A Two-Piece Property for Compact Minimal Surfaces in a Three-Sphere

ANTONIO ROS

ABSTRACT. We show that any 2-equator in a 3-sphere divides each embedded compact minimal surface in two connected pieces. We also shall see that closed regions in the sphere with mean convex boundary containing a nullhomologous great circle are the intersection of two closed halvespheres. As application of these results we prove that the normal surface of an embedded minimal torus is also embedded. Finally we show that the Clifford torus is the only embedded minimal torus which admits certain symmetries.

Introduction. In this paper we consider embedded compact minimal surfaces in the unit three sphere. Known examples give a finite number of such surfaces for any orientable fixed topology; see Lawson [L] and Karcher, Pinkall and Sterling [KPS]. An argument of Hopf shows that the only genus zero surface is the totally geodesic equator; see [L]. Frankel [F] showed that any two surfaces of this type must intersect (in particular, they are always connected) and Choi and Schoen [CS] have shown that for any fixed genus, the space of those surfaces is compact in the C^k topology, $k \geq 1$. The equators have Morse index equal to one and Urbano [U] has shown that, for the other surfaces, the smallest Morse index is five and corresponds just to the Clifford torus. Among known open problems for this kind of surfaces we mention the following (see for instance [Y]):

- (i) *For any given genus, show that there are only finitely many noncongruent surfaces.* This would hold, in particular, if any Jacobi field on a surface in our family would come from a Killing field of the ambient sphere.
- (ii) (Lawson) *Show that the only torus is the Clifford torus.*
- (iii) (Yau) *Show that for this kind of surface the first nonzero eigenvalue of the Laplacian is equal to 2.* We remark that the linear functions in $\mathbb{R}^4$ restrict to our surfaces as eigenfunctions with eigenvalue equal to 2.

It is proved by Choi and Wang [CW] that this eigenvalue is always bigger than one. Montiel and Ros [MR] have shown that Yau's conjecture implies Lawson's.

Another geometric consequence of conjecture (iii) is, by using Courant nodal arguments [C], that *any equator of the 3-sphere divides embedded compact minimal surfaces in exactly two open connected pieces*. One of the purposes of this paper is to give a direct proof of this fact.

Solomon [S] has shown that a branched compact minimal surface in a 3-sphere either meets or links each geodesic circle. We will also obtain a stronger version of this result, which states that *any mean convex closed region in the 3-sphere that contains a nulhomologous geodesic circle, is the intersection of two closed halvespheres. In particular, if the region has smooth boundary, it must be a halvesphere*.

Finally, we show two results about minimal tori: given such a surface, it is known that its normal surface (defined by the unit normal vector in S^3 of the first surface) is an immersed conformal minimal surface; see [L]. We will prove that *given an embedded minimal torus in S^3 , its normal surface is again embedded*. The second result is related to a uniqueness result for harmonic maps obtained by Ritorè [R]. We shall show that *the Clifford torus is the only embedded minimal torus in S^3 that is symmetric with respect to four pairwise othogonal hyperplanes in $\mathbb{R}^4$* .

The theorems. Let $S^3 = \{p/|p| = 1\} \subset \mathbb{R}^4$ be the three dimensional unit sphere. For any unit vector $v \in S^3$ we consider the totally geodesic equator $S^2(v) = \{p \in S^3 / \langle v, p \rangle = 0\}$ and the half spheres $H_+(V) = \{p \in S^3 / \langle v, p \rangle > 0\}$ and $H_-(v) = \{p \in S^3 / \langle v, p \rangle < 0\}$ bounded by $S^2(v)$.

Given a compact minimal surface M embedded in S^3 , other than an equator, it is a simple fact that M cannot be contained in a closed halvesphere. More strongly, as linear functions of M are eigenfunctions of its Laplacian, if we consider the intersection $\Gamma(v) = M \cap S^2(v)$, it is known that $\Gamma(v)$ is a finite nonempty embedded graph with smooth edges and that, at each vertex, a positive even number of edges meet in a equiangular way; see for example Cheng [C].

