \in U_{\lambda_n}^j$ such that

$$(2.18) \quad \begin{aligned} I_{\lambda_n}(u_n) &\rightarrow c, \\ \int |\nabla u_n|^2 - \lambda_n u_n^2 &= \int Q(x)|u_n|^{2^*}. \end{aligned}$$

It then follows easily that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and, in particular, $\lambda_n \int u_n^2 \rightarrow 0$ as $n \rightarrow \infty$. By Hölder's and Sobolev's inequalities we can fix $\nu > 0$ such that

$$\int |\nabla u_n|^2, \int Q(x)|u_n|^{2^*} \geq \nu > 0$$

for all $n = 1, 2, \dots$. Therefore, we may choose $t_n > 0$ so that $v_n = t_n u_n$ satisfies

$$\int |\nabla v_n|^2 = \int Q_M |v_n|^{2^*},$$

and

$$t_n = \left(\frac{\int Q(x)|u_n|^{2^*} + \lambda_n \int u_n^2}{\int Q_M |u_n|^{2^*}} \right)^{(N-2)/4}$$

are bounded. Suppose $t_n \rightarrow t_0$ as $n \rightarrow \infty$. Then $t_0 \leq 1$ since $Q(x) \leq Q_M$ and $\lambda_n \int u_n^2 \rightarrow 0$ as $n \rightarrow \infty$. In fact $t_0 = 1$. This follows easily from

$$\begin{aligned} \frac{S^{N/2}}{NQ_M^{(N-2)/2}} &\leq \lim_{n \rightarrow \infty} \frac{1}{N} \int |\nabla v_n|^2 = \lim_{n \rightarrow \infty} \frac{t_n^2}{N} \int |\nabla u_n|^2 \\ &= \lim_{n \rightarrow \infty} t_n^2 \frac{1}{N} \int (|\nabla u_n|^2 - \lambda_n u_n^2) \\ &= \lim_{n \rightarrow \infty} t_n^2 I_{\lambda_n}(u_n) = t_0^2 c \\ &\leq t_0^2 \frac{S^{N/2}}{NQ_M^{(N-2)/2}}. \end{aligned}$$

The inequalities above also show that

$$(2.19) \quad c = \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \quad \text{and} \quad \lim_{n \rightarrow \infty} \int |\nabla v_n|^2 = \frac{S^{N/2}}{Q_M^{(N-2)/2}}.$$

Set $w_n = v_n / (\int |v_n|^{2^*})^{1/2^*}$. It is easy to verify that

$$\int |\nabla w_n|^2 \rightarrow S, \quad \text{as } n \rightarrow \infty.$$

That is, $\{w_n\}$ is a minimizing sequence for the problem

$$S = \inf \left\{ \int |\nabla u|^2 : u \in H_0^1(\Omega), \int |u|^{2^*} = 1 \right\}.$$

We now use a result of P. L. Lions [16] to conclude that we can find a point $x_0 \in \bar{\Omega}$ and a subsequence, still denoted by $\{w_n\}$, such that

$$(2.20) \quad \lim_{n \rightarrow \infty} \int_{\Omega} v |w_n|^{2^*} = v(x_0)$$

for any $v \in C(\bar{\Omega})$. In particular, we have

$$g^i(u_n) = \frac{\int x_i |u_n|^{2^*}}{\int |u_n|^{2^*}} = \frac{\int x_i |w_n|^{2^*}}{\int |w_n|^{2^*}} \rightarrow (x_0)_i,$$

as $n \rightarrow \infty$, and since $g(u_n) \in S_\ell(a^j)$, $x_0 \in S_\ell(a^j)$. Using (2.19) and (2.20) we also have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int Q(x) |u_n|^2 &= \lim_{n \rightarrow \infty} \int Q(x) |v_n|^{2^*} \\ &= \frac{Q(x_0)}{Q_M} \lim_{n \rightarrow \infty} \int Q_M |v_n|^{2^*} \\ &= \frac{Q(x_0)}{Q_M} \lim_{n \rightarrow \infty} \int |\nabla v_n|^2 \\ &= \frac{Q(x_0)}{Q_M} \frac{S^N}{Q_M^{(N-2)/2}}, \end{aligned}$$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} I_{\lambda_n}(u_n) &= \frac{1}{N} \lim_{n \rightarrow \infty} \int Q(x) |u_n|^{2^*} \\
 &= \frac{1}{N} \lim_{n \rightarrow \infty} \int Q(x) |v_n|^{2^*} \\
 &= \frac{Q(x_0)}{NQ_M} \lim_{n \rightarrow \infty} \int Q_M |v_n|^{2^*} \\
 &= \frac{Q(x_0)}{Q_M} \frac{S^{N/2}}{NQ_M^{(N-2)/2}} < \frac{S^{N/2}}{NQ_M^{(N-2)/2}},
 \end{aligned}$$

contradicting (2.19). Hence (i) follows.

