Global Existence and Asymptotic Stability to the Full 1D Hydrodynamic Model for Semiconductor Devices

BO ZHANG

ABSTRACT. The full one-dimensional hydrodynamic model with small viscosity for semiconductor devices is studied. The model is self-consistent in the sense that the electric field, which forms a forcing term in the momentum and energy equations, is determined by the coupled Poisson equation. Global existence is demonstrated for the model, which is shown to be equivalent to a non-standard integro-differential equations. Finally, the asymptotic behavior of the smooth solution to the model is investigated.

1. Formulation and summary of results. We consider the following system of partial differential equations:

$$(1) \quad \rho_t + v\rho_x = -v_x\rho,$$

$$(2) \quad v_t + vv_x + \frac{1}{m\rho}(\rho T)_x - \frac{\varepsilon}{\rho}v_{xx} = \frac{e}{m}\varphi_x - \frac{v}{\tau_p},$$

$$(3) \quad T_t + vT_x + \frac{2}{3}Tv_x - \frac{2}{3\rho}(\kappa T_x)_x = \frac{mv^2(2\tau_w - \tau_p)}{3\tau_p\tau_w} - \frac{T - T^{(0)}}{\tau_w},$$

$$(4) \quad \frac{1}{e}\varphi_{xx} = \rho - C(x, t),$$

where ε is a positive parameter which measures the “viscosity.”

We view the system of the equations (1)–(4) as a singular perturbation [5] of the full one-dimensional hydrodynamic equations for semiconductor devices in

the case of one carrier type (e.g., electrons) [2, 9]:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ p_t + (pv + \rho T)_x = e\rho\varphi_x - \frac{p}{\tau_p}, \\ W_t + (vW + \rho vT)_x - (\kappa T_x)_x = e\rho v\varphi_x - \frac{3\rho(T - T^{(0)}) + m\rho v^2}{2\tau_w}, \\ \varphi_{xx} = e(\rho - C(x, t)), \end{cases}$$

where $\rho(x, t)$, $v(x, t)$, $T(x, t)$, and $\varphi(x, t)$ denote the electron density, velocity, temperature, and electrostatic potential, respectively; p and W are called the momentum density and energy density, respectively. They are given by the following equations:

$$\begin{aligned} p &= m\rho v, \\ W &= \frac{3}{2}\rho T + \frac{1}{2}m\rho v^2, \end{aligned}$$

where the positive constants e and m are the electronic charge and the effective electron mass, respectively. The dielectric constant has been set to 1. The thermal conductivity κ is given by a smooth function $\kappa = \kappa(\rho) > 0$ on $\mathcal{R}^+$. The positive constant $T^{(0)}$ is the ambient device temperature; $\tau_p = \tau_p(\rho, v, T)$ and $\tau_w = \tau_w(\rho, v, T)$ are given functions on $\mathcal{R}^+ \times \mathcal{R} \times \mathcal{R}^+$ called the momentum and energy relaxation times. The device domain is the x -interval $(0, 1)$. The given function $C(x, t)$ is the doping profile (background ion density).

The hydrodynamic model [2] for semiconductor devices plays an important role in simulating the behavior of the charge carrier in submicron semiconductor devices, since this model exhibits velocity overshoot which is not accounted for in the classical drift-diffusion model [1]. Some of its computational and physical aspects have been discussed by [6, 19]. For steady-state subsonic electron flow in the full one-dimensional hydrodynamic model, Gardner et al. [11] proved the existence of solutions and the convergence of Newton's method. Also, Gardner [9] first simulated an electron shock wave in the case of steady-state of the full one-dimensional hydrodynamic model for submicrometer semiconductor devices. A mathematical analysis of a simplified one-dimensional hydrodynamic model for semiconductors, where the energy equation (3) is replaced by a pressure-density relationship, has been presented by Gamba [8] in steady-state and by Zhang [21] in evolution. In this paper we investigate the regularized system of equations (1)–(4), which is a set of compressible Navier-Stokes equations [3, 12, 15, 18] coupled to the Poisson equation. Novel elements in (2)–(3) include an electric force term and relaxation approximations to represent averaged collisions. Notice that the electric force term is nonlocal.

From a mathematical point of view, we prescribe the following impermeably insulated initial-boundary value conditions:

$$(5) \quad (\rho, v, T)|_{t=0} = (\rho_0(x), v_0(x), T_0(x)), \quad x \in [0, 1],$$

$$(6) \quad v(0, t) = v(1, t) = 0, \quad T_x(0, t) = T_x(1, t) = 0, \quad t \geq 0,$$

$$(7) \quad \varphi_x(0, t) = 0, \quad \varphi(1, t) = \varphi_1, \quad t \geq 0,$$

where φ_1 is the applied bias.

We give the following notations that are used in this paper. For an arbitrary time $S \in (0, \infty)$, set $Q = (0, 1) \times (0, S]$ and let

$$C^{n+\alpha, 1+\alpha/2}(\bar{Q}) := \left\{ f(x, t) : \|f\|_{k+\alpha, \bar{Q}} = \sum_{i+nj=0}^n |\partial_x^i \partial_t^j f|_{\alpha, \bar{Q}} < \infty \right\}$$

for $n = 1, 2, 3$, where

$$|f|_{\alpha, \bar{Q}} = \sup_{\bar{Q}} |f| + \sup_{(x, t), (x', t') \in \bar{Q}} \frac{|f(x, t) - f(x', t')|}{|x - x'|^\alpha + |t - t'|^{\alpha/2}}, \quad 0 < \alpha < 1.$$

We also set

$$H_\alpha(\bar{Q}) := \left\{ f(x, t) : f, f_x \in C^{\alpha, \alpha/2}(\bar{Q}) \right\},$$

normed by

$$\|f\|_{H_\alpha(\bar{Q})} = |f|_{\alpha, \bar{Q}} + |f_x|_{\alpha, \bar{Q}}.$$

In order to obtain a solution of (1)–(4) with the desired regularity, we need the following assumptions:

(A₁) Relaxation times τ_p and τ_w satisfy uniform Lipschitz conditions on $\mathcal{R}^+ \times \mathcal{R} \times \mathcal{R}^+$, $0 < \delta \leq \tau_p$, $\tau_w \leq L$ for some positive constants δ and L , and $0 \leq 2\tau_w - \tau_p$; $\kappa \in C^2(\mathcal{R}^+)$ and $\delta \leq \kappa(\rho) \leq L$ on $\mathcal{R}^+$. Moreover, $C(x, t) \in C^0(\bar{Q})$.

(A₂) $\rho_0(x) \in C^{1+\alpha}[0, 1]$, $v_0(x) \in C^{2+\alpha}[0, 1]$, $T_0(x) \in C^{2+\alpha}[0, 1]$, $0 < \delta \leq \rho_0(x) \leq L$, $\delta \leq T_0(x) \leq L$ on $[0, 1]$. The initial-boundary value conditions satisfy the following compatibility conditions:

$$v_0(i) = T'_0(i) = 0 \quad \text{for } i = 0, 1,$$

where $(\cdot)' = \frac{d}{dx}(\cdot)$.

The assumptions on τ_p and τ_w are reasonable in semiconductor devices. For example, τ_p and τ_w satisfy the assumption (A₁) for a MOSFET (Metal Oxide Semiconductor Field Effect Transistor [10]) device.

The main results of this paper are as follows.

Theorem 1. *Assume that (A₁) and (A₂) hold. Then, for an arbitrary time $S \in (0, \infty)$, there exists a unique solution (ρ, v, T, φ) of the initial-boundary value problem (1)–(7) satisfying*

$$\begin{aligned} \rho &\in C^{1+\alpha, 1+\alpha/2}(\bar{Q}), & v &\in C^{2+\alpha, 1+\alpha/2}(\bar{Q}), \\ T &\in C^{2+\alpha, 1+\alpha/2}(\bar{Q}), & \varphi &\in C^{3+\alpha, 1+\alpha/2}(\bar{Q}), \\ \rho(x, t) &> 0, & T(x, t) &> 0, \quad (x, t) \in \bar{Q}. \end{aligned}$$

Theorem 2. *Assume that the conditions of Theorem 1 hold and that $\rho_0(x) = C(x)$ for $x \in [0, 1]$. Then, the solution (ρ, v, T, φ) of the initial-boundary problem (1)–(7) is asymptotically stable in the following sense:*

$$\begin{aligned} (8) \quad & \|\rho\|_{L^1(0,1)}(t) = \|\rho_0\|_{L^1(0,1)}, & t &\geq 0, \\ (9) \quad & v^2(x, t) \rightarrow 0, \quad x \in [0, 1], & t &\rightarrow \infty, \\ (10) \quad & T(x, t) \rightarrow T^{(0)}, \quad x \in [0, 1], & t &\rightarrow \infty, \\ (11) \quad & \|\varphi_x\|_{L^2(0,1)}^2(t) = f(t), & t &\geq 0, \end{aligned}$$

where $f(t) = \int_0^1 \left(\int_0^t \rho v dt \right)^2 dx$ satisfying $f(\infty) < \infty$.

The plan of this paper is as follows. In Section 2 we reduce the system of the equations (1)–(4) to a pair of integro-differential equations (29)–(30) by using the characteristics method. We obtain *a priori* estimates by applying the energy method and techniques used in Section 3. We then show the existence of a globally smooth solution of (29)–(30) by using the Leray-Schauder fixed point theorem in Section 4, thereby proving Theorem 1. In Section 5 we show that the solution of (1)–(4) converges to a steady-state solution as $t \rightarrow \infty$.