Lemma 1. *Let $\psi : M \longrightarrow S^3$ be a compact orientable immersed stable minimal surface with piecewise smooth boundary. Assume that there is a vector $v \in S^3$ such that $\psi(\partial M) \subset S^2(v)$. Then M is totally geodesic.*

Proof. The stability of M means that for any sufficiently smooth function u on M with $u = 0$ at ∂M , we have that

$$-\int_M \{u\Delta u + (2 + |\sigma|^2)u^2\} dA \geq 0,$$

where Δ is the Laplacian of M and $|\sigma|$ is the length of its second fundamental form (this is equivalent to saying that ψ minimizes area up to the second order). Hence we can consider this integral inequality for the linear function $u(p) = \langle v, p \rangle$. It is well known that, M being minimal, this function verifies the equation $\Delta u + 2u = 0$ and, thus, we get

$$-\int_{\Sigma} |\sigma|^2 u^2 dA \geq 0.$$

If ψ were not totally geodesic (i.e., $|\sigma| > 0$) around some point, we could conclude that $u = 0$ in this neighbourhood and, so, the immersion is a piece of an equator. This contradiction shows the lemma. $\square$

Remark. The lemma above extends to orientable stable minimal surfaces $\psi : M \rightarrow S^3$ with finite area and such that, for any $\varepsilon > 0$, $M_\varepsilon = \psi^{-1}([\varepsilon, \infty))$ is a compact subset of $\text{int}(M)$. To get that, use the test function $u_\varepsilon = |\langle v, \psi \rangle| - \varepsilon$ (on M_ε), $u_\varepsilon = 0$ (on $M - M_\varepsilon$) in the stability integral inequality and take limits when ε goes to zero.

Theorem 2. *Let M be a compact minimal surface embedded in S^3 . Then, for any $v \in S^3$, $M \cap H_+(v)$ and $M \cap H_-(v)$ are connected.*

Proof. If M is an equator, the statement above is trivial. In the following we exclude this case. Assume, reasoning by contradiction, that, for some v , $M \cap H_+(v)$ is a disjoint union of two nonempty open subsurfaces M_1 and M_2 , M_1 being connected. The boundary $\Gamma = \partial M_1$ is a nonempty subgraph of $\Gamma(v)$ that is not necessarily connected. Denote by W and W' the closures of the two components of $S^3 - M$. They are compact domains with mean convex boundary. Consider a compact surface Σ in W that minimizes area subject to the restriction $\partial \Sigma = \Gamma$; see for instance [M] and [MY]. Thus Σ is an embedded stable minimal surface, free of singularities out of the vertices of Γ . Moreover, by the maximum principle, we get that any component of Σ is either contained in ∂W , or meets ∂W only at points of Γ . In the first case the component would be orientable and, since we are assuming that M is not totally geodesic, this would contradict Lemma 1. As only the latter case is possible, we conclude that $M_1 \cup \Sigma$ encloses a region in W . Hence, Σ is orientable and Lemma 1 (or the remark after this lemma if you want to avoid possible singularities at the vertices) gives that Σ is totally geodesic, i.e. any component of Σ is a piece of some equator. From elementary geometry of spheres, any component of Σ which is not bounded by a geodesic circle of S^3 is contained in $S^2(v)$. If some component of Σ were bounded by a great circle $S^1 \subset \Gamma$, then the theorem below would imply that M is an equator of S^3 , which is not the case. Thus $\Sigma \subset S^2(v) \cap W$ and $\Sigma \cap M = \Gamma$.

Changing W by W' , we can construct another compact subsurface Σ' of $S^2(v)$ with $\partial\Sigma' = \Gamma$ and such that $\Sigma' \subset W'$ and $\Sigma' \cap M = \Gamma$. Note that $\Sigma \cup \Sigma'$ is a subsurface without boundary of $S^2(v)$ and, so, $\Sigma \cup \Sigma' = S^2(v)$. In particular, $M \cap S^2(v) = \Gamma$, which implies that $M_2 = M \cap H_+(v) - M_1$ has no boundary points and, so, M is not connected. As this is impossible (see [F]), the theorem is proved. $\square$

Solomon [S] has proven that if a great circle $S^1 \subset S^3$ does not meet a branched compact minimal surface M , then M and S^1 link. In particular, this result applies to the normal surface of M , which is again a branched minimal surface in S^3 . The following theorem is a generalization of this result. We consider domains W that are piecewise smooth and mean convex (i.e., with nonnegative mean curvature at ∂W in the sense of [MY]). The argument of Frankel in [F] implies that in this case ∂W is connected.

Theorem 3. *Let $W \subset S^3$ be a connected closed region with (nonempty) mean convex boundary. If W contains a great circle S^1 , and S^1 is nullhomologous in W , then W is the intersection of two closed halfspheres and $S^1 \subset \partial W$. In particular, if ∂W is smooth, then W is a closed halfsphere.*

Remark. This theorem extends the result in [S] because if M is a branched compact minimal surface in S^3 , then the closure of each component of $S^3 - M$ has mean convex boundary.