The proof of (ii) is similar to the proof above. We argue, as before, by way of contradiction. If, to the contrary, we have a sequence $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, and $u_n \in W_{\lambda_n}^j$ satisfying

$$\begin{aligned}
 I_{\lambda_n}(u_n) &\rightarrow \bar{C} \leq \frac{2S^{N/2}}{NQ_M^{(N-2)/2}}, \\
 \int |\nabla u_n^\pm|^2 - \lambda_n |u_n^\pm|^2 &= \int Q(x) |u_n^\pm|^{2^*},
 \end{aligned}$$

then

$$\begin{aligned}
 I_{\lambda_n}(u_n^+) &\rightarrow C_1, \\
 I_{\lambda_n}(u_n^-) &\rightarrow C_2, \\
 C_1 + C_2 &= \bar{C}, \quad \text{and} \\
 C_1 \text{ or } C_2 &\leq \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.
 \end{aligned}$$

As in the proof of part (i), we can show that C_1 and $C_2 = S^{N/2}/NQ_M^{(N-2)/2}$, which leads to a contradiction, as in the proof of part (i). $\square$

Lemma 2.3. Assume condition (Q) holds and $\lambda \in (0, \lambda_1)$. Then

(i) Any sequence $\{u_n\} \subset \Sigma_\lambda$ satisfying

$$\begin{aligned}
 I_\lambda(u_n) &\rightarrow c < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}, \\
 I'_\lambda(u_n) &\rightarrow 0, \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

is relatively compact in $H_0^1(\Omega)$.

(ii) Any sequence $(v_n) \subset V_\lambda^j$ satisfying

$$I_\lambda(v_n) \rightarrow c < m_\lambda^j + \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$

$$I'_\lambda(v_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

is relatively compact in $H_0^1(\Omega)$.

Proof. (i): Since $\lambda \in (0, \lambda_1)$, it is easy to show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and, therefore, we may assume that for some $u_0 \in H_0^1(\Omega)$,

$$u_n \rightarrow u_0 \text{ weakly in } H_0^1(\Omega),$$

$$u_n \rightarrow u_0 \text{ a.e. in } \Omega,$$

$$\int |\nabla u_0|^2 - \lambda u_0^2 = \int Q(x) |u_0|^{2^*},$$

$$I_\lambda(u_0) = \frac{1}{N} \int Q(x) |u_0|^{2^*} > 0 \text{ if } u_0 \not\equiv 0.$$

Let $w_n = u_n - u_0$. By Brezis-Lieb Lemma [5] we have

$$\int Q(x) |u_n|^{2^*} = \int Q(x) |u_0|^{2^*} + \int Q(x) |w_n|^{2^*} + o(1),$$

$$\int |\nabla u_n|^2 = \int |\nabla u_0|^2 + \int |\nabla w_n|^2 + o(1).$$

If $\|w_n\| \rightarrow 0$ we are done, so assume $\|w_n\| \rightarrow L > 0$. It follows from

$$\begin{aligned} \int |\nabla w_n|^2 &= \int |\nabla u_n|^2 - \int |\nabla u_0|^2 + o(1) \\ &= \lambda \int |u_n|^2 + \int Q(x) |u_n|^{2^*} - \int |\nabla u_0|^2 + o(1) \\ &= \lambda \int |u_n|^2 + \int Q(x) |w_n|^{2^*} + \int Q(x) |u_0|^{2^*} - \int |\nabla u_0|^2 + o(1) \\ &= \int Q(x) |w_n|^{2^*} + o(1), \end{aligned}$$

that we can choose $t_n > 0$ so that

$$\int |\nabla t_n w_n|^2 = \int Q_M |t_n w_n|^{2^*}.$$

In fact,

$$t_n^{4/(N-2)} = \frac{\int Q(x)|w_n|^{2^*} + o(1)}{\int Q_M|w_n|^{2^*}},$$

and $t_n \rightarrow t_0 \leq 1$. Therefore,

$$\begin{aligned} \frac{1}{N} \lim_{n \rightarrow \infty} \int |\nabla w_n|^2 &\geq \frac{t_0^2}{N} \lim_{n \rightarrow \infty} \int |\nabla w_n|^2 \\ &= \frac{1}{N} \lim_{n \rightarrow \infty} \int |\nabla t_n w_n|^2 \\ &\geq \frac{S^{N/2}}{NQ_M^{(N-2)/N}}, \end{aligned}$$

where the last inequality follows from the definition of S . We then have

$$\lim_{n \rightarrow \infty} \int |\nabla w_n|^2 \geq \frac{S^{N/2}}{Q_M^{(N-2)/2}},$$

and therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} I_\lambda(u_n) &= I_\lambda(u_0) + \lim_{n \rightarrow \infty} I_\lambda(w_n) \\ &= I_\lambda(u_0) + \frac{1}{N} \lim_{n \rightarrow \infty} \int |\nabla w_n|^2 \\ &\geq I_\lambda(u_0) + \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \\ &\geq \frac{S^{N/2}}{NQ_M^{(N-2)/2}}, \end{aligned}$$

contradicting the hypothesis in (i). Therefore $\|w_n\| \rightarrow L > 0$ is impossible, and (i) follows.

(ii): Since $\{v_n\} \subset V_\lambda^j$, we have $v_n^\pm \in O_\lambda^j$ and $I_\lambda(v_n^\pm) \geq m_\lambda^j$. We also have

$$\begin{aligned} \int |\nabla v_n^\pm|^2 - \lambda |v_n^\pm|^2 &= \int Q(x)|v_n^\pm|^{2^*}, \\ I_\lambda(v_n) &= I_\lambda(v_n^+) + I_\lambda(v_n^-). \end{aligned}$$

Since $I_\lambda(v_n) \rightarrow c < m_\lambda^j + S^{N/2}/NQ_M^{(N-2)/2}$, and $I_\lambda(v_n^\pm) \geq m_\lambda^j$, as $n \rightarrow \infty$, we must have

$$\lim_{n \rightarrow \infty} I_\lambda(v_n^\pm) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$