2. Characteristic method. In this section we use the characteristic method [13, 20] to reduce the system of equations (1)–(4) to a system of parabolic equations. Consider the following characteristic equation for (1) as a linear equation with respect to ρ :

$$(12) \quad \begin{cases} \frac{d}{d\sigma} \bar{y}(\sigma; x, t) = v(\bar{y}(\sigma; x, t), \sigma), & 0 \leq \sigma \leq t \leq S, \\ \bar{y}(t; x, t) = x. \end{cases}$$

Let $\bar{y}(\sigma; x, t)$ be the solution of (12). We then have a coordinate transformation as follows:

$$(13) \quad y = y(x, t) \equiv \bar{y}(0; x, t), \quad s = s(x, t) \equiv t.$$

This transformation $(x, t) \mapsto (y(x, t), s)$ is a one-to-one mapping from $\bar{Q}$ onto itself and keeps the same boundaries $\{0\} \times [0, T] \cup \{1\} \times [0, T]$ since $v(0, t) = v(1, t) = 0$. We use (y, s) as the characteristic coordinate, x and t are the inverse expressed by

$$(14) \quad x = x(y, s) \equiv \bar{y}(s; y, 0), \quad s = s(x, t) \equiv t.$$

For the function $v(x, t)$, we introduce a notation $\hat{v}(y, s) \equiv v(x(y, s), s)$, then

$$(15) \quad \begin{cases} \frac{d}{ds} x(y, s) = \hat{v}(y, s) \\ x(y, 0) = y \end{cases}.$$

It follows that

$$x(y, s) = y + \int_0^s \hat{v}(y, \sigma) d\sigma.$$

Then,

$$(16) \quad \frac{\partial x}{\partial y} = 1 + \int_0^s \hat{v}_y(y, \sigma) d\sigma.$$

For simplicity, we denote

$$\theta = \theta(\hat{v}) \equiv \int_0^s \hat{v}_y(y, \sigma) d\sigma.$$

Differentiating (12) with respect to x , we have

$$\begin{cases} \frac{d}{d\sigma} \bar{y}_x(\sigma; x, t) = v_x(\bar{y}(\sigma; x, t)) \bar{y}_x(\sigma; x, t), \\ \bar{y}_x(t; x, t) = 1. \end{cases}$$

Then,

$$\bar{y}_x(0; x, t) = e^{-\int_0^t v_x(\bar{y}(\sigma; x, t)) d\sigma}.$$

By (13) and (16), we have

$$(17) \quad \frac{\partial y}{\partial x} = \frac{1}{1 + \theta} = e^{-\int_0^t v_x(\bar{y}(\sigma; x, t)) d\sigma} > 0.$$

Let $(x, t) = (x(y, s), s)$. From the chain rule, the system of equations (1)–(4) reads:

$$(18) \quad \hat{\rho} = -\frac{\hat{v}_y}{1+\theta}\hat{\rho},$$

$$(19) \quad \hat{v}_s + \frac{1}{m\hat{\rho}(1+\theta)}(\hat{\rho}\hat{T})_y - \frac{\varepsilon}{\hat{\rho}(1+\theta)}\left(\frac{\hat{v}_y}{1+\theta}\right)_y = \frac{e}{m}\frac{\hat{\varphi}_y}{1+\theta} - \frac{\hat{v}}{\tau_p},$$

$$(20) \quad \begin{aligned} \hat{T}_s + \frac{2}{3(1+\theta)}\hat{v}_y\hat{T} - \frac{2}{3\hat{\rho}(1+\theta)}\left(\kappa\frac{\hat{T}_y}{1+\theta}\right)_y \\ = \frac{m\hat{v}^2(2\tau_w - \tau_p)}{3\tau_p\tau_w} - \frac{\hat{T} - T^{(0)}}{\tau_w}, \end{aligned}$$

$$(21) \quad \frac{1}{1+\theta}\left(\frac{\varphi_y}{1+\theta}\right)_y = e(\rho - C).$$

The initial-boundary value conditions (5)–(7) are translated into similar conditions:

$$(22) \quad (\hat{\rho}, \hat{v}, \hat{T})|_{s=0} = (\rho_0(y), v_0(y), T_0(y)), \quad y \in [0, 1],$$

$$(23) \quad \hat{v}(0, s) = \hat{v}(1, s) = 0, \quad \hat{T}_y(0, s) = \hat{T}_y(1, s) = 0, \quad s \geq 0,$$

$$(24) \quad \hat{\varphi}_y(0, s) = 0, \quad \hat{\varphi}(1, s) = \varphi_1(s), \quad s \geq 0.$$

There exists a unique solution of (18) and (22):

$$(25) \quad \hat{\rho}(y, s) = \frac{1}{1+\theta}\rho_0(y).$$

Substituting (25) for $\hat{\rho}$ in (21) and solving (21) and (24), we have

$$(26) \quad \frac{\hat{\varphi}_y}{1+\theta} = e(\bar{\rho}_0 - D)$$

where $\bar{\rho} = \int_0^y \rho_0(\xi) d\xi$ and $D = \int_0^y C(\xi, s) d\xi$.

Finally, substituting (25) and (26) for $\hat{\rho}$ and $\hat{\varphi}_y/(1+\theta)$ in the equations (19)–(20), we obtain

$$(27) \quad \hat{v}_s + \frac{1}{m\rho_0}\left(\frac{f_0}{1+\theta}\hat{T}\right)_y - \frac{\varepsilon}{\rho_0}\left(\frac{\hat{v}_y}{1+\theta}\right)_y = \frac{e^2}{m}(\bar{\rho}_0 - D) - \frac{\hat{v}}{\tau_p},$$

$$(28) \quad \hat{T}_s + \frac{2\hat{v}_y\hat{T}}{3(1+\theta)} - \frac{2}{3\rho_0}\left(\kappa\frac{\hat{T}_y}{1+\theta}\right)_y = \frac{m\hat{v}^2(2\tau_w - \tau_p)}{3\tau_p\tau_w} - \frac{\hat{T} - T^{(0)}}{\tau_w}.$$

Thus, the system of equations (1)–(4) is reduced to a system of nonlinear integro-differential equations (27)–(28). Since $1/(1+\theta) > 0$, we may visualize the integro-differential equations (27)–(28) as a couple of parabolic equations. For simplicity, we drop the hat “^” in the subsequent sections.

3. A priori estimates. We consider the following initial-boundary value problems in the region $Q = (0, 1) \times (0, S]$:

$$(29) \quad v_s + \frac{1}{m\rho_0} \left(\frac{\rho_0}{1+\theta} T \right)_y - \frac{\varepsilon}{\rho_0} \left(\frac{v_y}{1+\theta} \right)_y = \frac{e^2}{m} (\bar{\rho}_0 - D) - \frac{v}{\tau_p},$$

$$(30) \quad T_s + \frac{2v_y T}{3(1+\theta)} - \frac{2}{3\rho_0} \left(\frac{\kappa T_y}{1+\theta} \right)_y = \frac{mv^2(2\tau_w - \tau_p)}{3\tau_p\tau_w} - \frac{T - T^{(0)}}{\tau_w},$$

$$(31) \quad v(y, 0) = v_0(y), \quad T(y, 0) = T_0(y),$$

$$(32) \quad v(0, s) = v(1, s) = 0, \quad T_y(0, s) = T_y(1, s) = 0,$$

Let $\{v, T\}$ be a solution of (29)–(32). From the energy method, we have the following estimates.

Lemma 1. *There is a positive constant M_1 depending only on initial data such that*

$$(33) \quad \int_0^1 \rho_0 v^2 dy + \frac{3}{m} \int_0^1 \rho_0 T dy \leq M_1,$$

$$(34) \quad \varepsilon \int_0^s \int_0^1 \frac{v_y^2}{1+\theta} dy ds \leq M_1.$$

Proof. Multiplying (29) and (30) by $\rho_0 v$ and $\frac{3}{2m} \rho_0$, respectively, and integrating over $(0, 1)$, we have

$$(35) \quad \frac{1}{2} \frac{d}{ds} \int_0^1 \rho_0 v^2 dy - \frac{1}{m} \int_0^1 \frac{\rho_0 v_y T}{1+\theta} dy + \int_0^1 \left(\frac{\varepsilon v_y^2}{1+\theta} + \frac{\rho_0 v^2}{\tau_p} \right) dy$$

$$= \frac{e^2}{m} \int_0^1 \rho_0 (\bar{\rho}_0 - D) v dy,$$

$$(36) \quad \frac{3}{2m} \frac{d}{ds} \int_0^1 \rho_0 T dy + \frac{1}{m} \int_0^1 \frac{\rho_0 v_y T}{1+\theta} dy - \int_0^1 \frac{\rho_0 v^2 (2\tau_w - \tau_p)}{2\tau_p\tau_w} dy$$

$$= -\frac{3}{2m} \int_0^1 \frac{\rho_0 (T - T^{(0)})}{\tau_w} dy.$$

Adding (36) to (35), we obtain

$$\begin{aligned}
 (37) \quad & \frac{1}{2} \int_0^1 \rho_0 \left(v^2 + \frac{3}{m} T \right) dy + \int_0^s \int_0^1 \left(\frac{\varepsilon v_y^2}{1+\theta} + \frac{\rho_0 v^2}{2\tau_w} + \frac{3\rho_0 T}{2m\tau_w} \right) dy ds \\
 &= \frac{1}{2} \int_0^1 \rho_0 \left(v_0^2 + \frac{3}{m} T_0 \right) dy + \frac{3T^{(0)}}{2m} \int_0^s \int_0^1 \frac{dy ds}{\tau_w} \\
 &\quad + \frac{e^2}{m} \int_0^s \int_0^1 \rho_0 (\bar{\rho}_0 - D) v dy ds.
 \end{aligned}$$

The following, slightly weaker, inequality follows from (37):

$$\begin{aligned}
 (38) \quad & \frac{1}{2} \int_0^1 \rho_0 \left(v^2 + \frac{3}{m} T \right) dy + \int_0^s \int_0^1 \frac{\varepsilon v_y^2}{1+\theta} dy ds \\
 &\quad + \frac{1}{4} \int_0^s \int_0^1 \rho_0 \left(\frac{\rho_0 v^2}{\tau_w} + \frac{6\rho_0 T}{m\tau_w} \right) dy ds \\
 &\leq \frac{1}{2} \int_0^1 \rho_0 \left(v_0^2 + \frac{3}{m} T_0 \right) dy + \frac{3T^{(0)}}{2m} \int_0^s \int_0^1 \frac{dy ds}{\tau_w} \\
 &\quad + \frac{e^2}{m} \int_0^s \int_0^1 \rho_0 (\bar{\rho}_0 - D)^2 \tau_w dy ds \equiv M_1.
 \end{aligned}$$

Therefore, (33) and (34) hold. $\square$

Lemma 2. *There exists a positive constant M_2 independent of ε such that*

$$\begin{aligned}
 (39) \quad & \int_0^1 \rho_0 [(1+\theta) - 1 - \ln(1+\theta)] dy + \int_0^1 \rho_0 (T - 1 - \ln T) dy \\
 &\quad + \int_0^s \int_0^1 \frac{\kappa T_y^2}{(1+\theta)T^2} dy ds \leq M_2.
 \end{aligned}$$

Proof. First, note that

$$\begin{aligned}
 \frac{1}{m} \int_0^1 \rho_0 \theta dy &= \frac{1}{m} \int_0^1 \rho_0 \int_0^s v_y d\sigma dy \\
 &\leq \frac{2}{m^2} \int_0^s \int_0^1 \frac{(\rho'_0)^2 \tau_w}{\rho_0} dy ds + \frac{1}{8} \int_0^s \int_0^1 \frac{\rho_0 v^2}{\tau_w} dy ds.
 \end{aligned}$$

Now, multiplying (30) by $3\rho_0/(2mT)$ and integrating over $(0, 1) \times (0, s)$, we have

$$(40) \quad \frac{1}{m} \int_0^1 \rho_0 \left(\ln(1 + \theta) + \frac{3}{2} \ln T \right) dy - \frac{1}{m} \int_0^s \int_0^1 \frac{\kappa T_y^2}{(1 + \theta)T^2} dy ds$$

$$= \int_0^s \int_0^1 \left(\frac{\rho_0 v^2 (2\tau_w - \tau_p)}{2\tau_p \tau_w T} - \frac{3\rho_0(T - T^{(0)})}{2m\tau_w T} \right) dy ds + \frac{3}{2m} \int_0^1 \rho_0 \ln T_0 dy.$$