Proof. By moving continuously the great circle, if necessary, we can assume that $S^1 \cap \partial W$ is nonempty. As W has mean convex boundary, there exists a compact surface $\Sigma \subset W$, with $\partial\Sigma = S^1$, which minimizes area among compact surfaces in W bounded by S^1 , [M] and [MY]. The surface Σ is embedded, minimal and stable and either $\Sigma \subset \partial W$ or $\Sigma \cap \partial W \subset S^1$. If $\Sigma \cap \partial W \subset S^1$, then rotating Σ around S^1 (in the two possible directions) until the last time it remains in W (these last time exist because S^1 meets ∂W) it follows from the (interior and boundary version of the) maximum principle for surfaces with nonnegative mean curvature that the last rotated images of Σ are both contained in ∂W . In particular, Σ must be orientable and so, using Lemma 1, Σ is a half equator. As the union of the rotated images of Σ is a surface without boundary it must coincide with the whole ∂W (recall that ∂W is connected). Now the theorem follows directly.

The assertion (iii) below is similar to a result obtained by Hoffman and Meeks [HM] for proper disjoint minimal surfaces in $\mathbb{R}^3$.

Corollary 4. *Let $M \subset \bar{H}_+(v) = \{p \in S^3 / \langle p, v \rangle \geq 0\}$ be a compact embedded minimal surface with C^2 boundary $\partial M \subset S^2(v)$ and $\text{int}(M) \subset H_+(v)$, for some $v \in S^3$. Then:*

- (i) *Area $(M) \geq 2\pi$ and the equality holds only if M is a half-equator.*
- (ii) *If ∂M lies in some closed half-equator of $S^2(v)$, M is a half-equator.*
- (iii) *If M' is another surface in the same hypothesis that M (for the same v) and $\text{int}(M) \cap \text{int}(M') = \emptyset$, it follows that M and M' are two embedded half-equators both bounded by the same great circles $S^1 \subset S^2(v)$.*

Proof.

- (i) The closure W and W' of each one of the two components of $\bar{H}_+(v) - M$ is a mean convex region. If M is not a half-equator, then the arguments above show that the compact least area surface Σ (resp. Σ') in W (resp. W') bounded by ∂M is $\Sigma = W \cap S^2(v)$ (resp. $\Sigma' = W' \cap S^2(v)$). In particular, $\text{Area}(M) > \text{Area } \Sigma$, $\text{Area}(\Sigma')$ and the assertion follows from the equalities $\Sigma \cup \Sigma' = S^2(v)$, $\Sigma \cap \Sigma' = \Gamma$ and $\text{Area}(S^2(v)) = 4\pi$.
- (ii) With the notation of (i), either W or W' contains a closed half-equator and, so, there is a nulhomologous great circle in one of these regions. Thus, the assertion follows directly from Theorem 3.
- (iii) The closure W'' of the connected component of $\bar{H}_+(v) - M \cup M'$ that contains M and M' at its boundary is a mean convex region. The surface Σ of least area in W'' with $\partial \Sigma = \partial M$ separates M and M' and, so, Σ is not contained in $S^2(v)$. As in the proof of Theorem 3, we show that Σ must be a half-equator. As $\partial \Sigma \subset W''$ is a great circle, we conclude the proof applying Theorem 3 to W'' . $\square$

Remark. Concerning embedded orientable compact minimal surfaces in S^3 with nonempty boundary, the following results are known: (i) Hardt and Simon [HS] show that the only ones that bound a great circle are the half equators, and (ii) Hardt and Rosenberg [HR] prove that if the boundary consists in a pair of great circles lying in two orthogonal planes, then the surface must be a piece of a Clifford torus.

If $\psi : M \rightarrow S^3$ is an immersed oriented minimal surface, then the *normal surface* $\psi^\perp : M \rightarrow S^3$, which applies each point into its unit normal vector, viewed as a vector in $\mathbb{R}^4$, is a conformal branched minimal immersion and $(\psi^\perp)^\perp = \psi$. The branch points of $\psi^\perp$ coincide with the umbilic points of ψ and, thus, if M is a torus, its normal surface is an unbranched minimal surface; see [L].