It is easy to show that $\{v_n\}$ is bounded in $H_0^1(\Omega)$, and we may assume that

$$\begin{aligned} v_n &\rightarrow v_0 \text{ weakly in } H_0^1(\Omega), \\ v_n &\rightarrow v_0 \text{ a.e. in } \Omega, \\ v_n^+ &\rightarrow v_0^+ \text{ weakly in } H_0^1(\Omega), \\ v_n^- &\rightarrow v_0^- \text{ weakly in } H_0^1(\Omega). \end{aligned}$$

Let $w_n^\pm = v_n^\pm - v_0^\pm$. If $\|w_n^\pm\| \rightarrow L^\pm > 0$, we establish the contradiction

$$\lim_{n \rightarrow \infty} I_\lambda(v_n^\pm) \geq \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$

as in part (i). Hence $\|w_n\| \rightarrow 0$, as $n \rightarrow \infty$, which implies that $v_n \rightarrow v_0$ strongly in $H_0^1(\Omega)$. Thus Lemma 2.3 is proved. $\square$

Lemma 2.4. *Assume condition (Q) holds. Then there exists a $\lambda_0 \in (0, \lambda_1)$ and a sequence $\{u_n^j\} \subset O_\lambda^j$, for each $j = 1, 2, \dots, k$, satisfying $u_n \geq 0$,*

$$(2.21) \quad I_\lambda(u_n^j) \rightarrow m_\lambda^j,$$

$$(2.21) \quad I'_\lambda(u_n^j) \rightarrow 0,$$

as $n \rightarrow \infty$, for $\lambda \in (0, \lambda_0]$.

Proof. We first notice that for $\lambda < \lambda_1$ there is a positive constant $\delta = \delta(\lambda_1 - \lambda)$ such that $\|u\|_{L^{2^*}(\Omega)} \geq \delta > 0$ for all $u \in O_\lambda^j$. Therefore, $\bar{O}_\lambda^j = O_\lambda^j \cup U_\lambda^j$ and U_λ^j is the boundary of $\bar{O}_\lambda^j$ for $\lambda < \lambda_1$ and each $j = 1, 2, \dots, k$.

Using Lemmas 2.1 and 2.2 we see that there exists $\lambda_0 \in (0, \lambda_1)$ such that

$$(2.22) \quad m_\lambda^j < \bar{m}_\lambda^j$$

for $\lambda \in (0, \lambda_0], j = 1, \dots, k$. It follows that

$$(2.23) \quad m_\lambda^j = \inf\{I_\lambda(u) : u \in \bar{O}_\lambda^j\}.$$

Fix $\lambda \in (0, \lambda_0]$, and let $\{u_n^j\} \subset \bar{O}_\lambda^j$ be a minimizing sequence for (2.23). By replacing u_n^j with $|u_n^j|$, if necessary, we may assume that $u_n^j \geq 0$. By applying Ekeland's variational principle [13] we construct a minimizing sequence $\{v_n\} \subset \bar{O}_\lambda^j$, for each $j = 1, 2, \dots, k$, with the properties

$$(2.24) \quad \begin{aligned} (a) \quad & I_\lambda(v_n) \leq I_\lambda(u_n^j) < m_\lambda^j + \frac{1}{n}, \\ (b) \quad & \|v_n - u_n^j\| \leq \frac{1}{n}, \\ (c) \quad & I_\lambda(v_n) < I_\lambda(w) + \frac{1}{n} \|w - v_n\| \text{ for each } w \neq v_n \text{ in } \bar{O}_\lambda^j. \end{aligned}$$

Using (2.22) we may assume that $v_n \in O_\lambda^j$ for sufficiently large n . We may now employ the argument in [21] to construct, for each v_n , an $\varepsilon_n > 0$ and a functional $t^n(w)$ defined for $w \in H_0^1(\Omega)$, $\|w\| \leq \varepsilon_n$ such that $t^n(w)(v_n - w) \in O_\lambda^j$, and

$$(2.25) \quad \langle (t^n)'(0), v \rangle = \frac{2 \int (\nabla v_n \cdot \nabla v - \lambda v_n v) - 2^* \int Q(x) |v_n|^{2^*-2} v_n v}{\int |\nabla v_n|^2 - \lambda v_n^2 - (2^* - 1) \int Q(x) |v_n|^{2^*}}.$$

Choose $0 < \delta < \varepsilon_n$. Let $0 \neq u \in H_0^1(\Omega)$ and let $\omega_\delta = \delta u / \|u\|$. Fix n and let $z_\delta = t^n(w_\delta)(v_n - w_\delta)$. Since $z_\delta \in O_\lambda^j$ by the properties of $t^n(w_\delta)$,

$$I_\lambda(z_\delta) - I_\lambda(v_n) \geq -\frac{1}{n} \|z_\delta - v_n\|$$

follows from (2.24). The mean value theorem then gives

$$\langle I'_\lambda(v_n), z_\delta - v_n \rangle + o(\|z_\delta - v_n\|) \geq -\frac{1}{n} \|z_\delta - v_n\|.$$

Hence

$$\langle I'_\lambda(v_n), (v_n - w_\delta) + (t^n(w_\delta) - 1)(v_n - w_\delta) - v_n \rangle \geq -\frac{1}{n} \|z_\delta - v_n\| + o(\|z_\delta - v_n\|),$$

which implies that

$$(2.26) \quad \begin{aligned} & -\langle I'_\lambda(v_n), w_\delta \rangle + (t^n(w_\delta) - 1) \langle I'_\lambda(v_n), v_n - w_\delta \rangle \\ & \geq -\frac{1}{n} \|z_\delta - v_n\| + o(\|z_\delta - v_n\|). \end{aligned}$$