Subtracting (40) from (38), we obtain

$$(41) \quad \frac{3}{2m} \int_0^1 \rho_0(T - 1 - \ln T) dy + \frac{1}{m} \int_0^1 \rho_0[(1 + \theta) - 1 - \ln(1 + \theta)] dy$$

$$+ \frac{1}{2} \int_0^1 \rho_0 v^2 dy + \int_0^s \int_0^1 \left(\frac{\varepsilon v_y^2}{1 + \theta} + \frac{\kappa T_y^2}{m(1 + \theta)T^2} \right) dy ds$$

$$+ \int_0^s \int_0^1 \left(\frac{\rho_0 v^2}{4\tau_w} + \frac{3\rho_0 T}{2m\tau_w} + \frac{\rho_0 v^2 (2\tau_w - \tau_p)}{2\tau_p \tau_w T} + \frac{3\rho_0 T^{(0)}}{2m\tau_w T} \right) dy ds$$

$$\leq M_1 + \frac{3}{2m} \int_0^s \int_0^1 \frac{\rho_0}{\tau_w} dy ds + \frac{2}{m^2} \int_0^s \int_0^1 \frac{(\rho'_0)^2 \tau_w}{\rho_0} dy ds$$

$$- \frac{3}{2m} \int_0^1 \rho_0 \ln T_0 dy + \frac{1}{8} \int_0^s \int_0^1 \frac{\rho_0 v^2}{\tau_w} dy ds$$

$$\equiv M_2 + \frac{1}{8} \int_0^s \int_0^1 \frac{\rho_0 v^2}{\tau_w} dy ds.$$

Notice that the function $h(z) = z - 1 - \ln z$ is nonnegative for $z \in \mathcal{R}^+$. It follows from (41) that (39) holds. $\square$

Lemma 3. *There exists a positive constant M_3 depending on ε such that*

$$(42) \quad \frac{1}{M_3} \leq 1 + \theta \leq M_3.$$

Proof. We consider the function $F(y, s)$ defined by

$$F(y, s) = \frac{1}{\varepsilon} \int_0^y \rho_0(\xi) (v_0(\xi) - v(\xi, s)) d\xi + \int_0^s \left(\frac{\rho_0 T}{\varepsilon m(1 + \theta)} - \frac{v_y}{1 + \theta} \right) \Big|_{y=0} d\sigma.$$

Then,

$$(43) \quad F_s = -\frac{v_y}{1+\theta} + \frac{1}{\varepsilon m} \frac{\rho_0 T}{1+\theta} + g,$$

where

$$g = \frac{1}{\varepsilon} \int_0^y \rho_0 \left(\frac{v}{\tau_p} - \frac{e^2}{m} (\bar{\rho}_0 - D) \right) dy.$$

From the fact that

$$v_y = (1+\theta)_s$$

and (43), we have the following ordinary differential equation:

$$(44) \quad \begin{cases} \frac{d}{ds}(1+\theta) + (F_s - g)(1+\theta) = \frac{1}{\varepsilon m} \rho_0 T, \\ (1+\theta)|_{s=0} = 1. \end{cases}$$

It follows that

$$(45) \quad 1+\theta = e^{-F(y,s) + \int_0^s g \, ds} \left(1 + \frac{\rho_0}{\varepsilon m} \int_0^s T e^{F(y,s) - \int_0^s g \, d\sigma} \, ds \right).$$

In order to get (42), we first estimate $F(y, s)$ and $\int_0^s g \, d\sigma$. By Lemma 1, we have

$$\begin{aligned} \sup \left| \int_0^s g \, d\sigma \right| &\leq C(\varepsilon), \\ |F(y, s) - F(1, s)| &= \frac{1}{\varepsilon} \left| \int_y^1 \rho_0 (v_0 - v) \, dy \right| \leq C(\varepsilon). \end{aligned}$$

We claim that $|F(1, s)|$ is bounded. In fact, it follows from (44) and Lemma 1 that

$$\begin{aligned} (46) \quad &\int_0^s \int_0^1 \left[\frac{d}{ds}(1+\theta) + (1+\theta)F_s \right] dy \, ds \\ &= \int_0^s \int_0^1 \left[\frac{1}{\varepsilon m} \rho_0 T + g(1+\theta) \right] dy \, ds \\ &= \int_0^s \int_0^1 \left(\frac{1}{\varepsilon m} \rho_0 T + g - g_y \int_0^s v \, d\sigma \right) dy \, ds \leq C(\varepsilon). \end{aligned}$$

The left-hand side of (46) can be rewritten as follows:

$$\begin{aligned}
 (47) \quad & \int_0^s \int_0^1 \left[\frac{d}{ds}(1 + \theta) + (1 + \theta)F_s \right] dy ds \\
 &= \int_0^s \int_0^1 v_y(1 - F) dy ds + \int_0^1 (1 + \theta)F dy \\
 &= \int_0^1 F dy + \int_0^1 \left(\int_0^s v ds \right)_y F dy + \int_0^s \int_0^1 v F_y dy ds \\
 &= F(1, s) - \frac{1}{\varepsilon} \int_0^1 \rho_0(v_0 - v) \left(y + \int_0^s v ds \right) dy \\
 &\quad + \frac{1}{\varepsilon} \int_0^s \int_0^1 \rho_0 v(v_0 - v) dy ds.
 \end{aligned}$$

It follows from (46)–(47) and Lemma 1 that

$$|F(1, s)| \leq C(\varepsilon), \quad 0 \leq s \leq S.$$

Thus,

$$(48) \quad \sup_{\bar{Q}} |F(y, s)| \leq \sup_{\bar{Q}} |F(y, s) - F(1, s)| + \sup_{0 \leq s \leq 1} |F(1, s)| \leq C(\varepsilon).$$

It follows from (45) and (48) that

$$1 + \theta \geq e^{-C(\varepsilon)} \left(1 + \frac{\rho_0}{\varepsilon m} \int_0^s e^{-C(\varepsilon)T} ds \right) \geq e^{-C(\varepsilon)}.$$

Similarly, we obtain the upper bound of $1 + \theta$. By Lemma 2, for every $s \in [0, S]$, there exists a $y(s) \in [0, 1]$ such that

$$R_1 \leq T(y(s), s) \leq R_2,$$

where R_1 and R_2 are the two positive roots of the equation $z - 1 - \ln z = M_2/\delta$. Then,

$$\begin{aligned}
 (49) \quad \sqrt{T(y, s)} &= \sqrt{T(y(s), s)} + \frac{1}{2} \int_{y(s)}^y \frac{T_y}{\sqrt{T}} dy \\
 &\leq \sqrt{R_2} + \frac{1}{2} \int_0^1 \left| \frac{\sqrt{\kappa} T_y}{\sqrt{1 + \theta} T} \frac{\sqrt{1 + \theta} \sqrt{T}}{\sqrt{\kappa}} \right| dy \leq
 \end{aligned}$$

$$\begin{aligned}
&\leq \sqrt{R_2} + \frac{1}{2} \left(\int_0^1 \frac{\kappa T_y^2}{(1+\theta)T^2} dy \right)^{1/2} \left(\int_0^1 \frac{(1+\theta)T}{\kappa} dy \right)^{1/2} \\
&\leq \sqrt{R_2} + \frac{1}{2} \left(\frac{M_1}{\delta} U(s) \max_{0 \leq y \leq 1} (1+\theta) \right)^{1/2} \quad (\text{Lemma 1}),
\end{aligned}$$

where

$$U(s) = \int_0^1 \frac{\kappa T_y^2}{(1+\theta)T^2} dy.$$

It follows from (45) and (49) that there are two positive constants N_1 and N_2 depending on ε such that

$$\max_{0 \leq y \leq 1} (1+\theta) \leq N_1 + N_2 \int_0^s U(s) \max_{0 \leq y \leq 1} (1+\theta) ds.$$

By Gronwall's inequality and Lemma 2, we have

$$\max_{0 \leq y \leq 1} (1+\theta) \leq N_1 e^{N_2 \int_0^s U(s) ds} \leq N_1 e^{C_2 M_2}.$$

Let $M_3 = \max\{N_1 e^{N_2 M_2}, e^{C(\varepsilon)}\}$, then (42) holds. $\square$

Lemma 4. *There exists a positive constant M_4 depending on ε such that*

$$(50) \quad \int_0^1 (v^4 + T^2) dy + \int_0^s \int_0^1 (v^2 v_y^2 + T_y^2) dy ds \leq M_4.$$

Proof. From (29) and (30), we have

$$\begin{aligned}
&\left(\frac{1}{2} \rho_0 v^2 + \frac{3}{2m} \rho_0 T \right)_s + \frac{1}{m} \left(\frac{\rho_0}{1+\theta} v T \right)_y - \varepsilon \left(\frac{v_y}{1+\theta} \right)_y v - \frac{1}{m} \left(\frac{\kappa}{1+\theta} T_y \right)_y \\
&= \frac{e^2}{m} \rho_0 v (\bar{\rho}_0 - D) - \frac{\rho_0 v^2}{2\tau_w} - \frac{3\rho_0 T}{2m\tau_w} + \frac{3\rho_0 T^{(0)}}{2m\tau_w}.
\end{aligned}$$

Multiplying the equality above by

$$\left(\frac{1}{2} \rho_0 v^2 + \frac{3}{2m} \rho_0 T \right)$$

and using integration by parts, we obtain

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{ds} \int_0^1 \left(\frac{1}{2} \rho_0 v^2 + \frac{3}{2m} \rho_0 T \right)^2 dy - \frac{1}{m} \int_0^1 \frac{\rho_0 v T}{1 + \theta} \left(\frac{1}{2} \rho_0 v^2 + \frac{3}{2m} \rho_0 T \right)_y dy \\
 & + \varepsilon \int_0^1 \frac{v_y}{1 + \theta} \left(\frac{1}{2} \rho_0 v^3 + \frac{3}{2m} \rho_0 v T \right)_y dy + \frac{1}{m} \int_0^1 \frac{\kappa T_y}{1 + \theta} \left(\frac{1}{2} \rho_0 v^2 + \frac{3}{2m} \rho_0 T \right)_y dy \\
 & = \int_0^1 \left[\frac{e^2}{m} \rho_0 v (\bar{\rho}_0 - D) - \frac{\rho_0 v^2}{2\tau_w} - \frac{3\rho_0 T}{2m\tau_w} + \frac{3\rho_0 T^{(0)}}{2m\tau_w} \right] \left(\frac{1}{2} \rho_0 v^2 + \frac{3}{2m} \rho_0 T \right) dy.
 \end{aligned}$$

Differentiate out the terms in the integrals, use the bounds on ρ_0 and $1 + \theta$, and apply Lemma 1, along with the usual inequalities for the product of two terms. After this tedious calculation, it can be seen that

$$\begin{aligned}
 (51) \quad & \int_0^1 (v^4 + T^2) dy + \int_0^s \int_0^1 (v^2 v_y^2 + T_y^2 + T v_y^2) dy ds \\
 & \leq N_1 + N_2 \int_0^s \int_0^1 [v^2 v_y^2 + v^4 + (1 + |v|)^2 T^2] dy,
 \end{aligned}$$

where N_1 and N_2 depend on ε and the data.