Theorem 5. *Let $M \subset S^3$ be an embedded compact minimal torus. Then its normal surface is also embedded.*

Proof. Suppose that two different points $p_1, p_2 \in M$ have the same normal vector $\psi^\perp(p_1) = \psi^\perp(p_2) = v$. Thus, the equator $S^2(v)$ meets M tangentially at the points p_1 and p_2 . Moreover, as minimal tori have no umbilical points (see [L]), around p_i , $i=1,2$, $\Gamma = M \cap S^2(v)$ consists in a pair of smooth curves that meet orthogonally at p_i . Let M_1 and M_2 be the two components (according to Theorem 1) of $M - \Gamma$. The following standard argument allows to control the topology of Γ and M_i , $i=1,2$. These surfaces have piecewise smooth boundary and from the Gauss-Bonnet theorem we get

$$\int_{M_i} K dA = 2\pi\chi(M_i) - \int_{\partial M_i} k_g ds - \sum_{j_i} \vartheta_{j_i},$$

where k and k_g are the Gauss curvature and the geodesic curvature, respectively, and ϑ_{j_i} denote the external angles of ∂M_i . Summing up the equations above, the geodesic curvature terms cancel and we obtain, using again the Gauss-Bonnet theorem on the whole torus,

$$0 = \int_M K dA = 2\pi \sum_i \chi(M_i) - \sum_j \vartheta_j,$$

where the last term is the sum of all the external angles of $M - \Gamma$. Recall that the Euler characteristic of a surface with non empty boundary is at most 1 and that the equality holds only for the disk. Each vertex p_i contributes exactly 2π to the sum of external angles and thus

$$4\pi \geq 2\pi \sum_i \chi(M_i) = \sum_j \vartheta_j \geq 4\pi.$$

Consequently, the surfaces M_i , $i = 1,2$, are disks, and $\Gamma \subset S^2(v)$ is a graph with vertices just at p_1 and p_2 . Around each vertex Γ consists in a pair of embedded smooth curves that meet orthogonally.

To show that this situation is topologically impossible, we take stereographic projection $S^3 - p_3 \rightarrow \mathbb{R}^3$ from a point p_3 of $S^2(v) - \Gamma$. Thus we obtain a torus M' embedded in $\mathbb{R}^3$ and a plane P (the image of $S^2(v) - p_3$), which meet in a graph $\Gamma' = M' \cap P$ with exactly two vertices p'_1 and p'_2 , the intersection being transversal outside of the vertices. Around each of these points Γ' looks like two orthogonal straitlines, and $M' - \Gamma'$ has two connected components that are both disks. If n' denotes the Gauss map of M' , then the equalities $\psi^\perp(p_1) = \psi^\perp(p_2) = v$ transform into $n'(p'_1) = n'(p'_2) = w$, where w is a unit vector normal to P . Take $\varepsilon > 0$ small enough so that the closed slab $P(\varepsilon) \subset \mathbb{R}^3$ of points whose distance to P is at most ε cut the surface M' in a piece $M'(\varepsilon) = M' \cap P(\varepsilon)$ with smooth boundary verifying the following restrictions:

- (i) The only points of $M'(\varepsilon)$ where the Gauss map n' takes the values $\pm w$ are p'_1 and p'_2 (this is possible because the Gauss curvature of M' at the points P'_i is non zero).
- (ii) $M'(\varepsilon)$ meets (transversally) each one of the two components of $\partial P(\varepsilon)$ just in a Jordan curve (we can obtain that because the part of M' which lies at each side of P is a disk). In particular, $M'(\varepsilon)$ has genus one.

Let $M'' \subset \mathbb{R}^3$ be a smooth embedded compact surface without boundary obtained by glueing smoothly $M'(\varepsilon)$ with a pair of caps (disks) in $\mathbb{R}^3 - P(\varepsilon)$ along $\partial M'(\varepsilon)$. We can also assume that each one of these caps has only one point where the tangent plane is normal to the vector w and that the Gauss curvature at those points is positive. Clearly, they will be the maximum and the minimum on M'' of the height function $p \rightarrow \langle p, w \rangle$, and, thus, the Gauss map n'' of M'' takes the value w at one of the points and $-w$ at the other. In particular, there is only one point in M'' that applies to $-w$ by n'' , and the Jacobian at this point is positive. It follows that n'' has degree one and, as M'' is a torus, this contradicts the Gauss-Bonnet theorem. So, the theorem is proved. $\square$

Finally, we use an argument in [R] to give a characterization of the Clifford torus.