Since $t^n(w_\delta)(v_n - w_\delta) \in O_\lambda^j, \langle I'_\lambda(z_\delta),$

$$t^n(w_\delta)(v_n - w_\delta) \rangle = 0.$$

Thus it follows from (2.26) that

$$\begin{aligned} & -\delta \left\langle I'_\lambda(v_n), \frac{u}{\|u\|} \right\rangle + \frac{t^n(w_\delta) - 1}{t^n(w_\delta)} \langle I'_\lambda(z_\delta), t^n(w_\delta)(u_n - w_\delta) \rangle \\ & + (t^n(w_\delta) - 1) \langle I'_\lambda(v_n) - I'_\lambda(z_\delta), v_n - w_\delta \rangle \\ & \geq -\frac{1}{n} \|z_\delta - v_n\| + o(\|z_\delta - v_n\|). \end{aligned}$$

Hence

$$\begin{aligned} (2.27) \quad \left\langle I'_\lambda(v_n), \frac{u}{\|u\|} \right\rangle & \leq \frac{1}{n} \frac{\|z_\delta - v_n\|}{\delta} + \frac{o(\|z_\delta - v_n\|)}{\delta} \\ & + \frac{(t^n(w_\delta) - 1)}{\delta} \langle I'_\lambda(v_n) - I'_\lambda(z_\delta), u_n - w_\delta \rangle. \end{aligned}$$

But $\|z_\delta - v_n\| \leq \delta + |t^n(w_\delta) - 1|C$,

$$\lim_{\delta \rightarrow 0} \frac{|t^n(w_\delta) - 1|}{\delta} \leq \|t^n(0)\| \leq C$$

for some constant $C > 0$, independent of δ , as can be easily verified from (2.25).

For fixed n , letting $\delta \rightarrow 0$ in (2.27), we obtain

$$\left\langle I'_\lambda(v_n), \frac{u}{\|u\|} \right\rangle \leq \frac{C}{n},$$

which implies that $I'_\lambda(v_n) \rightarrow 0$, as $n \rightarrow \infty$, and by (2.24)(b) we conclude that $I'_\lambda(u_n^j) \rightarrow 0$ for $\lambda \in (0, \lambda_0)$. This completes the proof of Lemma 2.4. $\square$

3. Existence of Positive and Nodal Solutions.

Proof of Theorem 1.1. By combining Lemmas 2.3 and 2.4 we see that there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0]$, $j = 1, 2, \dots, k$, we have a minimizing sequence $\{u_n^j\} \subset O_\lambda^j$ such that

$$u_n^j \geq 0,$$

$$I_\lambda(u_n^j) \rightarrow m_\lambda^j,$$

$$I'_\lambda(u_n^j) \rightarrow 0,$$

$$u_n^j \rightarrow u^j \text{ strongly in } H_0^1(\Omega).$$

It then follows that $u^j \not\equiv 0$ is a weak solution of (1.1) and $u^j \geq 0$. By standard regularity argument and the strong maximum principle, we obtain $u^j(x) > 0$ in Ω , and since $g(u^j) \in B_\ell(a^j)$ and $B_\ell(a^j)$ are disjoint for $j = 1, 2, \dots, k$, we conclude that u^j , $j = 1, \dots, k$, are distinct positive solutions of (1.1). This completes the proof of Theorem 1.1. $\square$

The next two lemmas are needed for the proof of Theorem 1.2.

Lemma 3.1. *Suppose condition (Q) holds and $N \geq 7$. Then*

$$(3.1) \quad M_\lambda^j < m_\lambda^j + \frac{S^{N/2}}{NQ_M^{(N-2)/2}}$$

for $\lambda \in (0, \lambda_1)$.

Proof. The inequality (3.1) follows from the definition of M_λ^j if we show that:

- (i) For each $\lambda \in (0, \lambda_1)$ and $\varepsilon > 0$ sufficiently small, there exist $\alpha_\varepsilon, \beta_\varepsilon \in [\frac{1}{2}, 2]$ such that $(\alpha_\varepsilon u^j - \beta_\varepsilon u_\varepsilon)^\pm \in O_\lambda^j$, where u^j , for each $j = 1, 2, \dots, k$, is the positive solution constructed in Theorem 1.1, and

$$u_\varepsilon \equiv u_{\varepsilon, a^j} = \frac{C_N \varepsilon^{(N-2)/2} \varphi(x - a^j)}{(\varepsilon^2 + |x - a^j|^2)^{(N-2)/2}},$$

where φ is a radial nonnegative function, $\varphi \in C_0^2(\mathbb{R}^N)$, and

$$\varphi(x) = \begin{cases} 1 & x \in B_\rho(0), \\ 0 & x \notin B_{2\rho}(0), \end{cases}$$

where ρ is a fixed number such that $0 < \rho < \ell/2$.

(ii)

$$\sup_{\alpha, \beta \in [1/2, 2]} I_\lambda(\alpha u^j - \beta u_\varepsilon) < m_\lambda^j + \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

Proof of (i).