Next, multiply (29) by $\rho_0 v^3$ and integrate it over $(0, 1)$ to see that

$$\begin{aligned}
 & \frac{1}{4} \int_0^1 \rho_0 v^4 dy + \int_0^s \int_0^1 \left(\frac{3\varepsilon v^2 v_y^2}{1 + \theta} - \frac{3\rho_0 T v^2 v_y}{m(1 + \theta)} + \frac{\rho_0 v^4}{\tau_p} \right) dy ds \\
 & = \int_0^1 \rho_0 v_0^4 dy + \frac{e^2}{m} \int_0^s \int_0^1 \rho_0 v^3 (\bar{\rho}_0 - D) dy ds.
 \end{aligned}$$

By a similar argument, there exist positive constants N_3 and N_4 depending on ε and the data such that

$$(52) \quad \int_0^1 v^4 dy + \int_0^s \int_0^1 v^2 v_y^2 dy ds \leq N_3 + N_4 \int_0^s \int_0^1 (v^4 + v^2 T^2) dy ds.$$

It follows from (51) and (52) that

$$\begin{aligned}
 (53) \quad & \int_0^1 (v^4 + T^2) dy + \int_0^s \int_0^1 (v^2 v_y^2 + T_y^2 + T v_y^2) dy ds \\
 & \leq N_5 + N_6 \int_0^s \int_0^1 \left(v^4 + (1 + v^2) T^2 \right) dy ds.
 \end{aligned}$$

Recall the interpolation inequality

$$T^2 \leq \beta \int_0^1 T_y^2 dy + C_\beta \int_0^1 T^2 dy.$$

As a consequence of it, (53), and Lemma 1, we have

$$\begin{aligned} & \int_0^1 (v^4 + T^2) dy + \int_0^s \int_0^1 (v^2 v_y^2 + T_y^2 + T v_y^2) dy ds \\ & \leq N_7 + N_8 \int_0^s \int_0^1 (v^4 + T^2) dy ds. \end{aligned}$$

It follows from Gronwall's inequality that (50) holds. $\square$

Lemma 5. *There exists a positive constant M_5 depending on ε such that*

$$(54) \quad \begin{aligned} \int_0^s \int_0^1 v_y^2 dy ds & \leq M_5, \\ \int_0^1 \theta_y^2 dy & \leq M_5. \end{aligned}$$

Proof. Multiplying (29) by $\rho_0 v$ and using integration by parts, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{ds} \int_0^1 \rho_0 v^2 dy + \varepsilon \int_0^1 \frac{v_y^2}{1+\theta} dy + \int_0^1 \frac{\rho_0 v^2}{\tau_p} dy \\ & = \frac{e^2}{m} \int_0^1 \rho_0 v (\bar{\rho}_0 - D) dy + \frac{1}{m} \int_0^1 \frac{\rho_0 v_y}{1+\theta} T dy. \end{aligned}$$

There exist positive constants N_1 and N_2 such that

$$\begin{aligned} \int_0^1 v^2 dy + \int_0^s \int_0^1 (\varepsilon v_y^2 + v^2) dy ds & \leq N_1 + N_2 \int_0^s \int_0^1 T^2 dy ds \\ & \leq N_1 + N_2 M_4, \end{aligned}$$

by Lemma 4.

In order to prove (54), we rewrite (29) as follows:

$$(55) \quad \frac{\varepsilon}{\rho_0} \left(\frac{\theta_y}{1+\theta} \right)_s + \frac{T}{m(1+\theta)} \left(\frac{\theta_y}{1+\theta} \right) =$$

$$= v_s + \frac{(\rho_0 T)_y}{m \rho_0 (1 + \theta)} - \frac{e^2}{m} (\bar{\rho}_0 - D) + \frac{v}{\tau_p}.$$

Multiplying (55) by $\theta_y/(1 + \theta)$ and integrating over $(0, 1) \times (0, s)$, we have

$$\begin{aligned} (56) \quad & \frac{\varepsilon}{2} \int_0^1 \frac{1}{\rho_0} \left(\frac{\theta_y}{1 + \theta} \right)^2 dy + \frac{1}{m} \int_0^s \int_0^1 \frac{T \theta_y^2}{(1 + \theta)^3} dy ds \\ &= \int_0^s \int_0^1 \left[\frac{(\rho_0 T)_y}{m \rho_0 (1 + \theta)} - \frac{e^2}{m} (\bar{\rho}_0 - D) + \frac{v}{\tau_p} \right] \frac{\theta_y}{1 + \theta} dy ds \\ & \quad + \int_0^s \int_0^1 v_s (\ln(1 + \theta))_y dy ds \\ &= \int_0^s \int_0^1 \left[\frac{(\rho_0 T)_y}{m \rho_0 (1 + \theta)} - \frac{e^2}{m} (\bar{\rho}_0 - D) + \frac{v}{\tau_p} \right] \frac{\theta_y}{1 + \theta} dy ds + \int_0^1 v \frac{\theta_y}{1 + \theta} dy \\ & \quad + \int_0^s \int_0^1 \frac{v_y^2}{1 + \theta} dy ds \\ &\leq \int_0^s \int_0^1 \left[\frac{(\rho_0 T)_y}{m \rho_0 (1 + \theta)} - \frac{e^2}{m} (\bar{\rho}_0 - D) + \frac{v}{\tau_p} \right] \frac{\theta_y}{1 + \theta} dy ds \\ & \quad + \int_0^s \int_0^1 \frac{v_y^2}{1 + \theta} dy ds + \frac{\varepsilon}{2} \int_0^1 \frac{1}{\rho_0} \left(\frac{\theta_y}{1 + \theta} \right)^2 dy + C(\varepsilon) \int_0^1 \rho_0 v^2 dy. \end{aligned}$$

It follows from (56), Lemma 1, Lemma 3, and Lemma 4 that there exist two positive constants N_3 and N_4 depending on ε such that

$$\int_0^1 \theta_y^2 dy \leq N_3 + N_4 \int_0^s \int_0^1 \theta_y^2 dy ds.$$

Use of Gronwall's inequality completes the proof of Lemma 5. $\square$

Lemma 6. *There exist positive constants M_6 and M_7 depending on ε such that*

$$\begin{aligned} \int_0^1 v_y^2 dy + \int_0^s \int_0^1 v_{yy}^2 dy ds &\leq M_6, \\ \int_0^1 T_y^2 dy + \int_0^s \int_0^1 T_{yy}^2 dy ds &\leq M_7. \end{aligned}$$

Proof. Multiplying (29) by v_{yy} and integrating over $(0, 1)$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{ds} \int_0^1 v_y^2 dy + \varepsilon \int_0^1 \frac{v_{yy}^2}{\rho_0(1+\theta)} dy \\ &= \varepsilon \int_0^1 \frac{\theta_y v_y v_{yy}}{\rho_0(1+\theta)^2} dy + \frac{1}{m} \int_0^1 \left(\frac{(\rho_0 T)_y}{\rho_0(1+\theta)} - \frac{T \theta_y}{(1+\theta)^2} \right) v_{yy} dy \\ & \quad - \int_0^1 \left[\frac{e^2}{m} (\bar{\rho}_0 - D) - \frac{v}{\tau_p} \right] v_{yy} dy. \end{aligned}$$

Using interpolation inequalities and Lemma 3, we obtain

$$\begin{aligned} & \int_0^1 v_y^2 dy + \int_0^s \int_0^1 v_{yy}^2 dy ds \leq N_1 + N_2 \int_0^s \int_0^1 (v_y^2 + T^2) \theta_y^2 dy ds \\ & \leq N_1 + N_2 \int_0^s \int_0^1 \left[\beta \int_0^1 v_{yy}^2 dy + C_\beta \int_0^1 v_y^2 dy \right. \\ & \quad \left. + 2 \int_0^1 T_y^2 dy + 2 \left(\int_0^1 T dy \right)^2 \right] \theta_y^2 dy ds. \end{aligned}$$

It follows from Lemma 4 and Lemma 5 that

$$\begin{aligned} \int_0^1 v_y^2 dy ds + \int_0^s \int_0^1 v_{yy}^2 dy ds & \leq 2N_1 + 2N_2(C_\beta M_5 + 2M_4 + 2M_1^2)M_5 \\ & \equiv M_6. \end{aligned}$$

Multiplying (30) by T_{yy} and integrating over $(0, 1)$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \int_0^1 T_y^2 dy + \frac{3}{2} \int_0^1 \frac{\kappa T_{yy}^2}{\rho_0(1+\theta)} dy &= \frac{2}{3} \int_0^1 \left(\frac{v_y T}{1+\theta} - \frac{1}{\rho_0} \left(\frac{\kappa}{1+\theta} \right)_y T_y \right) T_{yy} dy \\ & \quad + \int_0^1 \left(\frac{T - T^{(0)}}{\tau_w} - \frac{mv^2(2\tau_w - \tau_p)}{3\tau_p \tau_w} \right) T_{yy} dy. \end{aligned}$$

Then, it follows from the interpolation inequalities and Lemma 3, Lemma 4, and Lemma 5 that

$$\begin{aligned}
 & \int_0^1 T_y^2 dy + \int_0^s \int_0^1 T_{yy}^2 dy ds \\
 \leq & N_1 + N_2 \int_0^s \int_0^1 (T^2 v_y^2 + \theta_y^2 T_y^2) dy ds \\
 \leq & N_1 + 2N_2 \int_0^s \int_0^1 \left(\int_0^1 T_y^2 dy + \left(\int_0^1 T dy \right)^2 \right) v_y^2 dy ds \\
 & + N_2 \int_0^s \int_0^1 \left(\beta \int_0^1 T_{yy}^2 dy + C_\beta \int_0^1 T_y^2 dy \right) \theta_y^2 dy ds \\
 \leq & N_1 + 2N_2 M_6 (M_4 + M_1^2) + C_\beta N_2 M_4 M_5 + \beta N_2 \int_0^1 T_{yy}^2 dy \\
 \equiv & \frac{1}{2} M_7 + \frac{1}{2} \int_0^1 T_{yy}^2 dy.
 \end{aligned}$$

This completes the proof of Lemma 6. $\square$

Lemma 7. *The velocity v and temperature T are bounded; that is,*

$$|v| \leq (M_6)^{1/2}, \quad 0 < T \leq M_1 + (M_7)^{1/2}.$$

Proof. From Lemma 6, we have

$$v^2(x, t) = \left(\int_0^y v_y dy \right)^2 \leq \int_0^1 v_y^2 dy \leq M_6.$$

Note that

$$(T(y, s) - \int_0^1 T(z, s) dz)^2 \leq \left(\int_0^1 |T_y| dy \right)^2 \leq M_7$$

by Lemma 6. By Lemma 1,

$$\int_0^1 T(y, s) dy \leq M_1,$$

and the upper bound for T follows. $\square$

From the a priori estimates above and the standard method applied in [14, 16], we obtain the following Schauder estimates.