Theorem 6. *Let $M \subset S^3$ be an embedded minimal torus that is symmetric with respect to four pairwise orthogonal hyperplanes of $\mathbb{R}^4$. Then M is the Clifford torus.*

Assume that these hyperplanes are the coordinate hyperplanes $P_i = \{x_i = 0\}$, $i = 1, \dots, 4$. In M we consider the closed domain $M_{i+} = M \cap \{x_i \geq 0\}$ and $x_i, n_i : M_{i+} \rightarrow \mathbb{R}$, where x_i is the i -th coordinate function of the position vector and n_i is the i -th linear function of the unit normal vector of M , $n = (n_1, \dots, n_4)$. From the minimality of M we get

$$(1) \quad \Delta x_i + 2x_i = 0 \quad \text{and} \quad \Delta n_i + |\sigma|^2 n_i = 0.$$

As x_i does not change sign on M_{i+} , we conclude that the first eigenvalue of the Laplacian, for the Dirichlet boundary condition, on M_{i+} is equal to 2. The symmetry of M implies the symmetry of the normal vector N and, in particular, using the embeddedness of M , we get that $n_i = 0$ at the boundary of M_{i+} . As a consequence, we have

$$(2) \quad \int_{M_{i+}} |\sigma|^2 n_i^2 dA = - \int n_i \Delta n_i dA \geq 2 \int n_i^2 dA.$$

The argument above also works in $M_{i-} = M \cap \{x_i \leq 0\}$ and, summing up these eight integral inequalities, and using that n is a unit vector, we get

$$(3) \quad \int_M |\sigma|^2 dA \geq 2 \text{Area}(M).$$

The Gauss formula gives $2K = 2 - |\sigma|^2$, K being the Gauss curvature of M . Hence the Gauss-Bonnet theorem implies that inequality (3) is in fact an equality and, thus, the equality holds in the eight inequalities in (2). This means that n_i is a first eigenfunction of M_{i+} and M_{i-} , i.e. $\Delta n_i + 2n_i = 0$ on M , $i = 1, \dots, 4$, which, together with (1), gives us $|\sigma|^2 = 2$ on M . Then M is a flat torus and, as a rule, the only minimal surface of this kind is the Clifford torus.

REFERENCES

- [C] S. Y. CHENG, *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), 43-55.
- [CS] H. CHOI & R. SCHOEN, *The space of minimal embeddings of a surface into a three dimensional manifold of positive Ricci curvature*, Int. Math. **81** (1985), 387-394.
- [CW] H. CHOI & A. N. WANG, *A first eigenvalue estimate for minimal hypersurfaces*, J. Differential Geom. **18**, 559-562.
- [F] T. FRANKEL, *On the fundamental group of a compact minimal submanifold*, Ann. of Math. **83** (1966), 68-73.
- [HM] D. HOFFMAN & W. H. MEEKS III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990), 373-377.
- [HR] R. HARDT & H. ROSENBERG, *Open book structures and unicity of minimal submanifolds*, Ann. Inst. Fourier, Grenoble **40** (1990), 701-708.
- [HS] R. HARDT & L. SIMON, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Ann. of Math. **110**, 439-486.
- [KPS] H. KARCHER, U. PINKALL & I. STERLING, *New minimal surfaces in S^3* , J. Differential Geom. **28** (1988), 169-185.
- [L] H. B. LAWSON, *Complete minimal surfaces in S^3* , Ann. of Math. **92** (1970), 335-374.
- [MY] W. H. MEEKS III & T. S. YAU, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. **179** (1982), 151-168.
- [MR] S. MONTIEL & A. ROS, *Minimal immersions of surfaces by the first eigenfunctions and conformal area*, Invent Math. **83** (1985), 153-166.
- [M] F. MORGAN, *Geometric Measure Theory (a beginner's guide)*, Academic Press, London, 1988.
- [R] M. RITORÉ, *Examples of constant mean curvature surfaces obtained from harmonic maps to the two sphere*, Preprint.
- [S] B. SOLOMON, *Harmonic maps to spheres*, J. Differential Geom. **21**, 151-162.
- [U] F. URBANO, *Minimal surfaces with low index in the three-dimensional sphere*, Proc. Amer. Math. Soc. **108** (1990), 988-992.
- [Y] S. T. YAU, *Problem section. Seminar on Differential Geometry*, Ann. Math. Stud. **102** (1982).

This work was partially supported by DGICYT grant # PB94-0796.

Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, SPAIN

Received: April 10th, 1995.