Set

$$d = \min\{u^j(x) : |x - a^j| \leq 2\rho\},$$

$$d' = \max\{u^j(x) : |x - a^j| \leq 2\rho\},$$

$$A = 2 \left(\frac{4C_N}{d} \right)^{1/(N-2)},$$

$$B = \frac{1}{2} \left(\frac{C_N}{4d'} \right)^{1/N-2}.$$

It is easy to check, using the definition of u_{ε, a^j} , that $|x - a^j| \geq \sqrt{\varepsilon}A$ implies that $\frac{1}{2}u^j(x) \geq 2u_{\varepsilon}(x)$; and $|x - a^j| \leq \sqrt{\varepsilon}B$ implies that $2u^j(x) \leq \frac{1}{2}u_{\varepsilon}(x)$. Now set

$$\Omega_{\varepsilon}^{+} = \{x \in \Omega : \alpha u^j(x) \geq \beta u_{\varepsilon}(x)\},$$

$$\Omega_{\varepsilon}^2 = \{x \in \Omega : |x - a^j| \geq \sqrt{\varepsilon}A\},$$

$$\Omega_{\varepsilon}^3 = \{x \in \Omega : |x - a^j| \geq \sqrt{\varepsilon}B\}.$$

It then follows that

$$\Omega_{\varepsilon}^2 \subset \Omega_{\varepsilon}^{+} \subset \Omega_{\varepsilon}^3.$$

Next we estimate

$$\int_{\Omega} Q(x) |(\alpha u^j - \beta u_{\varepsilon})^{\pm}|^{2^*}, \quad \int_{\Omega} |(\alpha u^j - \beta u_{\varepsilon})^{\pm}|^2, \quad \int_{\Omega} |\nabla(\alpha u^j - \beta u_{\varepsilon})^{\pm}|^2.$$

We have

$$\begin{aligned} (3.2) \quad & \int_{\Omega} Q(x) |(\alpha u^j - \beta u_{\varepsilon})^{+}|^{2^*} \\ &= \int_{\Omega_{\varepsilon}^{+}} Q(x) (\alpha u^j - \beta u_{\varepsilon})^{2^*} \\ &\geq \int_{\Omega_{\varepsilon}^2} Q(x) (\alpha u^j - \beta u_{\varepsilon})^{2^*} \\ &\geq \int_{\Omega_{\varepsilon}^2} Q(x) (\alpha u^j)^{2^*} - 2^* \alpha^{2^*-1} \beta \int_{\Omega_{\varepsilon}^2} Q(x) |u^j|^{2^*-1} u_{\varepsilon} \\ &\geq \int_{\Omega} Q(x) |\alpha u^j|^{2^*} - \int_{\Omega \setminus \Omega_{\varepsilon}^2} Q(x) |\alpha u^j|^{2^*} - 2^* \alpha^{2^*-1} \beta \int_{\Omega_{\varepsilon}^2} Q(x) |u^j|^{2^*-1} u_{\varepsilon}, \end{aligned}$$

where we used the inequality $(a - b)^{2^*} \geq a^{2^*} - 2^* a^{2^*-1} b$, for $a \geq b$.

We also have

$$\begin{aligned} & \int_{\Omega} Q(x) |(\alpha u^j - \beta u_{\varepsilon})^{+}|^{2^*} \leq \int_{\Omega} Q(x) |\alpha u^j|^{2^*}, \\ & \int_{\Omega \setminus \Omega_{\varepsilon}^2} Q(x) |\alpha u^j|^{2^*} \leq C \varepsilon^{N/2}, \\ & \int_{\Omega_{\varepsilon}^2} Q(x) |u^j|^{2^*-1} u_{\varepsilon} \leq C_1 \varepsilon^{(N-2)/2}, \end{aligned}$$

for some positive constants C, C_1 , depending only on $Q_M, \|u^j\|_{L^\infty(\Omega)}$, as can be easily verified. From the inequalities above we see that

$$(3.3) \quad \left| \int_{\Omega} Q(x) |(\alpha u^j - \beta u_\varepsilon)^+|^{2^*} - \int_{\Omega} Q(x) |\alpha u^j|^{2^*} \right| \leq O(\varepsilon^{(N-2)/2}).$$

Similarly we have

$$\begin{aligned} \int_{\Omega} |\nabla(\alpha u^j - \beta u_\varepsilon)^+|^2 &= \int_{\Omega_\varepsilon^+} |\nabla(\alpha u^j - \beta u_\varepsilon)^+|^2 \\ &\geq \int_{\Omega_\varepsilon^+} |\nabla(\alpha u^j)|^2 - 2\alpha\beta \int_{\Omega_\varepsilon^+} |\nabla u^j| |\nabla u_\varepsilon| \\ &\geq \int_{\Omega} |\nabla(\alpha u^j)|^2 - \int_{\Omega \setminus \Omega_\varepsilon^+} |\nabla(\alpha u^j)|^2 \\ &\quad - 8 \left(\int_{\Omega} |\nabla u^j|^2 \right)^{1/2} \left(\int_{\Omega_\varepsilon^+} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ &\geq \int_{\Omega} |\nabla(\alpha u^j)|^2 - 4 \|\nabla u^j\|_{L^\infty(\Omega)} \omega_N \varepsilon^{N/2} \\ &\quad - 8 \|\nabla u^j\|_{L^2} \left(\int_{\Omega_\varepsilon^+} |\nabla u_\varepsilon|^2 \right)^{1/2}, \end{aligned}$$

$$\begin{aligned} \int_{\Omega} |\nabla(\alpha u^j - \beta u_\varepsilon)^+|^2 &\leq \int_{\Omega_\varepsilon^3} |\nabla(\alpha u^j - \beta u_\varepsilon)^+|^2 \\ &\leq \int_{\Omega_\varepsilon^3} |\nabla(\alpha u^j)|^2 + \int_{\Omega_\varepsilon^3} \beta^2 |\nabla u_\varepsilon|^2 + 8 \int_{\Omega_\varepsilon^3} |\nabla u^j| |\nabla u_\varepsilon| \\ &\leq \int_{\Omega} |\nabla(\alpha u^j)|^2 + 4 \int_{|x-a^j| \geq \sqrt{\varepsilon}B} |\nabla u_\varepsilon|^2 \\ &\quad + 8 \|\nabla u^j\|_{L^2(\Omega)} \left(\int_{|x-a^j| \geq \sqrt{\varepsilon}B} |\nabla u_\varepsilon|^2 \right)^{1/2}. \end{aligned}$$

Easy calculations shows that $\int_{|x-a^j| \geq \sqrt{\varepsilon}B} |\nabla u_\varepsilon|^2 \leq C\varepsilon^{N/2}$ for some positive constant C , depending only on $\|\nabla u^j\|_{L^\infty(\Omega)}$.