Lemma 8. *There exists a positive constant M_8 depending on ε such that*

$$\|v\|_{H_\alpha(Q)} + \|T\|_{H_\alpha(Q)} \leq M_8, \quad 0 < \alpha < 1.$$

We omit the proof of Lemma 8. For details, see Theorem 2 of [16].

Lemma 9. *There exists a positive constant M_9 such that*

$$T(y, s) \geq M_9 > 0, \quad (y, s) \in \bar{Q}.$$

Proof. Let $\omega = T^{-1}$ and multiply (30) by $-\omega^2$; we have

$$\begin{aligned} (57) \quad \omega_s &= \frac{2}{3\rho_0} \left(\frac{\kappa\omega_y}{1+\theta} \right)_y + \left(\frac{1}{\tau_w} + \frac{2v_y}{3(1+\theta)} \right) \omega \\ &\quad - \left[\frac{4\kappa\omega_y^2}{3\rho_0\omega^3(1+\theta)} + \frac{mv^2(2\tau_w - \tau_p)}{3\tau_p\tau_w} + \frac{T^{(0)}}{\tau_w} \right] \omega^2 \\ &\leq \frac{2}{3\rho_0} \left(\frac{\kappa\omega_y}{1+\theta} \right)_y + \left(\frac{1}{\tau_w} + \frac{2v_y}{3(1+\theta)} \right) \omega. \end{aligned}$$

For $\ell = 1, 2, 3, \dots$, multiply (57) by $2\ell\omega^{2\ell-1}$ to obtain

$$\begin{aligned} (58) \quad (\omega^{2\ell})_s &\leq 2\ell \left(\frac{1}{\tau_w} + \frac{2v_y}{3(1+\theta)} \right) \omega^{2\ell} + \frac{2}{3\rho_0} \left(\frac{\kappa(\omega^{2\ell})_y}{1+\theta} \right)_y \\ &\quad - 4\ell(2\ell-1) \frac{\kappa\omega_y^2}{3\rho_0(1+\theta)} \omega^{2(\ell-1)} \\ &\leq 2\ell \left(\frac{1}{\tau_w} + \frac{2v_y}{3(1+\theta)} \right) \omega^{2\ell} + \frac{2}{3\rho_0} \left(\frac{\kappa(\omega^{2\ell})_y}{1+\theta} \right)_y. \end{aligned}$$

Therefore,

$$2\ell \|\rho_0^{1/(2\ell)} \omega\|_{L^{2\ell}}^{2\ell-1} \frac{d}{ds} \|\rho_0^{1/(2\ell)} \omega\|_{L^{2\ell}} \leq 2\ell \int_0^1 \rho_0 \left(\frac{1}{\tau_w} + \frac{2v_y}{3(1+\theta)} \right) \omega^{2\ell} dy;$$

that is,

$$\frac{d}{ds} \|\rho_0^{1/(2\ell)} \omega\|_{L^{2\ell}}(s) \leq (\delta^{-1} + M_8 M_3) \|\rho_0^{1/(2\ell)} \omega\|_{L^{2\ell}}(s).$$

It follows from Gronwall's inequality that

$$\|\rho_0^{1/(2\ell)} \omega\|_{L^{2\ell}}(s) \leq \|\rho_0^{1/(2\ell)} \omega_0\|_{L^{2\ell}} e^{(\delta^{-1} + M_3 M_8)S}.$$

Letting $\ell \rightarrow \infty$, we obtain

$$\frac{1}{\min T(y, s)} \leq \frac{1}{\delta} e^{(\delta^{-1} + M_3 M_8)S}.$$

Thus,

$$\min T(y, s) \geq \delta e^{-(\delta^{-1} + M_3 M_8)S} \equiv M_9 > 0.$$

This completes the proof of Lemma 8. $\square$

4. Existence and uniqueness. In this section we apply the Leray-Schauder fixed point theorem to the integro-differential equations (29)–(30). First, we define a set

$$(59) \quad \mathbf{F} = \left\{ (\tilde{v}, \tilde{T}) : \frac{1}{M_3} \leq 1 + \tilde{\theta} \leq M_3, |\tilde{v}| \leq M_6, 0 \leq \tilde{T} \leq M_1 + (M_7)^{1/2}, \right. \\ \left. \|\tilde{v}, \tilde{T}\|_{H_\alpha(\bar{Q})} \leq M_8, \tilde{v}(y, 0) = v_0(y), \tilde{T}(y, 0) = T_0(y) \right\},$$

where

$$\tilde{\theta} = \int_0^s \tilde{v}_y d\sigma$$

and constants M_i have been determined in the previous section. It is clear that $\mathbf{F}$ is a closed subset of the Banach space $H_\alpha(\bar{Q})$.

We define an operator

$$\Sigma : (\tilde{v}, \tilde{T}, \mu) \mapsto (v, T)$$

by solving the following linear integro-differential equations:

(60)

$$\left\{ \begin{aligned} v_s - \frac{\varepsilon}{\rho_0(1+\tilde{\theta})}v_{yy} + \frac{v}{\tilde{\tau}_p} &= \frac{\mu(\rho_0\tilde{T} - m\varepsilon)}{m\rho_0(1+\tilde{\theta})^2}\theta(v_y) \\ &\quad + \mu \left[\frac{e^2}{m}(\tilde{v}\rho_0 - D) - \frac{(\rho_0\tilde{T})_y}{m\rho_0(1+\tilde{\theta})} \right], \\ T_s - \frac{2\tilde{\kappa}}{3\rho_0(1+\tilde{\theta})}T_{yy} + \frac{T - \mu T^{(0)}}{\tilde{\tau}_w} &= -\frac{2\mu\tilde{T}_y(\rho_0\tilde{\kappa}' + \tilde{\kappa}(1+\tilde{\theta}))}{3\rho_0(1+\tilde{\theta})^3}\theta(v_y) \\ &\quad + \mu \left[\frac{2\rho'_0\tilde{T}_y\tilde{\kappa}'}{3\rho_0(1+\tilde{\theta})^2} - \frac{2\tilde{T}\tilde{v}_y}{3(1+\tilde{\theta})} + \frac{m\tilde{v}^2(2\tilde{\tau}_w - \tilde{\tau}_p)}{3\tilde{\tau}_p\tilde{\tau}_w} \right], \end{aligned} \right.$$

with the following initial-boundary conditions:

$$\begin{aligned} v(y, 0) &= \mu v_0(y), \quad T(y, 0) = \mu T_0(y), \\ v(0, s) &= v(1, s) = 0, \quad T_y(0, s) = T_y(1, s) = 0, \end{aligned}$$

where $\mu \in [0, 1]$, $(\tilde{v}, \tilde{T}) \in \mathbf{F}$, $\tilde{\kappa} = \kappa(\rho_0/(1+\tilde{\theta}))$, $\tilde{\tau}_p = \tau_p(\rho_0/(1+\tilde{\theta}), \tilde{v}, \tilde{T})$, and $\tilde{\tau}_w = \tau_w(\rho_0/(1+\tilde{\theta}), \tilde{v}, \tilde{T})$.

We prove the following theorem to see that the linear operator Σ is well defined from the closed subset $\mathbf{F}$ of $H_\alpha(\bar{Q})$ into $C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ for every fixed $\mu \in [0, 1]$.

Theorem 3. *Assume that (A₁) and (A₂) hold. Then, for an arbitrary time $S \in (0, \infty)$, there exists a unique solution $(v, T) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ of (60) satisfying the following estimate:*

$$(61) \quad \|v\|_{2+\alpha, \bar{Q}} + \|T\|_{2+\alpha, \bar{Q}} \leq \overline{M}(\|v_0\|_{w+\alpha} + \|T_0\|_{2+\alpha} + \|\tilde{v}\|_{2+\alpha, \bar{Q}} + \|\tilde{T}\|_{2+\alpha, \bar{Q}}),$$

where constant $\overline{M}$ is independent of (v, T) .

Proof. For simplicity, we write the equations of (60) as follows:

$$\left\{ \begin{aligned} v_s - a_1 v_{yy} + \frac{v}{\tilde{\tau}_p} &= b_1 \int_0^s v_{yy} d\sigma + f_1, \\ T_s - a_2 T_{yy} + \frac{T - \mu T^{(0)}}{\tilde{\tau}_w} &= b_2 \int_0^s v_{yy} d\sigma + f_2, \end{aligned} \right.$$

where

$$\begin{aligned} a_1 &= \frac{\varepsilon}{\rho_0(1+\tilde{\theta})}, & a_2 &= \frac{2\tilde{\kappa}}{3\rho_0(1+\tilde{\theta})}, \\ b_1 &= \frac{\mu(\rho_0\tilde{T} - m\varepsilon)}{m\rho_0(1+\tilde{\theta})^2}, & b_2 &= -\frac{2\mu\tilde{T}_y(\rho_0\tilde{\kappa}^1 + \tilde{\kappa}(1+\tilde{\theta}))}{3\rho_0(1+\tilde{\theta})^3}, \\ f_1 &= \mu \left[\frac{e^2}{m}(\tilde{v}\rho_0 - D) - \frac{(\rho_0\tilde{T})_y}{m\rho_0(1+\tilde{\theta})} \right], \\ f_2 &= \mu \left[\frac{2\rho'_0\tilde{T}_y\tilde{\kappa}'}{3\rho_0(1+\tilde{\theta})^2} - \frac{2\tilde{T}\tilde{v}_y}{3(1+\tilde{\theta})} + \frac{m\tilde{v}^2(2\tilde{\tau}_w - \tilde{\tau}_p)}{3\tilde{\tau}_p\tilde{\tau}_w} \right]. \end{aligned}$$

Notice that a_i , b_i , f_i , $i = 1, 2$, are Hölder continuous by assumptions on the given functions and $(\tilde{v}, \tilde{T})$.