The inequalities above then imply that

$$(3.4) \quad \left| \int_{\Omega} |\nabla(\alpha u^j - \beta u_\varepsilon)^+|^2 - \int_{\Omega} |\nabla(\alpha u^j)|^2 \right| \leq C\varepsilon^{N/4},$$

for some constant C , depending only on Q , Q_M , $\|\nabla u^j\|_{L^\infty(\Omega)}$.

Similar calculations show that

$$(3.5) \quad \left| \int_{\Omega} |(\alpha u^j - \beta u_\varepsilon)^+|^2 - \int_{\Omega} |\alpha u^j|^2 \right| \leq C\varepsilon^{N/4}.$$

Define:

$$(3.6) \quad h^+(\alpha, \beta) = \int_{\Omega} |\nabla(\alpha u^j - \beta u_\varepsilon)^+|^2 - \lambda |(\alpha u^j - \beta u_\varepsilon)^+|^2 \\ - \int_{\Omega} Q(x) |(\alpha u^j - \beta u_\varepsilon)^+|^{2^*}.$$

Since $u^j \in O_\lambda^j$, $\int_{\Omega} |\nabla u^j|^2 - \lambda \int_{\Omega} |u^j|^2 - \int_{\Omega} Q(x) |u^j|^{2^*} = 0$, it then follows from (3.3), (3.4), (3.5) and (3.6) that

$$(3.7) \quad h^+(\alpha, \beta) = \int_{\Omega} |\nabla \alpha u^j|^2 - \lambda |\alpha u^j|^2 - \int_{\Omega} Q(x) |\alpha u^j|^{2^*} + O(\varepsilon^{N/4}),$$

and therefore, for small $\varepsilon > 0$ we have

$$(3.8) \quad h^+\left(\frac{1}{2}, \beta\right) > 0, \quad h^+(2, \beta) < 0, \quad \text{for any } \beta \in \left[\frac{1}{2}, 2\right].$$

Now define:

$$(3.9) \quad h^-(\alpha, \beta) = \int_{\Omega} |\nabla(\alpha u^j - \beta u_\varepsilon)^-|^2 - \lambda |(\alpha u^j - \beta u_\varepsilon)^-|^2 \\ - \int_{\Omega} Q(x) |(\alpha u^j - \beta u_\varepsilon)^-|^{2^*}.$$

Then

$$h^-(\alpha, \beta) = \int_{\Omega} |\nabla(\beta u_\varepsilon - \alpha u^j)^+|^2 - \lambda |(\beta u_\varepsilon - \alpha u^j)^+|^2 - \int_{\Omega} Q(x) |(\beta u_\varepsilon - \alpha u^j)^+|^{2^*}.$$

Set $\Omega_\varepsilon^- = \{x \in \Omega : \beta u_\varepsilon > \alpha u^j\}$, so $\Omega_\varepsilon^- = \Omega \setminus \Omega_\varepsilon^+$, which implies that $\Omega \setminus \Omega_\varepsilon^3 \subset \Omega_\varepsilon^- \subset \Omega \setminus \Omega_\varepsilon^2$.

Using similar calculations to the above, we obtain

$$\left| \int_{\Omega} Q(x) |(\beta u_\varepsilon - \alpha u^j)^+|^{2^*} - \int_{\Omega} Q(x) |\beta u_\varepsilon|^{2^*} \right| \leq C\varepsilon^{(N-2)/4}, \\ \left| \int_{\Omega} |\nabla(\beta u_\varepsilon - \alpha u^j)^+|^2 - \int_{\Omega} |\nabla \beta u_\varepsilon|^2 \right| \leq C\varepsilon^{N/4}, \\ \left| \int_{\Omega} |(\beta u_\varepsilon - \alpha u^j)^+|^2 - \int_{\Omega} |\beta u_\varepsilon|^2 \right| \leq C\varepsilon^{N/4},$$

and

$$\begin{aligned}
 (3.10) \quad h^-(\alpha, \beta) &= \int_{\Omega} |\nabla(\beta u_{\varepsilon})|^2 - \lambda \int (\beta u_{\varepsilon})^2 - \int_{\Omega} Q(x)(\beta u_{\varepsilon})^{2^*} + O(\varepsilon^{(N-2)/4}) \\
 &= \beta^2 \int_{\Omega} |\nabla u_{\varepsilon}|^2 - \lambda \beta^2 \int |u_{\varepsilon}|^2 - \beta^{2^*} \int_{\Omega} Q(x)(u_{\varepsilon})^{2^*} + O(\varepsilon^{(N-2)/4}).
 \end{aligned}$$

We now use (3.10) and the estimates in (2.7), (2.8), (2.9) to obtain (here $U_{1,0}$ is as in (2.7))

$$\begin{aligned}
 (3.11) \quad h^-(\alpha, \beta) &= \beta^2 \left\| |\nabla U_{1,0}| \right\|_{L^2(\mathbb{R}^N)}^2 - \beta^{2^*} Q_M \|U_{1,0}\|_{L^2(\mathbb{R}^N)}^{2^*} + O(\varepsilon^{N-2}) \\
 &\quad - \lambda K_3 \varepsilon^2 + O(\varepsilon^{(N-2)/4}),
 \end{aligned}$$

and since $\left\| |\nabla U_{1,0}| \right\|_2^2 = Q_M \|U_{1,0}\|_{2^*}^{2^*}$, (3.11) implies that, for ε sufficiently small,

$$(3.12) \quad h^-\left(\alpha, \frac{1}{2}\right) > 0, \quad h^-(\alpha, 2) < 0, \quad \text{for any } \alpha \in \left[\frac{1}{2}, 2\right].$$

From (3.8), (3.12), and a theorem by Miranda [18], we conclude that there exist $\alpha_{\varepsilon}, \beta_{\varepsilon} \in [\frac{1}{2}, 2]$ such that

$$h^+(\alpha_{\varepsilon}, \beta_{\varepsilon}) = h^-(\alpha_{\varepsilon}, \beta_{\varepsilon}) = 0,$$

which means that $(\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^{\pm} \in \Sigma_{\lambda}$.

To complete the proof of (i) we show next that

$$(3.14) \quad g((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^{\pm}) \in B_{\ell}(a^j),$$

for small ε .

We consider $g((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+)$ first. We have

$$|(\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+|^{2^*} \leq |\alpha_{\varepsilon} u^j|^{2^*},$$

$$\text{and} \quad |(\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+|^{2^*} \geq |\alpha_{\varepsilon} u^j|^{2^*} - 2^* \alpha_{\varepsilon}^{2^*-1} \beta_{\varepsilon} |u^j|^{2^*-1} u_{\varepsilon},$$

on Ω_{ε}^+ . Therefore,

$$0 \geq |(\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+|^{2^*} - |\alpha u^j|^{2^*} \geq -2^* \alpha_{\varepsilon}^{2^*-1} \beta_{\varepsilon} |u^j|^{2^*-1} u_{\varepsilon},$$

and for any x_i we have

$$(3.15) \quad \left| \int_{\Omega} x_i [|(\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+|^{2^*} - |\alpha u^j|^{2^*}] \right| \leq 2^* \alpha_{\varepsilon}^{2^*-1} \beta_{\varepsilon} \int |u^j|^{2^*-1} u_{\varepsilon} |x_i|,$$

$$(3.16) \quad \left| \int_{\Omega} |(\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+|^{2^*} - \int_{\Omega} |\alpha_{\varepsilon} u^j|^{2^*} \right| \leq 2^* \alpha_{\varepsilon}^{2^*-1} \beta_{\varepsilon} \int_{\Omega} |u^j|^{2^*-1} u_{\varepsilon}.$$

From (3.15) and (3.16) we have

$$(3.17) \quad \begin{aligned} g^i((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+) &\leq \frac{\int_{\Omega} x_i |\alpha_{\varepsilon} u^j|^{2^*} + 2^* 2^{2^*} \int_{\Omega} |u^j|^{2^*-1} u_{\varepsilon} |x_i|}{\int_{\Omega} |\alpha_{\varepsilon} u^j|^{2^*} - 2^* 2^{2^*} \int_{\Omega} |u^j|^{2^*-1} u_{\varepsilon}}, \\ g^i((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+) &\geq \frac{\int_{\Omega} x_i |\alpha_{\varepsilon} u^j|^{2^*} - 2^* 2^{2^*} \int_{\Omega} |u^j|^{2^*-1} u_{\varepsilon} |x_i|}{\int_{\Omega} |\alpha_{\varepsilon} u^j|^{2^*} + 2^* 2^{2^*} \int_{\Omega} |u^j|^{2^*-1} u_{\varepsilon}}. \end{aligned}$$

Using (3.17) and the estimate

$$\int_{\Omega} |u^j|^{2^*-1} u_{\varepsilon} \leq \|u^j\|_{L^{\infty}(\Omega)}^{2^*-1} \int_{\Omega} u_{\varepsilon} \leq C \varepsilon^{(N-2)/2},$$

we deduce that

$$\begin{aligned} g^i(u^j) - C_1 \varepsilon^{(N-2)/2} &= \frac{\int_{\Omega} x_i |u^j|^{2^*} - C \varepsilon^{(N-2)/2}}{\int_{\Omega} |u^j|^{2^*} + C \varepsilon^{(N-2)/2}} \\ &\leq g^i((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+) \\ &\leq \frac{\int_{\Omega} x_i |u^j|^{2^*} + C \varepsilon^{(N-2)/2}}{\int_{\Omega} |u^j|^{2^*} - C \varepsilon^{(N-2)/2}} \\ &= g^i(u^j) + C_2 \varepsilon^{(N-2)/2}, \end{aligned}$$

for some positive constants C_1, C_2 , independent of ε , which implies, since $u^j \in O_{\lambda}^j$ and O_{λ}^j is open subset, that $g((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^+) \in B_{\ell}(a^j)$. The proof for $g((\alpha_{\varepsilon} u^j - \beta_{\varepsilon} u_{\varepsilon})^-) \in B_{\ell}(a^j)$ is similar. This completes the proof of (i).