Suppose that $(v, T) \in C^{2+\alpha, 1+\alpha/2}$ satisfies (60). We can visualize the integro-differential equations of (60) as a couple of parabolic equations [4] with two source terms:

$$b_i \int_0^s v_{yy} d\sigma + f_i, \quad i = 1, 2.$$

Then, for every rectangle $Q_{s_0} \equiv (0, 1) \times (0, s_0)$, $s_0 \in (0, S]$, the following estimate [7, 17] holds:

$$\begin{aligned} (62) \quad \|v\|_{2+\alpha, \bar{Q}_{s_0}} + \|T\|_{2+\alpha, \bar{Q}_{s_0}} &\leq \overline{M}_1 \left(\|v_0\|_{2+\alpha} + \|T_0\|_{2+\alpha} + \sum_{i=1}^2 |f_i|_{\alpha, \bar{Q}_{s_0}} \right. \\ &\quad \left. + \left| \int_0^s v_{yy} d\sigma \right|_{\alpha, \bar{Q}_{s_0}} \sum_{i=1}^2 |b_i|_{\alpha, \bar{Q}_{s_0}} \right), \end{aligned}$$

where $\overline{M}_1$ is independent of (v, T) .

By assumption on $(\tilde{v}, \tilde{T})$, there exists a $s_0 > 0$ independent of (v, T) such that

$$\overline{M}_1 \left(\sum_{i=1}^2 |b_i|_{\alpha, \bar{Q}} \right) (s_0 + s_0^{1-\alpha/2}) \leq \frac{1}{2}.$$

For such a $s_0 > 0$, it follows from (62) that

$$\|v\|_{2+\alpha, \bar{Q}_{s_0}} + \|T\|_{2+\alpha, \bar{Q}_{s_0}} \leq 2\bar{M}_1 \left(\|v_0\|_{2+\alpha} + \|T_0\|_{2+\alpha} + \sum_{i=1}^2 |f_i|_{\alpha, \bar{Q}_{s_0}} \right).$$

In particular,

$$\|v(\cdot, s_0)\|_{2+\alpha} + \|T(\cdot, s_0)\|_{2+\alpha} \leq 2\bar{M}_1 \left(\|v_0\|_{2+\alpha} + \|T_0\|_{2+\alpha} + \sum_{i=1}^2 |f_i|_{\alpha, \bar{Q}_{s_0}} \right).$$

Repeating the same procedure for the rectangles $(0, 1) \times (ks_0, (k+1)s_0)$, $k = 1, 2, \dots, 1 + [\frac{S}{s_0}]$, we have

$$(63) \quad \begin{aligned} & \|v\|_{2+\alpha, \bar{Q}} + \|T\|_{2+\alpha, \bar{Q}} \\ & \leq (1 + 2\bar{M}_1)^{1+[S/s_0]} \left(\|v_0\|_{2+\alpha} + \|T_0\|_{2+\alpha} + \sum_{i=1}^2 |f_i|_{\alpha, \bar{Q}} \right). \end{aligned}$$

It follows from (63) that (61).

The uniqueness of the solution of (60) follows immediately from (63). To show the existence of the solution (60), we consider the one parameter family of operators for $\lambda \in [0, 1]$:

$$\begin{aligned} L_{1,\lambda}(v) &:= v_s - \left(\lambda a_1 + (1-\lambda) \frac{\varepsilon}{n_0} \right) v_{yy} + \lambda \left(\frac{v}{\tilde{\tau}_p} - b_1 \int_0^s v_{yy} d\sigma \right), \\ L_{2,\lambda}(T) &:= T_s - \left(\lambda a_2 + (1-\lambda) \frac{2\tilde{\kappa}}{3n_0} \right) T_{yy} + \lambda \left(\frac{T - \mu T^{(0)}}{\tilde{\tau}_w} - b_2 \int_0^s v_{yy} d\sigma \right). \end{aligned}$$

Let Ω be the set of all λ for which the problem

$$(64) \quad \begin{cases} L_{1,\lambda}(v) = f_1, \\ L_{2,\lambda}(T) = f_2, \\ v(y, 0) = \mu v_0(y), \quad T(y, 0) = \mu T_0(y), \quad y \in [0, 1], \\ v(0, s) = v(1, s) = 0, \\ T_y(0, s) = T_y(1, s) = 0, \end{cases}$$

has a unique solution $(v, T) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ for every couple of functions $(f_1, f_2) \in C^{\alpha, \alpha/2}(\bar{Q})$.

It is easy to check that the compatibility conditions of problem (64) hold for $\lambda = 0$. By the results of the standard parabolic equations, we have $0 \in \Omega$.
eject

To this end, we show that Ω is both open and closed. Suppose $\lambda_0 \in \Omega$. Given

$(v, T) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$, we define a linear operator $(\psi_1, \psi_2) = H_\lambda(v, T)$ to be the unique solution of the following problem:

$$(65) \quad \begin{cases} L_{1, \lambda_0}(\psi_1) = L_{1, \lambda_0}(v) - L_{1, \lambda}(v) + f_1, \\ L_{2, \lambda_0}(\psi_2) = L_{2, \lambda_0}(T) - L_{2, \lambda}(T) + f_2, \\ \psi_1(y, 0) = \mu v_0(y), \quad \psi_2(y, 0) = \mu T_0(y), \quad y \in [0, 1], \\ \psi_1(0, s) = \psi_1(1, s) = 0, \\ \psi_{2, y}(0, s) = \psi_{2, y}(1, s) = 0. \end{cases}$$

The operator $(\psi_1, \psi_2) = H_\lambda(v, T)$ is well defined since

$$L_{1, \lambda_0}(v) - L_{1, \lambda}(v) + f_1, \quad L_{2, \lambda_0}(T) - L_{2, \lambda}(T) + f_2 \in C^{\alpha, \alpha/2}(\bar{Q}).$$

Notice that (65) is of the form (60). As in (63), there exists a constant $\bar{M}_2$ independent of (ψ_1, ψ_2) such that

$$\begin{aligned} & \|\psi_1\|_{2+\alpha, Q} + \|\psi_2\|_{2+\alpha, Q} \\ & \leq \bar{M}_2 \left\{ \|v_0\|_{2+\alpha} + \|T_0\|_{2+\alpha} + \sum_{i=1}^2 |f_i|_{\alpha, \bar{Q}} + |\lambda - \lambda_0| (\|v\|_{2+\alpha, \bar{Q}} + \|T\|_{2+\alpha, \bar{Q}}) \right\}. \end{aligned}$$

We consider the following closed subset X of $C^{2+\alpha, 1+\alpha/2}(\bar{Q})$:

$$X := \left\{ (v, T) : \|v, T\|_{2+\alpha, \bar{Q}} \leq 2\bar{M}_2 \left(\|v_0\|_{2+\alpha} + \|T_0\|_{2+\alpha} + \sum_{i=1}^2 |f_i|_{\alpha, \bar{Q}} \right) \right\}.$$

Then H_λ maps X into itself if $|\lambda - \lambda_0| < 1/(2\bar{M}_2)$. Furthermore, if $(v_1, T_1), (v_2, T_2) \in X$, then

$$\|H_\lambda(v_2, T_2) - H_\lambda(v_1, T_1)\|_{2+\alpha, \bar{Q}} \leq \frac{1}{2} \|v_2 - v_1\|_{2+\alpha, \bar{Q}} + \|T_2 - T_1\|_{2+\alpha, \bar{Q}}.$$

It follows from the contraction principle that there exists a unique fixed point $(v, T) = H_\lambda(v, T)$ in X . Thus, $\lambda \in \Omega$ and Ω is open.

Suppose $\lambda \in \bar{\Omega}$. Then there exists a sequence $\lambda_\ell \in \Omega$ such that $\lambda_\ell \rightarrow \lambda$ as $\ell \rightarrow \infty$. By assumption, the equation (64) has a unique solution $(v_\ell, T_\ell) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ for every λ_ℓ . As in (63), we can prove that $\{(v_\ell, T_\ell)\}$ is a Cauchy sequence in $C^{2+\alpha, 1+\alpha/2}(\bar{Q})$. Then there exists a unique (v, T) such that $(v_\ell, T_\ell) \rightarrow (v, T)$ in $C^{2+\alpha, 1+\alpha/2}(\bar{Q})$ as $\ell \rightarrow \infty$. Letting $\ell \rightarrow \infty$ in (64), we have that (v, T) is a unique solution of (64) corresponding to λ . Thus, $\lambda \in \Omega$ and Ω is closed. Therefore, $1 \in \Omega$.

This completes the proof of Theorem 3. $\square$

4.1. Proof of Theorem 1. For the set $\mathbf{F}$ and mapping $(v, T) = \Sigma(\tilde{v}, \tilde{T}, \mu)$ defined by (59) and (60), we can easily verify all conditions required in the Leray-Schauder fixed point theorem by a standard argument. Thus, there exists at least a fixed point (v, T) of $\Sigma(\cdot, \cdot, 1)$ in $\mathbf{F}$. We omit the details of verification of conditions for the Leray-Schauder fixed point theorem. From the regularity property of the solution of linear parabolic equations, we conclude that $(v, T) \in C^{2+\alpha, 1+\alpha/2}(\bar{Q})$. By the equivalence of (1)–(7) and (29)–(32), there exists a solution (ρ, v, T, φ) of (1)–(7), where the electron density ρ and electrostatic potential φ are defined by (24) and (25), respectively.

Now we show the uniqueness. Suppose that there are two solutions

$$(\rho_i, v_i, T_i, \varphi_i), \quad i = 1, 2.$$

Let

$$(\rho, v, T, \varphi) = (\rho_2 - \rho_1, v_2 - v_1, T_2 - T_1, \varphi_2 - \varphi_1).$$

Then (ρ, v, T, φ) satisfies the following initial-boundary value problem:

$$(66) \quad \rho_t + (v_2\rho + \rho_1v)_x = 0,$$

$$(67) \quad \begin{aligned} & \rho_2v_t - \varepsilon v_{xx} + \frac{1}{2}\rho_2((v_2 + v_1)v)_x + \frac{1}{m}(\rho_2T + T_1\rho)_x + \left(v_{1t} + \frac{1}{2}(v_1^2)_x\right)\rho \\ &= \frac{e}{m}(\rho_2\varphi_x + \varphi_1\rho) - \frac{(v_2\rho - \rho_1v)\tau_p^{(1)} - \rho_1v_1(\tau_p^{(2)} - \tau_p^{(1)})}{\tau_p^{(1)}\tau_p^{(2)}}, \end{aligned}$$

$$(68) \quad \begin{aligned} & \rho_2T_t - \frac{2}{3}(\kappa_2T_x)_x - \frac{2}{3}((\kappa_2 - \kappa_1)T_{1x})_x \\ & \quad + T_{1t}\rho + (\rho_2v_2)T_x + (\rho_2T_{1x})v + (v_1T_{1x})\rho \\ & \quad + \frac{2}{3}((\rho_2T_2)v_x + (\rho_2v_{1x})T + (T_1v_{1x})\rho) \\ &= \frac{mv_2^2 + 3(T_1 + T^{(0)})}{3\tau_w^{(1)}\tau_w^{(2)}}(\tau_w^{(2)} - \tau_w^{(1)}) \\ & \quad + \frac{2mv_1^2}{\tau_p^{(1)}\tau_p^{(2)}}(\tau_p^{(2)} - \tau_p^{(1)}) + \frac{m(v_2 + v_1)(2\tau_w^{(1)} - \tau_p^{(1)})}{3\tau_p^{(2)}\tau_2^{(1)}}v - \frac{T}{\tau_w^{(2)}}, \end{aligned}$$

$$(69) \quad \varphi_{xx} = e\rho,$$

$$(70) \quad \rho(x, 0) = v(x, 0) = T(x, 0) = 0, \quad x \in [0, 1],$$

$$v(0, t) = v(1, t) = T_x(0, t) = T_x(1, t) = 0,$$

$$(71) \quad \varphi_x(0, t) = \varphi(1, x) = 0, \quad t \geq 0,$$

where $\kappa_i = \kappa(\rho_i)$, $\tau_p^{(i)} = \tau_p(\rho_i, v_i, T_i)$, and $\tau_w^{(i)} = \tau_w(\rho_i, v_i, T_i)$, $i = 1, 2$.

Multiplying (66), (67), and (68) by ρ , v , and T , respectively, we have

$$(72) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^2 dx = - \int_0^1 \left(\frac{1}{2} v_{2x} \rho^2 + (\rho_1 v)_x \rho \right) dx \\ & \leq \frac{\varepsilon}{6} \int_0^1 v_x^2 dx + N_1 \int_0^1 (\rho^2 + \rho_2 v^2) dx, \end{aligned}$$

$$(73) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \rho_2 v^2 dx + \varepsilon \int_0^1 v_x^2 dx \\ & \leq \frac{\varepsilon}{6} \int_0^1 v_x^2 dx + N_2 \int_0^1 (\rho^2 + \rho_2 (v^2 + T^2)) dx, \end{aligned}$$

$$(74) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \rho_2 T^2 dx + \frac{2}{3} \int_0^1 \kappa_2 T_x^2 dx \\ & \leq \frac{1}{3} \int_0^1 \left(\kappa_2 T_x^2 + \frac{\varepsilon}{2} v_x^2 \right) dx + N_3 \int_0^1 (\rho^2 + \rho_0 (v^2 + T^2)) dx. \end{aligned}$$

where N_i , $i = 1, 2, 3$, are some positive constants depending on M_k , $k = 1, \dots, 8$.

By (72)–(74), we have

$$\frac{d}{dt} \int_0^1 [\rho^2 + \rho_2 (v^2 + T^2)] dx + \int_0^1 \left(\varepsilon v_x^2 + \frac{2\kappa_2}{3} T_x^2 \right) dx \leq N \int_0^1 [\rho^2 + \rho_2 (v^2 + T^2)] dx.$$

It follows from Gronwall's inequality and (70)–(71) that

$$\rho = v = T \equiv 0.$$

And $\varphi \equiv 0$ follows easily from (69) and (71). This completes the proof of Theorem 1. $\square$

5. Asymptotic stability. In the previous section we proved that there exists a unique smooth solution of the initial-boundary problem (1)–(7) for an arbitrary time $S \in (0, \infty)$. Now we can investigate the asymptotic behavior of the

solution of the problem (1)–(7) for the special data $\rho_0(x) = C(x)$, $x \in [0, 1]$. We first consider the following one-dimensional steady-state hydrodynamic equations with small viscosity:

$$(75) \quad (\rho v)' = 0,$$

$$(76) \quad \varepsilon v'' = \rho v v' + \frac{1}{m}(\rho T)' - \frac{e}{m}\rho\varphi' + \frac{\rho v}{\tau_p},$$

$$(77) \quad (\kappa T')' = \frac{3}{2}\rho v T' + \rho T v' - \frac{m\rho v^2(2\tau_w - \tau_p)}{2\tau_p\tau_w} + \frac{3\rho(T - T^{(0)})}{2\tau_w},$$

$$(78) \quad \varphi'' = e(\rho - C(x)),$$

$$(79) \quad v(0) = v(1) = 0, \quad T'(0) = T'(1) = 0,$$

$$(80) \quad \varphi'(0) = 0, \quad \varphi(1) = \varphi_1,$$

where $(\cdot)' = \frac{d}{dx}(\cdot)$.

We also assume that

$$\int_0^1 \rho(x) dx = \int_0^1 C(x) dx.$$

Theorem 4. *There exists a solution*

$$(81) \quad (\rho, v, T, \varphi) = \left(\frac{\int_0^1 C(x) dx}{\int_0^1 \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx} \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right), 0, T^{(0)}, \varphi(x) \right)$$

of the boundary value problem (75)–(80), where $\varphi(x)$ is a solution by solving

$$\begin{cases} \frac{1}{\varepsilon}\varphi''(x) = \frac{\int_0^1 C(x) dx}{\int_0^1 \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx} \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) - C(x), \\ \varphi'(0) = 0, \quad \varphi(1) = \varphi_1. \end{cases}$$

Proof. From (75) and the boundary data (79), we have

$$(\rho v)(x) = 0, \quad x \in [0, 1].$$

Suppose that $\rho(x) > 0$ for $x \in [0, 1]$, then

$$v(x) = 0, x \in [0, 1].$$

The equation (77) then reads:

$$(82) \quad (\kappa T')' = \frac{3\rho(T - T^{(0)})}{2\tau_w}.$$

Multiplying (82) by $T - T^{(0)}$ and using integration by parts, we obtain

$$\int_0^1 \kappa(T')^2 dx + \int_0^1 \frac{3\rho(T - T^{(0)})^2}{2\tau_w} dx = 0.$$

It follows that

$$T(x) = T^{(0)}, \quad x \in [0, 1].$$

The equation (76) then reads:

$$(83) \quad \rho' = \frac{e}{T^{(0)}} \rho \varphi'.$$

The unique solution of (83) is given by

$$(84) \quad \rho(x) = \rho(1) \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right).$$

We claim that $\rho(1)$ is a positive constant depending only on $\varphi(x)$. By (78)–(80) and assumption on $\rho(x)$, we have

$$\varphi'(1) = \varphi'(0) + e\left(\int_0^1 \rho(x) dx - \int_0^1 C(x) dx\right) = 0.$$

Substituting the expression (84) for $\rho(x)$ in (78) and integrating over $(0, 1)$, we have

$$\rho(1) \int_0^1 \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx - \int_0^1 C(x) dx = \varphi'(1) - \varphi'(0) = 0.$$

Thus,

$$\rho(1) = \frac{\int_0^1 C(x) dx}{\int_0^1 \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx} > 0.$$

To this end, we have to show the existence of a solution of the following boundary value problem:

$$(85) \quad \begin{cases} \frac{1}{e} \varphi''(x) = \frac{\int_0^1 C(x) dx}{\int_0^1 \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx} \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) - C(x), \\ \varphi'(0) = 0, \quad \varphi(1) = \varphi_1. \end{cases}$$

For a special case $C(x) = C = \text{const.} > 0$, there exists a trivial solution $\varphi = \varphi_1$ of (85). Then $(\rho, v, T, \varphi) = (C, 0, T^{(0)}, \varphi_1)$ is a trivial solution of (1)–(7), or (75)–(80).

Using the boundary condition, we have

$$(86) \quad |\varphi'(x)| = e \left| \frac{\int_0^x \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx}{\int_0^1 \exp\left(\frac{e(\varphi(x) - \varphi_1)}{T^{(0)}}\right) dx} \int_0^1 C(x) dx - \int_0^x C(x) dx \right|$$

$$\leq 2e \int_0^1 C(x) dx \equiv C_1,$$

$$(87) \quad |\varphi(x)| = \left| \int_1^x \varphi'(x) dx + \varphi_1 \right| \leq C_1 + |\varphi_1| \equiv C_2.$$

We then define the following bounded, closed, and convex subset of $C^0[0, 1]$:

$$X := \{\psi(x) : |\psi'| \leq C_1, |\psi| \leq C_2, \psi'(0) = 0, \psi(1) = \varphi_1\}.$$

We also consider the operator $\Sigma : \psi \rightarrow \varphi$ defined by solving

$$(88) \quad \begin{cases} \frac{1}{e} \varphi''(x) = \frac{\int_0^1 C(x) dx}{\int_0^1 \exp\left(\frac{e(\psi(x) - \varphi_1)}{T^{(0)}}\right) dx} \exp\left(\frac{e(\psi(x) - \varphi_1)}{T^{(0)}}\right) - C(x), \\ \varphi'(0) = 0, \quad \varphi(1) = \varphi_1, \end{cases}$$

where $\psi(x) \in X$.

The linear equation (88) is uniformly elliptic. Thus, the smoothness assumption on $C(x)$ guarantees the existence of a unique solution $\varphi = \Sigma(\psi)$ in $C^2[0, 1]$.

It is easy to check that the solution $\varphi = \Sigma(\psi)$ also satisfies (86)–(87); that is, $\varphi = \Sigma(\psi) \in X$. Thus, $\{\Sigma(\psi)\}$ is compact in $C^0[0, 1]$ by the Ascoli–Arzela theorem. By a standard continuity argument, we conclude the existence of a fixed point of Σ from the Schauder fixed point theorem. This completes the proof of Theorem 4. $\square$

5.1. Proof of Theorem 2. Multiplying (2) by ρv and using integration by parts, we have

$$(89) \quad \int_0^1 \rho v^2 dx + 2 \int_0^t \int_0^1 \left(\varepsilon v_x^2 + \frac{\rho v^2}{\tau_p} \right) dx dt + \frac{2}{m} (I_1 - e I_2) = \int_0^1 \rho_0 v_0^2 dx,$$

where

$$\begin{aligned} I_1 &= \int_0^t \int_0^1 v(\rho T)_x dx dt, \\ I_2 &= \int_0^t \int_0^1 \rho v \varphi_x dx dt. \end{aligned}$$

For I_1 , we have

$$\begin{aligned} I_1 &= - \int_0^t \int_0^1 T(\rho v_x) dx dt \\ &= \int_0^t \int_0^1 T(\rho_t + v \rho_x) dx dt \quad (\text{by (1)}) \\ &= \int_0^t \int_0^1 (\rho T)_t dx dt + \int_0^t \int_0^1 (T v \rho_x - \rho T_t) dx dt \\ &= \int_0^t \int_0^1 \left(v(\rho T)_x + \frac{2}{3} \rho T v_x - \frac{2}{3} (\kappa T_x)_x - \frac{m \rho v^2 (2\tau_w - \tau_p)}{3\tau_p \tau_w} \right. \\ &\quad \left. + \frac{\rho(T - T^{(0)})}{\tau_w} \right) dx dt + \int_0^1 (nT - n_0 T_0) dx \quad (\text{by (3)}) \\ &= \frac{1}{3} I_1 + \int_0^t \int_0^1 \left(\frac{\rho(T - T^{(0)})}{\tau_w} - \frac{m \rho v^2 (2\tau_w - \tau_p)}{3\tau_p \tau_w} \right) dx dt \\ &\quad + \int_0^1 (\rho T - \rho_0 T_0) dx. \end{aligned}$$