Proof of (ii).

$$I_{\lambda}(\alpha w^j - \beta u_{\varepsilon})$$

$$\begin{aligned}
 &= I_\lambda(\alpha u^j) + \frac{\beta^2}{2} \int (|\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2) \\
 &\quad - \alpha \beta \int Q(x)(u^j)^{2^*-1} u_\varepsilon - \frac{1}{2^*} \int Q(x)[|\alpha u^j - \beta u_\varepsilon|^{2^*} - (\alpha u^j)^{2^*}] \\
 &= I_\lambda(\alpha u^j) + \frac{\beta^2}{2} \int |\nabla u_\varepsilon|^2 - \frac{1}{2^*} \int Q_M |\beta u_\varepsilon|^{2^*} \\
 &\quad - \lambda \frac{\beta^2}{2} \int u_\varepsilon^2 + \frac{1}{2^*} \int |Q_M - Q(x)| |\beta u_\varepsilon|^{2^*} \\
 &\quad - \frac{1}{2^*} \int Q(x)[|\alpha u^j - \beta u_\varepsilon|^{2^*} - (\alpha u^j)^{2^*} - (\beta u_\varepsilon)^{2^*} + 2^* \alpha \beta (u^j)^{2^*-1} u_\varepsilon].
 \end{aligned}$$

Using the estimates in (2.7), (2.12) we also have $0 \leq \int (Q_M - Q(x)) |\beta u_\varepsilon|^{2^*} = o(\varepsilon^2)$,

$$\begin{aligned}
 &\left| \int |\alpha u^j - \beta u_\varepsilon|^{2^*} - (\alpha u^j)^{2^*} - (\beta u_\varepsilon)^{2^*} + 2^* \alpha \beta (u^j)^{2^*-1} u_\varepsilon \right| \\
 &\leq C \left(\int |\alpha|^{2^*-1} \beta |u^j|^{2^*-1} u_\varepsilon + |\alpha| |\beta|^{2^*-1} u^j u_\varepsilon^{2^*-1} \right) \\
 &\leq C_1 \left(\int u_\varepsilon^{2^*-1} + u_\varepsilon \right) \\
 &= O(\varepsilon^{(N-2)/2}).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \sup_{\alpha, \beta \in [1/2, 2]} I_\lambda(\alpha u^j - \beta u_\varepsilon) &\leq m_\lambda^j + \sup_{\beta \geq 0} \left[\frac{\beta^2}{2} \int (|\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2) - \frac{\beta^{2^*}}{2^*} \int Q_M u_\varepsilon^{2^*} \right] \\
 &\quad + o(\varepsilon^2) + O(\varepsilon^{(N-2)/2}).
 \end{aligned}$$

But we can show, as in Section 2, that

$$\sup_{\beta \geq 0} \left[\frac{\beta^2}{2} \int (|\nabla u_\varepsilon|^2 - \lambda u_\varepsilon^2) - \frac{\beta^{2^*}}{2^*} \int Q_M u_\varepsilon^{2^*} \right] \leq \frac{S^{N/2}}{NQ_M^{(N-2)/2}} - \lambda C \varepsilon^2 + O(\varepsilon^{N-2}),$$

for $N \geq 5$.

Therefore, we have

$$\begin{aligned}
 \sup_{\alpha, \beta \in [1/2, 2]} I_\lambda(\alpha u^j - \beta u_\varepsilon) &\leq m_\lambda^j + \frac{S^{N/2}}{NQ_M^{(N-2)/2}} - \lambda C \varepsilon^2 + O(\varepsilon^{N-2}) \\
 &\quad + o(\varepsilon^2) + O(\varepsilon^{(N-2)/2}),
 \end{aligned}$$

which shows that, for $N \geq 7$, and ε small,

$$\sup_{\alpha, \beta \in [1/2, 2]} I_\lambda(\alpha u^j - \beta u_\varepsilon) < m_\lambda^j + \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

This completes the proof of (ii). $\square$

By Lemmas 2.2 and 3.1 it follows that there exists a $\lambda_0 \in (0, \lambda_1)$ such that

$$\bar{M}_\lambda^j > M_\lambda^j$$

for all $\lambda \in (0, \lambda_0]$. A similar argument to the one used in Lemma 2.4 may be used to establish the following lemma.

Lemma 3.2. *Assume condition (Q) holds. Then there exists a $\lambda_0 \in (0, \lambda_1)$ and a sequence $\{u_n^j\} \subset U_\lambda^j$, for each $j = 1, 2, \dots, k$, satisfying*

$$(3.18) \quad \begin{aligned} I_\lambda(u_n^j) &\rightarrow M_\lambda^j, \\ I'_\lambda(u_n^j) &\rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, for $\lambda \in (0, \lambda_0)$ and $j = 1, 2, \dots, k$.

Proof of Theorem 1.2. From Lemma 2.3 and Lemma 3.2 we conclude that there exist a $\lambda_0 \in (0, \lambda_1)$ and a sequence $\{u_n^j\} \subset U_\lambda^j$, for each $j = 1, 2, \dots, k$, satisfying (3.18) and

$$(3.19) \quad u_n^j \rightarrow u^j \quad \text{strongly in } H_0^1(\Omega).$$

From (3.18) and (3.19) we have that u^j , $j = 1, \dots, k$, is a weak solution of (1.1). Since for $\lambda < \lambda_1$ there exists a positive constant δ , independent of u , such that $\|u^\pm\| \geq \delta > 0$ for all $u^\pm \in \Sigma_\lambda$, $(u^j)^\pm \neq 0$. Finally, u^j , $j = 1, 2, \dots, k$, are distinct nodal solutions since $g(u^j) \in B_\ell(a^j)$ and the $B_\ell(a^j)$ are disjoint. This completes the proof of Theorem 1.2. $\square$

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