Thus,

$$(90) \quad \begin{aligned} \frac{2}{m} I_1 &= \int_0^t \int_0^1 \left(\frac{3\rho(T - T^{(0)})}{m\tau_w} + \frac{\rho v^2}{\tau_w} \right) dx dt \\ &\quad - 2 \int_0^t \int_0^1 \frac{\rho v^2}{\tau_p} dx dt + \frac{3}{m} \int_0^1 (\rho T - \rho_0 T_0) dx. \end{aligned}$$

For I_2 , starting from (1), we have

$$\int_0^x \rho dx = \int_0^x \rho_0 dx - \int_0^t \rho v dt.$$

Thus,

$$(91) \quad \varphi_x = \int_0^x (\rho - C(x)) dx = - \int_0^t \rho v dx \quad (\text{by } \rho_0(x) = C(x)).$$

It follows that

$$(92) \quad \begin{aligned} I_2 &= - \int_0^t \int_0^1 (\rho v) \left(\int_0^t \rho v dt \right) dx dt - \frac{1}{2} \int_0^1 \left(\int_0^t \rho v dt \right)^2 dx \\ &\equiv - \frac{1}{2} f(t). \end{aligned}$$

Substituting (90) and (92) for I_1 and I_2 in (89), we obtain

$$(93) \quad \begin{aligned} &\int_0^1 \left(\frac{3\rho T}{m} + \rho v^2 \right) dx + \frac{e}{m} f(t) \\ &\quad + \int_0^t \int_0^1 \left(2\varepsilon v_x^2 + \frac{m\rho v^2 + 3\rho(T - T^{(0)})}{m\tau_w} \right) dx dt \\ &= \int_0^1 \rho_0 \left(\frac{3T_0}{m} + v_0^2 \right) dx. \end{aligned}$$

It follows from (93) that

$$(94) \quad \begin{aligned} &\int_0^1 \rho v^2 dx \in L^\infty(0, \infty) \cap L^1(0, \infty), \\ &\int_0^1 v_x^2 dx \in L^1(0, \infty), \\ &f(t) = \int_0^1 \left(\int_0^\infty \rho v dt \right)^2 dx \in L^\infty(0, \infty). \end{aligned}$$

Multiplying (3) by ρ/T and using integration by parts, we have the following expression from the left-hand side:

$$\begin{aligned}
 \text{L. H. S.} &= \int_0^1 \rho(\ln T)_t dx + \int_0^1 \rho v(\ln T)_x dx + \frac{2}{3} \int_0^1 (\rho v_x - \frac{1}{T}(\kappa T_x)_x) dx \\
 &= \frac{d}{dt} \int_0^1 \rho \ln T dx + \int_0^1 (\rho v(\ln T)_x - (\ln T)\rho_t) dx + \frac{2}{3} \int_0^1 \rho v_x dx \\
 &\quad - \frac{2}{3} \int_0^1 \kappa \left(\frac{T_x}{T} \right)^2 dx \\
 &= \frac{d}{dt} \int_0^1 \rho \ln T dx + \frac{2}{3} \int_0^1 \rho v_x dx - \frac{2}{3} \int_0^1 \kappa \left(\frac{T_x}{T} \right)^2 dx \quad (\text{by (1)}).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (95) \quad &\int_0^1 \rho \ln T dx + \int_0^t \int_0^1 \left[\frac{\rho(T - T^{(0)})}{T\tau_w} \frac{m\rho v^2(2\tau_w - \tau_p)}{3T\tau_p\tau_w} - \frac{2\kappa}{3} \left(\frac{T_x}{T} \right)^2 \right] dx dt \\
 &= \int_0^1 \rho_0 \ln T_0 dx - \frac{2}{3} \int_0^t \int_0^1 \rho v_x dx.
 \end{aligned}$$

On the other hand, we multiply (3) by $1/T$ and use integration by parts to obtain

$$\begin{aligned}
 \text{L. H. S.} &= \frac{3}{2} \int_0^1 (\rho(\ln \rho)_t + \rho(\ln T)_t) dx + \frac{m}{2} \int_0^1 \left(\frac{v}{T}(\rho v)_t + \frac{\rho v}{T}v_t \right) dx \\
 &\quad + \int_0^1 \frac{1}{T} \left(\frac{5}{2}\rho v T + \frac{1}{2}m\rho v^3 \right)_x dx \\
 &\quad - \int_0^1 \kappa \left(\frac{T_x}{T} \right)^2 dx - \varepsilon m \int_0^1 \frac{v v_{xx}}{T} dx \\
 &= \frac{3}{2} \frac{d}{dt} \int_0^1 \rho(\ln \rho + \ln T) dx - \frac{3}{2} \int_0^1 (\ln \rho + \ln T)\rho_t dx \\
 &\quad - \frac{3}{2} \int_0^1 (\ln T)(\rho v)_x dx + \int_0^1 \rho v_x dx \\
 &\quad - \int_0^1 \left[\kappa \left(\frac{T_x}{T} \right)^2 + \frac{m\rho v^2}{T\tau_p} - \frac{e\rho v}{T}\varphi_x \right] dx \quad (\text{by (2)}) =
 \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{2} \frac{d}{dt} \int_0^1 \rho (\ln \rho + \ln T) dx - \int_0^1 \left[\kappa \left(\frac{T_x}{T} \right)^2 + \frac{m\rho v^2}{T\tau_p} - \frac{e\rho v}{T} \varphi_x \right] dx \\
&\quad + \frac{5}{2} \int_0^1 \rho v_x dx \quad (\text{by (1)}).
\end{aligned}$$

Thus,

$$\begin{aligned}
(96) \quad &\frac{2}{5} \int_0^1 \rho (\ln \rho + \ln T) dx - \frac{4}{15} \int_0^t \int_0^1 \left[\kappa \left(\frac{T_x}{T} \right)^2 + \frac{m\rho v^2}{T\tau_p} \right] dx dt \\
&= \frac{2}{5} \int_0^1 \rho_0 (\ln \rho_0 + \ln T_0) dx - \frac{2}{15} \int_0^t \int_0^1 \frac{3n(T - T^{(0)}) + m\rho v^2}{T\tau_w} dx dt \\
&\quad - \frac{2}{3} \int_0^t \int_0^1 \rho v_x dx dt.
\end{aligned}$$

Subtracting (96) from (95), we obtain

$$\begin{aligned}
(97) \quad &\int_0^1 \frac{1}{m} \left(3\rho \ln T + 2\rho \ln \frac{1}{\rho} \right) dx \\
&\quad - \frac{1}{m} \int_0^t \int_0^1 \left[2\kappa \left(\frac{T_x}{T} \right)^2 + \frac{m\rho v^2(2\tau_w - \tau_p)}{T\tau_p\tau_w} - \frac{3\rho(T - T^{(0)})}{T\tau_w} \right] dx dt \\
&= \int_0^1 \frac{1}{m} \left(3\rho_0 \ln T_0 + 2\rho_0 \ln \frac{1}{\rho_0} \right) dx.
\end{aligned}$$

Subtracting (97) from (93), we obtain the following inequality:

$$\begin{aligned}
&\frac{3}{m} \int_0^1 \rho (T - 1 - \ln T) dx + \frac{2}{m} \int_0^1 \rho \left(\frac{1}{\rho} - 1 - \ln \frac{1}{\rho} \right) dx \\
&\quad + 2 \int_0^t \int_0^1 \left(\frac{\kappa}{m} \left(\frac{T_x}{T} \right)^2 + \varepsilon v_x^2 \right) dx dt + \frac{e}{m} f(t) + \int_0^1 \rho v^2 dx \\
&\quad + \int_0^t \int_0^1 \left[\frac{3\rho(T - T^{(0)})}{m\tau_w} + \frac{\rho v^2}{\tau_w} \left(1 + \frac{2\tau_w - \tau_p}{T\tau_p} \right) \right] dx dt \\
&= \int_0^1 \frac{\rho_0}{m} [3(T_0 - \ln T_0) + 2 \ln \rho_0 + m v_0^2 - 5] dx + \int_0^t \int_0^1 \frac{3\rho(T - T^{(0)})}{mT\tau_w} dx dt
\end{aligned}$$

$$\begin{aligned} \leq & \frac{1}{m} \int_0^1 \rho_0 [3(T_0 + |\ln T_0|) + 2|\ln \rho_0| + mv_0^2 + 5] dx \\ & + \frac{1}{T^{(0)}} \int_0^1 \rho_0 \left(\frac{3}{m} T_0 + v_0^2 \right) dx \quad (\text{by (93)}). \end{aligned}$$

It follows that

$$(98) \quad \int_0^1 \left(\frac{T_x}{T} \right)^2 dx \in L^1(0, \infty).$$

From boundary value conditions, we have

$$\begin{aligned} \frac{d}{dt} \int_0^1 v_x^2 dx &= -2 \int_0^1 v_{xx} v_t dx, \\ \frac{d}{dt} \int_0^1 (\ln T)_x^2 &= -2 \int_0^1 \left(\frac{T_{xx}}{T^2} - \frac{T_x^2}{T^3} \right) T_t dx. \end{aligned}$$

Then, it follows from Lemma 6 that

$$\int_0^\infty \left(\left| \frac{d}{dt} \|v_x\|_{L^2(0,1)}^2(t) \right| + \left| \frac{d}{dt} \|(\ln T)_x\|_{L^2(0,1)}^2(t) \right| \right) dt \leq C.$$

Thus, $\|v_x\|_{L^2}^2(t)$ and $\|(\ln T)_x\|_{L^2}^2(t)$ are uniformly continuous on $[0, \infty)$. (94) and (98) imply that

$$\begin{aligned} \|v_x\|_{L^2(0,1)}^2(t) &\rightarrow 0, \quad t \rightarrow \infty, \\ \|(\ln T)_x\|_{L^2(0,1)}^2(t) &\rightarrow 0, \quad t \rightarrow \infty, \end{aligned}$$

Therefore,

$$\begin{aligned} v^2(x, t) &= \left(\int_0^x v_x dx \right)^2 \leq \|v_x\|_{L^2(0,1)}^2(t) \rightarrow 0, \quad t \rightarrow \infty, \\ \left(\ln \frac{T}{T^{(0)}} \right)^2(x, t) &= \left(\int_0^x (\ln T)_x dx \right)^2 \leq \|(\ln T)_x\|_{L^2(0,1)}^2(t) \rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

Finally, (8) and (11) follow easily from the integrations of the equations (1) and (91), respectively.

This completes the proof of Theorem 2. $\square$

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Department of Mathematics
Stanford University
Stanford, California 94305

Received: April 10th, 1995.